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Partial differential analogs of ordinary differential equations and systems

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Dedicated to Professor Zbigniew Ciesielski on the occasion of his 70th birthday

Abstract. In this paper we develop a multivariate setting which includes natural partial differential analogs of the well-known first-order normal system of ordinary differential equations and the nth order normal differential equation. The multivariate counterparts of both of the above examples are overdetermined normal systems of PDEs: a first-order one, which is the well-known Pfaff system of PDEs and a higher order (HO) system of PDEs, with a single unknown function, which was introduced by the authors (see Hakopian and Tonoyan, 1998b and 2002b). We generalize the well-known results, including the connection of equivalence which is rather unexpected for the theory of PDEs. Among other generalized results are the method of variation of constants, fundamental set of solutions, Wronskian, Liouville's formula. The equivalence, in view of the existence and uniqueness result of the Pfaff system, yields a similar result for the HO system.

Also the linear constant coefficient cases and an algebraic system, arising from the multivariate characteristic polynomials, are studied. This algebraic system is a multivariate counterpart of the univariate polynomial equation. Interestingly a multivariate analog of the fundamental theorem of algebra (FTA) holds for the algebraic system (see Hakopian and Tonoyan, 1998a, 2000 and 2002b, and Hakopian, 2003a), which allows us to find a fundamental set of solutions of the HO systems of PDEs, similarly to the univariate case.

Contents

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1. Introduction: The Pfaff system of PDEs and the Frobenius theorem

In this paper we consider two systems of PDEs, first order and higher order (HO), which are multivariate analogs of the first-order normal system of ordinary differential equations:

$$\mathbf{y}' = \mathbf{F}(x, \mathbf{y}), \tag{1}$$

where $\mathbf{y} = (y_1, \ldots, y_n)^T$ is the vector of unknown functions, and the nth order normal differential equation:

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}). \tag{2}$$

Here $\mathbf{F}$ is a vector function and f is a function.

The connection of equivalence which reduces the higher order equation (2) to the first order system (1) is well-known.

The multivariate setting we present also is based on an equivalence relation, described in Section 2, between the above mentioned two systems of PDEs. This is rather unexpected for the theory of PDEs. As F. G. Tricomi mentions in ([T54, Section 2.7]) the absolute absence of the connection of equivalence in the theory of PDEs is a main reason for its not satisfactorily advancing and a major difference from the theory of ODEs.

While the first-order PDE system is the well-known Pfaff system of PDEs, the other, HO system of PDEs, which was a missed ring of the chain, was introduced by the authors (see [HT98b], [HT02b]). On the basis of the equivalence and the existence and uniqueness result of Pfaff system, we get such a result for the HO system (Section 2).

Our multivariate setting includes also natural analogs of other well-known univariate results concerning the system (1) and the equation (2), such as: the method of variation of constants, fundamental set of solutions, Wronskian's properties, Liouville's formula. This is presented in Sections 3 and 5 for the Pfaff and HO systems, respectively.

Note that in the multivariate case we have two systems instead of a system and an equation (1)–(2) in the univariate case. The analogy here is that, unlike the first-order system, the HO system, as the equation (2), contains a single unknown function.

Sections 4 and 6 deal with the constant coefficient cases of the Pfaff and HO linear systems, respectively. We consider an algebraic system of characteristic equations to solve the HO system of PDE, i.e., to determine a fundamental set of solutions. The algebraic system is the multivariate counterpart of the univariate polynomial equation:

$$x^n - a_{n-1}x^{n-1} - \cdots - a_0 = 0. \tag{3}$$

This is substantiated by the fact that a multivariate fundamental theorem of algebra (MFTA) holds for the algebraic system. We bring two versions of MFTA [HT98a], [HT00], [HT02b], [H03a] (see also [M99, Theorem 3.1], [X94, Theorem 3.4]). The first version: MFTA1 characterizes the case when the algebraic system has maximal number of distinct solutions. The second version: MFTA2 concerns the general case of multiple solutions.

Since the systems of PDEs, as well as the algebraic system, are overdetermined, we deal with certain consistency conditions. These conditions for HO system in the homogeneous constant coefficient case coincide with those of the algebraic system of characteristic equations.

In the last Section 7 we consider some applications of the systems of PDEs considered in this paper. We present sketches of proofs of MFTA2 based on a result on HO systems of PDEs and the Bezout theorem based on MFTA2. It is worth mentioning that the multiplicities in the Bezout theorem are characterized solely by means of partial differential operators arisen by some spaces of polynomials. Also we discuss a result in ideal theory. At the end it is pointed out how other systems of PDEs can be reduced to HO systems of PDEs.

Note that the main results of this paper were announced in [HT02a].

We start with the Pfaff system of PDEs. Consider a vector function

$$\mathbf{z} : \mathbb{R}^n \to \mathbb{R}^m, \quad \mathbf{z} := \mathbf{z}(\mathbf{x}) = (z_1(\mathbf{x}), \ldots, z_m(\mathbf{x}))^T ,$$

of n real variables $\mathbf{x} = (x_1, \ldots, x_n)$, where n, m are positive integers.

As a multivariate analog of the normal system of differential equations of first order, we study the following Pfaff system of PDEs with respect to the unknown vector function $\mathbf{z}$ (m unknown scalar functions):

$$\frac{\partial \mathbf{z}}{\partial x_i} = \mathbf{f}_i(\mathbf{x}, \mathbf{z}), \quad i = 1, \ldots, n, \tag{4}$$

where $\mathbf{f}_i = (f_{i,1}, \ldots, f_{i,m})^T$ is a vector function defined on a domain $G \subset \mathbb{R}^n \times \mathbb{R}^m$. The initial condition is given by

$$\mathbf{z}(\mathbf{x}^0) = \mathbf{z}^0, \tag{5}$$

where $(\mathbf{x}^0, \mathbf{z}^0) \in G$.

As in the case of ordinary differential equations, the discussion of the linear problem remains valid in the case of complex valued functions and complex initial values.

The linear case of the system (4) is the following:

$$\frac{\partial \mathbf{z}}{\partial x_i} = \mathbf{A}_i \mathbf{z} + \mathbf{b}_i, \quad i = 1, \ldots, n. \tag{6}$$

Here $\mathbf{A}_i = \mathbf{A}_i(\mathbf{x})$ is the $(m \times m)$ matrix of coefficients and $\mathbf{b}_i = \mathbf{b}_i(\mathbf{x})$ is the $(m \times 1)$ column of free terms. The elements of these matrices are n-variate real (or complex valued) functions defined on some domain $D \subset \mathbb{R}^n$. The initial condition is

$$\mathbf{z}(\mathbf{x}^0) = \mathbf{z}^0, \tag{7}$$

where $\mathbf{x}^0 \in D, \;\; \mathbf{z}^0 \in \mathbb{R}^m$ (or $\mathbf{z}^0 \in \mathbb{C}^m$).

In order to guarantee that the solutions of the linear problem (6)–(7) are defined on the whole domain D (see Theorem 1.1, below) we assume that D is *s*urface-simply connected, i.e., for any closed contour L in D there is a two-dimensional surface in D whose boundary coincides with L.

To present the Frobenius uniqueness and existence theorem concerning the Pfaff system of PDEs let us denote by

$$\frac{\partial \mathbf{f}_i}{\partial \mathbf{z}} \text{ the } (m \times m) \text{ matrix with columns } \frac{\partial \mathbf{f}_i}{\partial z_j}, \quad j = 1, \ldots, m,$$

where $\mathbf{f}_i := \mathbf{f}_i(\mathbf{x}, \mathbf{z})$ is the $(m \times 1)$ column given in (4).

In the above described linear case, the matrix $\partial \mathbf{f}_i / \partial \mathbf{z}$ simply coincides with $\mathbf{A}_i$.

Theorem 1.1 (Frobenius [F77]). *Assume that the functions $\mathbf{f}_i(\mathbf{x}, \mathbf{z})$ are continuously differentiable with respect to $\mathbf{x}$ and $\mathbf{z}$ in a domain $G \subset \mathbb{R}^n \times \mathbb{R}^m$. Then the system* (4) *has a solution for arbitrary initial data* (5) *if and only if the following consistency conditions are satisfied:*

$$\frac{\partial \mathbf{f}_i}{\partial x_j} + \frac{\partial \mathbf{f}_i}{\partial \mathbf{z}} \mathbf{f}_j = \frac{\partial \mathbf{f}_j}{\partial x_i} + \frac{\partial \mathbf{f}_j}{\partial \mathbf{z}} \mathbf{f}_i, \quad i, j = 1, \ldots, n, \ i \neq j. \tag{8}$$

In addition, the solution is unique on the domain where it is defined. In the linear case, the solution is defined on the whole domain $D \subset \mathbb{R}^n$, where the coefficients and free terms are defined, provided the domain is surface-simply connected.

Note that in the linear case, real or complex, the conditions (8) can be expressed in terms of coefficients and free terms as follows:

$$\mathbf{A}_i \mathbf{A}_j + \frac{\partial}{\partial x_j} \mathbf{A}_i = \mathbf{A}_j \mathbf{A}_i + \frac{\partial}{\partial x_i} \mathbf{A}_j, \quad \mathbf{A}_i \mathbf{b}_j + \frac{\partial}{\partial x_j} \mathbf{b}_i = \mathbf{A}_j \mathbf{b}_i + \frac{\partial}{\partial x_i} \mathbf{b}_j, \tag{9}$$

where $i, j = 1, \ldots, n, \ i \neq j$.

Let us mention that the smoothness assumption of the theorem in the linear case means that the coefficients and free terms are continuously differentiable with respect to $\mathbf{x}$.

For the sake of completeness we give:

Sketch of proof of Theorem 1.1. (Cf. [F77], [S72, Chapter 2, Section 2.5].) Let $\mathbf{z}$ be a solution of (4). Then we have

$$\frac{\partial}{\partial x_j} \frac{\partial \mathbf{z}}{\partial x_i} = \frac{\partial}{\partial x_j} \mathbf{f}_i = \frac{\partial \mathbf{f}_i}{\partial x_j} + \frac{\partial \mathbf{f}_i}{\partial \mathbf{z}} \frac{\partial \mathbf{z}}{\partial x_j} = \frac{\partial \mathbf{f}_i}{\partial x_j} + \frac{\partial \mathbf{f}_i}{\partial \mathbf{z}} \mathbf{f}_j.$$

Now, since by the hypothesis the initial data (5) is arbitrary, we get the necessity of the consistency conditions (8) by changing the order of differentiation in the first expression above. This can be done since the second order derivatives of the solution are continuous in view of the smoothness assumption.

For sufficiency we use induction on n — the number of variables. The case of $n = 1$ is well-known. Note that in this case there are no consistency conditions and therefore the functions $\mathbf{f}_i(\mathbf{x}, \mathbf{z})$ can be assumed merely to be continuous with respect to $\mathbf{x}$ while still continuously differentiable with respect to $\mathbf{z}$.

