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Geometric quasi-isometric embeddings into Thompson's group F

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Abstract. We use geometric techniques to investigate several examples of quasi-isometrically embedded subgroups of Thompson's group F. Many of these are explored using the metric properties of the shift map ϕ in F. These subgroups have simple geometric but complicated algebraic descriptions. We present them to illustrate the intricate geometry of Thompson's group F as well as the interplay between its standard finite and infinite presentations. These subgroups include those of the form $F^m \times \mathbb{Z}^n$, for integral $m, n \geq 0$, which were shown to occur as quasi-isometrically embedded subgroups by Burillo and Guba and Sapir.

Contents

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1. Introduction

Bridson asked the question of whether a quasi-isometry exists between Thompson's group F and the group $F \times \mathbb{Z}$. Burillo [3] provides an example of a quasi-isometric embedding of $F \times \mathbb{Z}$ into F as possible evidence towards a negative answer to this question. While investigating this question, we came across some interesting examples of quasi-isometric embeddings into F which we describe below. These quasi-isometric embeddings all have simple geometric interpretations, which are often easier to express than the corresponding algebraic or group theoretic definitions. We present these examples to illustrate the beautiful geometry evident in Thompson's group F. Our examples are based on the interaction between the finite and infinite presentations of F and the representation of elements of F as pairs of binary rooted trees. These embeddings use shift maps of F and provide concrete geometric realizations of subgroups of F of the form $F^m \times \mathbb{Z}^n$ which are known to be quasi-isometrically embedded by work of Burillo [3] and Guba and Sapir [7, 8].

2. Thompson's group *F*

Thompson's group F has both finite and infinite presentations; it is usually presented finitely as

$$\mathcal{F} = \langle x_0, x_1 \mid [x_0x_1^{-1}, x_0^{-1}x_1x_0], [x_0x_1^{-1}, x_0^{-2}x_1x_0^2] \rangle$$

and infinitely as

$$\mathcal{P} = \langle x_k,\ k \geq 0 \mid x_i^{-1}x_jx_i = x_{j+1} \ \text{ if } i < j \rangle.$$

The infinite presentation provides a set of normal forms for elements of F. Namely, each $w \in F$ can be written $x_{i_1}^{r_1}x_{i_2}^{r_2}\dots x_{i_k}^{r_k}x_{j_l}^{-s_l}\dots x_{j_2}^{-s_2}x_{j_1}^{-s_1}$ where $r_i, s_i > 0$, and $i_1 < i_2 \dots < i_k$ and $j_1 < j_2 \dots < j_l$. To obtain a unique normal form for each element, we add the condition that when both x_i and x_i^{-1} occur, so does x_{i+1} or x_{i+1}^{-1}, as discussed by Brown and Geoghegan [2]. We will always mean unique normal form when we refer to a word w in normal form. We give a brief introduction to F below; for a more detailed and comprehensive description, we refer the reader to Cannon, Floyd and Parry [4] and Fordham [6].

Elements of F can be thought of equivalently in three different forms: in either presentation above, as piecewise-linear homeomorphisms of $[0, 1]$ whose break points are dyadic rationals and whose slopes are powers of two, and as pairs of finite rooted binary trees, each with the same number of exposed leaves. For the equivalence of these representations, we refer the reader to Cannon, Floyd and Parry [4]. We choose this last interpretation for work below, and begin with some basic vocabulary related to these trees.

Let T be a rooted binary tree. An *exposed leaf* of T ends in a vertex of valence 1, and the exposed leaves are numbered from left to right, beginning with 0. A node together with its two leaves is called a *caret*. A caret C may have a *right child*; that is, a caret which is attached to the right leaf of C. We can similarly define the *left child* of the caret C. In a pair (T_-, T_+) of rooted binary trees, the tree T_- is called the *negative tree* and T_+ the *positive tree*.

