New York Journal of Mathematics

New York J. Math. **11** (2005) 291–302.

When is Galois cohomology free or trivial?

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Abstract. Let p be a prime, F a field containing a primitive pth root of unity, and E/F a cyclic extension of degree p. Using the Bloch–Kato Conjecture we determine precise conditions for the cohomology group $H^n(E) := H^n(G_E, \mathbb{F}_p)$ to be free or trivial as an $\mathbb{F}_p[\mathrm{Gal}(E/F)]$-module, and we examine when these properties for $H^n(E)$ are inherited by $H^k(E)$, $k > n$. By analogy with cohomological dimension, we introduce notions of cohomological freeness and cohomological triviality, and we give examples of $H^n(E)$ free or trivial for each $n \in \mathbb{N}$ with prescribed cohomological dimension.

Contents

Let p be a prime and F a field containing a primitive pth root of unity ξ_p. Let E/F be a cyclic extension of degree p, G_E the absolute Galois group of E, and $G = \mathrm{Gal}(E/F)$. In [LMS] we determined the structure of $H^n(E) := H^n(G_E, \mathbb{F}_p)$, $n \in \mathbb{N}$, as an $\mathbb{F}_p[G]$-module. In this paper we study more closely the question of when $H^n(E)$ is free or trivial as an $\mathbb{F}_p[G]$-module.

Received October 26, 2004.

Mathematics Subject Classification. 12G05 (primary), 19D45 (secondary).

Key words and phrases. Galois cohomology, Milnor K-theory, Bloch–Kato conjecture, pro-p-group, free product.

The first author was supported in part by NSERC grant R3276A01.

The second author was supported in part by NSERC grant R0370A01, by the Institute for Advanced Study, Princeton, by a Distinguished Professorship during 2004–2005 at the University of Western Ontario, and by the Mathematical Sciences Research Institute, Berkeley.

The third author was supported in part by NSA grant MDA904-02-1-0061.

ISSN 1076-9803/05

1. Free cohomology

Let $a \in F$ satisfy $E = F(\sqrt[p]{a})$. We write $\operatorname{ann}_n x$ for the annihilator of $x \in H^m(F)$, $m \in \mathbb{N}$, under the cup-product operation on $H^n(F)$. Let $(f) \in H^1(F)$ denote the class of f under the Kummer isomorphism of $H^1(F)$ with the pth-power classes of $F^\times := F \setminus \{0\}$, and let $(f, g) \in H^2(F)$ denote the cup-product of (f) and $(g) \in H^1(F)$.

Theorem 1. *Let $n \in \mathbb{N}$. If $p > 2$, then the following are equivalent:*

(1) $H^n(E)$ *is a free* $\mathbb{F}_p[G]$*-module.*
(2) $H^{n-1}(F) = \operatorname{ann}_{n-1}(a)$.
(3) $\operatorname{res}\colon H^n(F) \to H^n(E)$ *is injective.*
(4) $\operatorname{cor}\colon H^{n-1}(E) \to H^{n-1}(F)$ *is surjective.*

If $p = 2$, then the following are equivalent:

(1) $H^n(E)$ *is a free* $\mathbb{F}_2[G]$*-module.*
(2) $\operatorname{ann}_{n-1}(a) = \operatorname{ann}_{n-1}(a, -1)$ *and* $H^n(F) = \operatorname{cor} H^n(E) + (a) \cup H^{n-1}(F)$.
(3) $\operatorname{ann}_{n-1}(a) = \operatorname{ann}_{n-1}(a, -1)$ *and* $H^n(F) = \operatorname{ann}_n(a) + (a) \cup H^{n-1}(F)$.
(4) $H^n(F) = \operatorname{ann}_n(a) \oplus (a) \cup H^{n-1}(F)$.

We prove this theorem using the sequence of results below.

For $i \geq 0$, let K_iF denote the ith Milnor K-group of the field F, with standard generators denoted by $\{f_1, \ldots, f_i\}$, $f_1, \ldots, f_i \in F^\times$. For $\alpha \in K_iF$, we denote by $\bar{\alpha}$ the class of α modulo p, and we abbreviate K_nF/pK_nF by k_nF. Throughout the paper we use the Bloch–Kato Conjecture in identifying k_nF and $H^n(F)$, as well as Hilbert 90 for Milnor K-theory. (See [V1, Lemma 6.11 and §7] and [V2, §6 and Theorem 7.1]. For further expositions of the work of Rost and Voevodsky on Bloch–Kato Conjecture, see [Ro] and [Su].) The image of an element $\alpha \in K_iF$ in $H^i(F)$ we also denote by α. We omit the bars over elements $\overline{\{a\}}$, $\overline{\{\xi_p\}}$, $\overline{\{a, a\}}$, $\overline{\{a, \xi_p\}}$, and $\overline{\{\sqrt[p]{a}\}}$. We denote the norm map $K_nE \to K_nF$ by $N_{E/F}$, the map $K_nF \to K_nE$ induced by the inclusion map by i_E, and multiplication by $\{a\}$ from $K_{n-1}F$ to K_nF by $\{a\} \cdot -$. We use the same notation for the analogous maps of K-theory modulo p.

The following theorem is a strengthening of [V1, Prop. 5.2].

Theorem 2 ([LMS, Thm. 5])**.** *Let F be a field containing a primitive pth root of unity. Then for any cyclic extension E/F of degree p and $m \geq 1$ the sequence*

$$k_{m-1}E \xrightarrow{N_{E/F}} k_{m-1}F \xrightarrow{\{a\}\cdot -} k_mF \xrightarrow{i_E} k_mE$$

is exact.

Corollary. *Assume the same hypotheses. The following are equivalent for each $n \in \mathbb{N}$:*

(1) $\operatorname{ann}_{n-1}\{a\} = \operatorname{ann}_{n-1}\{a, -1\}$ *and* $k_nF = N_{E/F}k_nE + \{a\} \cdot k_{n-1}F$.
(2) $\operatorname{ann}_{n-1}\{a\} = \operatorname{ann}_{n-1}\{a, -1\}$ *and* $k_nF = \operatorname{ann}_n\{a\} + \{a\} \cdot k_{n-1}F$.
(3) $k_nF = \operatorname{ann}_n\{a\} \oplus \{a\} \cdot k_{n-1}F$.

Proof. (1) $\implies$ (2) follows from $N_{E/F}k_nE = \operatorname{ann}_n\{a\}$.

(2) $\implies$ (3) Let $\bar{\alpha} \in (\{a\} \cdot k_{n-1}F) \cap \operatorname{ann}_n\{a\}$. Then $\bar{\alpha} = \{a\} \cdot \bar{f}$ for some $f \in K_{n-1}F$, and $0 = \{a, a\} \cdot \bar{f} = \{a, -1\} \cdot \bar{f}$. By the first hypothesis, $\{a\} \cdot \bar{f} = 0$. Then $\bar{\alpha} = 0$ and the sum is direct.