Assume that the theorem is true in the case of $(n-1)$ variables. Consider the following $(n-1)$-variate problem in the hyperplane $h = \{x_n = x_n^0\}$:

$$\begin{aligned} &\frac{\partial \mathbf{z}^h}{\partial x_i} = \mathbf{f}_i(x_1, \ldots, x_{n-1}, x_n^0, \mathbf{z}^h), \quad i = 1, \ldots, n-1, \\ &\mathbf{z}^h(x_1^0, \ldots, x_{n-1}^0) = \mathbf{z}^0. \end{aligned} \tag{10}$$

By the induction assumption, this problem has a unique solution in a neighborhood $V_1 = \{(x_1, \ldots, x_{n-1}, x_n^0) : |x_i - x_i^0| \leq \delta_1, \ \ i = 1, \ldots, n-1\}$.

Next, for each $(x'_1, \dots, x'_{n-1}, x^0_n) \in V_1 \subset h$, consider the univariate Cauchy problem on the line $l = \{x_1 = x'_1, \cdots, x_{n-1} = x'_{n-1}\}$:

$$\frac{\partial \mathbf{z}^l}{\partial x_n} = \mathbf{f}_n(x'_1, \dots, x'_{n-1}, x_n, \mathbf{z}^l), \tag{11}$$
$$\mathbf{z}^l(x^0_n) := \mathbf{z}^l_{x'_1,\dots,x'_{n-1}}(x^0_n) = \mathbf{z}^h(x'_1, \dots, x'_{n-1}).$$

This problem also has a unique solution in a neighborhood $V_2 = \{x_n : |x_n - x^0_n| \le \delta_2\}$. By a continuity argument for this ordinary equation we can assume that $\mathbf{z}^l$ is differentiable with respect to the parameters $x'_1, \dots, x'_{n-1}$, and that δ_2 is independent of them.

In view of the induction hypothesis the solution of (4)–(5), if it exists, coincides necessarily with $\mathbf{z}^h$ on h. Therefore, in view of a uniqueness result for ODE, it coincides also with the corresponding $\mathbf{z}^l$, on each l. Thus the solution is unique on $V := V_1 \times V_2$ whenever it exists.

Now, what is left is to show that

$$\mathbf{z}(x_1, \dots, x_n) = \mathbf{z}^l_{x_1,\dots,x_{n-1}}(x_n), \ (x_1, \dots, x_n) \in V,$$

is a solution of (4). More precisely, since $\mathbf{z}^l$ here is a solution of ODE (11), we need to check only that

$$\mathbf{y}_i := \frac{\partial \mathbf{z}}{\partial x_i} - \mathbf{f}_i(\mathbf{x}, \mathbf{z}) = 0,$$

for each fixed $i = 1, \dots, n-1$.

To this end we differentiate $\mathbf{y}_i$, with respect to x_n, and make use of (11) to obtain

$$\frac{\partial}{\partial x_n}\mathbf{y}_i = \frac{\partial}{\partial x_i}\frac{\partial \mathbf{z}}{\partial x_n} - \frac{\partial}{\partial x_n}\mathbf{f}_i = \frac{\partial}{\partial x_i}\mathbf{f}_n - \frac{\partial \mathbf{f}_i}{\partial x_n} - \frac{\partial \mathbf{f}_i}{\partial \mathbf{z}}\frac{\partial \mathbf{z}}{\partial x_n}$$
$$= \frac{\partial \mathbf{f}_n}{\partial x_i} + \frac{\partial \mathbf{f}_n}{\partial \mathbf{z}}\frac{\partial \mathbf{z}}{\partial x_i} - \frac{\partial \mathbf{f}_i}{\partial x_n} - \frac{\partial \mathbf{f}_i}{\partial \mathbf{z}}\mathbf{f}_n.$$

Next, in view of (8), we can interchange the indices n and i in the terms with $(-)$ sign and get finally

$$\frac{\partial}{\partial x_n}\mathbf{y}_i = \frac{\partial \mathbf{f}_n}{\partial \mathbf{z}}\mathbf{y}_i.$$

Hence the column vector $\mathbf{y}_i$ satisfies a homogeneous linear system of ordinary differential equations. On the other hand, according to the second relation of (11) and the first relation of (10), we have the following initial conditions:

$$\mathbf{y}_i(x_1, \dots, x_{n-1}, x^0_n) = 0.$$

Therefore $\mathbf{y}_i = 0$, which is the desired conclusion.

Note that in the linear case the above $(n-1)$-variate system and the ordinary equation are linear too. This establishes the last statement of Theorem if D is a rectangular parallelepiped. The general case of a surface-simply connected domain D can be obtained then by standard arguments. □

Let us mention that in this paper we consider also type (6) linear systems for which the consistency conditions are not necessarily satisfied. As can be readily seen from the above proof, the following assertion holds for such systems:

The solution of problem (6)–(7) *on any domain D is unique wherever it exists, provided the coefficients and free terms of the system* (6) *are continuous on D.*

2. Higher order (HO) systems of PDEs and the equivalence relation

In this section we study the multivariate counterpart of the normal differential equation of nth order (2), which is the HO system of PDEs with a single unknown function.

Let us start with some multivariate notation.

For $\mathbf{x} = (x_1, \dots, x_n),\ \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ and for multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ in $\mathbb{Z}_+^n$ we put

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i, \quad \mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad |\alpha| := \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!,$$

$$D^\alpha z := \frac{\partial^{|\alpha|} z}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

We denote by $e_i,\ i = 1, \dots, n$, the multiindex with all entries zero except the ith which is one and by $\overline{0} = (0, \dots, 0)$ the zero multiindex. For multiindices $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_k)$ the inequality $\alpha \le \beta$ means that $\alpha_i \le \beta_i$, for $i = 1, \dots, n$.

We call the index set $I \subset \mathbb{Z}_+^k$ *down connected* (*to zero*) (see [M99]) if for any $\alpha \in I$ there exists $i_1, \dots, i_n$ with $\alpha = e_{i_1} + \cdots + e_{i_n}$ and $e_{i_1} + \cdots + e_{i_m} \in I$ for $m = 1, \dots, n$. This means that by adding successively suitable e_i and staying in I we can reach any element of I.

In earlier versions of our results instead we were using *lower* set I, which means $\alpha \in I$ and $\beta \le \alpha$ imply $\beta \in I$. Evidently lower sets are down connected to zero.

From now on, we assume that the set I is down connected.

The following is a set of leading multiindices of I:

$$\delta(I) = \cup_{i=1}^{n} (I + e_i) \setminus I.$$

In particular, we have in the total degree case:

$$\delta(I) = \{\alpha : |\alpha| = k + 1\} \quad \text{if} \quad I = \{\alpha : |\alpha| \le k\}.$$

For the brevity we put

$$\{y_*\} := \{y_\alpha\}_* := \{y_\alpha\}_{\alpha \in I}.$$

Similarly we set

$$\{D^* y\} := \{D^\alpha y\}_* := \{D^\alpha y\}_{\alpha \in I}.$$

Next we bring the HO system of partial differential equations with the single unknown scalar function $z : \mathbb{R}^n \to \mathbb{R}$ (see [HT98b], [HT02b]):

$$D^\alpha z = f_\alpha(\mathbf{x}, \{D^* z\}), \quad \alpha \in \delta(I), \tag{12}$$

where the functions $f_\alpha(\mathbf{x}, \{z_*\})$ are continuously differentiable on a domain $G \subset \mathbb{R}^n \times \mathbb{R}^\mu$, with $\mu = \#I$.

For example in the case when $I = \{(0,0);(1,0);(0,1)\}$ the HO system is the following:

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= f_{2,0}\left(x,y,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right),\\ \frac{\partial^2 z}{\partial x \partial y} &= f_{1,1}\left(x,y,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right),\\ \frac{\partial^2 z}{\partial y^2} &= f_{0,2}\left(x,y,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right). \end{aligned}$$

The initial conditions for the system (12) are the following:

$$D^\alpha z(\mathbf{x}^0) = z_\alpha^0, \quad \alpha \in I, \tag{13}$$

where $(\mathbf{x}^0, \{z_\alpha^0\}_*) \in G$.

The linear case of problem (12)–(13) is the following:

$$D^\alpha z = \sum_{\beta \in I} a_{\alpha,\beta} D^\beta z + b_\alpha, \quad \alpha \in \delta(I), \tag{14}$$

$$D^\alpha z(\mathbf{x}^0) = z_\alpha^0, \quad \alpha \in I, \tag{15}$$

where $\mathbf{x}^0 \in D$, $\{z_\alpha^0\}_* \in \mathbb{R}^\mu$ (or $\{z_\alpha^0\}_* \in \mathbb{C}^\mu$). Here the coefficients $a_{\alpha,\beta} = a_{\alpha,\beta}(\mathbf{x})$ and $b_\alpha = b_\alpha(\mathbf{x})$, are continuously differentiable real (or complex valued) functions on a surface-simply connected domain $D \subset \mathbb{R}^n$.

Next we discuss the basic connection of the Pfaff system (4) and HO system (12): the equivalence. The HO (higher order) system (12) will be reduced to an equivalent Pfaff system of first order (4). This will be done essentially in the same way as in the univariate case the nth order differential equation (2) is reducing to the first order system of differential equations (1).

For this end, on the basis of unknown z of the system (12), we introduce $\nu = \#I$ new unknown scalar functions $z_\alpha = D^\alpha z$, $\alpha \in I$, and consider the following Pfaff system with respect to these unknowns:

$$\frac{\partial \{z_*\}}{\partial x_i} = \mathbf{F}_i(\mathbf{x}, \{z_*\}), \quad i = 1, \ldots, n, \tag{16}$$

where the vector function $\mathbf{F}_i = \{F_{i,*}\}$ is given by

$$F_{i,\alpha}(\mathbf{x}, \{z_*\}) = \begin{cases} z_{\alpha+e_i}, & \text{if } \alpha + e_i \in I \\ f_{\alpha+e_i}(\mathbf{x}, \{z_*\}), & \text{if } \alpha + e_i \in \partial(I). \end{cases} \tag{17}$$

The initial conditions for the system (16) are the following:

$$z_\alpha(\mathbf{x}^0) = z_\alpha^0, \quad \alpha \in I, \tag{18}$$

where $(\mathbf{x}^0, \{z_\alpha^0\}_*) \in G$.

