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L^2-index theorems, KK-theory, and connections

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Abstract. Let M be a compact manifold and D a Dirac type differential operator on M. Let A be a C^*-algebra. Given a bundle W (with connection) of A-modules over M, the operator D can be twisted with this bundle. One can then use a trace on A to define numerical indices of this twisted operator. We prove an explicit formula for these indices. Our result does complement the Mishchenko–Fomenko index theorem valid in the same situation. We establish generalizations of these explicit index formulas if the trace is only defined on a dense and holomorphically closed subalgebra $\mathcal{B}$.

As a corollary, we prove a generalized Atiyah L^2-index theorem if the twisting bundle is flat.

There are actually many different ways to define these numerical indices. From their construction, it is not clear at all that they coincide. A substantial part of the paper is a complete proof of their equality. In particular, we establish the (well-known but not well-documented) equality of Atiyah's definition of the L^2-index with a K-theoretic definition.

In case A is a von Neumann algebra of type 2, we put special emphasis on the calculation and interpretation of the center valued index. This completely contains all the K-theoretic information about the index of the twisted operator.

Some of our calculations are done in the framework of bivariant KK-theory.

Contents

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1. Introduction

Let M be a closed smooth manifold, $D\colon \Gamma(E) \to \Gamma(E)$ a generalized Dirac operator on the finite-dimensional (graded) Dirac bundle E over M.

Assume that A is a C^*-algebra and W a smooth bundle of finitely generated projective modules over A equipped with a connection ∇_W. In this situation, one can define the twisted Dirac operator D_W (compare (5.17)). The resulting operator is an elliptic A-operator in the sense of Mishchenko–Fomenko [14]. In particular, its index $\operatorname{ind}(D_W) \in K_0(A)$ as an element of the K-theory of A is defined. Mishchenko and Fomenko prove a formula for this index (or rather its rationalization). Improvements to a K-theoretic level, and even equivariant generalizations, can be found, e.g., in [21, 23, 22]. However, for certain purposes these formulas are rather inexplicit.

The main goal of this paper is an explicit index formula in this context, in terms of the curvature of the twisting bundle W. This can not be done directly for the index. However, whenever we have a trace $\tau\colon A \to Z$ with values in any commutative C^*-algebra (e.g., the complex numbers), it induces a homomorphism $\tau\colon K_0(A) \to Z$, and we get the explicit formula

$$\tau(\operatorname{ind}(D_W)) = \langle \operatorname{ch}(\sigma(D)) \cup \operatorname{Td}(T_{\mathbb{C}}M) \cup \operatorname{ch}_\tau(W), [TM]\rangle \in Z \tag{1.1}$$

in Theorem 5.9, where the crucial term $\operatorname{ch}_\tau(W)$ can be calculated directly from the curvature of W.

Such a Chern–Weil approach to higher index theorems can replace heat equation proofs. This is, e.g., remarked by Mathai in [13, p. 14] and can be used to simplify his proof of the Novikov conjecture for low-dimensional cohomology classes. Applications of the explicit index formulas to the study of manifolds with metrics of positive scalar curvature are obtained by Hanke and Schick in [6]. There, we also use generalizations of Equation (1.1) to situations where a trace is only defined on a dense subalgebra $\mathcal{B}$ of A which is closed under holomorphic functional calculus, as explained in Section 8.

Corollary 1.2. *Assume that M is connected. Then the index formula* (1.1) *implies that for a* flat *bundle W*

$$\tau(\mathrm{ind}(D_W)) = \mathrm{ind}(D) \cdot d,$$

where $d := \dim_\tau(W_x)$ is the "fiber dimension", the trace of the projection onto the (*finitely generated projective*) *fiber W_x of W over an arbitrary point $x \in M$.*

If A is a finite von Neumann algebra and τ is its center valued trace, $\tau(\mathrm{ind}(D_W))$ contains as much information as $\mathrm{ind}(D_W)$.

There are several other ways to define an index for D twisted with W. The most direct is probably given by the Kasparov product of a KK-element defined by W with the index element D defined in $KK(C(M), \mathbb{C})$. In Theorem 5.22 we show that this coincides with the index defined directly using the Mishchenko–Fomenko calculus.

If A is a finite von Neumann algebra, it is more popular to twist with A-Hilbert space bundles, where the fibers are ordinary Hilbert spaces, but with an appropriate action of the von Neumann algebra A. We show that we can assign such a bundle $l^2(W)$ to W as above (and vice versa), and that the twisted indices obtained both ways are essentially equal (compare Theorem 7.30 and Corollary 7.31). Here we need the additional assumption that the trace is normal.

A special situation occurs if $A = C^*\Gamma$ is the C^*-algebra of a discrete group and W is the flat bundle associated to a unitary representation $\pi_1(M) \to C^*\Gamma$ induced from a group homomorphism $\pi_1(M) \to \Gamma$. Associated to this homomorphism is a Γ-covering space $\widetilde{M} \to M$, and we can lift D to an operator $\widetilde{D}$ on $\widetilde{M}$. Atiyah defines the L^2-index $\mathrm{ind}_{(2)}(\widetilde{D})$ of $\widetilde{D}$ in terms of sections on $\widetilde{M}$ and proves his L^2-index theorem in [1]. We show in Theorem 7.19 that there is a direct correspondence between this L^2-index (and generalizations thereof) and the index of D twisted with the flat $C^*\Gamma$-module bundle W as above. In particular,

$$\mathrm{ind}_{(2)}(\widetilde{D}) = t(\mathrm{ind}(D_W)),$$

where $W = \widetilde{M} \times_\Gamma C^*\Gamma$ is the flat bundle with fiber $C^*\Gamma$ associated to the Γ-covering $\widetilde{M}$ and $t\colon C^*\Gamma \to \mathcal{N}\Gamma \to \mathbb{C}$ is the canonical trace (producing the coefficient of the trivial element and factoring through the group von Neumann algebra $\mathcal{N}\Gamma$).

Finally, we consider the situation where $\mathcal{B}$ is a dense subalgebra of the C^*-algebra A with a trace $\tau\colon \mathcal{B} \to Z$. The prototypical situation is the algebra of trace class operators in the algebra of compact operators on a separable Hilbert space, with the ordinary trace. If $\mathcal{B}$ is closed under holomorphic functional calculus, then τ induces a homomorphism $\tau\colon K_0(A) \to Z$. In this situation, if the Hilbert A-module bundle W is induced up from a bundle $\mathcal{W}$ of finitely generated projective $\mathcal{B}$-modules, then we can define and use the curvature of $\mathcal{W}$ to give an explicit expression for $\mathrm{ch}_\tau(W) = \mathrm{ch}_\tau(\mathcal{W})$. We then prove the index formula

$$\tau(\mathrm{ind}(D_W)) = \langle \mathrm{ch}(\sigma(D)) \cup \mathrm{Td}(T_{\mathbb{C}}M) \cup \mathrm{ch}_\tau(\mathcal{W}), [TM] \rangle.$$

Our proof of the index formula for Hilbert A-module bundles works by just using a number of crucial properties of the K-theory of A and of $C(M) \otimes A$. Since $\mathcal{B}$ is closed under holomorphic functional calculus, its K-theory shares these properties.