A pair of trees (T_-, T_+) is *unreduced* if both T_- and T_+ contain a caret with two exposed leaves numbered m and $m + 1$. There are many tree pair diagrams representing the same element of F but each element has a unique reduced tree

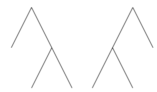

FIGURE 1. The tree pair diagram for the generator x_0 of F.

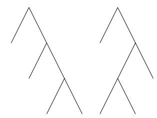

FIGURE 2. The tree pair diagram for the generator x_1 of F.

pair diagram representing it. When we write (T_-, T_+) to represent an element of F, we are assuming that the tree pair is reduced. The tree pair diagrams for the generators x_0 and x_1 are given in Figures 1 and 2.

Group multiplication of $w = (T_-, T_+)$ and $v = (S_-, S_+)$ is accomplished by creating temporary unreduced representatives (T'_-, T'_+) of w and (S'_-, S'_+) of v in which $T'_+ = S'_-$. The product wv is defined to be the tree pair (T'_-, S'_+) which may be unreduced. This method is used to compute the distance with respect to the word metric induced by the standard finite generating set $\{x_0, x_1\}$ between elements w and v, namely, $d(w, v) = |w^{-1}v|$.

2.1. Tree pair diagrams and the normal form. There is a bijective correspondence between the tree pair diagrams described above and the normal form of an element. The *leaf exponent* of an exposed leaf numbered n in T_- or T_+ is defined to be the length of the maximal path consisting entirely of left edges from n which does not reach the right side of the tree, and is written $E(n)$. Note that $E(n) = 0$ for an exposed leaf labelled n which is a right leaf of a caret, as there is no path consisting entirely of left edges originating from n. In Figure 3, the exponents of the left-hand tree T_- are as follows: $E(0) = 1, E(1) = 0, E(2) = E(3) = 1$ and $E(4) = E(5) = 0$.

Once the exponents of the leaves in T_- and T_+ have been computed, the normal form of the element $w = (T_-, T_+)$ is easily obtained. The positive part of the normal form of w is

$$x_0^{E(0)} x_1^{E(1)} \cdots x_m^{E(m)}$$

where m is the number of exposed leaves in either tree, and the exponents are obtained from the leaves of T_+. The negative part of the normal form of w is similarly found to be

$$x_m^{-E(m)} x_{m-1}^{-E(m-1)} \cdots x_0^{-E(0)}$$

where the exponents are now computed from the leaves of T_-. Note that many of the exponents in the normal form as given above may be zero.

2.2. Word length in $\mathcal{F}$. Fordham [6] presents a remarkable method of calculating the word length of an element $w \in F$ with respect to the standard finite generating set $\{x_0, x_1\}$ based solely on the trees representing w. In [5] we prove the following theorem, which follows from Fordham's result, and approximates the word length of $w \in F$ using only the number of carets in either tree of the reduced tree pair diagram representing w.

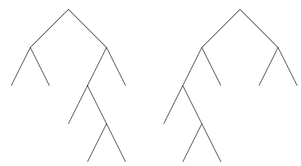

Figure 3. Tree pair diagram for the element $w = x_0^2 x_1 x_3^{-1} x_2^{-1} x_0^{-1}$.

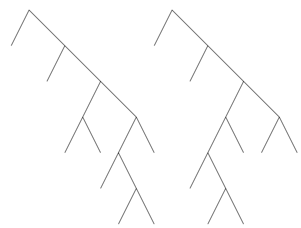

Figure 4. Tree pair diagram for $\phi^2(w) = x_2^2 x_3 x_5^{-1} x_4^{-1} x_2^{-1}$, where the word $w = x_0^2 x_1 x_3^{-1} x_2^{-1} x_0^{-1}$ is depicted in Figure 3.