In the linear case the problem (16)–(18) has the following form:

$$\begin{aligned} \frac{\partial \{z_*\}}{\partial x_i} &= \mathcal{A}_i \{z_*\} + b_i, \quad i = 1, \ldots, n, \\ z_\alpha(\mathbf{x}^0) &= z_\alpha^0, \quad \alpha \in I, \end{aligned} \tag{19}$$

where $\mathbf{x}^0 \in D$, $\{z_\alpha^0\}_* \in R^\nu$ (or $\{z_\alpha^0\}_* \in C^\nu$). Here the matrix $\mathcal{A}_i = \left\{\widetilde{a}_{\alpha,\beta}^{(i)}\right\}$, is given by

$$\widetilde{a}_{\alpha,\beta}^{(i)} = \begin{cases} 0, & \text{if } \alpha + e_i \in I, \quad \beta \neq \alpha + e_i \\ 1, & \text{if } \alpha + e_i \in I, \quad \beta = \alpha + e_i \\ a_{\alpha+e_i,\beta}, & \text{if } \alpha + e_i \in \delta(I). \end{cases} \tag{20}$$

and $b_i := \{\widetilde{b}_\alpha^{(i)}\}$, with

$$\widetilde{b}_\alpha^i = \begin{cases} 0, & \text{if } \alpha + e_i \in I \\ b_{\alpha+e_i}, & \text{if } \alpha + e_i \in \partial(I). \end{cases} \tag{21}$$

Here and above also we use the lexicographical order.

The structure of the above matrix $\mathcal{A}_i$ is as follows: some its rows coincide with coefficients of equations of the system (14), and the remaining rows consist of 0s and one 1.

The systems (16) and (19) will be referred to as the systems associated with the system (12) and (14), respectively.

Now, using that the set I is down connected, we obtain that the problems (16)–(18) and (12)–(13) are equivalent by the following one to one correspondence of solutions:

Theorem 2.1 (Equivalence)**.** *To each solution z of the problem* (12)–(13) *there corresponds a solution $\{z_\alpha\}_* = \{D^\alpha z\}_*$ of the problem* (16)–(18) *and vice versa, i.e., each solution $\{z_\alpha\}_*$ of* (16)–(18) *has the form $\{z_\alpha\}_* = \{D^\alpha z\}_*$ where z is a solution of* (12)–(13).

Proof. The equivalence of initial conditions (13) and (18) for the above solutions is obvious. Let $\bar{z}$ be a solution of the equation (12). Then let us verify that $\{\bar{z}_\alpha\}_* = \{D^\alpha \bar{z}\}_*$ is a solution of (16). Indeed, the equations of (16) with $\alpha + e_i \in I$, i.e., corresponding to the first line of (17) are satisfied identically due to the form of $\{\bar{z}_\alpha\}_*$. While the remaining equations, with $\{z_\alpha\}_*$ substituted by $\{D^\alpha \bar{z}\}_*$, are identical with the corresponding equations of (12) with $\bar{z}$. Now suppose $\{\bar{z}_\alpha\}_*$ is a solution of (16). Then the equations of (16) with $\alpha + e_i \in I$, i.e., corresponding to the first line of (17) imply

$$\frac{\partial \{z_\alpha\}_*}{\partial x_i} = z_{\alpha+e_i}, \qquad \text{if } \alpha + e_i \in I.$$

By using the down connectivity of I this gives that

$$\{z_\alpha\}_* = \{D^\alpha z_{\overline{0}}\}_*,$$

i.e., it has the required form. It remains to use that the remaining equations of (16), as above, are identical with the corresponding ones of (12), and therefore they are satisfied by $z = \bar{z}_{\overline{0}}$. □

It follows from this equivalence that the consistency conditions for the system (12) can be obtained from those of the associated system (16), i.e., (8), and can be expressed as in the following two cases below:

$$\frac{\partial f_{\alpha+e_i}}{\partial x_j} + \frac{\partial f_{\alpha+e_i}}{\partial \mathbf{z}}\mathbf{F}_j = \frac{\partial f_{\alpha+e_j}}{\partial x_i} + \frac{\partial f_{\alpha+e_j}}{\partial \mathbf{z}}\mathbf{F}_i, \quad i,j = 1,\dots,n, \tag{22}$$

where $\alpha \in I$ and $i, j,\ 1 \le i \neq j \le n$ are such that both $\alpha + e_i$ and $\alpha + e_j$ belong to $\delta(I)$;

$$\frac{\partial f_{\alpha+e_i}}{\partial x_j} + \frac{\partial f_{\alpha+e_i}}{\partial \mathbf{z}} \mathbf{F}_j = \psi_{\alpha+e_i+e_j}, \quad i, j = 1, \ldots, n, \tag{23}$$

where $\alpha \in I$ and $i, j,\ 1 \le i \neq j \le n$ are such that $\alpha + e_i \in \delta(I),\ \ \alpha + e_j \in I$, and

$$\psi_\beta = \begin{cases} z_\beta, & \text{if } \beta \in I \\ f_\beta, & \text{if } \beta \in \delta(I). \end{cases}$$

In the case of linear HO system (14), in view of (9), the consistency conditions are:

$$\mathcal{A}_i \mathcal{A}_j + \frac{\partial}{\partial x_j} \mathcal{A}_i = \mathcal{A}_j \mathcal{A}_i + \frac{\partial}{\partial x_i} \mathcal{A}_j, \quad \mathcal{A}_i b_j + \frac{\partial}{\partial x_j} b_i = \mathcal{A}_j b_i + \frac{\partial}{\partial x_i} b_j, \tag{24}$$

where $i, j = 1, \ldots, n,\ i \neq j$.

Since many entries of above matrices are 0s, it makes sense to express these conditions, in view of (20) and (21), through the coefficients and free terms of the system (14) as it is done in the following two cases ($\triangleright$) below, where $\alpha, \beta \in I,\ 1 \le i \neq j \le n,$ and for brevity we set

$$\eta_{i,j} := \eta_{i,j}(\alpha, \beta) := \sum_{\gamma + e_j \in \delta(I)} a_{\alpha+e_i,\gamma} a_{\gamma+e_j,\beta} + \chi(\beta - e_j) a_{\alpha+e_i,\beta-e_j} + \frac{\partial a_{\alpha+e_i,\beta}}{\partial x_j},$$

$$\xi_{i,j} := \xi_{i,j}(\alpha) := \sum_{\gamma + e_j \in \delta(I)} a_{\alpha+e_i,\gamma} b_{\gamma+e_j} + \frac{\partial b_{\alpha+e_i}}{\partial x_j} :$$

- $\triangleright$ $\eta_{i,j} = \eta_{j,i}$, $\xi_{i,j} = \xi_{j,i}$, if $\alpha + e_i,\ \alpha + e_j \in \delta(I)$;
- $\triangleright$ $\widetilde{a}^{(j)}_{\alpha+e_i,\beta} = \eta_{j,i}$, $\xi_{j,i} = 0$, if $\alpha + e_i \in I,\ \alpha + e_j \in \delta(I)$.

Note that χ above is the characteristic function of the set I.

Now from the Frobenius Theorem 1.1, in view of Theorem 2.1 on equivalence, we get the following existence and uniqueness theorem for the HO system:

Theorem 2.2. *Assume that the functions $f_\alpha(\mathbf{x}, \{z_*\})$ are continuously differentiable with respect to $\mathbf{x}$ and $\{z_*\}$ in a domain $G \subset \mathbb{R}^n \times \mathbb{R}^\mu$. Then problem* (12)–(13) *has a solution for arbitrary initial data if and only if the consistency conditions* (22)–(23) *are satisfied. In addition, the solution is unique on the domain where it is defined. In the linear case, the solution is defined on the whole domain $D \subset \mathbb{R}^n$ where the coefficients and free terms are defined, provided the domain is surface-simply connected.*

Let us mention that the smoothness assumption of the theorem in the linear case means that the coefficients and free terms are continuously differentiable with respect to $\mathbf{x}$.

3. The Wronskian and the method of variation of constants in the case of the Pfaff linear systems

We start this section by considering the case of a linear homogeneous system

$$\frac{\partial \mathbf{z}}{\partial x_i} = \mathbf{A}_i \mathbf{z}, \quad i = 1, \ldots, n, \tag{25}$$

where the coefficients are not necessarily constants.

Note that the consistency conditions for (25), in view of (9), are

$$\mathbf{A}_i\mathbf{A}_j + \frac{\partial}{\partial x_j}\mathbf{A}_i = \mathbf{A}_j\mathbf{A}_i + \frac{\partial}{\partial x_i}\mathbf{A}_j, \tag{26}$$

where $i, j = 1, \ldots, n,\ i \neq j$.

Let us introduce the Wronskian with respect to a set of functions, $\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(m)}$, in the following way:

$$W(\mathbf{x}) := W(\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(m)}) = \det \left[z_j^{(i)}(\mathbf{x})\right]_{i,j=1,\ldots,m}.$$

It readily follows from Frobenius' theorem that W, with $\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(m)}$ being solutions of (25), is either identically zero — when the solutions are linearly dependent or never vanishes on G — when they are independent, i.e., they form a fundamental set of solutions.

Let us mention that, similarly to the univariate case, for any given matrix

$$G(\mathbf{x}) = \left[g_j^{(i)}(\mathbf{x})\right]_{i,j=1,\ldots,m}$$

for which $\det G(\mathbf{x}) \neq 0$, $\mathbf{x} \in G$, and whose entries have continuous partial derivatives of second order, one can find uniquely matrices of coefficients of the system (25) such that the row vector functions: $\mathbf{g}^{(i)}$ are its fundamental solutions. In fact, this means

$$\frac{\partial \mathbf{g}^{(j)}}{\partial x_i} = \mathbf{A}_i\mathbf{g}^{(j)}, \quad i = 1, \ldots, n,\ j = 1, \ldots, m.$$

In view of this relation we can determine, for each fixed i and k ($1 \leq k \leq m$), the kth row of $\mathbf{A}_i$, i.e., the vector $\{a_{k,l}^{(i)}\}_{l=1}^m$, from the following linear algebraic system:

$$\frac{\partial g_k^{(j)}}{\partial x_i} = \sum_{l=1}^m a_{k,l}^{(i)} g_l^{(j)}, \quad j = 1, \ldots, m.$$

Moreover, the coefficients are continuously differentiable in view of the smoothness assumptions on $G(\mathbf{x})$. Note also that this gives a method of construction of homogeneous systems (25) satisfying the consistency conditions (26).

Now let us mention the following multivariate analog of Liouville's formula:

$$W(\mathbf{x}) = W(\mathbf{x}^0)\exp[\sigma(\mathbf{x})], \tag{27}$$

where

$$\sigma(\mathbf{x}) = \int_{\mathbf{x}^0}^{\mathbf{x}} \sum_{i=1}^n s_i(\mathbf{x})dx_i, \tag{28}$$

with $s_i(\mathbf{x}) = \text{Trace}\,[\mathbf{A}_i]$. Here of course we suppose that consistency conditions of the system (25) are satisfied and the functions in the Wronskian are its solutions.

Note that the equality (28) is equivalent to the following system:

$$\frac{\partial \sigma}{\partial x_i} = s_i, \quad i = 1, \ldots, n,$$

with the initial condition $\sigma(\mathbf{x}^0) = 0$.