We will therefore only briefly describe where changes in the first proof are necessary to obtain the second result.

Along the way, we solve a number of related questions, in particular the following:

(1) We develop a Chern–Weil calculus for connections on Hilbert A-module bundles.
(2) We prove existence and uniqueness of smooth structures on Hilbert A-module bundles, and show how K-theory of a manifold with coefficients in A is described using smooth bundles.
(3) The index $\operatorname{ind}(D_W) \in K_0(A)$ has to be defined in a complicated way, since kernel and cokernel of D_W are in general not finitely generated projective over A. If A is a von Neumann algebra, we prove that this caution is not necessary and that one can use the naive definition of the index.
(4) We prove that for a finite von Neumann algebra, Hilbert A-modules and A-Hilbert spaces are equivalent categories, and that the same is true for bundles with corresponding fibers.
(5) We establish a one-to-one correspondence between section of bundles on a Γ-covering space and of the associated flat $\mathcal{N}\Gamma$-bundle.

1.1. Notation and conventions. Throughout this paper, A denotes a unital C^*-algebra. Much of the theory can be carried out for nonunital C^*-algebras, but for quite a few statements, the existence of a unit is crucial, and they would have to be considerably reformulated in the nonunital case. In our applications, we are interested mainly in the reduced C^*-algebra and the von Neumann algebra of a discrete group, which are unital.

For some of our constructions, we will have to restrict to the case where A is a von Neumann algebra.

2. Hilbert modules and their properties

In this section, we recall the notion of a Hilbert C^*-module and its basic properties. A good and more comprehensive introduction to this subject is, e.g., [10] or [24, Chapter 15].

Definition 2.1. A *Hilbert A-module V* is a right A-module V with an A-valued "inner product" $\langle \cdot, \cdot \rangle \colon V \times V \to A$ with the following properties:

(1) $\langle v_1, v_2 a\rangle = \langle v_1, v_2\rangle a \qquad \forall v_1, v_2 \in V,\ a \in A.$
(2) $\langle v_1 + v_2, v_3\rangle = \langle v_1, v_3\rangle + \langle v_2, v_3\rangle \quad \forall v_1, v_2, v_3 \in V.$
(3) $\langle v_1, v_2\rangle = (\langle v_2, v_1\rangle)^* \qquad \forall v_1, v_2 \in V.$
(4) $\langle v, v\rangle$ is a nonnegative self-adjoint element of the C^*-algebra A for each $v \in V$, and $\langle v, v\rangle = 0$ if and only if $v = 0$.
(5) The map $v \mapsto |\langle v, v\rangle|_A^{1/2}$ is a norm on V, and V is a Banach space with respect to this norm.

Given two Hilbert A-modules V and W, a *Hilbert A-module morphism* $\Phi\colon V \to W$ is a continuous (right) A-linear map which has an adjoint $\Phi^*\colon W \to V$, i.e.,

$$\langle \Phi(v), w\rangle_W = \langle v, \Phi^*(w)\rangle_V$$

for all $v \in V$, $w \in W$. The vector space of all such maps is denoted $\operatorname{Hom}_A(V, W)$.

$\mathrm{Hom}_A(V,W)$ is an $\mathrm{End}_A(W)$-left-$\mathrm{End}_A(V)$-right module (but is not equipped with an inner product in general). The Hilbert A-module V itself is an $\mathrm{End}_A(V)$-A-bimodule.

Example 2.2. The most important example of a Hilbert A-module is A^n with inner product $\langle (a_i),(b_i)\rangle = \sum_{i=1}^n a_i^* b_i$.

In this case, $\mathrm{Hom}_A(A^n,A^m) \cong M(m\times n, A)$, where the matrices act by multiplication from the left.The adjoint homomorphism is given by taking the transpose matrix and the adjoint of each entry. In particular, $\mathrm{End}_A(A)\cong A$ as C^*-algebra.

We also consider H_A, the standard countably generated Hilbert A-module. It is the completion of $\bigoplus_{i\in\mathbb{N}} A$ with respect to the norm $|(a_i)| = \left|\sum_{i\in\mathbb{N}} a_i^* a_i\right|_A^{1/2}$ and with the corresponding A-valued inner product.

Given two Hilbert A-modules V and W, their direct sum $V\oplus W$ is a Hilbert A-module with $\langle (v_1,w_1),(v_2,w_2)\rangle = \langle v_1,v_2\rangle_V + \langle w_1,w_2\rangle_W$.

In [10, page 8] the following result is proved:

Lemma 2.3. *Assume that V and W are Hilbert A-modules. Then* $\mathrm{Hom}_A(V,W)$ *is a Banach space with the operator norm, and* $\mathrm{End}_A(V) := \mathrm{Hom}_A(V,V)$ *is a C^*-algebra.*

If A is a von Neumann algebra we get the following stronger result:

Proposition 2.4. *If A is a von Neumann algebra, so is* $\mathrm{End}_A(H_A)$.

Proof. This follows from the isomorphism $\mathrm{End}_A(H_A)\cong \mathcal{B}(H)\otimes A$ (spacial tensor product), since $\mathcal{B}(H)$ is a von Neumann algebra, and (spacial) tensor products of von Neumann algebras are von Neumann algebras. $\square$

Example 2.5. Let $V = A^n$ and $W = A^m$. Then we can identify $\mathrm{Hom}_A(V,W)$ with $M(n\times m, A)$, matrices acting by multiplication from the left. On the other hand, $M(n\times m,A) = A^{nm}$ is itself a Hilbert A-module (if A is not commutative, this A-module structure is of course not compatible with the action of $\mathrm{Hom}_A(V,W)$ on the A-modules V and W).

