

On unit-regular ideals

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ABSTRACT. In this paper we introduce the notion of unit-regular ideals for unital rings, which is a natural generalization of unit-regular rings. It is shown that every square matrix over unit-regular ideals admits a diagonal reduction. We also prove that a regular ideal of a unital ring is unit-regular if and only if pseudo-similarity via the ideal is similarity.

Let I be an ideal of a unital ring R . We say that I is regular in case for every $x \in I$ there exists $y \in I$ such that $x = xyx$. Following Goodearl [7], a unital ring R is unit-regular provided that for every $x \in R$ there exists $u \in U(R)$ such that $x = xux$. Unit-regular rings play an important role in the structure theory of regular rings. In this paper we introduce the notion of unit-regular ideals for unital rings, which is a natural generalization of unit-regular rings. We say that an ideal I of a unital ring R is unit-regular in case for every $x \in I$, there exists $u \in U(R)$ such that $x = xux$.

Let D be a division ring, V a countably generated infinite dimensional vector space over D . Let $I = \{x \in \text{End}_D V \mid \dim_D(xV) < \infty\}$. Clearly, I is an ideal of $\text{End}_D V$. Given any $x \in I$, we have right D -module split exact sequences $0 \rightarrow \text{Ker } x \rightarrow V \rightarrow xV \rightarrow 0$ and $0 \rightarrow xV \rightarrow V \rightarrow V/xV \rightarrow 0$. Then $V \cong xV \oplus \text{Ker } x \cong V/xV \oplus xV$; hence, $\dim_D(\text{Ker } x) = \dim_D(V/xV) = \infty$ because $\dim_D(xV) < \infty$. By [5, Corollary], $x \in \text{End}_D(V)$ is unit-regular. Therefore I is a unit-regular ideal of $\text{End}_D(V)$, while $\text{End}_D(V)$ is not a unit-regular ring by [5, Corollary]. This shows that the notion of unit-regular ideal is a nontrivial generalization of unit-regularity for regular rings.

An $m \times n$ matrix A over a unital ring R is called to admit a diagonal reduction if there exist $P \in \text{GL}_m(R)$ and $Q \in \text{GL}_n(R)$ such that PAQ is a diagonal matrix. It is well-known that every square matrix over unit-regular rings admits a diagonal reduction by invertible matrices (cf. [9, Theorem 3]). But Henriksen's method can not be extend to unit-regular ideals. P. Ara et al. have extended this result to separative exchange rings (cf. [1, Theorem 2.4]). Let D be a division ring, V an infinite dimensional vector space over D . Set $R = \text{End}_D(V)$. Then R is one-sided unit-regular, so it is a separative regular ring. Given any $A \in M_n(R)$, by [1, Theorem 2.5], A admits a diagonal reduction. So we can find $U, V \in \text{GL}_n(R)$ such that $UAV = \text{diag}(r_1, \dots, r_n)$. Assume now that all $r_i \in R$ are idempotents.

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Let $E = \text{diag}(r_1, \dots, r_n)$. Then $A = U^{-1}EV^{-1}$, whence $AVUA = A$. That is, $M_n(R)$ is unit-regular. This shows that R is unit-regular, a contradiction. This infers that there exists some square matrix over R which doesn't admit a diagonal reduction with idempotent entries. In other words, we may not reduce some square matrices over unit-regular ideals to diagonal matrices with idempotent entries by Ara's technique. In this paper, we will prove that every square matrix over unit-regular ideals admits a diagonal reduction with idempotent entries. We also prove that a regular ideal of a unital ring is unit-regular if and only if pseudo-similarity via the ideal is similarity, which give a nontrivial generalization of [8, Theorem].

Throughout, all rings are associative with identity and all modules are right modules. $U(R)$ denotes the set of all units of R and $GL_n(R)$ denotes the general linear group of R . The notation $FP(I)$ stands for the set of all finitely generated projective right R -modules P such that $P = PI$.