(3) $\Longrightarrow$ (1) The second claim follows from the fact that $\mathrm{ann}_n\{a\} = N_{E/F}k_nE$. For the first, suppose $\{a,-1\}\cdot\bar{f} = 0$ for $f \in K_{n-1}F$. Because $\{a,-1\} = \{a,a\}$, we have $\{a\}\cdot\bar{f} \in (\mathrm{ann}_n\{a\}) \cap (\{a\}\cdot k_{n-1}F) = \{0\}$. Hence $\bar{f} \in \mathrm{ann}_{n-1}\{a\}$ and $\mathrm{ann}_{n-1}\{a\} = \mathrm{ann}_{n-1}\{a,-1\}$. □

For any $\mathbb{F}_p[G]$-module W we denote by W^G the submodule of W consisting of elements fixed by G. For $\gamma \in K_nE$, let $l(\gamma)$ denote the dimension of the cyclic $\mathbb{F}_p[G]$-submodule $\langle\bar{\gamma}\rangle$ of k_nE generated by $\bar{\gamma}$. Then we have, for $l(\gamma) \geq 1$,

$$(\sigma-1)^{l(\gamma)-1}\langle\bar{\gamma}\rangle = \langle\bar{\gamma}\rangle^G \neq 0 \quad \text{and} \quad (\sigma-1)^{l(\gamma)}\langle\bar{\gamma}\rangle = 0.$$

We denote by N the map $(\sigma-1)^{p-1}$ on k_nE. Because

$$(\sigma-1)^{p-1} = 1+\sigma+\cdots+\sigma^{p-1}$$

in $\mathbb{F}_p[G]$, we shall use $i_EN_{E/F}$ and N interchangeably on k_nE.

Lemma 1. *Let $n \in \mathbb{N}$. Suppose that either*

- *$p > 2$ and $N_{E/F}\colon k_{n-1}E \to k_{n-1}F$ is surjective, or*
- *$p = 2$ and $k_nF = N_{E/F}k_nE \oplus \{a\}\cdot k_{n-1}F$.*

Then we have:

(1) *For each $\gamma \in K_nE$, there exists $\alpha \in K_nE$ with $\langle N\bar{\alpha}\rangle = \langle\bar{\gamma}\rangle^G$.*
(2) *$(k_nE)^G = i_EN_{E/F}k_nE = (\sigma-1)^{p-1}k_nE = i_Ek_nF$.*

Proof. (1) Assume first that $p > 2$. By hypothesis, $N_{E/F}\colon k_{n-1}E \to k_{n-1}F$ is surjective, and then using the projection formula ([FW, p. 81]) we see that $N_{E/F}\colon k_nE \to k_nF$ is also surjective. Hence if $\bar{\gamma} \in i_Ek_nF$ then there exists $\bar{\alpha} \in k_nE$ such that $N\bar{\alpha} = \bar{\gamma}$ and we are done. Otherwise, let $l = l(\gamma)$ and suppose $\bar{\gamma} \notin i_Ek_nF$ and $1 \leq l \leq i \leq p$.

If $l \geq 2$ we show by induction on i that there exists $\alpha_i \in K_nE$ such that $\langle(\sigma-1)^{i-1}\bar{\alpha}_i\rangle = \langle\bar{\gamma}\rangle^G$. Then setting $\alpha := \alpha_p$, the proof will be complete in the case when $2 \leq l$. Assume then that $l \geq 2$. If $i = l$ then $\alpha_i = \gamma$ suffices. Assume now that $2 \leq l \leq i < p$ and that our statement is true for i. Set $c = N_{E/F}\alpha_i$. Since $i_E\bar{c} = N\bar{\alpha}_i = (\sigma-1)^{p-1}\bar{\alpha}_i$ and $i < p$, $i_E\bar{c} = 0$. By our hypothesis and Theorem 2, we have $\bar{c} = 0$, that is, $c = pf$ for some $f \in K_nF$. Hence $N_{E/F}(\alpha_i - i_E(f)) = 0$. By Hilbert 90, there exists $\omega \in K_nE$ such that $(\sigma-1)\omega = \alpha_i - i_E(f)$. Since $l(\alpha_i) > 1$ we have $(\sigma-1)^2\bar{\omega} = (\sigma-1)\bar{\alpha}_i \neq 0$. Therefore $\langle(\sigma-1)^i\bar{\omega}\rangle = \langle\bar{\gamma}\rangle^G$ and we can set $\alpha_{i+1} = \omega$. We have proved that if $l \geq 2$ then there exists $\alpha \in K_nE$ such that $N\bar{\alpha} = (\sigma-1)^{l(\gamma)-1}\bar{\gamma}$.

Now assume that $l(\gamma) = 1$ but $\bar{\gamma} \notin i_Ek_nF$. Then $\bar{\gamma} = \bar{\alpha}_1$ and proceeding as above we have $(\sigma-1)\bar{\omega} = \bar{\alpha}_1 - i_E(\bar{f}) \neq 0$. Thus $l(\omega) = 2$ and our argument above shows that there exists $\beta \in K_nE$ such that $N\bar{\beta} = (\sigma-1)\bar{\omega} = \bar{\alpha}_1 - i_E(\bar{f})$. As we observed at the beginning of our proof there exists an element $\delta \in K_nE$ such that $N\bar{\delta} = i_E(\bar{f})$. Therefore we have $N(\bar{\beta}+\bar{\delta}) = \bar{\alpha}_1 = \bar{\gamma}$. Hence for each $\gamma \in K_nE$, there exists $\alpha \in K_nE$ such that $\langle N\bar{\alpha}\rangle = \langle\bar{\gamma}\rangle^G$.

Now consider the case $p = 2$. In this case from our hypothesis $k_nF = N_{E/F}k_nE + \{a\}\cdot k_{n-1}F$ we again have $i_EN_{E/F}k_nE = i_Ek_nF$. Therefore if $\bar{\gamma} \in i_Ek_nF$ our statement follows. Assume that $\bar{\gamma} \in k_nE \setminus i_Ek_nF$. (We use this hypothesis only to exclude the trivial case $\bar{\gamma} = 0$.) Since $p = 2$ we see that $l(\gamma) \leq 2$, and if $l(\gamma) = 2$ we may set $\alpha = \gamma$ and (1) follows again. Next we shall assume that $l(\gamma) = 1$ and therefore $\bar{\gamma} \in (k_nE)^G$. Set $c = N_{E/F}\gamma$. Then $i_E\bar{c} = 0$. From Theorem 2, we conclude that $c = \{a\} \cdot g + 2f$ for $g \in K_{n-1}F$ and $f \in K_nF$. Hence from the projection formula, $N_{E/F}(\gamma - \{\sqrt{a}\} \cdot i_E(g) - i_E(f)) = \{-1\} \cdot g$. Using Theorem 2 again, we obtain that $\{a, -1\} \cdot \bar{g} = 0$. Our hypothesis and the Corollary imply that $\{a\} \cdot \bar{g} = 0$. Hence $\{a\} \cdot g = 2h$ for some $h \in K_nF$ and $N_{E/F}\gamma = 2(h + f)$. Thus $N_{E/F}(\gamma - i_E(h + f)) = 0$. Then by Hilbert 90 there exists $\alpha \in K_nE$ such that $(\sigma - 1)\alpha = \gamma - i_E(h + f)$. Hence $\bar{\gamma} = (\sigma - 1)\bar{\alpha} + i_E(\overline{h + f}) \in Nk_nE$.