However, as Hilbert A-module $\mathrm{Hom}_A(V,W)$ inherits the structure of a Banach space. The corresponding Banach norm $|\cdot|$ is in general not equal to the operator Banach norm $\|\cdot\|$ from Lemma 2.3. But it is always true that the two norms are equivalent. For $\Phi\in\mathrm{Hom}_A(A^n,A^m)$, represented by the matrix

$$(a_{ij})\in M(n\times m, A),$$

with $e_i = (0,\dots,0,1,0,\dots,0)$ (1 at the ith position), and for arbitrary $v\in V$,

$$\begin{aligned}|\Phi(v)| &= \left|\sum_{j=1}^m e_j\langle \Phi_j, v\rangle\right| \le \sum_{j=1}^m |\langle\Phi_j,v\rangle| \\ &\le \sum_{j=1}^m |\Phi_j|\cdot|v| \le \sqrt{m}\,|\Phi|\cdot|v|\,,\end{aligned}$$

where Φ_j is the adjoint of the jth row of Φ. Since this holds for arbitrary $v\in V$,

$$\|\Phi\| \le \sqrt{m}\,|\Phi|\,.$$

On the other hand

$$|\Phi|^2 = \left|\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^* a_{ij}\right| \le \sum_{i=1}^{n}\left|\sum_{j=1}^{m} a_{ij}^* a_{ij}\right| = \sum_{i=1}^{n} |\Phi(e_i)|^2 \le n\,\|\Phi\|^2 .$$

Remark 2.6. In particular, when we are looking at functions defined on a smooth manifold with values in $\mathrm{Hom}_A(V,W)$, the smooth ones are unambiguously defined, using either of the two norms to define a Banach space structure on $\mathrm{Hom}_A(V,W)$.

Lemma 2.7. *Assume that V is a Hilbert A-module. The map*

$$\alpha\colon V \to \mathrm{Hom}_A(V,A); v \mapsto (x \mapsto \langle v, x\rangle)$$

is an A-sesquilinear isomorphism. A-sesquilinear means that $\alpha(va) = a^*\alpha(v)$ for all $v \in V$ and $a \in A$. Recall that $\mathrm{Hom}_A(V,A)$ is a left A-module (even a left Hilbert A-module) because of the identification $\mathrm{End}_A(A) \cong A$.

Proof. [10, page 13]. □

Definition 2.8. A *finitely generated projective* Hilbert A-module V is a Hilbert A-module which is isomorphic as Hilbert A-module to a (closed) orthogonal direct summand of A^n for suitable $n \in \mathbb{N}$. In other words, there is a Hilbert A-module W such that $V \oplus W \cong A^n$. The corresponding projection $p\colon A^n \to A^n$ with range V and kernel W is a projection in $M(n \times n, A)$, i.e., satisfies $p = p^2 = p^*$. On the other hand, the range of each such projection is a finitely generated projective Hilbert A-module.

We will also consider tensor products of the modules we are considering. Assume, e.g., that V is a Hilbert A-module, and that W is a left A-module. Then we consider the algebraic tensor product $V \otimes_A W$, still an $\mathrm{End}_A(V)$ left module. In general, it would not be appropriate to consider only the algebraic tensor product, but we would have to find suitable completions. However, we will apply this construction only to finitely generated projective modules, where no such completions are necessary.

Example 2.9. Let V be a Hilbert A-module. Then $\mathrm{Hom}_A(V,A)$ is an A-$\mathrm{End}_A(V)$ bimodule (since $\mathrm{End}_A(A) \cong A$). Consequently, we can consider $V \otimes_A \mathrm{Hom}_A(V,A)$ as an $\mathrm{End}_A(V)$ bimodule. It is even an algebra, with multiplication map

$$\begin{aligned}(V \otimes_A \mathrm{End}_A(V,A)) \otimes_{\mathrm{End}_A(V)} (V \otimes_A \mathrm{End}_A(V,A)) &\to V \otimes_A \mathrm{End}_A(V,A)\\ (v_1 \otimes \phi_1) \otimes (v_2 \otimes \phi_2) &\mapsto v_1(\phi_1(v_2)) \otimes \phi_2.\end{aligned}$$

The map $\iota\colon V \otimes_A \mathrm{Hom}_A(V,A) \to \mathrm{End}_A(V)$ which sends $v \otimes \phi$ to the endomorphism $x \mapsto v\phi(x)$ is a ring homomorphism which respects the $\mathrm{End}_A(V)$ bimodule structure.

Definition 2.10. Let X be a locally compact Hausdorff space. A *Hilbert A-module bundle* E over X is a topological space E with projection $\pi\colon E \to X$ such that each fiber $E_x := \pi^{-1}(x)$ $(x \in X)$ has the structure of a Hilbert A-module, and with local trivializations $\phi\colon E|_U \xrightarrow{\cong} U \times V$ which are fiberwise Hilbert A-module isomorphisms.

If X is a smooth manifold, a *smooth structure* on a Hilbert A-module bundle E is an atlas of local trivializations such that the transition functions

$$x \mapsto \phi_2 \circ \phi_1^{-1}(x) \colon U_1 \cap U_2 \to \mathrm{Hom}_A(V_1, V_2)$$

are smooth maps with values in the Banach space $\mathrm{Hom}_A(V_1, V_2)$.

Given smooth Hilbert A-module bundles W and W_2 on X, then $\mathrm{Hom}_A(W, W_2)$ (constructed fiberwise) also carries a canonical smooth structure.

A Hilbert A-module bundle is called *finitely generated projective*, if the fibers are finitely generated projective Hilbert A-modules, i.e., if they are direct summands in finitely generated free Hilbert A-modules.

We also define *finitely generated projective A-module bundles* (not Hilbert A-module bundles!), which are locally trivial bundles of left A-modules which are direct summands in A^n. Using a partition of unity and convexity of the space of A-valued inner products, we can choose a Hilbert A-module bundle structure on each such finitely generated projective A-module bundle.

Definition 2.11. The smooth sections of a bundle W on a smooth manifold M are denoted by $\Gamma(W)$. If V is a Hilbert A-module, then we sometimes write $C^\infty(M, V) := \Gamma(M \times V)$ for the smooth sections of the trivial bundle $M \times V$.

For the continuous sections we write $C(M, V)$.

The space of smooth differential forms is denoted $\Omega^*(M) = \Gamma(\Lambda^* T^* M)$. By definition, differential forms with values in a Hilbert A-module bundle W are the sections of $\Lambda^* T^* M \otimes W$. We sometimes write $\Omega^*(M; W) := \Gamma(\Lambda^* T^* M \otimes W)$. Note that the wedge product of differential forms induces a map

$$\Omega^p(M; W) \otimes \Omega^q(M; W_2) \to \Omega^{p+q}(M; W \otimes W_2).$$

Lemma 2.12. *Given two finitely generated projective Hilbert A-module bundles W and W_2 on a locally compact Hausdorff space X which are isomorphic as A-module bundles, then there is an isomorphism which preserves the inner products as well.*

If X is a smooth manifold and both bundles carry smooth structures and the given isomorphism preserves the smooth structure, we can arrange for the new isomorphism to preserve the smooth structure and the inner product at the same time.