Lemma 1. *Let I be a regular ideal of a unital ring R . Then the following are equivalent:*

- (1) I is unit-regular.
- (2) If $aR + bR = R$ with $a \in I$, then there exists $y \in R$ such that $a + by \in U(R)$.
- (3) If $Ra + Rb = R$ with $a \in I$, then there exists $z \in R$ such that $a + zb \in U(R)$.

Proof. (1) \Rightarrow (2) Suppose that $aR + bR = R$ with $a \in I$. Then $ax + bz = 1$ for some $x, z \in R$. Since $a \in I$, we have $u \in U(R)$ such that $a = aua$. Set $au = e$. Then $e \in R$ is an idempotent. Furthermore, we have $eu^{-1}x + bz = 1$; hence $e + bz(1 - e) = 1 - eu^{-1}x(1 - e) \in U(R)$. Let $y = z(1 - e)u^{-1}$. We see that $a + by = (1 - eu^{-1}x(1 - e))u^{-1} \in U(R)$, as asserted.

(2) \Rightarrow (1) Given any $x \in I$, we have $y \in R$ such that $x = xyx$. From $xy + (1 - xy) = 1$, we have $z \in R$ such that $x + (1 - xy)z \in U(R)$. By [6, Lemma 3.1], we have $s \in R$ such that $y + s(1 - xy) = u \in U(R)$. Therefore $x = xyx = x(y + s(1 - xy))x = vux$, as required.

(1) \Leftrightarrow (3) By symmetry, we get the result. □

For any $\alpha, \beta, a, b \in R$, we set

$$[\alpha, \beta] = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad B_{12}(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad B_{21}(b) = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}.$$

In [4, Proposition 2], the first author and F. Li showed that ideal-stable range conditions are invariant under matrix extensions. Now we give an analogue for unit-regular ideals.

Theorem 2. *Let I be a unit-regular ideal of a unital ring R . Then $M_n(I)$ is a unit-regular ideal of $M_n(R)$.*

Proof. Let I be a unit-regular ideal of a unital ring R . By [2, Lemma 2], $M_n(I)$ is a regular ideal of $M_n(R)$. Suppose that $AX + B = I_n$ with $A = (a_{ij}) \in M_n(I)$ and $X = (x_{ij}), B = (b_{ij}) \in M_n(R)$. Then $\begin{pmatrix} A & B \\ -I_n & X \end{pmatrix} = \begin{pmatrix} X & XA - I_n \\ I_n & A \end{pmatrix}^{-1} \in GL_2(M_n(R))$. Since $a_{11}R + \dots + a_{1n}R + b_{11}R + \dots + b_{1n}R = R$ with $a_{11} \in I$, by Lemma 1, we can find $y_2, \dots, y_n, z_1, \dots, z_n \in R$ such that

$$a_{11} + a_{12}y_2 + \dots + a_{1n}y_n + b_{11}z_1 + \dots + b_{1n}z_n = u_1 \in U(R).$$

Thus

$$\begin{pmatrix} A & B \\ -I_n & X \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0}_{1 \times (2n-1)} \\ y_2 & \\ \vdots & \\ y_n & \\ z_1 & \mathbf{I}_{2n-1} \\ \vdots & \\ z_n & \end{pmatrix} = \begin{pmatrix} u_1 & a_{12} & \dots & a_{1n} & b_{11} & \dots & b_{1n} \\ a'_{21} & a_{22} & \dots & a_{2n} & b_{21} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & & \vdots & \ddots & \vdots \\ ** & a_{n2} & \dots & a_{nn} & b_{n1} & \dots & b_{nn} \\ ** & 0 & \dots & 0 & x_{11} & \dots & x_{1n} \\ \vdots & \vdots & \ddots & & \vdots & \ddots & \vdots \\ ** & 0 & \dots & -1 & x_{n1} & \dots & x_{nn} \end{pmatrix};$$