To prove (27) it is enough, just as in the univariate case, to differentiate the Wronskian and to find the system for which it is a solution. In this way, using the differentiation rule for determinants, we arrive to the system

$$\frac{\partial W}{\partial x_i} = s_i W, \quad i = 1, \ldots, n,$$

solving which we get (27). As a corollary we get that the above traces s_i satisfy the consistency conditions

$$\frac{\partial s_i}{\partial x_j} = \frac{\partial s_j}{\partial x_i}, \quad i, j = 1, \ldots, n, \ i \neq j.$$

This could also have been checked directly from the first equality of (9).

Our next aim is to generalize the well-known method of variation of constants to this multivariate case. Consider the non-homogeneous system (6), where the consistency conditions (9) are satisfied. Suppose that $\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(m)}$ are fundamental solutions of the corresponding homogeneous system (25). Then $W(\mathbf{x}) \neq 0$. Therefore the unique solution of problem (6)–(7), just as any function, can be presented in the form

$$\mathbf{z}(\mathbf{x}) = \sum_{k=1}^{m} c_k(\mathbf{x}) \mathbf{z}^{(k)}(\mathbf{x}), \tag{29}$$

where the coefficients are uniquely determined. Indeed, we get this by considering (29) as a linear algebraic system, with respect to unknown coefficients for each fixed $\mathbf{x}$.

Now we are going to point out a way of finding these coefficients. For this end we denote

$$\frac{\partial c_k}{\partial x_i} = q_k^{(i)}, \quad i = 1, \ldots, n, \ k = 1, \ldots, m. \tag{30}$$

Substituting for $\mathbf{z}(\mathbf{x})$ the expression (29) in the system (6) and using the fact that $\mathbf{z}^{(k)}$ satisfies (25) we obtain linear algebraic systems for determining the functions in the right-hand side of the above equalities. Namely, the system for the functions with fixed $i, \ 1 \leq i \leq m$, is the following:

$$\sum_{k=1}^{m} q_k^{(i)}(\mathbf{x}) \mathbf{z}^{(k)}(\mathbf{x}) = \mathbf{b}_i(\mathbf{x}). \tag{31}$$

On the other hand, by setting $\mathbf{x} = \mathbf{x}^0$ in (29) and using the initial condition (7) for $\mathbf{z}(\mathbf{x})$ we get a linear algebraic system

$$\sum_{k=1}^{m} c_k(\mathbf{x}^0) \mathbf{z}^{(k)}(\mathbf{x}^0) = \mathbf{z}^0, \tag{32}$$

for determining the initial conditions

$$c_k(\mathbf{x}^0) = c_k^0, \ k = 1, \ldots, m. \tag{33}$$

Therefore the coefficients $c_k(\mathbf{x}), \ 1 \leq k \leq m$, can be found uniquely from the problem (30)–(33). As a corollary we get that the above functions $q_k^{(i)}$ satisfy the consistency conditions

$$v_{i,j}^k := \frac{\partial q_k^{(i)}}{\partial x_j} = \frac{\partial q_k^{(j)}}{\partial x_i}, \quad i,j = 1,\dots,n,\ i \neq j,\ k = 1,\dots,m. \tag{34}$$

It will be useful to derive these conditions directly from the second equality of (9). Besides, this gives another proof of the Frobenius theorem for the linear nonhomogeneous case, assuming it holds in the homogeneous case.

Indeed, $\mathbf{z}(\mathbf{x})$ given by (29) is a solution of (6)–(7) if and only if the relations (29)–(31) are satisfied for some coefficients $c_k(\mathbf{x})$. On the other hand, by verifying (34), where $q_k^{(i)}$ is determined from (31), we establish that problem (30)–(33) is (uniquely) solvable, i.e., there are coefficients $c_j(\mathbf{x})$ for which (31) and (32) hold.

We will prove (34) by checking that $v_{i,j}^k$ and $v_{j,i}^k$ (for each fixed i and j, $1 \leq i \neq j \leq n$, and $\mathbf{x}$) satisfy the same system of linear equations. To this end we differentiate (31) to obtain

$$\sum_{k=1}^m \frac{\partial q_k^{(i)}}{\partial x_j}\mathbf{z}^{(k)} + \sum_{k=1}^m q_k^{(i)} \frac{\partial \mathbf{z}^{(k)}}{\partial x_j} = \frac{\partial \mathbf{b}_i}{\partial x_j}. \tag{35}$$

Now, notice that the second sum in the left side is equal to

$$\sum_{k=1}^m q_k^{(i)} \mathbf{A}_j \mathbf{z}^{(k)},$$

or, in view of (31), to $\mathbf{A}_j \mathbf{b}_i$. Therefore (35) reduces to the following system:

$$\sum_{k=1}^m v_{i,j}^k \mathbf{z}^{(k)} = \frac{\partial \mathbf{b}_i}{\partial x_j} - \mathbf{A}_j \mathbf{b}_i.$$

The right side of this linear algebraic system, in view of the second relation of (9), is symmetric with respect to i and j. This ends the proof.

Now let us denote by $\mathbf{H}(\mathbf{x})$ the $(m \times m)$ matrix with the columns $\mathbf{z}^{(i)}(\mathbf{x})$, $i = 1,\dots,n$, and set $\mathbf{c}(\mathbf{x}) = (c_1(\mathbf{x}),\dots,c_m(\mathbf{x}))^T$, $\mathbf{q}^{(i)}(\mathbf{x}) = \left(q_1^i(\mathbf{x}),\dots,q_m^i(\mathbf{x})\right)^T$.

Then we can rewrite (29), (30), and (31) as follows:

$$\mathbf{z}(\mathbf{x}) = \mathbf{H}(\mathbf{x})\mathbf{c}(\mathbf{x}), \quad \frac{\partial \mathbf{c}(\mathbf{x})}{\partial x_i} = \mathbf{q}^{(i)}(\mathbf{x}), \quad \text{and} \quad \mathbf{b}_i(\mathbf{x}) = \mathbf{H}(\mathbf{x})\mathbf{q}^{(i)}(\mathbf{x}), \quad i = 1,\dots,n.$$

Therefore

$$\mathbf{c}(\mathbf{x}) = \mathbf{c}^0 + \int_{\mathbf{x}^0}^{\mathbf{x}} \sum_{i=1}^n \mathbf{H}(\mathbf{x})^{-1}\mathbf{b}_i(\mathbf{x})dx_i,$$

where $\mathbf{c}^0 = \mathbf{H}(\mathbf{x}^0)^{-1}\mathbf{z}^0$.

Finally, we get the following explicit expression for the solution of the nonhomogeneous problem (6)–(7), quite similar to the univariate case:

$$\mathbf{z}(\mathbf{x}) = \mathbf{H}(\mathbf{x})\left(\mathbf{c}^0 + \int_{\mathbf{x}^0}^{\mathbf{x}} \sum_{i=1}^n \mathbf{H}(\mathbf{x})^{-1}\mathbf{b}_i(\mathbf{x})dx_i\right).$$

4. Pfaff homogeneous linear systems with constant coefficients

In this section we consider the case of system (25)

$$\frac{\partial \mathbf{z}}{\partial x_i} = \mathbf{A}_i \mathbf{z}, \quad i = 1, \ldots, n, \tag{36}$$

where the entries of the matrices $\mathbf{A}_i$ are constants.

The initial condition is

$$\mathbf{z}(\mathbf{x}^0) = \mathbf{z}^0, \tag{37}$$

where $\mathbf{x}^0 \in \mathbb{C}^n,\ \mathbf{z}^0 \in \mathbb{C}^m$. It is easily seen that the consistency conditions (26) are reducing to the following commuting relations:

$$\mathbf{A}_i \mathbf{A}_j = \mathbf{A}_j \mathbf{A}_i \quad i, j = 1, \ldots, n,\ i \neq j.$$

The following representation of the solution of the above problem follows readily from the univariate counterpart:

Proposition 4.1. *Let $\mathbf{A}_i$, $i = 1, \ldots, n$, be commuting matrices. Then the solution of the problem* (36)–(37) *can be expressed in the following form:*

$$\mathbf{z} = \exp\left(\sum_{i=1}^{n} (x_i - x_i^0) \mathbf{A}_i\right) \mathbf{z}^0.$$

Proof. Notice that whenever the matrices $\mathbf{A}$ and $\mathbf{B}$ commute then

$$\exp(\mathbf{A} + \mathbf{B}) = \exp(\mathbf{A}) \cdot \exp(\mathbf{B}) \quad \text{and} \quad \exp(\mathbf{A}) \cdot \mathbf{B} = \mathbf{B} \cdot \exp(\mathbf{A}).$$

In view of these relations and the familiar univariate case we get

$$\begin{aligned}
\frac{\partial \mathbf{z}}{\partial x_j} &= \left[\frac{\partial}{\partial x_j} \exp\left(\sum_{i=1}^{n} (x_i - x_i^0) \mathbf{A}_i\right)\right] \mathbf{z}^0 \\
&= \exp\left(\sum_{i=1}^{j-1} (x_i - x_i^0) \mathbf{A}_i\right) \left[\frac{\partial}{\partial x_j} \exp\left((x_j - x_j^0) \mathbf{A}_j\right)\right] \exp\left(\sum_{i=j+1}^{n} (x_i - x_i^0) \mathbf{A}_i\right) \mathbf{z}^0 \\
&= \mathbf{A}_j \exp\left(\sum_{i=1}^{n} (x_i - x_i^0) \mathbf{A}_i\right) \mathbf{z}^0 = \mathbf{A}_j \mathbf{z},
\end{aligned}$$

for $1 \leq j \leq n$. Obviously, the initial condition (37) is satisfied too. □

Now we are going to find a simpler formula for solutions. Let us start with the following:

Proposition 4.2. *The function*

$$\mathbf{z} = \exp(\boldsymbol{\lambda} \cdot \mathbf{x})\, \mathbf{h},$$

with $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_n\}$, $\mathbf{h} \in \mathbb{C}^m$, *is a solution of the system* (36) *if and only if* $\mathbf{h}$ *is a common eigenvector of the matrices* $\mathbf{A}_i$ *with the corresponding eigenvalues* λ_i, $i = 1, \ldots, n$.

Proof. Let $\mathbf{h}$ be a common eigenvector. Then, for the given $\mathbf{z}$, and any fixed $1 \le i \le n$, we have

$$\frac{\partial \mathbf{z}}{\partial x_i} = \lambda_i \exp(\boldsymbol{\lambda} \cdot \mathbf{x})\,\mathbf{h} = \exp(\boldsymbol{\lambda} \cdot \mathbf{x})\,\mathbf{A}_i \mathbf{h} = \mathbf{A}_i \mathbf{z}.$$

In other words $\mathbf{z}$ is a solution of (36). Now let this latter take place. Then we have

$$\exp(\boldsymbol{\lambda} \cdot \mathbf{x})\,\mathbf{A}_i \mathbf{h} = \mathbf{A}_i \mathbf{z} = \frac{\partial \mathbf{z}}{\partial x_i} = \lambda_i \exp(\boldsymbol{\lambda} \cdot \mathbf{x})\,\mathbf{h}.$$

Therefore $\mathbf{A}_i \mathbf{h} = \lambda_i \mathbf{h}$. □

The next theorem is based on the following well-known result (see, e.g., [L69, Theorem 8.6.1]) on semisimple matrices, i.e., matrices which have a complete set of eigenvectors:

Every family of commuting semisimple matrices possesses a complete set of common eigenvectors.