Proof. We use the property that the inclusion of the isometries into all invertible operators is a homotopy equivalence.

More precisely, assume that $\Phi \in C(X, \mathrm{Hom}_A(W, W_2))$ is an isomorphism. Then we can decompose $\Phi = U\,|\Phi|$ with $|\Phi|\,(x) = \sqrt{\Phi(x)^* \Phi(x)} \in \mathrm{End}_A(W_x)$, using the fact that $\mathrm{End}_A(W_x)$ is a C^*-algebra by Lemma 2.3 and $U(x) = \Phi\,|\Phi|^{-1}$. Then U and $|\Phi|$ are continuous sections of the corresponding endomorphism bundles, and $U(x)$ is an isometry for each $x \in X$, i.e., provides the desired isomorphism which preserves the inner products.

Of course we use that multiplication, taking the adjoint, taking the inverse, and $a \mapsto |a|$ are all continuous operations for A-linear adjointable operators.

In case we have smooth structures, the isomorphism being smooth translates to Φ being a smooth section of $\mathrm{Hom}_A(W, W_2)$. The new isomorphism will be smooth since all operations involved, namely multiplication, taking the adjoint, taking the inverse, and $a \mapsto |a| = \sqrt{a^* a}$ are smooth, even analytic, operations for A-linear adjointable invertible operators. $\square$

Theorem 2.13. *Let V_1 and V_2 be two smooth Hilbert A-module bundles on a paracompact manifold M which are topologically isomorphic (but the isomorphism is not necessarily smooth). Then there is also a smooth isomorphism between the two bundles.*

In other words, up to isomorphism there is at most one smooth structure on a given Hilbert A-module bundle.

Proof. An isomorphism between V_1 and V_2 is the same as a continuous section s of the bundle $\mathrm{Hom}_A(V_1, V_2)$ which takes values in the subset of invertible elements $\mathrm{Iso}_A(V_1, V_2)$ of each fiber. The fact that $\mathrm{End}_A(V_i)$ are C^*-algebra bundles (and a von Neumann series argument) shows that the invertible elements form an *open* subset of $\mathrm{End}_A(V_1, V_2)$.

The smooth structures on V_1 and V_2 induce a smooth structure on $\mathrm{End}_A(V_1, V_2)$, and s is a smooth section if and only if the corresponding bundle isomorphism is smooth.

Observe that the inverse morphism s^{-1} is obtained by taking fiberwise the inverse: $s^{-1}(x) = s(x)^{-1}$. The map $\mathrm{Iso}_A(V_1, V_2) \to \mathrm{Iso}_A(V_2, V_1);\ s \mapsto s^{-1}$ is smooth (even analytic), in particular continuous. This is the reason why it suffices to consider s alone.

Assume for the moment that M is compact. Then, to the given s we find $\epsilon > 0$ such that $|s(x) - y| < \epsilon$ implies that $y \in \mathrm{Iso}_A((V_1)_x, (V_2)_x)$. Using the continuity of s we can find a finite collection $\{x_i\} \subset M$ of points, and a smooth partition of unity ϕ_i with support in some neighborhood U_i of x_i, with smooth trivialization ψ_i of our bundles over U_i, such that

$$t(x) := \sum_i \phi_i(x)\psi_i^{-1} s(x_i)$$

satisfies $|t(x) - s(x)| < \epsilon$ for all $x \in M$. Observe that $s(x_i)$ is mapped to nearby fibers (on U_i) using the trivializations. The section $t \in \mathrm{End}_A(V_1, V_2)$ is by its definition smooth, and invertible by the choice of ϵ.

This method generalizes to paracompact manifolds in the usual way, replacing ϵ by a function $\epsilon(x) > 0$, and the finite partition of unity by a locally finite partition of unity. □

2.1. Structure of finitely generated projective bundles. By definition, finitely generated projective Hilbert A-modules are direct summands of modules of the form A^n. We know that, on compact spaces, complex vector bundles are direct summands of trivial vector bundles. We now put these two observations together.

Theorem 2.14. *Let X be a compact Hausdorff space and $\pi\colon W \to X$ a finitely generated projective Hilbert A-module bundle.*

1. *Then W is isomorphic (as Hilbert A-module bundle preserving the inner product) to a direct summand of a trivial bundle $X \times A^n$ for suitable n (with orthogonal complement bundle $W^\perp$ such that $W \oplus W^\perp = X \times A^n$).*
2. *In other words, there is a projection valued function $\varepsilon\colon X \to M(n \times n, A)$ such that W is isomorphic to the fiberwise image of ε.*
3. *Conversely, the image of every such projection valued function is a finitely generated projective Hilbert A-module bundle.*

(4) *If ε_1 and ε_2 are two projection valued functions as above, then, for some $\delta > 0$ determined by ε_1, if $|\varepsilon_1(x) - \varepsilon_2(x)| < \delta$ for each $x \in X$, then the two image bundles are isomorphic.*

(5) *If X is a smooth manifold and W is a smooth bundle, then the function ε can be chosen smooth. The image bundle inherits a canonical smooth structure, and W is isomorphic to this bundle as a smooth bundle.*

(6) *Every finitely generated projective Hilbert A-module bundle over a smooth compact base manifold admits a smooth structure. It is unique up to isomorphism.*

Proof. Assume that the situation of the theorem is given.

(1) Choose a covering $U_1, \ldots, U_k$ of X with trivializations $\alpha_i \colon W|_{U_i} \xrightarrow{\cong} U_i \times V_i$, and $\hat{V}_i$ with $V_i \oplus \hat{V}_i \cong A^n$ (of course, if X is connected, all the V_i are isomorphic). Choose a partition of unity $\phi_i^2 \geq 0$ subordinate to the covering $\{U_i\}$. Define the (isometric!) embedding

$$j \colon W \to X \times (A^n)^k \colon v \mapsto \Big(\sum \phi_i(\pi(v))\alpha_i(v)\Big)_{i=1,\cdots,n}.$$

Claim: the fiberwise orthogonal complements to W in $X \times A^{nk}$ form a Hilbert A-module bundle W_2 such that $W \oplus W_2 = X \times A^{nk}$. To prove the claim, first of all, we can study W_2 for each component of X separately, and therefore assume that all V_i are equal (to V with complement $\hat{V}$). Secondly, it suffices to find W_2 such that $W \oplus W_2 \cong X \times V^k$; then $W \oplus W_2 \oplus (X \times \hat{V}^k) \cong X \times A^{nk}$. Observe that the embedding j factors through an embedding (also called j)

$$j \colon W \hookrightarrow X \times V^k.$$

We claim that this embedding has an orthogonal complement $W^\perp$ with $j(W) \oplus W^\perp \cong X \times V^k$. Therefore we can use $W_2 := W^\perp$ to conclude that W has a complementary Hilbert A-module bundle.