hence,

$$\begin{pmatrix} * & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} A & B \\ -I_n & X \end{pmatrix} \begin{pmatrix} * & 0 \\ ** & I_n \end{pmatrix} = \begin{pmatrix} u_1 & a'_{12} & \dots & a'_{1n} & b'_{11} & \dots & b'_{1n} \\ 0 & a'_{22} & \dots & a'_{2n} & b'_{21} & \dots & b'_{2n} \\ \vdots & \vdots & \ddots & & \vdots & \ddots & \vdots \\ 0 & a'_{n2} & \dots & a'_{nn} & b'_{n1} & \dots & b'_{nn} \\ ** & 0 & \dots & 0 & x_{11} & \dots & x_{1n} \\ \vdots & \vdots & \ddots & & \vdots & \ddots & \vdots \\ ** & 0 & \dots & -1 & x_{n1} & \dots & x_{nn} \end{pmatrix},$$

where $a'_{22} = a_{22} - a'_{21}u_1^{-1}a_{12} \in I$. Analogously, we claim that

$$\begin{aligned} & [* , *] \begin{pmatrix} A & B \\ -I_n & X \end{pmatrix} B_{21}(*)[* , *] \\ & = \begin{pmatrix} u_1 & a_{12}^{(n)} & \dots & a_{1n}^{(n)} & b_{11}^{(n)} & \dots & b_{1n}^{(n)} \\ 0 & u_2 & \dots & a_{2n}^{(n)} & b_{21}^{(n)} & \dots & b_{2n}^{(n)} \\ \vdots & \vdots & \ddots & & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_n & b_{n1}^{(n)} & \dots & b_{nn}^{(n)} \\ ** & * & \dots & * & x_{11} & \dots & x_{1n} \\ \vdots & \vdots & \ddots & & \vdots & \ddots & \vdots \\ ** & * & \dots & * & x_{n1} & \dots & x_{nn} \end{pmatrix} = [* , *] B_{21}(*) B_{12}(*), \end{aligned}$$

where $u_1, u_2, \dots, u_n \in U(R)$. So $\begin{pmatrix} A & B \\ -I_n & X \end{pmatrix} = [* , *]B_{21}[*]B_{12}[*]B_{21}[*]$; and then $\begin{pmatrix} A & B \\ -I_n & X \end{pmatrix} B_{21}(Y) = [* , *]B_{21}[*]B_{12}[*]$ for a $Y \in M_n(R)$. This implies that $A + BY \in \text{GL}_n(R)$. It follows by Lemma 1 that $M_n(I)$ is unit-regular. \square

Corollary 3. *Let I be a unit-regular ideal of a unital ring R . Then every square matrix over I is a product of an idempotent matrix and an invertible matrix.*

Proof. Let $A \in M_n(I)$. In view of Theorem 2, there exists $U \in \text{GL}_n(R)$ such that $A = AUA$. Set $E = AU$. Then $E = E^2$ and $A = EU^{-1}$, as asserted. \square

Lemma 4. *Let I be a unit-regular ideal of a unital ring R . Suppose that $a, b \in I$. Then the following hold:*

- (1) *If $aR = bR$, then there exists $u \in U(R)$ such that $a = bu$.*
- (2) *If $Ra = Rb$, then there exists $u \in U(R)$ such that $a = ub$.*

Proof. Suppose that $aR = bR$ with $a, b \in I$. Then we have $x, y \in R$ such that $ax = b$ and $a = by$. Assume that $a = aa'a$. Replacing $a'a$ with x , we may assume that $x \in I$. Likewise, we may assume that $y \in I$. Obviously, $b = ax = byx$. From $yx + (1 - yx) = 1$, we have $z \in R$ such that $y + (1 - yx)z = u \in U(R)$ by Lemma 1. As a result, we get $a = by = b(y + (1 - yx)z) = bu$. The second statement is proved by the symmetry. \square

Theorem 5. *Let I be a regular ideal of a unital ring R . Then the following are equivalent:*