(2) Suppose $\bar{\gamma} \in (k_nE)^G$. Then by (1) we see that $\langle\bar{\gamma}\rangle = \langle N\bar{\alpha}\rangle$ for $\alpha \in K_nE$. Hence $(k_nE)^G \subset i_EN_{E/F}k_nE$. Then $i_EN_{E/F}k_nE \subset i_Ek_nF \subset (k_nE)^G \subset i_EN_{E/F}k_nE$, and so all inclusions are equalities. □

Proof of Theorem 1. First we show that for all p, k_nE free implies

$$i_EN_{E/F}k_nE = i_Ek_nF = (k_nE)^G.$$

If k_nE is free, then we have $(\sigma - 1)^{p-1}k_nE = (k_nE)^G$. Hence $i_EN_{E/F}k_nE = (\sigma - 1)^{p-1}k_nE = (k_nE)^G$. Then $i_EN_{E/F}k_nE \subset i_Ek_nF \subset (k_nE)^G$ and we have established our claim.

Assume first that $p > 2$. First we show (1) $\Longrightarrow$ (2). Let $f \in K_{n-1}F$ be arbitrary, and set $\alpha = \{\sqrt[p]{a}\} \cdot i_E(f)$. Now because $(k_nE)^G = i_EN_{E/F}k_nE$, there exists $\beta \in K_nE$ such that $i_EN_{E/F}\bar{\beta} = i_E(\{\xi_p\} \cdot \bar{f}) \in (k_nE)^G$. Set $\gamma = (\sigma - 1)^{p-2}\beta$. Since γ is in the image of $\sigma - 1$ we have $N_{E/F}\gamma = 0$. Then $N_{E/F}(\bar{\alpha} - \bar{\gamma}) = \{a\} \cdot \bar{f}$. Also, $(\sigma - 1)(\bar{\alpha} - \bar{\gamma}) = 0$ and therefore $\bar{\alpha} - \bar{\gamma} \in (k_nE)^G = i_Ek_nF$. But on i_Ek_nF the norm map $N_{E/F}$ is trivial. Therefore $\{a\} \cdot \bar{f} = 0$ and $\mathrm{ann}_{n-1}\{a\} = k_{n-1}F$. By Theorem 2, items (2), (3) and (4) are all equivalent. Now we show (4) $\Longrightarrow$ (1). Assume that $N_{E/F}\colon k_{n-1}E \to k_{n-1}F$ is surjective. By Lemma 1 we have

$$(k_nE)^G = (\sigma - 1)^{p-1}k_nE.$$

Hence $H^2(G, k_nE) = \{0\}$ and so k_nE is free. (See [La, p. 63].)

Now suppose that $p = 2$. By the Corollary, we need only show that (1) and (2) are equivalent. We show first that (1) $\Longrightarrow$ (2). We established that (1) implies $i_EN_{E/F}k_nE = i_Ek_nF$. Since $\ker i_E = \{a\} \cdot k_{n-1}F$, this equality is equivalent to $k_nF = N_{E/F}k_nE + \{a\} \cdot k_{n-1}F$, so we have the second part of (2). We now show that $\mathrm{ann}_{n-1}\{a, -1\} \subset \mathrm{ann}_{n-1}\{a\}$. Let $\bar{f} \in \mathrm{ann}_{n-1}\{a, -1\}$. Set $\bar{\alpha} = \{\sqrt{a}\} \cdot i_E(\bar{f})$. Since $\{a\} \cdot \{-1\} \cdot \bar{f} = 0$, Theorem 2 tells us that there exists $\beta \in K_nE$ such that $N_{E/F}\bar{\beta} = \{-1\} \cdot \bar{f}$. Now we calculate by the projection formula $N_{E/F}(\bar{\alpha} - \bar{\beta}) = \{-a\} \cdot \bar{f} - \{-1\} \cdot \bar{f} = \{a\} \cdot \bar{f}$. On the other hand, $(\sigma - 1)(\bar{\alpha} - \bar{\beta}) = \{-1\} \cdot i_E(\bar{f}) - \{-1\} \cdot i_E(\bar{f}) = 0$. Hence $\bar{\alpha} - \bar{\beta} \in (k_nE)^G$. Since k_nE is free, $(k_nE)^G = (\sigma - 1)k_nE$. But on $(\sigma - 1)k_nE$ the norm map $N_{E/F}$ is trivial. Hence $\{a\} \cdot \bar{f} = 0$, and $\bar{f} \in \mathrm{ann}_{n-1}\{a\}$, so $\mathrm{ann}_{n-1}\{a, -1\} \subset \mathrm{ann}_{n-1}\{a\}$. Now we show that (2) $\Longrightarrow$ (1). Assume that $\mathrm{ann}_{n-1}\{a, -1\} = \mathrm{ann}_{n-1}\{a\}$ and that $k_nF = N_{E/F}k_nE + \{a\} \cdot k_{n-1}F$. By Lemma 1 and the Corollary, $(k_nE)^G = (\sigma - 1)k_nE$ and so k_nE is free. □

If $p > 2$ then $N_{E/F}k_0E = \{0\} \neq k_0F \cong \mathbb{F}_p$. Therefore:

Corollary 1. *For $p > 2$, k_1E is never free.*

If $p = 2$ and $-1 \in F^{\times 2}$, then $\mathrm{ann}_0\{a, -1\} = k_0F \cong \mathbb{F}_2 \neq \mathrm{ann}_0\{a\} = \{0\}$ and hence:

Corollary 2. *For $p = 2$ and $\sqrt{-1} \in F$, k_1E is never free.*

2. Hereditary freeness

We say that a property of Milnor k-groups k_nE and k_nF is *hereditary* if the validity of the property for a given n implies the validity of the property for all integers greater than n. We consider the zero $\mathbb{F}_p[G]$-module $\{0\}$ to be a free $\mathbb{F}_p[G]$-module.

Theorem 3. *Suppose that either $p > 2$ or $p = 2$ and $a \in (F^2 + F^2) \setminus F^2$. Then free cohomology is hereditary: if $n \in \mathbb{N}$, then for all $m \geq n$, $H^n(E)$ is a free $\mathbb{F}_p[G]$-module $\implies$ $H^m(E)$ is a free $\mathbb{F}_p[G]$-module. Moreover, if $H^m(E)$, $m \in \mathbb{N}$, is a free $\mathbb{F}_p[G]$-module, then the sequence*

$$0 \to H^m(F) \xrightarrow{\text{res}} H^m(E) \xrightarrow{\text{cor}} H^m(F) \to 0.$$

is exact at the first and third terms. When $p = 2$ this sequence is exact.

Lemma 2. *Let $n \in \mathbb{N}$. The following are hereditary properties:*

(1) $k_{n-1}F = \mathrm{ann}_{n-1}\{a\} = \mathrm{ann}_{n-1}\{a, \xi_p\}$.
(2) $i_E \colon k_nF \to k_nE$ *is injective.*
(3) $N_{E/F} \colon k_{n-1}E \to k_{n-1}F$ *is surjective.*
(4) *for some fixed* $\alpha_1, \alpha_2 \in K_1F$, $\overline{\alpha}_1 \cdot k_{n-1}F \subset \overline{\alpha}_2 \cdot k_{n-1}F$.
(5) *for some fixed* $\alpha \in K_1E$, $k_nE = i_Ek_nF + \overline{\alpha} \cdot i_Ek_{n-1}F$.