In contrast to Hilbert spaces, not every Hilbert A-submodule does have an orthogonal complement. Therefore, we have to prove the above assertion. Observe that there is no problem in defining the complementary bundle

$$W^\perp := \{(x, v) \in X \times V^k \mid v \perp j(W_x)\}.$$

Positivity of the inner product implies $j(W) \cap W^\perp = X \times \{0\}$. It remains to prove that for each fiber $j(W_x) + W_x^\perp = V^k$. For this, observe that $j(W_x) = \{(\phi_1\alpha_1(v), \ldots, \phi_k\alpha_k(v)) \mid v \in W_x\}$, with $\phi_1, \ldots, \phi_k \in \mathbb{R}$ and not all $\phi_k = 0$, and $\alpha_i \colon W_x \to V$ Hilbert A-module isometries. Without loss of generality, $\phi_1 \neq 0$. Then

$$j(W_x) = \{(v, \beta_2(v), \cdots, \beta_k(v)) \mid v \in V\},$$

with $\beta_i = \phi_1^{-1}\phi_i\alpha_i \circ \alpha_1^{-1} \in \mathrm{End}_A(V)$. More precisely, they are real multiples (zero is possible) of Hilbert A-module isometries. Observe that an isometry is automatically adjointable, the inverse being the adjoint.

We claim that $W_x^\perp$ is the Hilbert A-submodule U_x of V^k generated by the elements

$$(-\beta_i^*(v), 0, \cdots, 0, v, 0, \cdot, 0), \quad \text{with } v \in V \text{ at the } i\text{th position } (i = 2, \ldots, k).$$

Because of the calculation of inner products

$$\begin{aligned}\langle (v,\beta_2(v),\dots,\beta_k(v)),&(-\beta_i^*(w),0,\dots,0,w,0,\dots,0)\rangle\\ &= \langle v,-\beta_i^*(w)\rangle_V + \langle \beta_i(v),w\rangle_V\\ &= -\langle \beta_i(v),w\rangle_V + \langle \beta_i(v),w\rangle_V\\ &= 0\end{aligned}$$

each of these elements is indeed contained in $W_x^\perp$. To show that the sum satisfies $j(W_x)+U_x = V^k$ we have for arbitrary $(v_1,\dots,v_k)\in V^k$ to find $w_1,\dots,w_k\in V^k$ with

$$\begin{aligned}w_1-\beta_2^*(w_2)-\dots-\beta_k^*(w_k) &= v_1\\ \beta_2(w_1)+w_2 &= v_2\\ &\dots\\ \beta_k(w_1)+w_k &= v_k.\end{aligned}$$

Equivalently (adding β_i^* of the lower equations to the first one),

$$\begin{aligned}w_1+\beta_2^*\beta_2(w_1)+\dots+\beta_k^*\beta_k(w_1) &= v_1+\beta_2^*(v_2)+\dots+\beta_k^*(v_k)\\ w_2 &= v_2-\beta_2(w_1)\\ \dots&\\ w_k &= v_k-\beta_k(w_1).\end{aligned}$$

Since $1+\beta_2^*\beta_2+\dots\beta_k^*\beta_k\geq 1$ is an invertible element of the C^*-algebra $\mathrm{End}_A(V)$, there is indeed a (unique) solution $(w_1,\dots,w_k)$ of our system of equations.

It remains to check that $W^\perp$ (with the A-valued inner product given by restriction) is really a locally trivial bundle of Hilbert A-modules. Because of our description of $W^\perp$, $W^\perp|_{\{\phi_1\neq 0\}}\to V^{k-1}\colon (v_1,\dots,v_k)\mapsto (v_2,\dots,v_k)$ is an isomorphism of right A-modules and therefore gives a trivialization of a right A-module bundle (finitely generated projective).

The transition functions (here between $\{\phi_1\neq 0\}$ and $\{\phi_i\neq 0\}$) are given by

$$\begin{aligned}&(v_2,\dots,v_k)\\ &\quad\mapsto \left(-\phi_1^{-1}\phi_2(\alpha_2\circ\alpha_1^{-1})^*(v_2)\cdots-\phi_1^{-1}\phi_k(\alpha_k\alpha_1^{-1})^*(v_k),v_2,\dots,v_k\right)\\ &\quad\mapsto \left(-\phi_1^{-1}\phi_2(\alpha_2\circ\alpha_1^{-1})^*(v_2)\cdots-\phi_1^{-1}\phi_k(\alpha_k\circ\alpha_1^{-1})^*(v_k),v_2,\dots,\hat{v}_i,\dots,v_k\right).\end{aligned}$$

Here, $\hat{v}_i$ means that this entry is left out.

In particular, we observe that in the case where X is a smooth manifold and W is a smooth bundle, if we choose a smooth partition of unity, the complementary bundle $W^\perp$ obtains a canonical smooth structure, as well. Moreover, the inclusions of W and $W^\perp$ into $X\times V^k$ are both smooth.

Then $W^\perp\oplus(X\times\hat{V}^k)$ also has a smooth structure, and again the inclusions are smooth.

By Lemma 2.12, from the noninner product preserving trivialization of $W^\perp$ we produce trivializations which respect the given inner product.

(2) Define now $\varepsilon\colon X\to M(nk\times nk,A)=\mathrm{Hom}_A(A^{nk},A^{nk})$ such that $\varepsilon(x)$ is the matrix representing the orthogonal projection from A^{nk} onto $j(W_x)$. ε can be written as the composition of three maps: the inverse of the isomorphism $W\oplus W_2\to X\times A^{nk}$ which is continuous, the projection $W\oplus W_2\twoheadrightarrow W$ (which, in a local

trivialization is constant, and therefore depends continuously on $x \in X$), and the inclusion of W into $X \times A^{nk}$, which is continuous. Altogether, $x \mapsto \varepsilon(x)$ is a continuous map.

Moreover, if X and the bundle W are smooth and we perform our construction using the smooth structure, then the above argument implies that ϵ is a smooth map.

(3) and (5): We now have to show that the images $W_x := \operatorname{im}(\varepsilon(x))$ of a projection valued map $\varepsilon\colon X \to M(n \times n, A)$ form a finitely generated projective Hilbert A-module bundle W, with a canonical smooth structure if X and ε are smooth. Evidently, each fiber is a finitely generated projective Hilbert A-module. But one still has to check (as before for $W^\perp$) that this is a locally trivial bundle.