Theorem 4.3. *Let the commuting matrices $\mathbf{A}_i$, $1 \le i \le n$, be semisimple and let $\{\mathbf{h}^{(i)},\ 1 \le i \le m\}$ be a complete set of common eigenvectors of these matrices. Let also $\boldsymbol{\lambda}^{(i)} = \{\lambda_1^{(i)}, \dots, \lambda_n^{(i)}\}$ be the collections of eigenvalues corresponding to $\mathbf{h}^{(i)}$. Then the functions*

$$\mathbf{z}^{(i)} = \mathbf{h}^{(i)} \exp\left(\boldsymbol{\lambda}^{(i)} \cdot \mathbf{x}\right), \quad i = 1, \dots, m,$$

form a fundamental set of solutions of the system (36)*.*

Proof. According to Proposition 4.2, above functions are solutions of (36). On the other hand we have that

$$\mathbf{z}^{(i)}(0) = \mathbf{h}^{(i)},$$

and the set of eigenvectors forms a basis of $\mathbb{C}^m$. This means that the linear combinations of the above functions $\mathbf{z}^{(i)}$ give uniquely solutions of (36) with arbitrary initial values $\mathbf{z}^0 \in \mathbb{C}^m$. Therefore, in view of Theorem 1.1, they give uniquely all the solutions. □

Note that in the case of real $\mathbf{A}$ it is not difficult to get a fundamental set of real valued solutions by using the Euler formula.

Let us mention that in Section 7 (see Corollary 7.4) we bring an extension of this result for the case when the above matrices $\mathbf{A}_i$ are arbitrary.

Here it is worth pointing out the following (see [HT02b, Remark 3(i)]):

Remark 4.4. Let the eigenvalues of some matrix $\mathbf{A}_{i_0}$, $1 \le i_0 \le n$, which commutes with all the others, be distinct. Then the other matrices are semisimple, commute with each other, and possess as a complete set of common eigenvectors those of $\mathbf{A}_{i_0}$.

Set, for the case of general commuting matrices,

$$d := d\{\mathbf{A}_i\}_{i=1}^n := \max_{1 \le i \le n} d_i,$$

where d_i is the number of distinct eigenvalues of $\mathbf{A}_i$.

The next proposition, in view of Proposition 4.2, follows from the following well-known result:

Every family of commuting matrices $\mathbf{A}_i,\ 1 \le i \le n$, possesses at least d common linearly independent eigenvectors which correspond to distinct collections of eigenvalues.

In [HT02b], after Theorem 4, we point out how these common eigenvectors can be found.

Proposition 4.5. *Let the matrices $\mathbf{A}_i,\ 1 \le i \le n$, be commuting. Then the system* (36) *has at least d linearly independent solutions of the form*

$$\mathbf{z} = \mathbf{h} \exp(\boldsymbol{\lambda} \cdot \mathbf{x}),$$

where $\mathbf{h}$ is a common eigenvector of the matrices and $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_n\}$ is the corresponding collection of eigenvalues. Moreover, the collections corresponding to distinct eigenvectors are distinct.

5. The Wronskian and the method of variation of constants in the case of the higher order systems

In this section we study the homogeneous case of system (14):

$$D^\alpha z = \sum_{\beta \in I} a_{\alpha,\beta} D^\beta z, \quad \alpha \in \delta(I), \tag{38}$$

where the coefficients are not necessarily constants. The consistency conditions, in view of (24), are

$$\mathcal{A}_i \mathcal{A}_j + \frac{\partial}{\partial x_j} \mathcal{A}_i = \mathcal{A}_j \mathcal{A}_i + \frac{\partial}{\partial x_i} \mathcal{A}_j, \tag{39}$$

where $i, j = 1, \dots, n,\ i \ne j$.

Let us introduce the Wronskian of a set of functions $\{z_\beta\}_*$ in the following way:

$$W_I(\mathbf{x}) = W_I(\{z_*\}) = \det\left[D^\alpha z_\beta(\mathbf{x})\right]_{\alpha,\beta \in I}.$$

Here and subsequently, in matrices, we use a fixed linear order for the set I, for example the lexicographical. Moreover, we will use an extension of this order also for sets of the following forms:

$I \cup \alpha$ — assuming that α takes the last place;
$I \backslash \beta \cup \alpha$ — assuming that α takes the place of β.

Now using Theorem 2.1 we get that W_I, with functions $\{z_*\}$ being solutions of the system (38), is either identically zero — when the solutions are linearly dependent, or never vanishes on G — when they are independent, i.e., they form a fundamental set of solutions.

Let us mention that, similarly to the univariate case (or to the case of the Pfaff system), for any given set of functions $\{g_*\}$ for which

$$W_I(\{g_*\}) \ne 0, \text{ for all } \mathbf{x} \in G, \text{ and}$$
$$D^{\alpha+e_i} g_\beta, \quad \text{is continuous for each } \alpha \in \delta(I),\ \beta \in I \text{ and } 1 \le i \le n, \tag{40}$$

one can find uniquely a type (38) system, satisfying the consistency conditions (39) for which the above functions $\{g_*\}$ form a set of fundamental solutions.

In fact, this system can be expressed in terms of Wronskians, in the following way:

$$\frac{W_{I\cup\alpha}(\{g_*\}\cup z)}{W_I(\{g_*\})}=0, \quad \alpha\in\delta(I).$$

Coefficients of this system will be continuous in view of the smoothness assumptions (40).

Now, in view of (27) and the equivalence of the systems (14) and (19), (or directly) we get the multivariate analog of Liouville's formula for this case:

$$W_I(\mathbf{x})=W_I(\mathbf{x}^0)\exp[\sigma(\mathbf{x})],$$

where

$$\sigma(\mathbf{x})=\int_{\mathbf{x}^0}^{\mathbf{x}}\sum_{i=1}^{n}\sum_{\alpha\in I,\ \alpha+e_i\notin I}a_{\alpha+e_i,\alpha}(\mathbf{x})dx_i.$$

Next, applying the method of variation of constants, described in Section 3, for the system (19), associated with the system (14), we get the analog of this method for the latter system. Namely, by means of the fundamental solutions $\{z_*\}$ of the homogeneous system (38), we get the solution z of the corresponding non-homogeneous system (14) in the following form:

$$z(\mathbf{x})=\sum_{\beta\in I}c_\beta(\mathbf{x})z_\beta(\mathbf{x}).$$

The algebraic system for determining the functions in the right-hand sides of

$$\frac{\partial c_\beta}{\partial x_i}=q_\beta^{(i)}, \quad \beta\in I,$$

with fixed $i=1,\dots,n$, is the following:

$$\sum_{\beta\in I}q_\beta^{(i)}(\mathbf{x})D^\alpha z_\beta(\mathbf{x})=\widetilde{b}_\alpha^{(i)}, \quad \alpha\in I.$$

Also the explicit representation of the solution of the problem (14)–(15) is:

$$z(\mathbf{x})=\mathcal{H}_0(\mathbf{x})\left(c^0+\int_{\mathbf{x}^0}^{\mathbf{x}}\sum_{i=1}^{n}\mathcal{H}(\mathbf{x})^{-1}b_i(\mathbf{x})dx_i\right).$$

Here $\mathcal{H}(\mathbf{x})$ is the $(\mu\times\mu)$ matrix $[D^\alpha z_\beta(\mathbf{x})]_{\alpha,\beta\in I}$, $\mathcal{H}_0(\mathbf{x})$ is the row with $\alpha=\overline{0}$ of this matrix and $c^0=\mathcal{H}(\mathbf{x}^0)^{-1}\{z_\alpha^0\}_*$.

At the end of this section let us generalize the following well-known result of Bôcher [B01] (see also [PS64, Chapter 7, Problem 60]) on the relation of linear dependence of functions and univariate Wronskians:

If $W(\phi_1,\dots,\phi_s)\neq 0$ *and* $W(\phi_1,\dots,\phi_s,\phi)\equiv 0$ *then* ϕ *is a linear combination of the functions* $\phi_1,\dots,\phi_s$.

Theorem 5.1. *Assume that*

$$W_{I\cup\alpha}(\{f_*\}\cup f)\equiv 0 \text{ for all } \alpha\in\delta(I) \quad\text{and}\quad W_I(\{f_*\})\neq 0 \text{ for all } \mathbf{x}\in G.$$

Then there are constants $c_\beta,\ \beta\in I$, *such that* $f=\sum_{\beta\in I}c_\beta f_\beta$.

Note that the restrictions of smoothness on the functions here are the minimal, in order to have only defined the Wronskians, i.e., we assume that $D^\alpha f$ and $D^\alpha f_\beta$ exist for each $\mathbf{x} \in G$, and $\alpha \in \delta(I)$. Let us mention that for more smooth functions, namely for functions with (40), this theorem is a consequence of Theorem 2.1. Indeed, as was mentioned after the relation (40), such functions form a fundamental set of solutions of a type (38) system to which f is a solution too.

Proof. (Cf. the proof of the univariate case [PS64, Chapter 7, Problem 60]). Determining the functions ψ_β from the following algebraic system:

$$D^\alpha f = \sum_{\gamma \in I} \psi_\gamma D^\alpha f_\gamma, \quad \alpha \in I,$$

we get

$$\psi_\gamma = \frac{W_I(\{f_*\} \setminus f_\gamma \cup f)}{W_I(\{f_*\})}, \quad \gamma \in I.$$

Then, by differentiating with respect to $x_i,\ i = 1, \ldots, n$, we get that $\partial/\partial x_i \psi_\gamma$ equals

$$\frac{(\partial/\partial x_i) W_I(\{f_*\} \backslash f_\gamma \cup f)\ W_I(\{f_*\}) - W_I(\{f_*\} \backslash f_\gamma \cup f)\ (\partial/\partial x_i) W_I(\{f_*\})}{W_I(\{f_*\})^2}.$$

Now, according to the determinant differentiation rule, the numerator equals

$$\sum_{\alpha \in I,\ \beta \notin I} \left[W_{I \backslash \alpha \cup \beta}(\{f_*\} \setminus f_\gamma \cup f) W_I(\{f_*\}) - W_I(\{f_*\} \setminus f_\gamma \cup f) W_{I \backslash \alpha \cup \beta}(\{f_*\}) \right],$$

where $\beta = \alpha + e_i$.