Proof. (1) $k_nF = k_{n-1}F \cdot k_1F$, and since $\mathrm{ann}_{n-1}\{a\} = k_{n-1}F$, we have $\mathrm{ann}_n\{a\} = k_nF$ as well. The other equality follows from $\mathrm{ann}_n\{a\} \subset \mathrm{ann}_n\{a, \xi_p\}$. The result follows by induction.

(2)–(3) By Theorem 2 the first three properties are equivalent, hence (2) and (3) are hereditary.

(4) K_nF is generated by elements of the form $\{f_1, f_2, \dots, f_n\} = \{f_1, \dots, f_{n-1}\} \cdot \{f_n\}$, $f_i \in F^\times$. For each such generator, we calculate $\overline{\alpha}_1 \cdot \overline{\{f_1, \dots, f_n\}} = \overline{\alpha}_2 \cdot \overline{g} \cdot \overline{\{f_n\}}$ for some $g \in K_{n-1}F$, whence $\overline{\alpha}_1 \cdot k_nF \subset \overline{\alpha}_2 \cdot k_nF$. The result follows by induction.

(5) The condition on k_nE gives us that $k_{n+1}E$ is generated by elements of the form $\overline{\gamma}_1 = \overline{\{\delta\}} \cdot i_E(\overline{\{f_1, \dots, f_n\}})$, $\delta \in E^\times$, $f_i \in F^\times$ and $\overline{\gamma}_2 = \overline{\{\delta\}} \cdot \overline{\alpha} \cdot i_E(\overline{\{f_1, \dots, f_{n-1}\}})$, $\delta \in E^\times$, $f_i \in F^\times$. If $n - 1 \geq 1$ then we see that $k_{n+1}E$ is generated by the elements in $k_nE \cdot i_Ek_1F$. By hypothesis $k_nE = i_Ek_nF + \overline{\alpha} \cdot i_Ek_{n-1}F$ and therefore $k_{n+1}E$ is generated by elements in $i_Ek_{n+1}F + \overline{\alpha} \cdot i_Ek_nF$. If instead $n = 1$, then using our hypothesis $k_1E = i_Ek_1F + \overline{\alpha} \cdot i_Ek_0F$ we may write the second type of generators $\overline{\gamma}_2$ of k_2E as $\overline{\gamma}_2 = (i_E(\overline{\{f\}}) + c\overline{\alpha}) \cdot \overline{\alpha} = -\overline{\alpha} \cdot i_E(\overline{\{f\}}) - \overline{\alpha} \cdot i_E(c\{-1\})$, $f \in F^\times$, $c \in \mathbb{Z}$. Thus in this case both types of generators of k_2E have the required form of elements in $i_Ek_2F + \overline{\alpha} \cdot i_Ek_1F$. The result now follows by induction as above. $\square$

Lemma 3. *Suppose that $p = 2$ and for some $n \in \mathbb{N}$, $\mathrm{ann}_{n-1}\{a\} = k_{n-1}F$. Then k_mE is a free $\mathbb{F}_2[G]$-module for all $m \geq n$.*

Proof. We show that the two conditions of part (2) of the $p = 2$ portion of Theorem 1 hold for K-theory degree at least n. From Lemma 2, part (1), we deduce that $k_mF = \mathrm{ann}_m\{a\} = \mathrm{ann}_m\{a, -1\}$ for all $m \geq n - 1$. By Theorem 2 and Lemma 2, we have $k_mF = N_{E/F}k_mE$ for all $m \geq n-1$ and therefore we see in particular that $k_mF = N_{E/F}k_mE + \{a\} \cdot k_{m-1}F$ for all $m \geq n - 1$. We conclude that k_mE is a free $\mathbb{F}_2[G]$-module for all $m \geq n$. $\square$

Proof of Theorem 3. For $p > 2$, the fact that free cohomology is hereditary follows from Lemma 2 and condition (2) in Theorem 1. The exactness of the first term of the sequence follows from Theorem 1, part (3) and Lemma 2, part (2), while the exactness at the third term follows from Theorem 1, part (4) and Lemma 2, part (3).

Assume then that $p = 2$ and $a = x^2 + y^2$ for some $x, y \in F^\times$. It is well-known that then $\{a, -1\} = 0 \in k_2F$ and hence $\mathrm{ann}_{n-1}\{a, -1\} = k_{n-1}F$. Now observe that since k_nE is a free $\mathbb{F}_2[G]$-module, by Theorem 1 we have $\mathrm{ann}_{n-1}\{a\} = \mathrm{ann}_{n-1}\{a, -1\}$, and so $\mathrm{ann}_{n-1}\{a\} = k_{n-1}F$. We deduce from Lemma 3 that k_mE is a free $\mathbb{F}_2[G]$-module for all $m \geq n$.

For the exact sequence for $p = 2$, we have shown that $k_{n-1}F = \mathrm{ann}_{n-1}\{a\}$, and so by Theorem 2 and Lemma 2, we have $k_mF = N_{E/F}k_mE$ for all $m \geq n - 1$. Furthermore, we conclude from Theorem 2 that i_E is injective from k_mF to k_mE for all $m \geq n$. The exactness of our sequence in the middle term follows from [Ar, Satz 4.5]. $\square$

Freeness is not generally hereditary when $p = 2$. For example, let $p = 2$, $F = \mathbb{Q}_2$, and $a = -1$, so $E = \mathbb{Q}_2(\sqrt{-1})$. Then $k_1F \cong F^\times/F^{\times 2} = \langle[-1], [2], [5]\rangle$. $N_{E/F}k_1E \cong N_{E/F}(E^\times)F^{\times 2}/F^{\times 2} = \langle[2], [5]\rangle$. Therefore $k_1F = N_{E/F}k_1E + \{-1\} \cdot k_0F$. Since $[-1] \notin N_{E/F}(E^\times)F^{\times 2}/F^{\times 2}$, we have $\{-1, -1\} \neq 0 \in k_2F$ and $\mathrm{ann}_0\{-1, -1\} = \{0\} = \mathrm{ann}_0\{-1\}$. Hence the conditions of part (2) of the $p = 2$ portion of Theorem 1 are satisfied, whence k_1E is a free $\mathbb{F}_2$-module. It is well-known, however, that $k_2E \cong \mathbb{F}_2$.

3. Examples of free cohomology

Define $\mathrm{cf}(E/F) \in \{0\} \cup \mathbb{N} \cup \{\infty\}$ by

$$\mathrm{cf}(E/F) = \sup\{n \in \mathbb{N} \cup \{0\} \mid H^n(E) \text{ is not a free } \mathbb{F}_p[G]\text{-module}\}.$$

Further for any pro-p-group T we denote by $\mathrm{cd}(T)$ the cohomological dimension of T.