Fix $x_0 \in X$. We claim that $\varepsilon(x_0)|_{W_x}\colon W_x \to W_{x_0}$ defines a trivialization of $W|_U$, for U a sufficiently small open neighborhood of x_0. To see this, precompose $\varepsilon(x_0)$ with $\varepsilon(x)$. For $x = x_0$, this is the identity map, and it depends continuously on x. Therefore $\varepsilon(x_0) \circ \varepsilon(x)\colon W_{x_0} \to W_x \to W_{x_0}$ is an isomorphism for x sufficiently small (the invertible endomorphisms being an open subset of all endomorphisms). More precisely, if $|\varepsilon(x_0) - \varepsilon(x)| < 1$, then $|\varepsilon(x_0) - \varepsilon(x_0)\varepsilon(x)| < 1$ and then, since $\varepsilon(x_0)$ is the identity on W_{x_0}, by the von Neumann series argument $\varepsilon(x_0)\varepsilon(x)$ is an isomorphism. In the same way, under the same assumption

$$\varepsilon(x) \circ \varepsilon(x_0)\colon W_x \to W_{x_0} \to W_x$$

is an isomorphism. This shows that we have indeed constructed local trivializations of W, which therefore is a Hilbert A-module bundle (we obtain other trivialization which preserve the inner product by Lemma 2.12).

Let $\alpha_x := (\varepsilon(x_0)\colon W_x \to W_{x_0})^{-1}\colon W_{x_0} \to W_x$ be the inverse of the trivialization isomorphism (where defined). We want to show that our trivializations define a smooth structure on W if $\varepsilon(x)$ is a smooth function. We have to show that $\varepsilon(x_1) \circ \alpha_x\colon W_{x_0} \to W_{x_1}$ depends smoothly on x (where defined). To do this, we precompose with the isomorphism $\varepsilon(x_0) \circ \varepsilon(x)\colon W_{x_0} \to W_x \to W_{x_0}$. By assumption, this and therefore automatically also its inverse depend smoothly on x. But the composition is $\varepsilon(x_1) \circ \varepsilon(x)$, which again is a smooth function of x. This establishes smoothness of W.

If we construct the subbundle W and the projection ε as in (1) and (3), we still have to check that the smooth structures coincide. The map $\alpha_i^{-1}\colon U_i \times V \to W|_{U_I}$ is (by definition of the smooth structure of W) a smooth map, the embedding $i\colon W \to X \times A^n$ is a smooth map, and the projection $\varepsilon(x_0)\colon W \to X \times W_{x_0}$ is a bounded linear map which (in the coordinates just constructed) does not depend on x and therefore is also smooth. The composition of these maps is therefore also smooth, and it is the map which changes from the old smooth bundle chart to the new smooth bundle chart. Therefore the inclusion gives an isomorphism of smooth bundles between W and the subbundle $i(W)$ which is the image bundle of ε.

(4) Given two projection valued functions ε_1 and ε_2 with image bundle W_1 and W_2, respectively, $\varepsilon_1\colon W_2 \to W_1$ will be an isomorphism (not preserving the inner products) if ϵ_1 and ϵ_2 are close enough because of exactly the same argument which showed in (3) that the projections can be used to get local trivializations.

(6) By Theorem 2.13 there is up to isomorphism at most one smooth structure on a Hilbert A-module bundle W. Therefore it suffices to prove that one smooth structure exists. To do this, embed a finitely generated projective Hilbert A-module

bundle W into $X \times A^n$ as in (1). Let $\varepsilon\colon X \to M(n, A)$ be the projection valued function such that the image bundle is (isomorphic to) W. Choose a smooth approximation ε' to ε, sufficiently close such that the image bundles are isomorphic by (4). Observe that we can approximate continuous functions to Banach spaces arbitrarily well by smooth function, and we can achieve that the new smooth function is projection valued by application of the holomorphic functional calculus (because of the analyticity, smoothness is preserved). Because ε' is smooth, the image bundle obtains a smooth structure by (5), and this does the job. □

Remark 2.15. The usual approximation results work for the infinite-dimensional bundles we are considering: if M is a compact manifold and W is a finitely generated projective Hilbert A-module bundle on X, then the space of smooth sections is dense for the C^k-topology in the space of k-times differentiable sections.

2.2. K-theory with coefficients in a C^*-algebra.

Definition 2.16. Let X be a compact Hausdorff space and A a C^*-algebra. The K-theory of X with coefficients in A, $K(X; A)$, is defined as the Grothendieck group of isomorphism classes of finitely generated projective Hilbert A-module bundles over X.

Proposition 2.17. *Let X be a compact Hausdorff space. Then*

$$K(X; A) \cong K_0(C(X, A)),$$

i.e., the K-theory group of Hilbert A-module bundles is isomorphic to the K-theory of the C^*-algebra of continuous A-valued functions on X. The isomorphism is implemented by the map which assigns to a Hilbert bundle the module of continuous sections of this bundle.

Observe also that $C(X, A) \cong C(X) \otimes A$, where we use the (minimal) C^*-algebra tensor product. (Actually, since $C(X)$ is continuous and therefore nuclear, there is only one option for the tensor product.)

Proof. By Theorem 2.14, every finitely generated projective Hilbert A-module bundle W has a complement W_2 such that $W \oplus W_2 \cong X \times A^n$ for a suitable n. Moreover,

$$C(X, W) \oplus C(X, W_2) \cong C(X, W \oplus W_2) \cong C(X, A^n) \cong (C(X, A))^n,$$

i.e., $C(X, W)$ is a direct summand in a finitely generated free $C(X, A)$-module and therefore is finitely generated projective.

An isomorphism $W \to W_2$ of Hilbert A-module bundles induces an isomorphism $C(X, W) \to C(X, W_2)$ of $C(X, A)$-modules. Moreover,

$$C(X, W \oplus W_2) \cong C(X, W) \oplus C(X, W_2)$$

as $C(X, A)$-modules. It follows that

$$s\colon K(X; A) \to K_0(C(X, A));\ W \mapsto C(X, W) \tag{2.18}$$

is a well-defined group homomorphism.