Theorem 2.1 ([5] Theorem 3.1)**.** *Let $w = (T_-, T_+)$ and $N(w)$ be the number of carets in T_-. Then*

$$N(w) - 2 \leq |w| \leq 4N(w) - 4.$$

3. Quasi-isometric embeddings

Let X and Y be metric spaces, and $K \geq 1$ and $C \geq 0$ be constants. A (K, C)-*quasi-isometric embedding* $f : X \to Y$ is a map satisfying the following property:

$$\frac{1}{K} d_X(x, y) - C \leq d_Y(f(x), f(y)) \leq K d_X(x, y) + C$$

for all $x, y \in X$, where d_X (resp. d_Y) represents the metric in X (resp. Y). When considering a quasi-isometric embedding between groups, we use the word metric on each group induced by a particular set of generators. A quasi-isometric embedding is a *quasi-isometry* if there is a constant C' so that $\mathrm{Nbhd}_{C'}(f(X)) = Y$.

Quasi-isometries need not be continuous maps, nor are they required to preserve any algebraic structure. Below we give several examples of quasi-isometric embeddings of $F^m \times \mathbb{Z}^n$ into F, for nonnegative integers m and n, which, though homomorphisms, are algebraically cumbersome but can easily be described geometrically in terms of tree pair diagrams. We call these *geometric* quasi-isometric embeddings.

3.1. The shift map ϕ. We begin with an example of a quasi-isometric embedding $F \to F$ which is easily understood either algebraically or geometrically. There is a *shift map* $\phi : F \to F$ defined on Thompson's group in the infinite presentation $\mathcal{P}$ which increases the index of each generator by 1. For example, if $w = x_3^2 x_5 x_{13} x_{10}^{-1} x_9^{-4}$, then $\phi(w) = x_4^2 x_6 x_{14} x_{11}^{-1} x_{10}^{-4}$. Thus if w is in normal form, so is $\phi(w)$. We show that any power of ϕ is a quasi-isometric embedding.

Theorem 3.1. *Any positive integral power $\phi^n : F \to F$ of the shift map $\phi : F \to F$ is a quasi-isometric embedding.*

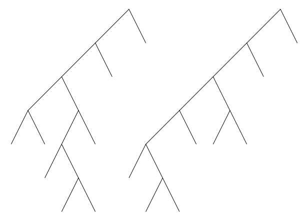

Figure 5. Tree pair diagram for $\psi^2(w) = x_0^4 x_1 x_4 x_3^{-1} x_2^{-2} x_0^{-3}$, where the word $w = x_0^2 x_1 x_3^{-1} x_2^{-1} x_0^{-1}$ is depicted in Figure 3.

Proof. Let $w = (T_-, T_+)$. Using the notion of leaf exponents discussed in §2.1, we see that if $\phi(w) = (S_-, S_+)$, then $S_\pm$ is the tree composed of a root caret with $T_\pm$ as the right subtree of the root caret. Thus if $v, w \in F$, the tree pair diagram representing the product $\phi(w)^{-1}\phi(v)$ has one more caret in each tree than the tree pair diagram representing the product $w^{-1}v$. Applying Theorem 2.1, we obtain

$$\frac{1}{4}|w^{-1}v| \leq |\phi(w)^{-1}\phi(v)| \leq 4|w^{-1}v| + 8.$$

Thus ϕ is a quasi-isometric embedding, and it follows that ϕ^n is as well. □

3.2. The reverse shift map ψ. Given the simple geometric representation of the shift map ϕ mentioned in the proof of Theorem 3.1, it is natural to define a "reverse shift map", which we denote ψ, in which the original trees become left subtrees of the root caret rather than right subtrees. See, for example, Figure 5, which depicts the tree pair diagram for $\psi^2(w)$. Geometrically, it is completely natural to consider ψ as well as ϕ. However, in the literature, ψ is rarely mentioned, likely because it is algebraically cumbersome, unlike ϕ, in the following way. If $w \in F$ is written in normal form, then the normal form of $\phi(w)$ is easily determined, whereas for ψ the situation is quite different. Let $w = (T_-, T_+)$ and $\psi(w) = (S_-, S_+)$. Then leaves with exponent 0 which are the exposed left leaves of right carets in T_- or T_+ will have leaf exponent 1 in S_- and S_+, respectively, and thus cause new generators to appear in the normal form of $\psi(w)$. It is difficult to predict solely from the normal form of w which additional generators will appear in the normal form of $\psi(w)$. For example if $w = x_0^2 x_1 x_5^2 x_4^{-1} x_3^{-1} x_1^{-1} x_0^{-1}$ then $\psi(w) = x_0^3 x_1 x_4 x_5^3 x_7^{-1} x_6^{-1} x_4^{-1} x_3^{-2} x_1^{-1} x_0^{-1}$.