Theorem 4. *Given $1 \leq n \leq m \in \mathbb{N} \cup \{\infty\}$ and a prime p, there exists a cyclic extension E/F of degree p with $\xi_p \in F$ such that:*

1. *G_F is a pro-p-group;*
2. *$\mathrm{cf}(E/F) = n$; and*
3. *$\mathrm{cd}(G_E) = m$.*

Let $\mathbb{Z}_{(p)} := \left\{\frac{c}{d} \in \mathbb{Q} \;\middle|\; c, d \in \mathbb{Z}, d \neq 0;\ \text{if } c \neq 0 \text{ then } (c, d) = 1, p \nmid d\right\}$, I be a well-ordered set of cardinality m, and Γ be a direct sum of m copies of $\mathbb{Z}_{(p)}$, indexed by I. We order Γ lexicographically. Then $m = \dim_{\mathbb{F}_p} \Gamma/p\Gamma$. It is well-known that the field

$$F_m := \mathbb{C}((\Gamma)) := \{f : \Gamma \to \mathbb{C} \mid \mathrm{supp}(f) \text{ is well-ordered}\}$$

is a henselian valued field with value group Γ and residue field $\mathbb{C}$. The absolute Galois group of F_m is known to be $\mathbb{Z}_p^m$, the topological product of m copies of $\mathbb{Z}_p$ [K, pp. 3–4].

Lemma 4 ([Wad, Thm. 3.6])**.** *For $m, n \in \mathbb{N} \cup \{\aleph_0\}$, $H^n(F_m) \cong \bigwedge^n H^1(\mathbb{Z}_p^m) \cong \bigwedge^n \oplus_m \mathbb{F}_p$, where the cup-product is sent to the wedge product.*

Lemma 5. *Suppose that m_1, m_2 are nonzero cardinal numbers, and let F_{m_1} and F_{m_2} be as above. There exists a field F_{m_1,m_2} of characteristic 0, containing a primitive p^2th root of unity ξ_{p^2}, such that the absolute Galois group $G_{F_{m_1,m_2}} \cong G_{F_{m_1}} \star_{pro\text{-}p} G_{F_{m_2}} \cong \mathbb{Z}_p^{m_1} \star_{pro\text{-}p} \mathbb{Z}_p^{m_2}$, where the free products are taken in the category of pro-p-groups, and the natural restriction maps $\mathrm{res}_\star \colon H^n(F_{m_1,m_2}) \to H^n(F_{m_1}) \oplus H^n(F_{m_2})$ are isomorphisms.*

Proof. The existence of a field F_{m_1,m_2} with

$$\mathrm{char}(F_{m_1,m_2}) = \mathrm{char}(F_{m_1}) = \mathrm{char}(F_{m_2}) = 0$$

and the given absolute Galois group follows from [EH, Prop. 1.3]. Additionally using the construction of F_{m_1,m_2} following [EH, proof of Prop. 1.3] we assume that F_{m_1,m_2} is the intersection of two henselian valued fields $(L_i, V_i), i = 1, 2$, with residue fields isomorphic to some fields $\widetilde{F}_{m_1}$ and $\widetilde{F}_{m_2}$ closely related to F_{m_1} and F_{m_2}, respectively. Here V_i is a henselian valuation on L_i. Then by Hensel's Lemma (see [Ri, pp. 12-13, condition (3)]) and by the fact that $\widetilde{F}_{m_1}$ and $\widetilde{F}_{m_2}$ have characteristic 0 and both contain a primitive p^2th root of unity, we see that F_{m_1,m_2} also contains a primitive p^2th root of unity. The fact that the restriction maps are isomorphisms follows from [N, Sätze (4.1),(4.2)]. □

Proof of Theorem 4. The case $m \in \mathbb{N}$:

(1) Let $F := F_{n,m}$ be a field of characteristic 0 with $G_F \cong \mathbb{Z}_p^n \star_{\text{pro-}p} \mathbb{Z}_p^m$ and $\xi_{p^2} \in F$, given by Lemma 5. Let $E = F(\sqrt[p]{a})$ for any $a \in F^\times$ such that under the restriction map on H^1, $\mathrm{res}_\star(a) = (a)_1 \oplus (a)_2$, $(a)_1 \neq 0$, $(a)_2 = 0$. We use here the fact that $\mathrm{res}_\star$ is an isomorphism, by Lemma 5. Observe that there exists an a with the required conditions because by Lemma 4, $H^1(F_n) \neq \{0\}$.

(2) We first show $H^n(E)$ is not free. We claim that $\mathrm{ann}_{n-1}(a) \neq H^{n-1}(F)$. If $n = 1$ this statement is true since $(a) \neq 0 \in H^1(F)$. Assume now that $n > 1$. Let $a_1 \in F_n^\times$ satisfy $(a_1) = (a)_1$, and extend $\{(a_1)\}$ to a basis $\{(a_1), (a_2), \cdots, (a_n)\}$ of $H^1(F_n)$. By Lemma 4, the element $(a_1) \cup (a_2) \cup \cdots \cup (a_n) \in H^n(F_n)$ is nontrivial, so that $0 \neq (a_2) \cup \cdots \cup (a_n) \in H^{n-1}(F_n)$. Let $b \in H^{n-1}(F)$ satisfy $b_1 = (a_2) \cup \cdots \cup (a_n) \in H^{n-1}(F_n)$, $b_2 = 0 \in H^{n-1}(F_m)$. Then since the cup-product commutes with $\mathrm{res}_\star$, $((a) \cup b)_1 = (a_1) \cup b_1 \neq 0 \in H^n(F_n)$, so that $(a) \cup b \neq 0 \in H^n(F)$ and hence $\mathrm{ann}_{n-1}(a) \neq H^{n-1}(F)$. If $p > 2$, we conclude by Theorem 1 that $H^n(F)$ is not free. If $p = 2$, observe that since $\sqrt{-1} \in F$, we have $\mathrm{ann}_{n-1}(a, -1) = H^{n-1}(F)$, so that $\mathrm{ann}_{n-1}(a, -1) \neq \mathrm{ann}_{n-1}(a)$. We deduce from Theorem 1 that $H^n(F)$ is not free.

We now show $H^k(E)$ is free for all $k \geq n+1$. We claim that $\mathrm{ann}_n(a) = H^n(F)$. Let $c \in H^n(F)$. Then since $H^{n+1}(F_n) = 0$ by Lemma 4,

$$\mathrm{res}_\star(a) \cup c = \big((a_1) \cup c_1\big) \oplus \big(0 \cup c_2\big) = \mathrm{res}_\star\, 0.$$

Hence $(a) \cup c = 0$ and $\mathrm{ann}_n(a) = H^n(F)$. If $p > 2$ then we conclude by Theorem 1 that $H^{n+1}(E)$ is free, and by Theorem 3, $H^k(E)$ is free for all $k \geq n+1$. If $p = 2$, observe that $\mathrm{ann}_n(a, -1) = H^n(F)$. By Theorem 2, $\mathrm{cor} \colon H^n(E) \to H^n(F)$ is surjective. Then by Lemma 3 and Theorem 1, we have that $H^k(E)$ is free for all $k \geq n+1$.

(3) First we claim that G_F does not contain an element of order p. By the Artin–Schreier theorem, finite subgroups of absolute Galois groups are either trivial or of order 2, and since $\sqrt{-1} \in F$ no element of order 2 exists in G_F. Then, by Serre's

theorem [S], we obtain $\mathrm{cd}(G_E) = \mathrm{cd}(G_F)$. From Lemmas 4 and 5 we find that $\mathrm{cd}(G_F) = \max\{\mathrm{cd}(F_n), \mathrm{cd}(F_m)\} = m$. Thus $\mathrm{cd}(G_E) = m$. (One can also conclude that G_F contains no element of order p from the fact that $\mathrm{cd}(G_F) = m < \infty$.)