We now explain how to construct the inverse homomorphism. Assume therefore that L is a finitely generated projective $C(X, A)$-module with complement L', i.e., $L \oplus L' = C(X, A)^n = C(X, X \times A^n)$. Define the set

$$W := \{(x, v) \in X \times A^n \mid \exists s \in L; s(x) = v\}.$$

We claim that W is a finitely generated Hilbert A-module bundle with $C(X,W) \cong L$, where $\pi\colon W \to X$ is given by $\pi(x,v) = x$. Let $p\colon C(X, X \times A^n) \to L$ be the projection along L'. We have to prove that W is a locally trivial bundle. Fix $x \in X$. Define

$$\alpha_x\colon X \times W_x \to W;\ (x,v) \mapsto (x, p(c_v)(x))$$

where $c_v \in C(X, A^n)$ is the constant section with value $v \in W_x \subset A^n$. Restricted to a sufficiently small neighborhood $U \subset X$ of x, this map is an isomorphism $U \times W_x \to W|_U$. This can be seen as follows: we compose α_x with the map $\beta\colon W \to X \times W_x$ with $(y,v) \mapsto (y, p(c_v)(x))$. Then $\beta \circ \alpha_x\colon X \times W_x \to X \times W_x$ is a continuous section of $\mathrm{End}_A(X \times W_x)$ and its value at x is id_{W_x}. By continuity, and since the set of invertible elements of the C^*-algebra $\mathrm{End}_A(W_x)$ is open, $\beta \circ \alpha_x(y)$ is invertible if y is close enough to x. By Lemma 2.12, we can turn this into an isomorphism $W|_U \to U \times W_x$ which preserves the inner products.

Consequently, W is a Hilbert A-module bundle. Using the local trivializations constructed above, we conclude also that indeed $C(X,W) = L$.

The same reasoning applies to show that $L' = C(X, W')$ with W' defined in the same way as W is defined, and $C(X,W) \oplus C(X,W') = C(X, A^n)$. From this, we conclude that $W \oplus W' = X \times A^n$, i.e., W is a finitely generated projective Hilbert A-module bundle.

Assume that $\rho\colon L \to N$ is an isomorphism of finitely generated projective $C(X,A)$-modules. Assume that $L \oplus L' \cong A^n$ and $N \oplus N' \cong A^m$. We can assume that there is an isomorphism $\rho'\colon L' \to N'$ (simply replace L' by $L' \oplus (N \oplus N')$ and N' by $N' \oplus (L \oplus L')$). Then our construction shows that ρ induces an isomorphism between the Hilbert A-module bundles associated to L and to N, respectively. Similarly, the Hilbert A-module bundle associated to $L \oplus N$ is the direct sum of the bundles associated to L and to N. Consequently, the construction defines a homomorphism

$$t\colon K_0(C(X,A)) \to K^0(X;A). \tag{2.19}$$

The maps s of (2.18) and t of (2.19) are by their construction inverse to each other. This concludes the proof of the proposition. □

For several reasons, in particular to be able to discuss Bott periodicity conveniently, it is useful to extend the definition of K-theory from compact to locally compact spaces. For the latter ones, we will restrict ourselves to compactly supported K-theory (which is the usual definition).

Definition 2.20. Let X be a locally compact Hausdorff space. Denote its one-point compactification X_+. Then

$$K^0_c(X;A) := \ker(K^0(X_+;A) \to K^0(\{\infty\};A)),$$

where the map is induced by the inclusion of the additional point $\infty \hookrightarrow X_+$.

Proposition 2.21. *Assume that X is a locally compact Hausdorff space. Then*

$$K^0_c(X;A) \cong K_0(C_0(X;A)).$$

Proof. The split exact sequence of C^*-algebras

$$0 \to C_0(X;A) \to C(X_+;A) \xrightarrow{ev_\infty} A \to 0$$

gives rise to the split exact sequence in K-theory

$$0 \to K_0(C_0(X;A)) \to K_0(C(X_+;A)) \to K_0(A) \to 0.$$

We know by the proof of Proposition 2.17 that $\ker(K_0(C_0(X_+;A) \to K_0(A)))$ is given by the Grothendieck group of finitely generated projective Hilbert A-module bundles over X_+, where the fiber over ∞ formally is zero. □

As in the case of a compact space X, we now show that $K_c^0(X;A)$ can be described in terms of finitely generated projective bundles over X.

Proposition 2.22. *Assume that X is a locally compact Hausdorff space. The group $K_c^0(X;A)$ is isomorphic to the group of stable isomorphism classes of tuples*

$$(W, W_2, \phi_W, \phi_{W_2})$$

where W and W_2 are finitely generated projective Hilbert A-module bundles on X and $\phi_W : W_{X\setminus K} \to (X \setminus K) \times P$, $\phi_{W_2} : W_{X\setminus K} \to (X \setminus K) \times P$ are trivializations of the restriction of W and W_2 to the complement of a compact subset K of X where the range for both these trivializations is equal.

Two such tuples $(W, \dots)$ and $(V, \dots)$ are defined to be stably isomorphic if there is a third one $(U, \dots)$ and isomorphisms $W \oplus U \to V \oplus U$, $W_2 \oplus U_2 \to V_2 \oplus U_2$ such that the induced isomorphisms of the trivializations on the common domain of definition $(X \setminus K) \times (P_W \oplus P_U) \to (X \setminus K) \times (P_V \oplus P_U)$ extends to an isomorphism $(X_+ \setminus K) \times (P_W \oplus P_U) \to (X_+ \setminus K) \times (P_V \oplus P_U)$, and correspondingly for $W_2, \dots$.

The sum is given by direct sum, where the trivializations have to be restricted to the common domain of definition.

Proof. We have shown that $K^0(X_+;A)$ is the Grothendieck group of finitely generated projective Hilbert A-module bundles over X_+. The kernel of the map to $K^0(\infty;A)$ is therefore given by formal differences of two Hilbert A-module bundles over X_+ with isomorphic fibers over ∞. A tuple $(W, W_2, \phi_W, \phi_{W_2})$ gives rise to such a formal difference by extending the bundles W and W_2 to X_+ using the trivialization on $X \setminus K$. The equivalence relation on the tuples is made exactly in such a way that this map is well-defined. On the other hand, a formal difference of two bundles W, W_2 on X_+ gives the first two entries of such a tuple by restriction to X, and trivializations $W|_{X_+\setminus K} \to (X_+ \setminus K) \times W_\infty$, $W_2|_{X_+\setminus K} \to (X_+ \setminus K) \times W_{2\infty}$ together with an identification of $(W_2)_\infty$ with W_∞ (which is possible since we assume that the two are isomorphic) give by restriction rise to the required isomorphisms. Again we see that our equivalence relation is made in such a way that different choices (including different choices of the trivializations) give rise to equivalent tuples.

The maps being well-defined, it is immediate from their definitions that they are inverse to each other. □

Recall that in this language it is possible to define the first K-theory group using "suspension" in the following way.