Each summand in above expression is a determinant of second order with entries being determinants of order μ which are cofactors of $W_{I \cup \{\alpha + e_i\}}(f \cup \{f_*\})$. Therefore (see [K01, pp. 80, 109]) the above sum equals

$$\sum_{\alpha \in I,\ \alpha + e_i \notin I} W_{I \backslash \{\alpha\}}(\{f_*\} \backslash f_\gamma) W_{I \cup \{\alpha + e_i\}}(\{f_*\} \cup f) = 0.$$

Thus ψ_β are constants, i.e., $f_\beta = c_\beta,\ \beta \in I$. □

6. HO homogeneous linear systems with constant coefficients and a multivariate analog of the fundamental theorem of algebra

Consider the HO linear homogeneous system

$$D^\alpha z = \sum_{\beta \in I} a_{\alpha,\beta} D^\beta z, \quad \alpha \in \delta(I), \tag{41}$$

where the coefficients $a_{\alpha,\beta}$ are constants (cf. the system of PDEs in [BHS93, Chapter 11, Sections 3,4] and [HS88]). Then the consistency conditions (39) reduce to

$$\mathcal{A}_i \mathcal{A}_j = \mathcal{A}_j \mathcal{A}_i, \quad i, j = 1, \ldots, n,\ i \neq j. \tag{42}$$

The system associated with the system (41) is (see (19))

$$\frac{\partial \{z_*\}}{\partial x_i} = \mathcal{A}_i \{z_*\}, \quad i = 1, \ldots, n. \tag{43}$$

Let us denote by Π^n the space of all polynomials in n variables. Consider the following subspace of Π^n, connected with the index set I:

$$\Pi_I^n = \left\{ p \in \Pi^n : p(\mathbf{x}) = \sum_{\alpha\in I} c_\alpha \mathbf{x}^\alpha \right\}.$$

Of course we have $\dim \Pi_I^n = \#I$.

In this section we discuss the system of characteristic equations of the system of PDEs (41):

$$\mathbf{x}^\alpha - \sum_{\beta\in I} a_{\alpha,\beta}\mathbf{x}^\beta =: \mathbf{x}^\alpha - P_\alpha(\mathbf{x}) = 0, \quad \alpha \in \delta(I), \tag{44}$$

where $P_\alpha \in \Pi_I^n$.

This algebraic system, as the system (41) of PDEs, is overdetermined.

Let us denote by $\mathcal{S}$ the set of all *distinct* solutions of the algebraic system (44).

The following theorem establishes some basic connections between the solutions of the systems of PDEs (41), (43) and the solutions of the characteristic algebraic system (44):

Theorem 6.1. *The following assertions are equivalent:*

- ▷ $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathcal{S}$, *i.e., it is a solution of the algebraic system* (44).
- ▷ $\boldsymbol{\lambda}$ *is a sequence of eigenvalues corresponding to some* (*nonzero*) *common eigenvector* $\mathbf{h}$ *for the matrices* $\mathcal{A}_i$, $1 \le i \le n$.
- ▷ $\mathbf{h} = \{\boldsymbol{\lambda}^\beta\}_*$ *is a common eigenvector for the matrices* $\mathcal{A}_i$, $1 \le i \le n$.
- ▷ *The function* $f = \exp(\boldsymbol{\lambda}\cdot\mathbf{x})$ *is a solution of the system* (41).
- ▷ $\mathbf{z} = \mathbf{h}\exp(\boldsymbol{\lambda}\cdot\mathbf{x})$ *is a solution of the system* (43), *with some vector* $\mathbf{h}$.

Moreover, $\mathbf{h} = c\{\boldsymbol{\lambda}^\beta\}_*$, *i.e., the common eigendirection is determined uniquely from the corresponding sequence of eigenvalues* $\boldsymbol{\lambda}$.

In view of Proposition 4.2 and the equivalence of the systems of PDEs (41) and (43) the proof is straightforward (see [HT02b, Theorems 2,10]).

As we mentioned in Section 1 the above algebraic system is a multivariate analog of the univariate polynomial equation (3). The important point to note here is that there is a multivariate analog of fundamental theorem of algebra (MFTA) for the algebraic system (44). According to the classic fundamental theorem of algebra the number of roots of (3) equals n, i.e., the number of coefficients a_i (no matter they are zero or not), or the number of monomials following the leading term. It turns out that the number of solutions of system (44), denoted by ν, does not exceed the number of coefficients in an equation there, i.e., $\nu \le \mu = \#I$. And there is a nice necessary and sufficient condition (coinciding with (42)) for the equality $\nu = \mu$ stated as MFTA2 [H03a] (see also [HT02b]). Also the case when all above solutions of the system (44) are distinct is characterized for the algebraic system in MFTA1 [HT98a], [HT00], [HT02b].

Let us mention the close relation of many of our results on algebraic system (44) with the corresponding results in earlier paper of Y. Xu [X94], where the special case of common zeros of multivariate orthogonal polynomials is considered. Also the basic statements of MFTA2, except one (see details in [H03a]), follow from a result on normal form algorithms of B. Mourrain ([M99, Theorem 3.1]) and a result from ideal theory on the codimension of a 0-dimensional polynomial ideal (see e.g., [BR91]).

It is worth pointing out that MFTA2 is an extension of MFTA1 to the case of arbitrary (multiple) solutions of the system (44). For the proof of MFTA1, stated first in [HT98a] (see also [HT98b]), we refer the reader to [HT02b]. In the next section we outline a proof of MFTA2 from [H03a] based on a result on HO systems of PDEs.

Theorem 6.2 (MFTA1)**.** *The number of distinct solutions of the algebraic system* (44) *is the maximal possible, which is* $\#I$, *if and only if the matrices* $\mathcal{A}_i,\ 1 \le i \le n$, *are commuting and each of them is semisimple.*

Moreover, these solutions form a poised set of points for Lagrange interpolation with Π^n_I. *And any poised set is the set of solutions of exactly one algebraic system of type* (44), *for which the associated matrices satisfy the above conditions.*

Theorem 6.3 (MFTA2)**.** *The number of solutions of the algebraic system* (44), *counting also the multiplicities, is the maximal possible, which is* $\#I$, *if and only if the matrices* $\mathcal{A}_i,\ 1 \le i \le n$, *commute.*

Moreover, these solutions form a poised set of points for Hermite interpolation with Π^n_I. *And any poised set is the set of solutions of exactly one algebraic system of type* (44), *for which the associated matrices commute.*

Let us mention that in the univariate case there is just one associated matrix. Thus the commuting condition falls and the above theorem turns into the classic fundamental theorem of algebra.

It is also worth mentioning that, as in the univariate case with Equation (3), the (multi)sets of solutions of the algebraic system (44) coincide exactly with the poised sets of multivariate Lagrange and Hermite interpolations.

Next we discuss the notion of (common) multiple zeros of multivariate polynomials mentioned in MFTA2. The derivative conditions at a multiple point $\boldsymbol{\lambda}$ are characterized by partial differential operators arisen by a space of polynomials $\mathcal{P}_\lambda$, called *m*ultiplicity space. Namely, we have

$$q(D)\,[\mathbf{x}^\alpha - P_\alpha(\mathbf{x})]\,|_{\mathbf{x}=\lambda} = 0, \quad \text{for all} \;\; q \in \mathcal{P}_\lambda,\ \alpha \in \delta(I).$$

Let as note that we consider multiple Hermite zero, i.e., result of coalescence of simple zeros. Then, as it is shown in [H03a], the multiplicity space $\mathcal{P}_\lambda$ necessarily is D-invariant:

$$p \in \mathcal{P}_\lambda \Rightarrow D^\alpha p \in \mathcal{P}_\lambda \quad \text{for all } \alpha \in \mathbb{Z}^n_+.$$

Let us mention that this approach to multiple zero readily becomes identical with one based on ideal interpolation schemes, pointed out by Marinari, Möller, and Mora [MMM91] as well as de Boor and Ron [BR92].

In view of this we have the characterization:

$$\mathcal{P}_\lambda = \{q \in \Pi^n : D^\beta q(D)\,[\mathbf{x}^\alpha - P_\alpha(\mathbf{x})]\,|_{\mathbf{x}=\lambda} = 0, \quad \text{for all} \;\; \alpha \in \delta(I),\ \beta \in \mathbb{Z}^n_+\}.$$

The dimension of the multiplicity space $\mathcal{P}_\lambda$ is called the (arithmetical) multiplicity of $\boldsymbol{\lambda}$ (see [MMM96]):

$$\mu_\lambda = \dim \mathcal{P}_\lambda.$$

Thus, according to MFTA2 the associated matrices commute if and only if

$$\sum_{\lambda \in \mathcal{S}} \mu_\lambda = \#I. \tag{45}$$

We bring a result from [H03a] (see also [HT02b]) that extends Theorem 6.1 to the case of multiple solutions. Also it shows that the concept of multiple zero considered above is natural from the point of view of PDEs.

Theorem 6.4 ([H03a]). *The following assertions are equivalent:*

- ▷ $\boldsymbol{\lambda}$ *is a solution of the algebraic system* (44), *with the multiplicity set containing the polynomial* q, *i.e.,* $q \in \mathcal{P}_\lambda$, *which means that*

$$\left[D^\beta q\right](D)\left[\mathbf{x}^\alpha - P_\alpha(\mathbf{x})\right]|_{\mathbf{x}=\lambda} = 0, \quad \text{for all } \ \alpha \in \delta(I),\ \beta \in \mathbb{Z}_+^n.$$

- ▷ *The function* $f = q(\mathbf{x})\exp(\boldsymbol{\lambda} \cdot \mathbf{x})$ *is a solution of the system* (41).

It follows from Theorem 6.4 that if the function

$$z = p(\mathbf{x})\exp(\boldsymbol{\lambda} \cdot \mathbf{x}),\ p \in \Pi^n, \tag{46}$$

is a solution of the system (41) then such are all the following functions:

$$z = [D^\alpha p]\,\mathbf{x})\exp(\boldsymbol{\lambda} \cdot \mathbf{x}),\ \alpha \in \mathbb{Z}_+^n.$$

In the case when all the solutions of the algebraic system are distinct (simple) then, according to Theorems 2.2 and 6.1, the general solution of the HO system of PDEs (41) can be presented as linear combination of functions of form

$$z = \exp(\boldsymbol{\lambda} \cdot \mathbf{x}).$$

Here we use also the linear independence of above functions for distinct values of $\boldsymbol{\lambda}$.

In view of this and MFTA2 one could expect, that whenever the consistency conditions are satisfied, the general solution of the system of PDEs (41) can be presented as linear combination of functions of form (46).

As it is proved in [H03a] the above statement remains valid even if the consistency conditions are not satisfied. Moreover this result yields MFTA2. We outline this in the next section (see Theorem 7.1).

Thus in view of these results, solving the PDE system (41) is reducing completely to the finding of common simple and multiple zeros of the characteristic algebraic system (44).