- (1) *I is unit-regular.*
- (2) *If $aR \cong bR$ with $a, b \in I$, then there exist $u, v \in U(R)$ such that $a = ubv$.*

Proof. (1) \Rightarrow (2) Suppose that $\psi : aR \cong bR$ with $a, b \in I$. Clearly, $\psi(a)R = bR$. Because of the regularity of I , we have an idempotent $e \in R$ such that $bR = eR$. Hence $\psi(a)R = eR$. This infers that $\psi(a) \in R$ is regular as well. So we can find $c \in R$ such that $\psi(a) = \psi(a)c\psi(a) = \psi(ac\psi(a))$. It follows that $a = ac\psi(a) \in R\psi(a)$, whence $Ra \subseteq R\psi(a)$. Inasmuch as $a \in I$ is regular, we have $a = ada$ for some $d \in R$. This implies that $\psi(a) = \psi(ada) = \psi(a)da \in Ra$; hence, $R\psi(a) \subseteq Ra$. So we see that $Ra = R\psi(a)$. Clearly, $\psi(a) \in I$. In view of Lemma 4, there exist $u, v \in U(R)$ such that $\psi(a) = ua$ and $b = \psi(a)v$. Therefore we conclude that $b = uav$.

(2) \Rightarrow (1) Given any $x \in I$, there exists $y \in R$ such that $x = xyx$. Set $e = xy$. Then we have $xR = eR$ with $x, e \in I$, so there are $u, v \in U(R)$ such that $x = uev$. We easily check that $x = x(v^{-1}u^{-1})x$, as required. \square

Lemma 6. *Let I be a regular ideal of a unital ring R . If $P \in \text{FP}(I)$, then there exist idempotents $e_1, \dots, e_n \in I$ such that $P \cong e_1R \oplus \dots \oplus e_nR$.*

Proof. Suppose that $P \in \text{FP}(I)$. Then we have a right R -module Q such that $P \oplus Q \cong nR$ for some $n \in \mathbb{N}$. Let $e : nR \rightarrow P$ be the projection onto P . Then $P \cong e(nR)$, whence $\text{End}_R(P) \cong eM_n(R)e$. Inasmuch as $P = PI$, we have $e(nR) = e(nR)I \subseteq nI$. Set $e = (\alpha_1, \dots, \alpha_n) \in M_n(R)$. We have $e(1, 0, \dots, 0)^T \in nI$. Hence $\alpha_1 \in nI$. Likewise, we have $\alpha_2, \dots, \alpha_n \in nI$. Therefore $e \in M_n(I)$. Since I is a regular ideal of R , by [2, Lemma 2], $M_n(I)$ is also regular. One directly checks

that $\text{End}_R(P)$ is a regular ring, hence an exchange ring. Thus P has the finitely exchange property. Set $M = P \oplus Q$. Then we have $M = P \oplus Q = \bigoplus_{i=1}^n R_i$ with all $R_i \cong R$. By the finite exchange property of P , we have $Q_i (1 \leq i \leq n)$ such that $M = P \oplus \left(\bigoplus_{i=1}^n Q_i\right)$, where all Q_i are direct summands of R_i respectively. Assume that $Q_i \oplus P_i = R_i$ for all i . Then $P \oplus \left(\bigoplus_{i=1}^n Q_i\right) = \left(\bigoplus_{i=1}^n P_i\right) \oplus \left(\bigoplus_{i=1}^n Q_i\right)$. Hence $P \cong P_1 \oplus \dots \oplus P_n$, where P_i is isomorphic to a direct summand of R as a right R -module for all i . So we have idempotents e_i such that $P_i \cong e_i R$. Clearly, $e_i R$ is a finitely generated projective right R -module. It follows from $P = PI$ that $P \otimes_R (R/I) = 0$; hence, $P_i \otimes_R (R/I) = 0$. That is, $(e_i R) \otimes_R (R/I) = 0$, so $e_i R = e_i R I \subseteq I$. Furthermore, we have $e_i \in I$ for all i . Therefore $P \cong e_1 R \oplus \dots \oplus e_n R$ with all $e_i \in I$. \square

Theorem 7. *Let I be a unit-regular ideal of a unital ring R . Then for any $A \in M_n(I)$, there exist invertible matrices $P, Q \in M_n(R)$ such that*

$$PAQ = \text{diag}(e_1, \dots, e_n)$$

for some idempotents $e_1, \dots, e_n \in I$.