Unsurprisingly from the geometric point of view, we obtain a theorem for ψ analogous to Theorem 3.1 for ϕ.

Theorem 3.2. *Any integral power $\psi^n : F \to F$ of the reverse shift map $\psi : F \to F$ defined above is a quasi-isometric embedding.*

Proof. We express ψ geometrically in terms of the tree pair diagrams using the shift map ϕ and the outer automorphism α of F, which performs a vertical reflection on each tree pair diagram. Brin [1] showed α to be the unique outer automorphism of F. It is easily seen that α maps the generators x_0 and x_1 of F to an alternate generating set $\{x_0^{-1}, x_0 x_1 x_0^{-2}\}$ of F. Thus α is a quasi-isometry of F. We note that the composition $\alpha \circ \phi \circ \alpha$ is exactly the reverse shift map ψ. Since α is a quasi-isometry and ϕ is a quasi-isometric embedding, the composition is again a quasi-isometric embedding, as is ψ^n for integral n. □

We note that Theorem 3.2 can be proven identically to Theorem 3.1, since the tree pair diagram representing $\psi(w)^{-1}\psi(v)$ will again have one more caret in each tree than the tree pair diagram representing $w^{-1}v$.

3.3. Clone subgroups of *F*. The maps ϕ^n and ψ^n are shown to be quasi-isometric embeddings because an element and its image are represented by tree pair diagrams which differ by a finite number of carets. This idea can be generalized as follows. Let p denote the composition

$$p = f_n \circ f_{n-1} \circ \cdots \circ f_1$$

where each f_i is either ϕ or ψ. A *clone subgroup* is the image of F under p. The simplest examples of clone subgroups are $\phi^n(F)$ and $\psi^n(F)$. Clone subgroups can be understood in a number of equivalent ways.

Let p be as above, and consider the clone subgroup $p(F)$. We can describe this subgroup using a binary address. A node in a binary tree is given an address using the following inductive method. The root node has the empty label. Given a node with label s, the left child of the node is labelled $s0$ and right child of the node labelled $s1$. For example, the right child of the right child of the left child of the right child of the root has address 1011.

Given p as above, and $w = (T_-, T_+) \in F$, let $p(w) = (S_-, S_+)$. From the definitions of ϕ, ψ and p, we know that the tree T_- will be a subtree of S_-, and T_+ will be a subtree of S_+. We consider the address $s = \epsilon_n \dots \epsilon_2\epsilon_1$ of the root caret of T_- as a subtree of S_- and use s to give an "address" for the clone subgroup $p(F)$. Namely, we can uniquely refer to $p(F)$ as C_s. For example, the subgroup C_{1011} is the image $\phi(\psi(\phi^2(F)))$.

We can also describe clone subgroups by their representation as piecewise linear homeomorphisms of the unit interval. Each dyadic subinterval of the form $[\frac{i}{2^n}, \frac{i+1}{2^n}]$ is affinely equivalent to the standard unit interval $[0,1]$ by an affine map with dyadic coefficients. For a fixed clone subgroup C_s where s has length n, there is a dyadic subinterval $I' \subseteq [0,1]$ which contains the x-coordinates of all the breakpoints of elements of C_s. The length of I' is 2^{-n} and the endpoints of $I'$are $\frac{i}{2^n}$ and $\frac{i+1}{2^n}$. The endpoints can be computed easily from the address s. Furthermore, any element of F whose breakpoints all lie in the interval $[\frac{i}{2^n}, \frac{i+1}{2^n}]$ will be an element of C_s. For example, in the subgroup C_{1011}, the dyadic interval containing the all breakpoints of its elements is $[\frac{11}{16}, \frac{3}{4}]$