Remark 6.5. Let us mention that Theorem 6.1 indicates a method of solving of type (44) multivariate algebraic systems, by reducing them to finding the common eigenvectors of the matrices $\mathcal{A}_i,\ 1 \le i \le n$.

Thus to solve (44) it is enough to solve just n separate univariate algebraic equations: to determine the eigenvalues of the matrices $\mathcal{A}_i,\ 1 \le i \le n$. After this, in view of Theorem 6.1(third assertion), we can find the common eigenvectors by simple checking, using all possible sequences of eigenvalues.

Moreover, if the case described in Remark 4.4 takes place with the matrices $\mathcal{A}_i,\ 1 \le i \le n$, then solving of the algebraic system (44) is reducing to solving just one nonlinear equation to find the distinct eigenvalues of a matrix, and several linear systems to find the corresponding eigenvectors.

Next, we discuss an example. Consider the following type (44) algebraic system:

$$r_{2,0} := x^2 = 0; \quad r_{1,1} := xy = 0; \quad r_{0,2} := y^2 - x = 0. \tag{47}$$

It can be checked easily that the corresponding matrices $\mathcal{A}_1$ and $\mathcal{A}_2$ here commute. This system has only one solution $(0,0)$ with the multiplicity conditions

$$q\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) r_{i,j}(0,0) = 0 \quad \text{for} \quad i+j \le 2,$$

where $q = 1, y, x + y^2$. Therefore, since $e^0 = 1$, these latter functions form a set of fundamental solutions of the corresponding second order system of PDEs of type (41), i.e.,

$$\frac{\partial^2 z}{\partial x^2} = 0, \quad \frac{\partial^2 z}{\partial x \partial y} = 0, \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial x}.$$

Let us mention also that the system (47) is the limiting case of the following system (as $\varepsilon \to 0$):

$$x^2 = \varepsilon^2 x, \quad xy = \varepsilon x, \quad y^2 = (1-\varepsilon)x + \varepsilon^2 y,$$

for which the corresponding matrices $\mathcal{A}_1$ and $\mathcal{A}_2$ also commute. This system has three distinct solutions: $(0,0)$; $(0,\varepsilon^2)$; $(\varepsilon^2, \varepsilon)$. The solutions of corresponding second order system of PDEs of type (41) will be 1, $\exp(\varepsilon^2 y)$, $\exp(\varepsilon^2 x + \varepsilon y)$.

7. Some applications of Pfaff and higher order systems of PDEs

The main application of the PDEs discussed in this paper is the characteristic algebraic system arisen from the constant coefficient case of the HO system. This establishes a connection with the multivariate fundamental theorem of algebra (MFTA) for the mentioned algebraic systems, similar to the classic univariate case.

As it turns out, in the multivariate case, in contrast to the univariate, one can readily derive the fundamental theorem of algebra (MFTA2) from the following result concerning the HO system of PDEs:

Theorem 7.1 ([H03a])**.** *The general solution of the* HO *system of* PDEs (41) *can be presented as*

$$z = \sum_{i=1}^{m} c_i p_i(\mathbf{x}) \exp(\boldsymbol{\lambda}_i \cdot \mathbf{x}), \tag{48}$$

where $p_i \in \Pi^n, \boldsymbol{\lambda}_i \in \mathbb{C}^n$.

Let us mention that this is more than needed for MFTA2. Namely, here we do not assume that the consistency conditions (42) hold. Note that $\boldsymbol{\lambda}_i$ above, and also in the forthcoming relation (53), are not necessarily distinct.

Now let us verify that Theorem 7.1 yields the main statements of MFTA2. Suppose that the number of solutions of the algebraic system (44), counting also multiplicities, equals $\#I$. In view of (45) this means

$$\sum_{\lambda \in \mathcal{S}} \#\{P \in \mathcal{B}[\mathcal{P}_\lambda]\} = \#I,$$

where $\mathcal{B}[\mathcal{P}_\lambda]$ is a basis of the multiplicity space $\mathcal{P}_\lambda$. (More precisely, it is so-called level basis, see [H03a].) We are to verify that then the corresponding solutions of the system of PDEs (41) (see Theorem 6.4):

$$\left\{P(\mathbf{x})\exp(\boldsymbol{\lambda}\cdot\mathbf{x}) : \boldsymbol{\lambda}\in\mathcal{S}, P\in\mathcal{B}[\mathcal{P}_\lambda]\right\}$$

are linearly independent and hence form a fundamental set of solutions, which, in view of Theorem 2.2 means that the associated matrices commute, and vice versa.

The above statements follow from the following relation between Wronskian and generalized Vandermonde determinant (see [H03a]):

$$W_I(\mathbf{x}|\mathcal{S},\mathcal{B}) = \prod_{\lambda\in\mathcal{S}} \exp(\mu_\lambda\boldsymbol{\lambda}\cdot\mathbf{x})V_I(\mathcal{S},\mathcal{B}). \tag{49}$$

They are defined as follows:

$$W_I(\mathbf{x}|\mathcal{S},\mathcal{B}) := \det||D^\alpha P(\mathbf{x})\exp(\boldsymbol{\lambda}\cdot\mathbf{x})||_{\alpha\in I,\lambda\in\mathcal{S},P\in\mathcal{B}[\mathcal{P}_\lambda]},$$

and

$$V_I(\mathcal{S},B) := \det||P(D)\{\boldsymbol{\lambda}^\alpha\}_*||_{\alpha\in I,\lambda\in\mathcal{S},P\in\mathcal{B}[\mathcal{P}_\lambda]},$$

where we use the lexicographical order for rows and some fixed order for the columns.

Here we are to recall, that $W_I \neq 0$ means that the corresponding set of solutions forms a fundamental set (Section 5), while $V_I \neq 0$ means that the number of (linearly independent) solutions of (44), counting also multiplicities, equals $\#I$.

The proof of (49) is straightforward in view of the relation (see [H03a])

$$D^\alpha\left[P(\mathbf{x})\exp(\boldsymbol{\lambda}\cdot\mathbf{x})\right] = \sum_\beta \frac{1}{\beta!}P^{(\beta)}(D)\,\boldsymbol{\lambda}^\alpha\mathbf{x}^\beta\exp(\boldsymbol{\lambda}\cdot\mathbf{x}), \qquad \alpha\in I.$$

and elementary operations of determinant applied to the Wronskian. The above formula is to be treated as an relation of column vectors with coordinates $\alpha\in I$.

Let us mention that the above elementary operations can be carried out also when the number of linearly independent solutions is less than $\#I$, i.e., instead of Wronskian and Vandermonde determinant we have matrices with the number of columns less than the number of rows. Then what we obtain is that the ranks of these matrices are equal. Hence the following statement is true, where also the consistency conditions (42) are not assumed to hold:

Corollary 7.2 ([H03a])**.** *The dimension of the linear space of solutions of the* HO *system of* PDEs (41) *equals the number of solutions of the algebraic system* (44)*, counting also the multiplicities.*

To establish Theorem 7.1 we need to prove first the following:

Lemma 7.3 ([H03a])**.** *Assume that the set of functions*

$$\mathcal{G} := \mathcal{G}_m := \{g_1,\ldots,g_m\}$$

is linearly independent. Then there is a set of points $\{\mathbf{x}_1,\ldots,\mathbf{x}_m\}$ *such that the respective Vandermonde determinant does not vanish, i.e.,*

$$V(\mathbf{x}_1,\ldots,\mathbf{x}_m;\mathcal{G}) := \det||g_i(\mathbf{x}_j)||_{i,j=1,\ldots,m} \neq 0.$$

Proof. Let us use induction on m. The case $m = 1$ is obvious. Assume the lemma is true for $m-1$, i.e., there is a set $\{\mathbf{x}_1^0, \ldots, \mathbf{x}_{m-1}^0\}$ such that

$$V(\mathbf{x}_1^0, \ldots, \mathbf{x}_{m-1}^0; \mathcal{G}_{m-1}) \neq 0. \tag{50}$$

Now let us verify that the determinant

$$V(\mathbf{x}_1^0, \ldots, \mathbf{x}_{m-1}^0, \mathbf{x}_m; \mathcal{G})$$

is not identically zero, which will complete the proof. Indeed, it is easily seen that this determinant is a nontrivial linear combination of set $\mathcal{G}$, since the coefficient of g_m there is $(-1)^m$ times the determinant in (50) and thus is not zero. □

Sketch of proof of Theorem 7.1. (See [H03a].) In view of Theorem 2.1 (equivalence) it is enough to prove the theorem with the system (41) replaced by the system (43) (or (36)). Correspondingly, we are to show that the first coordinate $z_{\overline{0}}$ of any solution of (43) has the form (48). We shall prove this by induction on n, i.e., on the dimension. If $n = 1$, then we have the case of system of ordinary differential equations which is well-known. Assume the theorem is true for the case $n-1$; we shall prove it for n. Suppose that $z(\mathbf{x})$ is any solution of (43). Then for any fixed last coordinate x_n this is also a solution of the system

$$\frac{\partial\{z_*\}}{\partial x_i} = \mathcal{A}_i\{z_*\}, \qquad i = 1, \ldots, n-1.$$

Therefore, according to the induction hypothesis we have

$$z_{\overline{0}}(\mathbf{x}) = \sum_{i=1}^{m} c_i(x_n) p_i(\widetilde{\mathbf{x}}) \exp(\widetilde{\boldsymbol{\lambda}}_i \cdot \widetilde{\mathbf{x}}), \tag{51}$$

where $p_i \in \Pi^{n-1}$, $\widetilde{\boldsymbol{\lambda}}_i \in \mathbb{C}^{n-1}$, $\widetilde{\mathbf{x}} = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. Without loss of generality we can assume that the set of functions

$$\widetilde{\mathcal{G}} = \{\widetilde{g}_i\}_{i=1}^m,$$

where $\widetilde{g}_i = p_i(\widetilde{\mathbf{x}}) \exp(\widetilde{\boldsymbol{\lambda}}_i \cdot \widetilde{\mathbf{x}})$, is linearly independent. Then we apply the previous lemma and get a set of points $\{\widetilde{\mathbf{x}}_1, \ldots, \widetilde{\mathbf{x}}_m\}$ such that

$$V(\widetilde{\mathbf{x}}_1, \ldots, \widetilde{\mathbf{x}}_m; \widetilde{\mathcal{G}}) \neq 0. \tag{52}$$

Now we notice that $z(\mathbf{x})$ with any fixed first $n-1$ coordinates is a solution of the system

$$\frac{\partial\{z_*\}}{\partial x_n} = \mathcal{A}_n\{z_*\}.$$

Therefore we have

$$\sum_{i=1}^{m} c_i(x_n) p_i(\widetilde{\mathbf{x}}_j) \exp(\widetilde{\boldsymbol{\lambda}}_i \cdot \widetilde{\mathbf{x}}_j) = \sum_{i=1}^{m'} c'_{ij} p'_i(x_n) \exp(\lambda_i x_n), \qquad j = 1, \ldots, m.$$

We consider this as a linear system with respect to unknowns $c_i(x_n)$, whose main determinant coincides with the one in (52) and thus is not zero. Since the entries of the main determinant are constants, we get that c_i, $i = 1, \ldots, m'$, are linear combinations of the functions in the right sides above. Therefore, by substituting them into (51) we come to the desired conclusion. □

Note that at the same time we have proved:

Corollary 7.4 ([H04]). *The general solution of the Pfaff system of* PDEs (36) *can be presented as*

$$\mathbf{z} = \sum_{i=1}^{m} c_i \mathbf{h}_i(\mathbf{x}) \exp(\boldsymbol{\lambda}_i \cdot \mathbf{x}), \tag{53}$$

where $\boldsymbol{\lambda}_i \in \mathbb{C}^n,\ \mathbf{h}_i = (p_{i1}, \dots, p_{im})$ *with* $p_{ij} \in \Pi^n$.