Proof. Since I is a unit-regular ideal of R , $M_n(I)$ is a unit-regular ideal of $M_n(R)$ by Theorem 2. Given any $A \in M_n(I)$, we have $B \in \text{GL}_n(R)$ such that $A = ABA$. Set $E = AB$. Then $E = E^2 \in M_n(I)$ and $AM_n(R) = EM_n(R)$. Clearly, $ER^n \in \text{FP}(I)$. From Lemma 6, we can find idempotents $e_1, \dots, e_n \in I$ such that $ER^n \cong e_1 R \oplus \dots \oplus e_n R \cong \text{diag}(e_1, \dots, e_n)R^n$ as right R -modules. Hence $ER^{n \times 1} \cong \text{diag}(e_1, \dots, e_n)R^{n \times 1}$, where $R^{n \times 1}$ consisting of all n -column vectors over R is a right R -module and a left $M_n(R)$ -module. Let $R^{1 \times n} = \{(x_1, \dots, x_n) \mid x_i \in R\}$. Then $R^{1 \times n}$ is a left R -module and a right $M_n(R)$ -module. One checks that $(ER^{n \times 1}) \otimes_R R^{1 \times n} \cong (\text{diag}(e_1, \dots, e_n)R^{n \times 1}) \otimes_R R^{1 \times n}$. In addition, $R^{n \times 1} \otimes_R R^{1 \times n} \cong M_n(R)$ as right $M_n(R)$ -modules. Thus,

$$AM_n(R) = EM_n(R) \cong \text{diag}(e_1, \dots, e_n)M_n(R).$$

According to Theorem 5, we have invertible matrices $P, Q \in M_n(R)$ such that $PAQ = \text{diag}(e_1, \dots, e_n)$, as asserted. \square

Let I be an ideal of a unital ring R . We say that I has stable range one provided that $aR + bR = R$ with $a \in 1 + I, b \in R$ implies that $a + by \in U(R)$ for a $y \in R$. It is well known that I having stable range one depends only on the ring structure of I and not on the ambient ring R . Let I and J be regular ideals of a unital ring R . If I has stable range one, then $I + J$ is unit-regular if and only if so is J .

Corollary 8. *Let R be a regular, right self-injective ring, and let $A \in M_n(R)$. If $AM_n(R)$ is directly finite, then there exist invertible matrices $P, Q \in M_n(R)$ such that $PAQ = \text{diag}(e_1, \dots, e_n)$ for some idempotents $e_1, \dots, e_n \in R$.*

Proof. Let $I = \{x \in R \mid xR \text{ is a directly finite right } R\text{-module}\}$. In view of [7, Corollary 9.21], I is an ideal of R . Given any idempotent $e \in I$, we know from [7, Corollary 9.3 and Theorem 9.17] that eRe is unit-regular; hence, I has stable range

one. This infers that I is unit-regular. Inasmuch as $AM_n(R)$ is directly finite, we deduce that $A \in M_n(I)$. Therefore we complete the proof by Theorem 7. \square

Let R be a unital ring, and let $A \in M_n(R)$. If $M_n(R)AM_n(R)$ is a unit-regular ideal of $M_n(R)$, we claim that there exist invertible matrices $P, Q \in M_n(R)$ such that $PAQ = \text{diag}(e_1, \dots, e_n)$ for some idempotents $e_1, \dots, e_n \in R$. Since $M_n(R)AM_n(R)$ is a unit-regular ideal of $M_n(R)$, we have an ideal J of R such that $M_n(J) = M_n(R)AM_n(R)$. Hence $A \in M_n(J)$. Clearly, J is a regular ideal of R ; hence, A is a regular matrix over J . Analogously to Theorem 7, the result follows. We say that a is pseudo-similar to b via I provided that there exist $x, y, z \in I$ such that $xay = b, zbx = a$ and $xyx = xzx = x$. We denote it by $a \approx b$ via I . Note that if $eR \cong fR$ for idempotents $e, f \in I$ then $e \approx f$ via I , where I is an ideal of R .