The case $n < m = \infty$: set $F_\infty := \mathbb{C}((\Gamma))$, where $m = \aleph_0$ and Γ is the direct sum of m copies of $\mathbb{Z}_{(p)}$. By Lemma 5, there exists a field $F := F_{n,\infty}$ such that $G_F \cong G_{F_n} \star_{\text{pro-}p} G_{F_\infty}$ and $\xi_{p^2} \in F$. Then set $E = F(\sqrt[p]{a})$ for any $a \in F^\times$ such that under the restriction map $\mathrm{res}_\star\colon H^1(F) \to H^1(F_n) \oplus H^1(F_\infty)$, we have $\mathrm{res}_\star(a) = (a)_1 \oplus 0$, $(a)_1 \neq 0$. Then $\mathrm{cd}(G_F) = \mathrm{cd}(G_E) = \infty$, and as above we see that $\mathrm{cf}(E/F) = n$.

The case $n = \infty = m$: let Γ be a direct sum of $\aleph_0$ copies of $\mathbb{Z}_{(p)}$. Then set $F := F_\infty = \mathbb{C}((\Gamma))$. Let $a \in F^\times$ such that $v(a) \in \Gamma \setminus p\Gamma$, where v is a natural valuation on F. Then from the description of Galois cohomology of p-henselian fields (see [Wad, Thm. 3.6]), we obtain $\mathrm{ann}_n(a) = (a) \cup H^{n-1}(F)$ and $(a) \cup H^{n-1}(F) \neq H^n(F)$ for all $n \in \mathbb{N}$. Setting $E = F(\sqrt[p]{a})$, we obtain $\mathrm{cf}(E/F) = \infty$. □

4. Trivial cohomology

By a trivial $\mathbb{F}_p[G]$-module we mean an $\mathbb{F}_p[G]$-module W such that $W = W^G$.

Theorem 5. *Let $n \in \mathbb{N}$. If $p > 2$, then the following are equivalent:*

(1) *$H^n(E)$ is a trivial $\mathbb{F}_p[G]$-module.*
(2) *$(\xi_p) \cup H^{n-1}(F) \subset (a) \cup H^{n-1}(F)$ and $\mathrm{ann}_n(a) = (a) \cup H^{n-1}(F)$.*
(3) *$(\xi_p) \cup H^{n-1}(F) \subset (a) \cup H^{n-1}(F)$ and*

$$H^n(E) = \mathrm{res}\, H^n(F) + (\sqrt[p]{a}) \cup \mathrm{res}\, H^{n-1}(F).$$

If $p = 2$, then the following are equivalent:

(1) *$H^n(E)$ is a trivial $\mathbb{F}_2[G]$-module.*
(2) *$\mathrm{ann}_n(a) \subset (a) \cup \mathrm{ann}_{n-1}(a, -1)$.*

In the $p = 2$ case, suppose additionally that $a \in (F^2 + F^2) \setminus F^2$. Then the conditions above are also equivalent to:

(3) *$H^n(E) = \mathrm{res}\, H^n(F) + (\delta) \cup \mathrm{res}\, H^{n-1}(F)$, where $(\delta) \in H^1(E)^G$ satisfies $\mathrm{cor}(\delta) = (a)$.*

Remark. For $p > 2$ and $n = 1$ the second condition in (3) was observed in [War, Lemma 3].

Lemma 6. *Let $n \in \mathbb{N}$. Suppose that $i_E(\{\xi_p\} \cdot k_{n-1}F) = i_E N_{E/F} k_n E = \{0\}$. Then $(k_n E)^G = k_n E$.*

Proof. Let $\gamma \in K_n E$. We show that $l(\gamma) > 1$ leads to a contradiction, whence we will have the result. Suppose that $l = l(\gamma) \geq 2$ and $2 \leq l \leq i \leq p$. We show by induction on i that there exists $\alpha_i \in K_n E$ such that $\langle (\sigma-1)^{i-1}\overline{\alpha}_i \rangle = \langle (\sigma-1)^{l-1}\overline{\gamma} \rangle$. If $i = l$ then $\alpha_i = \gamma$ suffices. Assume now that $l \leq i < p$ and that our statement is true for i. Set $c = N_{E/F}\alpha_i$. By our hypothesis, $i_E \bar{c} = 0$. By Theorem 2, $\bar{c} = \{a\} \cdot \bar{b}$ for some $b \in K_n F$. Hence $c = \{a\} \cdot b + pf$ for $f \in K_n F$. Since $2 \leq i < p$, $N_{E/F}(\alpha_i - \{\sqrt[p]{a}\} \cdot i_E(b) - i_E(f)) = 0$. By Hilbert 90, there exists $\omega \in K_n E$ such that

$$(\sigma - 1)\omega = \alpha_i - \{\sqrt[p]{a}\} \cdot i_E(b) - i_E(f).$$

Then

$$(\sigma-1)^2\omega = (\sigma-1)\alpha_i - i_E(\{\xi_p\}\cdot i_E(b)) = (\sigma-1)\alpha_i \neq 0,$$

and we can set $\alpha_{i+1} = \omega$. Observe that here we use our hypothesis

$$i_E(\{\xi_p\}\cdot k_{n-1}F) = \{0\}.$$

Hence by induction there exists $\alpha_p \in K_nE$ such that $\langle N\bar{\alpha}_p\rangle = \langle(\sigma-1)^{l-1}\bar{\gamma}\rangle$. But $i_E N_{E/F}\bar{\alpha}_p = 0$, whence $(\sigma-1)^{l-1}\bar{\gamma} = 0$, a contradiction. □

Lemma 7. *Suppose that $p = 2$. Then $\{a\}\cdot \operatorname{ann}_{n-1}\{a,-1\} \subset N_{E/F}k_nE$.*

Proof. Let $\bar{\beta} \in \operatorname{ann}_{n-1}\{a,-1\}$. Then $\{-1\}\cdot\bar{\beta} \in \operatorname{ann}_n\{a\} = N_{E/F}k_nE$ by Theorem 2. Let $\gamma \in K_nE$ such that $\{-1\}\cdot\bar{\beta} = N_{E/F}(\bar{\gamma})$. Then we have $\{a\}\cdot\bar{\beta} = N_{E/F}(\{\sqrt{a}\}\cdot i_E(\bar{\beta}) + \bar{\gamma})$. Thus $\{a\}\cdot\operatorname{ann}_{n-1}\{a,-1\} \subset N_{E/F}k_nE$. □

Proof of Theorem 5. We consider first the case $p > 2$.