Next, let us mention the following two applications of MFTA in algebraic geometry (see [H03b], [H04]):

The first one is the Bezout theorem for n polynomials $g_1, \dots, g_n \in k[x_1, \dots, x_n]$.

Notably the intersection multiplicities: $\mu_\lambda(\mathcal{G}),\ (\mathcal{G} = \{g_1, \dots, g_n\})$ as in MFTA, are characterized *j*ust by means of partial differential operators given by polynomials from D-invariant linear spaces:

$$\mathcal{P}_\lambda(\mathcal{G}) := \{p : [D^\alpha p]\,(D) g_i(\mathbf{x})|_{\mathbf{x}=\lambda} = 0, \quad i = 1, \dots, n, \text{ for all } \alpha \in \mathbb{Z}_+^n\},$$
$$\mu_\lambda(\mathcal{G}) := \dim \mathcal{P}_\lambda(\mathcal{G}).$$

Theorem 7.5 (Bezout, [H03b]). *Let k be an algebraically closed field. Suppose that there is no intersection at "infinity". Then*

$$\sum_\lambda \mu_\lambda(\mathcal{G}) = \begin{cases} \deg(g_1) \cdots \deg(g_n), & \text{or} \\ \infty. \end{cases}$$

The case of the intersection at "infinity" can be treated in a standard way.

The second consequence of MFTA is the following:

Theorem 7.6 ([H03b]). *Let k be an algebraically closed field and $f, f_1, \dots, f_s \in k[x_1, \dots, x_n]$. Then f belongs to the polynomial ideal $\langle f_1, \dots, f_s \rangle$ if and only if for any $g \in k[x_1, \dots, x_n]$,*

$$\begin{aligned} &[D^\alpha g]\,(D) f_i(x) = 0, \quad i = 1, \dots, s, \ \textit{for all } \alpha \in \mathbb{Z}_+^n, \quad \textit{implies} \\ &[D^\alpha g]\,(D) f(x) = 0, \quad \textit{for all } \alpha \in \mathbb{Z}_+^n. \end{aligned} \tag{54}$$

As it is mentioned in [H03b] one readily gets Nullstellensatz from here.

Next we bring the construction (see [H02], [H04]) which shows that actually any standard algebraic system, with finite set of solutions, can be reduced to a type (44) algebraic system. Infact, the proofs of Theorems 7.5 and 7.6 are based on this device.

For simplicity, we illustrate this in the bivariate case. Consider the following general system with finitely many solutions (see [H04] for details):

$$p(x, y) = 0, \quad q(x, y) = 0. \tag{55}$$

Suppose that p and q are irreducible polynomials of degree n and m, respectively. Thus the coordinate system can be chosen such that

(56) the leading terms of p and q have no common factor of degree ≥ 1.

Denote by C_n and C_m the curves given by the equations in (55), respectively. Let $\mathcal{S}_0$ be the set of solutions of this system. At the same time it is the set of intersection of above curves.

Choose sets $\mathcal{X} \subset C_n \backslash C_m$ and $\mathcal{Y} \subset C_m \backslash C_n$ such that they are poised for interpolation with Π^2_{n-2} and Π^2_{m-2}, respectively.

As such sets $\mathcal{X}$ and $\mathcal{Y}$ one can just take intersection of C_n and C_m with suitable sets of $n-1$ and $m-1$ lines. This argument goes back to Berzolari [B14], (see also [C59, p. 24]).

We reduce (55) to the following system of polynomial equations of total degree $n+m-1$:

$$\begin{aligned} &[x^i y^j - \psi_i(x,y)]\, p(x,y) = 0, \quad i+j = n-1, \\ &[x^k y^l - \phi_k(x,y)]\, q(x,y) = 0, \quad k+l = m-1, \end{aligned} \tag{57}$$

where ψ_i and ϕ_k are the interpolants of $x^i y^j$ and $x^k y^l$ with respect to the node sets $\mathcal{X}$ and $\mathcal{Y}$, respectively.

In view of (56) the leading terms, i.e., the terms of degree $n+m-1$, in the left-hand sides of Equations (57) are linearly independent. From other hand, there are $n+m$ equations. Therefore, the linear span of above leading terms contains all monomials of degree $n+m-1$. Thus by taking linear combinations of equations in the system (57) we can readily transform it to a type (44) system.

Moreover, it can be shown that for this resulted system the consistency conditions are satisfied automatically (see [H04]).

Now, notice that the set of solutions of (57) is exactly

$$\mathcal{S} := \mathcal{S}_0 \cup \mathcal{X} \cup \mathcal{Y}.$$

The cardinality of $\mathcal{S}$, according to MFTA2, equals $\binom{n+m}{2}$, while $\#\mathcal{X} = \binom{n}{2}$ and $\#\mathcal{Y} = \binom{m}{2}$.

Thus we get that

$$\#\mathcal{S}_0 = \binom{n+m}{2} - \binom{n}{2} - \binom{m}{2} = nm,$$

which is the Bezout theorem (with multiplicities described by partial differential operators) in bivariate case.

Next, we discuss briefly the proof of Theorem 7.6 in the bivariate case (see [H04]). We consider only the basic case $s = 2$ (cf. Noether's fundamental theorem [C59, pp. 29, 244]). Namely, let us show that

$$\langle p, q \rangle = \mathcal{N}, \tag{58}$$

where p, q are as above and $\mathcal{N}$ is the set (ideal) of polynomials f satisfying

$$\begin{aligned} &[D^\alpha g]\,(D) p(x) = 0 \text{ and } [D^\alpha g]\,(D) q(x) = 0 \text{ for all } \alpha \in \mathbb{Z}_+^n, \quad \text{implies} \\ &[D^\alpha g]\,(D) f(x) = 0, \quad \text{for all } \alpha \in \mathbb{Z}_+^n, \end{aligned} \tag{59}$$

for any polynomial g.

The part $\subseteqq$ in (58) is straightforward. For $\supseteqq$ suppose that $f \in \mathcal{N}$, i.e., (59) is satisfied for any polynomial g. We start with

$$f = (f - P_f) + P_f, \tag{60}$$

where $P_f := P_{f,\mathcal{S},n+m-2}$ is the interpolation polynomial (of degree $n+m-2$) of f with respect to $\mathcal{S}$, i.e., the solution set of (57).

Now, as in [HT02b, Proof of Proposition 1], we get $f - P_f \in \langle p, q \rangle$. Concerning the second summand, in view of $f \in \mathcal{N}$, we have

$$P_f = pP_{r,\mathcal{Y},m-2} + qP_{s,\mathcal{X},n-2} \in \langle p, q \rangle,$$

where $r = \frac{f}{p}, s = \frac{f}{q}$.

The above procedure with algebraic system (55) can be applied also for systems of PDEs. For example, given the system of two PDEs (see [H04] for details)

$$p\,(D_x, D_y)\,z = f(x,y), \quad q\,(D_x, D_y)\,z = g(x,y), \tag{61}$$

where p and q are the polynomials from (55). In the same way as above, we arrive to the following system:

$$\begin{aligned} &[(D_x)^i (D_y)^j - \psi_i\,(D_x, D_y)]p\,(D_x, D_y)\,\widetilde{z} = \widetilde{f}(x,y), \quad i+j = n-1, \\ &[(D_x)^k (D_y)^l - \phi_k\,(D_x, D_y)]q\,(D_x, D_y)\,\widetilde{z} = \widetilde{g}(x,y), \quad k+l = n-1, \end{aligned} \tag{62}$$

where

$$\widetilde{f} = [(D_x)^i (D_y)^j - \psi_i\,(D_x, D_y)]f, \quad \widetilde{g} = [(D_x)^k (D_y)^l - \phi_k\,(D_x, D_y)]g. \tag{63}$$

As above, this system can be transformed, by taking linear combinations of equations, to a type (14) HO system (of total degree). Moreover, the corresponding homogeneous system satisfies consistency conditions. This implies that the nonhomogeneous system (62) satisfies consistency conditions provided that (as we assume) the system of PDEs (61) has a solution. Therefore, after the finding the solutions of the homogeneous system we can solve the system (62), by using the method of variation of constants for HO systems of PDEs described in Section 5. Thus, we can find $\widetilde{z}$. To find the solution of the system (61): z, notice that, $\widetilde{z} - z$ satisfies the system homogeneous to (62), hence

$$\widetilde{z} = z + \sum_{(\lambda,\mu)\in\mathcal{X}} a_{\lambda,\mu} e^{\lambda x + \mu y} + \sum_{(\lambda,\mu)\in\mathcal{S}_0} b_{\lambda,\mu} e^{\lambda x + \mu y} + \sum_{(\lambda,\mu)\in\mathcal{Y}} c_{\lambda,\mu} e^{\lambda x + \mu y}. \tag{64}$$

Here the coefficients $b_{\lambda,\mu}$ are arbitrary, while $a_{\lambda,\mu}$ and $c_{\lambda,\mu}$ can be found from the relations

$$\begin{aligned} p\,(D_x, D_y)\,\widetilde{z} &= f + \sum_{(\lambda,\mu)\in\mathcal{Y}} c_{\lambda,\mu} p(\lambda,\mu) e^{\lambda x + \mu y}, \\ q\,(D_x, D_y)\,\widetilde{z} &= g + \sum_{(\lambda,\mu)\in\mathcal{X}} a_{\lambda,\mu} q(\lambda,\mu) e^{\lambda x + \mu y}, \end{aligned} \tag{65}$$

which we get readily from (64) by applying $p\,(D_x, D_y)$ and $q\,(D_x, D_y)$, respectively. And conversely, the latter applications also imply that z determined from (64), where the coefficients $a_{\lambda,\mu}$ and $c_{\lambda,\mu}$ are given by the relation (65), is a solution of the original system (61). Thus, the solving of the system of PDEs (61) completely reduces to the solving of the system of PDEs (62). In particular it is solvable if and only if the latter satisfies the consistency conditions.

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