It is easy to see that any clone subgroup $C_s = p(F)$ is isomorphic to F. If $w = (T_-, T_+)$ and $p(w) = (S_-, S_+)$, we know by definition that the tree pair (T_-, T_+) is reduced. From the definitions of ϕ and ψ, and thus p, no additional reduction can occur when the tree pair (S_-, S_+) is formed. Thus each element of F produces a unique element of C_s, and it is clear that $p(x_0)$ and $p(x_1)$ must generate C_s.

The following theorem is a consequence of Theorems 3.1 and 3.2.

Theorem 3.3. *Any clone subgroup of F is quasi-isometrically embedded.*

Proof. The clone subgroup C_s is determined by $p = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$ where each f_i is either the map ϕ or ψ. We know from Theorems 3.1 and 3.2 that ϕ and ψ are quasi-isometric embeddings into F; it immediately follows that the above composition is also a quasi-isometric embedding. □

Theorem 3.3 can be proved geometrically by noting that if $p = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$ as above, the trees representing $w = (T_-, T_+)$ and $p(w) = (S_-, S_+)$ differ by n carets. Theorem 2.1 can then be applied to show that p is a quasi-isometric embedding, and obtain the quasi-isometry constants.

3.4. Quasi-isometrically embedded products. We now show that F contains a family of quasi-isometrically embedded subgroups of the form $F^m \times \mathbb{Z}^n$ for $n, m \geq 0$. These examples include those of Burillo [3], who presents a family of subgroups of F of the form $F \times \mathbb{Z}^n$ which are quasi-isometrically embedded. Guba and Sapir [7, 8], using the approach of diagram groups, also show that $F^m \times \mathbb{Z}^n$ occur as quasi-isometrically embedded subgroups of F and furthermore show that all abelian subgroups and centralizers of elements are quasi-isometrically embedded. We present an alternate geometric proof that some concrete geometric realizations of the subgroups $F^m \times \mathbb{Z}^n$ are quasi-isometrically embedded, using the shift maps described above.

Theorem 3.4 ([3, 7, 8])**.** *For each integral pair $m, n \geq 0$, Thompson's group F contains an infinite family of quasi-isometrically embedded subgroups of the form $F^m \times \mathbb{Z}^n$.*

We first note that using Theorem 3.3 above and the clone subgroups C_0 and C_1, it is easy to see that $F \times F$ quasi-isometrically embeds in F. Theorem 3.4 is an immediate consequence of this fact and Burillo's examples of quasi-isometrically embedded subgroups of the form $F \times \mathbb{Z}^n$.

These quasi-isometrically embedded $F^n \times \mathbb{Z}^m$ subgroups are viewed geometrically as follows. Let $s_1, s_2, \cdots s_m, s_{m+1}$ be binary addresses of nodes in a binary rooted tree, subject to the condition that pairwise, no s_i is a prefix of s_j. Then for each i with $1 \leq i \leq m$, we have that C_{s_i} is a clone subgroup of F, and by construction, no two of these clone subgroups intersect in a nonidentity element. Thus the product $C_{s_1} \times \cdots \times C_{s_m}$ is isomorphic to F^m, and is quasi-isometrically embedded.

Let p_{m+1} be the product of maps ϕ and ψ corresponding to the binary address s_{m+1}. In the final clone subgroup $C_{m+1} = p_{m+1}(F)$, we produce a quasi-isometrically embedded copy of Z^m. Burillo [3] provides generators for the quasi-isometrically embedded $\mathbb{Z}^m$ which he describes. The image of these generators under the map p_{m+1} will produce a quasi-isometrically embedded copy of $\mathbb{Z}^m$ inside of C_{m+1}. Since the clone subgroups $C_1, \cdots, C_{m+1}$ are distinct, we have produced the required quasi-isometrically embedded copy of $F^n \times \mathbb{Z}^m$.

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