(1) $\Longrightarrow$ (3) Assume that k_nE is a trivial $\mathbb{F}_p[G]$-module. Suppose $f \in K_{n-1}F$, and set $\beta = \{\sqrt[p]{a}\}\cdot i_E(f)$. Then $(\sigma-1)\bar{\beta} = 0 \implies i_E(\{\xi_p\}\cdot\bar{f}) = 0$. But then by Theorem 2, $\{\xi_p\}\cdot\bar{f} \in \{a\}\cdot k_{n-1}F$. Now let $\gamma \in K_nE$ be arbitrary. Then $i_E N_{E/F}\bar{\gamma} = (\sigma-1)^{p-1}\bar{\gamma} = 0$ and so by Theorem 2, $N_{E/F}\bar{\gamma} = \{a\}\cdot\bar{f}$ for $f \in K_{n-1}F$. By the projection formula, $N_{E/F}(\{\sqrt[p]{a}\}\cdot i_E(\bar{f})) = \{a\}\cdot\bar{f}$. Then $N_{E/F}(\bar{\gamma} - \{\sqrt[p]{a}\}\cdot i_E(\bar{f})) = 0$, and hence $N_{E/F}(\gamma - \{\sqrt[p]{a}\}\cdot i_E(f)) = pg$, for some $g \in K_{n-1}F$. Set $\delta = \gamma - \{\sqrt[p]{a}\}\cdot i_E(f) - i_E(g)$. Then $N_{E/F}(\delta) = 0$. By Hilbert 90, there exists $\alpha \in K_nE$ such that $(\sigma-1)\alpha = \delta$. But since k_nE is fixed by G, $\bar{\delta} = 0$. Hence $k_nE = i_E(k_nF) + \{\sqrt[p]{a}\}\cdot i_E(k_{n-1}F)$.

(3) $\Longrightarrow$ (2) Since $p > 2$, $N_{E/F}(\{\sqrt[p]{a}\}\cdot i_E(\bar{f})) = \{a\}\cdot\bar{f}$ for $f \in K_{n-1}F$, and $N_{E/F}(i_E(\bar{g})) = 0$ for $g \in K_nF$. Hence $N_{E/F}k_nE = \{a\}\cdot k_{n-1}F$. Thus $\operatorname{ann}_n\{a\} = \{a\}\cdot k_{n-1}F$.

(2) $\Longrightarrow$ (1) Assume that $\{\xi_p\}\cdot k_{n-1}F \subset \{a\}\cdot k_{n-1}F$ and $\operatorname{ann}_n\{a\} = \{a\}\cdot k_{n-1}F$. Hence $\{\xi_p\}\cdot k_{n-1}F \subset N_{E/F}k_nE = \{a\}\cdot k_{n-1}F$. We then apply Lemma 6 to deduce that $k_nE = (k_nE)^G$.

Now we consider the case $p = 2$:

(1) $\Longrightarrow$ (2) Assume that k_nE is a trivial $\mathbb{F}_2[G]$-module. Let $\alpha \in K_nE$. Then $i_E N_{E/F}\bar{\alpha} = (\sigma-1)\bar{\alpha} = 0$ implies that $N_{E/F}\bar{\alpha} = \{a\}\cdot\bar{b}$ for some $b \in K_{n-1}F$. Now $\{a,-1\} = \{a,a\}$ in k_2F, and then $\{a,-1\}\cdot\bar{b} = \{a\}\cdot N_{E/F}\bar{\alpha} = 0$. Hence $\operatorname{ann}_n\{a\} \subset \{a\}\cdot\operatorname{ann}_{n-1}\{a,-1\}$.

(2) $\Longrightarrow$ (1) Assume that $\operatorname{ann}_n\{a\} \subset \{a\}\cdot\operatorname{ann}_{n-1}\{a,-1\}$. Then $N_{E/F}k_nE \subset \{a\}\cdot\operatorname{ann}_{n-1}\{a,-1\}$. By Lemma 7, $\{a\}\cdot\operatorname{ann}_{n-1}\{a,-1\} \subset N_{E/F}k_nE$ and hence $N_{E/F}k_nE = \{a\}\cdot\operatorname{ann}_{n-1}\{a,-1\}$. Let $\gamma \in K_nE$ be arbitrary. Then $N_{E/F}\bar{\gamma} = \{a\}\cdot\bar{b}$ for some $\bar{b} \in \operatorname{ann}_{n-1}\{a,-1\}$. Hence $(\sigma-1)\bar{\gamma} = i_E N_{E/F}\bar{\gamma} = i_E(\{a\}\cdot\bar{b}) = 0$. Hence $(\sigma-1)\bar{\gamma} = 0$, and $(k_nE)^G = k_nE$.

Now assume $p = 2$ and $a \in (F^2+F^2)\setminus F^2$. Then $-1 = N_{E/F}\gamma$ for some $\gamma \in E^\times$. Thus for $\delta = \gamma\sqrt{a}$ we have $N_{E/F}\delta = a$. Then $\overline{\{\delta\}} \in (k_1E)^G$ and $N_{E/F}\overline{\{\delta\}} = \{a\}$.

(1) $\Longrightarrow$ (3) Let $\gamma \in K_nE$ be arbitrary. Then, replacing $\sqrt[p]{a}$ by δ and using $N_{E/F}\overline{\{\delta\}} = \{a\}$ in the proof of the (1) $\Longrightarrow$ (3) portion of the $p > 2$ case, we obtain $\bar{\gamma} \in i_E(k_nF) + \overline{\{\delta\}}\cdot i_E(k_{n-1}F)$.

(3) $\implies$ (1). If $\beta \in K_nE$, then $\bar{\beta} = i_E(\bar{g}) + \overline{\{\delta\}} \cdot i_E(\bar{f})$ for $g \in K_nF$ and $f \in K_{n-1}F$. Then $(\sigma - 1)\bar{\beta} = 0$, which implies that $(k_nE)^G = k_nE$. □

Corollary 3. *Suppose $n \in \mathbb{N}$ and $(k_nE)^G = k_nE$. Then we have the following exact sequence:*

$$0 \to \mathrm{ann}_{n-1}\{a\} \to k_{n-1}F \xrightarrow{\{a\}\cdot -} k_nF \xrightarrow{i_E} k_nE \xrightarrow{N_{E/F}} \{a\} \cdot \mathrm{ann}_{n-1}\{a, \xi_p\} \to 0.$$

Proof. We first consider exactness at the fifth term. In the $p = 2$ case, Theorem 5 tells us that $N_{E/F}k_nE = \mathrm{ann}_n\{a\} \subset \{a\} \cdot \mathrm{ann}_{n-1}\{a, -1\}$. By Lemma 7 we have the reverse inclusion, so that $N_{E/F}k_nE = \{a\} \cdot \mathrm{ann}_{n-1}\{a, -1\}$. In the $p > 2$ case, observe that $\{\xi_p\} \cdot k_{n-1}F \subset \{a\} \cdot k_{n-1}F$ implies that $k_{n-1}F = \mathrm{ann}_{n-1}\{a, \xi_p\}$, since $\{a, a\} = 0$. Therefore, by part (2) of Theorem 5, we know $N_{E/F}k_nE = \mathrm{ann}_n\{a\} = \{a\} \cdot \mathrm{ann}_{n-1}\{a, \xi_p\}$. Hence the sequence is exact at the fifth term in the $p > 2$ case as well.