Lemma 9. *Let I be an ideal of a unital ring R . Then the following are equivalent:*

- (1) $a \approx b$ via I .
- (2) *There exist some $x, y \in I$ such that $a = xby, b = yax, x = xyx$ and $y = yxy$.*

Proof. (2) \Rightarrow (1) is trivial.

(1) \Rightarrow (2) As $a \approx b$ via I , there are $x, y, z \in I$ such that $b = xay, zbx = a$ and $x = xyx = xzx$. Then $xa(yxy) = xzbx(yxy) = xzb(xyxy) = xzbxy = xay = b$. Analogously, $(zxx)bx = a$. By replacing y with yxy and z with zxx , we can assume $y = yxy$ and $z = zxx$. Furthermore, we directly check that $xazxy = xzbxxzy = xzbxy = xay = b, zxybx = zxyxayx = zxyax = zbx = a, zxy = zxyxzy$ and $x = xzxyx$, thus yielding the result. \square

Theorem 10. *Let I be a regular ideal of a unital ring R . Then the following are equivalent:*

- (1) I is unit-regular.
- (2) *Whenever $a \approx b$ via I , there exists $u \in U(R)$ such that $a = ubu^{-1}$.*

Proof. (1) \Rightarrow (2) Suppose that $a \approx b$ via I . According to Lemma 9, there exist $x, y \in I$ such that $a = xby, b = yax, x = xyx$ and $y = yxy$. Since I is unit-regular, we have $v \in U(R)$ such that $y = yvy$. Let $u = (1 - xy - vy)v(1 - yx - yv)$. It is easy to verify that $(1 - xy - vy)^2 = 1 = (1 - yx - yv)^2$; hence, $u \in U(R)$. In addition, we have $au = a(1 - xy - vy)v(1 - yx - yv) = -av(1 - yx - yv) = -av + ax + av = ax$. Likewise, we have $xb = ub$. Clearly, $ax = xbyx = xyaxyx = xyax = xb$. Therefore $au = ub$, as required.

(2) \Rightarrow (1) Given any $x \in I$, there exists $y \in R$ such that $x = xyx$. Clearly, $\psi : (xy)R \cong (yx)R$ with idempotents $xy, yx \in I$. Hence $xy \approx yx$ via I , so we have $u \in U(R)$ such that $1 - xy = u(1 - yx)u^{-1}$. Set $a = (1 - xy)u(1 - yx)$ and $b = (1 - yx)u^{-1}(1 - xy)$. Then $1 - xy = ab$ and $1 - yx = ba$. Thus $\phi : (1 - xy)R \cong (1 - yx)R$. Define $u \in \text{End}_R(R)$ so that u restricts to $\psi : xR = (xy)R \cong (yx)R$ and u restricts to $\phi : (1 - xy)R \cong (1 - yx)R$. It is easy to verify that $x = xux$, as asserted. \square

Let $A, B \in M_n(R)$. If $M_n(R)AM_n(R) + M_n(R)BM_n(R)$ is a unit-regular ideal of $M_n(R)$, by Theorem 10 we deduce that $A \approx B$ if and only if there exists some $U \in \text{GL}_n(R)$ such that $A = UBU^{-1}$.

Corollary 11. *Let I be a regular ideal of a unital ring R . Then the following are equivalent:*

- (1) I is unit-regular.
- (2) Whenever $R = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1, A_2 \in \text{FP}(I)$ and $A_1 \cong A_2$, we have $B_1 \cong B_2$.
- (3) Whenever $aR \cong bR$ with $a, b \in I$, we have $R/aR \cong R/bR$.