For exactness at the fourth term, suppose $\gamma \in K_nE$ and $N_{E/F}\bar{\gamma} = 0$. Then there exists $f \in K_nF$ such that $N_{E/F}\gamma = pf$, and then $N_{E/F}(\gamma - i_E(f)) = 0$. By Hilbert 90, there exists $\alpha \in K_nE$ such that $(\sigma - 1)\bar{\alpha} = \bar{\gamma} - i_E(\bar{f})$. But $(\sigma - 1)\bar{\alpha} = 0$ because $(k_nE)^G = k_nE$. Hence $\bar{\gamma} = i_E(\bar{f})$. Since exactness in the first two terms is obvious and exactness at the third term follows from Theorem 2, our proof is complete. □

5. Hereditary triviality

Theorem 6. *Trivial $\mathbb{F}_p[G]$-module cohomology is hereditary: if $n \in \mathbb{N}$, then for all $m \geq n$, $H^n(E)^G = H^n(E) \implies H^m(E)^G = H^m(E)$.*

Proof. In the $p > 2$ case, the result on heredity follows from Theorem 5, part (3), together with two hereditary properties from Lemma 2: item (4), with $\alpha_1 = \{\xi_p\}$ and $\alpha_2 = \{a\}$, and item (5).

In the $p = 2$ case, since $m > n \geq 1$, by [BT, Cor. 5.3] K_mE is generated by symbols $\{u, f_1, \ldots, f_{m-1}\}$ where $u \in E^\times$ and $f_i \in F^\times$ for all $i = 1, \ldots, m-1$. Using the projection formula and a straightforward induction on m, we prove that if $H^n(E)^G = H^n(E)$ then $\mathrm{ann}_m\{a\} \subset \{a\} \cdot \mathrm{ann}_{m-1}\{a, -1\}$. Hence by Theorem 5 we conclude that $H^m(E)^G = H^m(E)$ as well. □

6. Examples of trivial cohomology

Define $\mathrm{ct}(E/F) \in \{0\} \cup \mathbb{N} \cup \{\infty\}$ by

$$\mathrm{ct}(E/F) = \sup\{n \in \mathbb{N} \cup \{0\} \mid H^n(E) \text{ is not a trivial } \mathbb{F}_p[G]\text{-module}\},$$

where we set $\sup \emptyset = 0$.

Theorem 7. *Given $1 \leq n \leq m \in \mathbb{N} \cup \{\infty\}$, or $0 = n < m \in \mathbb{N} \cup \{\infty\}$, and a prime p, there exists a cyclic extension E/F of degree p with $\xi_p \in F$ such that:*

(1) *G_F is a pro-p-group;*
(2) *$\mathrm{ct}(E/F) = n$; and*
(3) *$\mathrm{cd}(G_E) = m$.*

Proof. The case $m \in \mathbb{N}$:

(1) Let $F := F_{n,m}$ be a field of characteristic 0 with $G_F \cong \mathbb{Z}_p^n \star_{\text{pro-}p} \mathbb{Z}_p^m$ and $\xi_{p^2} \in F$, given by Lemma 5. (Observe that if $n = 0$ then the first factor is $\{1\}$.) Let $E = F(\sqrt[p]{a})$ where $a \in F^\times$ such that under the restriction map on H^1,

$\operatorname{res}_\star(a) = (a)_1 \oplus (a)_2$, $(a)_1 = 0$, $(a)_2 \neq 0$. Observe that there exists an a with the required conditions because by Lemma 4, $H^1(F_m) \neq \{0\}$.

(2) We first show that if $n > 0$ then $H^n(E)$ is not trivial. We claim that $\operatorname{ann}_n(a) \not\subset (a) \cup H^{n-1}(F)$. By Lemma 4, $H^n(F_n)$ contains a nonzero element c. Let $b \in H^n(F)$ be such that its components $b_1 \in H^n(F_n)$ and $b_2 \in H^n(F_m)$ are $b_1 = c$ and $b_2 = 0$. Then $b \neq 0$ and since the cup-product commutes with $\operatorname{res}_\star$, $\operatorname{res}_\star(a) \cup b = (0 \cup b_1) \oplus ((a)_2 \cup 0) = 0 \in H^{n+1}(F)$. Therefore $b \in \operatorname{ann}_n(a)$. Now let $f \in H^{n-1}(F)$ be arbitrary. Then $((a) \cup f)_1 = 0$ and therefore $b \notin (a) \cup H^{n-1}(F)$. Thus $\operatorname{ann}_n(a) \not\subset (a) \cup H^{n-1}(F)$. For the case $p > 2$, Theorem 5, part (2) implies that $H^n(E)$ is not trivial. In the case $p = 2$ we have $(-1) = 0$ since $\sqrt{-1} \in F^\times$. Thus $\operatorname{ann}_{n-1}(a, -1) = H^{n-1}(F)$ and $(a) \cup \operatorname{ann}_{n-1}(a, -1) = (a) \cup H^{n-1}(F)$. Hence by our claim above $\operatorname{ann}_n(a) \not\subset (a) \cup \operatorname{ann}_{n-1}(a, -1)$, and we can again apply Theorem 5 to conclude that $H^n(E)$ is not a trivial $\mathbb{F}_p[G]$-module.

We now show $H^k(E)^G = H^k(E)$ for all $k \geq n+1$. Let $a_1 \in F_m^\times$ satisfy $(a_1) = (a)_2$ and extend $\{(a_1)\}$ to a basis $\{(a_1), \ldots, (a_m)\}$ of $H^1(F_m)$. Recall that by Lemma 4, $H^k(F_m)$ is just the kth homogenous summand of the exterior algebra over $\mathbb{F}_p$ generated by $H^1(F_m)$. Using this fact and writing each element in $H^k(F_m)$ as a linear combination of elements of the form $(a_{i_1}) \cup \cdots \cup (a_{i_k})$, $1 \leq i_1 < i_2 < \cdots < i_k \leq m$, and also the fact that $H^k(F_n) = \{0\}$, we see that $\operatorname{ann}_k(a) = (a) \cup H^{k-1}(F)$. Now again using Theorem 5 as in the preceding paragraph, we conclude that $H^k(E)^G = H^k(E)$.

(3) By Lemmas 4 and 5, $\operatorname{cd}(G_F) = \max\{\operatorname{cd}(G_{F_n}), \operatorname{cd}(G_{F_m})\} = m$. By Serre's theorem [S, p. 413] we have $\operatorname{cd}(G_E) = \operatorname{cd}(G_F)$.

The case $m = \infty$: We first consider the subcase of this case when $n < \infty$. Set $F_\infty := \mathbb{C}((\Gamma))$, where $m = \aleph_0$ and Γ is the direct sum of m copies of $\mathbb{Z}_{(p)}$. By Lemma 5 we see that there exists a field $F := F_{n,\infty}$ such that $G_F \cong G_{F_n} \star_{\text{pro-}p} G_{F_\infty}$ and $\xi_{p^2} \in F$. Let $a \in F^\times$ such that under the restriction map $\operatorname{res}_\star : H^1(F) \to H^1(F_n) \oplus H^1(F_\infty)$ we have $\operatorname{res}_\star(a) = 0 \oplus (a)_2$, $(a)_2 \neq 0$. Then $\operatorname{cd}(F) = \infty$ and as above we see that $\operatorname{ct}(E/F) = n$. Finally we consider the case $n = \infty = m$. Define F_∞ as above and $F = F_{\infty,\infty}$. Also let $a \in F^\times$ such that $\operatorname{res}_\star(a) = 0 \oplus (a)_2$, $(a)_2 \neq 0$. Then as in (2) we see that $\operatorname{ct}(F) = \infty$. $\square$

Acknowledgement

We thank the anonymous referee for his or her suggestions, as a result of which we reached a more succinct and efficient exposition.

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This paper is available via http://nyjm.albany.edu:8000/j/2005/11-14.html.