Proof. (1) \Rightarrow (2) Suppose that $R = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1, A_2 \in \text{FP}(I)$ and $A_1 \cong A_2$. Then we have idempotents $e, f \in I$ such that $eR \cong fR$, and whence $e \approx f$ via I . By Theorem 10, there exists $u \in U(R)$ such that $e = ufu^{-1}$. Hence $1 - e = u(1 - f)u^{-1}$. Set $a = (1 - e)u(1 - f)$ and $b = (1 - f)u^{-1}(1 - e)$. Then $1 - e = ab$ and $1 - f = ba$. Therefore we get $B_1 \cong (1 - e)R \cong (1 - f)R \cong B_2$.

(2) \Rightarrow (3) Suppose that $aR \cong bR$ with $a, b \in I$. Since I is regular, we have idempotents $e, f \in I$ such that $aR = eR$ and $bR = fR$. Hence $R/aR \cong (1 - e)R \cong (1 - f)R \cong R/bR$.

(3) \Rightarrow (1) Given idempotents $e, f \in I$ such that $eR \cong fR$, then $(1 - e)R \cong R/eR \cong R/fR \cong (1 - f)R$. Analogously to Theorem 10, we complete the proof. \square

Recall that an ideal I of a unital ring is of bounded index if there is a positive integer n such that $x^n = 0$ for any nilpotent $x \in I$.

Corollary 12. *Every regular ideal of bounded index is unit-regular.*

Proof. Let R be a unital ring with a regular ideal I of bounded index. Suppose that $R = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1, A_2 \in \text{FP}(I)$ and $A_1 \cong A_2$. Then we have an idempotent $e \in I$ such that $A_1 \cong eR \cong A_2$. Since $\text{End}_R(eR) \cong eRe$ is a regular ring of bounded index, by [7, Corollary 7.11], it is unit-regular. Therefore we get $B_1 \cong B_2$ from [7, Proposition 4.13]. It follows from Corollary 11 that I is a unit-regular ideal of R . \square

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References

- [1] P. Ara, K.R. Goodearl, K.C. O'Meara and E.Pardo, *Diagonalization of matrices over regular rings*, Linear Algebra Appl., **265**(1997), 147–163, MR 1466896, Zbl 0883.15006.
- [2] B. Brown, N.H. McCoy, *The maximal regular ideal of a ring*, Proc. Amer. Math. Soc., **1**(1950), 165–171, MR 0034757, Zbl 0036.29702.
- [3] H. Chen, *On exchange QB-rings*, Comm. Algebra, **31**(2003), 831–841, MR 1968926.
- [4] H. Chen and F. Li, *Exchange rings having ideal-stable range one*, Sci. China (Series A), **44**(2001), 579–586, MR 1843753, Zbl 1004.19002.
- [5] G. Ehrlich, *Units and one-sided units in regular rings*, Trans. Amer. Math. Soc., **216** (1976), 81–90, MR 0387340, Zbl 0298.16012.
- [6] K.R. Goodearl, *Cancellation of low-rank vector bundles*, Pacific J. Math., **113**(1984), 289–302, MR 0749537, Zbl 0493.55007.
- [7] K.R. Goodearl, *Von Neumann regular rings*, Pitman, London, San Francisco, Melbourne, 1979; second ed., Krieger, Malabar, Fl., 1991, MR 1150975, Zbl 0749.16001.
- [8] R. Guralnick, C. Lanski, *Pseudosimilarity and cancellation of modules*, Linear Algebra and Appl., **47**(1982), 111–115, MR 0672734, Zbl 0493.15010.
- [9] M. Henriksen, *On a class of regular rings that are elementary divisor rings*, Arch. Math., **24**(1973), 133–141, MR 0379574, Zbl 0257.16015.

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