Definition 2.23. Assume that X is a compact Hausdorff space. Define

$$K^1(X;A) := K_c^0(X \times \mathbb{R};A).$$

In particular,

$$K_1(A) := K^1(\{*\};A) = K_c^0(\mathbb{R};A).$$

2.2.1. Bott periodicity. We now formulate Bott periodicity in our world of Hilbert A-module bundles.

Theorem 2.24. *Assume that X is a compact Hausdorff space. Then there is an isomorphism*

$$\beta\colon K^0(X;A) \to K_c^0(X\times\mathbb{R}^2;A);\ W\mapsto \pi_1^*W\otimes\pi_2^*B.$$

Here B is the Bott generator of $K_c^0(\mathbb{R};\mathbb{C})$. It corresponds under the identification with $\ker(K^0(S^2)\to K^0(\mathbb{C}))$ to the formal difference $H-1$ where H is the Hopf bundle and 1 the 1-dimensional trivial bundle. $\pi_1\colon X\times\mathbb{R}^2\to X$ and $\pi_2\colon X\times\mathbb{R}^2\to\mathbb{R}^2$ are the projections, and the tensor product, being a tensor product of a bundle of finitely generated projective Hilbert A-modules with a bundle of finite $\mathbb{C}$-vector spaces, is well-defined.

Proof. The result is of course perfectly well-known. For the convenience of the reader we show here that the general facts about Bott periodicity implies that our map does the job.

Our proof follows the idea of [24, Exercise 9.F]. The given map β is functorial in X and A. It is classical that for $A=\mathbb{C}$ it is the Bott periodicity isomorphism. Moreover, $K^0(X;A)=K_0(C(X)\otimes A)=K_0(pt;C(X)\otimes A)$, and this identification is compatible with β. Therefore we can assume that $X=\{pt\}$. Use now Morita equivalence $K_0(A)\cong K_0(M_n(A))$ which is induced by a (nonunital) C^*-algebra homomorphism $A\to M_n(A)$ and therefore compatible with β. For any $x\in K_0(A)$ we find $n\in\mathbb{N}$ and projections $p,q\in M_n(A)$ such that $x=[p]-[q]\in K_0(M_n(A))$, where we use Morita equivalence to view x as an element in $K_0(M_n(A))$. Define $c_p\colon\mathbb{C}\to M_n(A)$ by $c_p(1)=p$. By naturality, $\beta(p)=c_p(\beta(1))$, i.e., the natural transformation β is determined by the specific value $\beta(1)$ for $1\in K_0(\mathbb{C})$.

Since the usual Bott periodicity homomorphism coincides with β on $K_0(\mathbb{C})$ and is also a natural transformation, the two coincide for all C^*-algebras, proving the claim. □

Remark 2.25. Theorem 2.24 extends to locally compact Hausdorff spaces X. The proof has to be slightly modified, because $C_0(X)\otimes A$ is not unital, such that we haven't defined $K^0(\{pt\};C_0(X)\otimes A)$. Since we don't need the result in this paper, we omit the details.

2.3. Traces and dimensions of Hilbert A-modules.

Proposition 2.26. *Assume that V is a finitely generated projective Hilbert A-module. Then the map*

$$\iota\colon V\otimes_A \operatorname{Hom}_A(V,A)\to \operatorname{End}_A(V) \tag{2.27}$$

of Example 2.9 given by $v\otimes\phi\mapsto(x\mapsto v\phi(x))$ is a canonical isomorphism. Since the isomorphism is canonical, the corresponding result holds for any Hilbert A-module bundle W, i.e.,

$$W\otimes_A\operatorname{Hom}_A(W,A)\cong\operatorname{End}_A(W).$$

Proof. In general, the image of $V\otimes_A\operatorname{Hom}_A(V,A)$ in $\operatorname{End}_A(V)$ is (after completion) by definition the algebra of compact operators $K(V)$. Since V is finitely generated projective, $K(V)=\mathcal{B}(V)=\operatorname{End}_A(V)$, and it is not necessary to complete.

More explicitly, recall that V is a direct summand in A^n and let $p \in \mathrm{End}_A(A^n)$ be the projection with image V. Then $\mathrm{End}_A(V) = p\,\mathrm{End}_A(A^n)p$, $\mathrm{Hom}_A(V, A) = \mathrm{Hom}_A(A^n, A)p$, and $V = p(A^n)$ can be considered as submodules of $\mathrm{End}_A(A^n)$, $\mathrm{End}_A(A^n, A)$ and A^n, respectively.

Then

$$\begin{aligned} V \otimes_A \mathrm{Hom}_A(V, A) &= pA^n \otimes_A \mathrm{Hom}_A(A^n, A)p = p(A^n \otimes_A A^n)p \\ &= p\,\mathrm{End}_A(A^n)p = \mathrm{End}_A(V). \end{aligned}$$

The identifications are given by the maps we have to consider. □

Definition 2.28. For each algebra A let $[A, A]$ be the subspace of A generated by commutators $[a, b] := ab - ba$ for $a, b \in A$.

Lemma 2.29. *Given a finitely generated projective Hilbert A-module V, there is a canonical linear homomorphism $ev\colon \mathrm{End}_A(V) \to A/[A, A]$, given by the composition*

$$ev\colon \mathrm{End}_A(V) \xleftarrow{\cong} V \otimes_A \mathrm{Hom}_A(V, A) \xrightarrow{v\otimes\phi\mapsto\phi(v)+[A,A]} A/[A, A].$$

Since this homomorphism is canonical, it extends to a finitely generated projective Hilbert A-module bundle W, to give rise to a bundle homomorphism

$$ev\colon \mathrm{End}_A(W) \to M \times (A/[A, A]).$$

This homomorphism does have the trace property, i.e., for all endomorphisms Φ_1 and Φ_2,

$$ev(\Phi_1 \circ \Phi_2) - ev(\Phi_2 \circ \Phi_1) = [A, A] = 0 \in A/[A, A]. \tag{2.30}$$

Proof. The first assertion is true by Proposition 2.26. Observe that $va \otimes \phi$ is mapped to $\phi(v)a + [A, A]$, whereas $v \otimes a\phi$ is mapped to $a\phi(v) + [A, A]$. Clearly, $\phi(v) \cdot a - a \cdot \phi(v) \in [A, A]$.

For the trace property, observe that for $\phi_1, \phi_2 \in \mathrm{Hom}_A(V, A)$ and $v_1, v_2 \in A$

$$\iota(v_1 \otimes \phi_1) \circ \iota(v_2 \otimes \phi_2) = \iota(v_1(\phi_1(v_2)) \otimes \phi_2),$$

with ι of (2.27). It follows that

$$ev(\iota(v_1 \otimes \phi_1) \circ \iota(v_2 \otimes \phi_2)) = \phi_2(v_1) \cdot \phi_1(v_2) + [A, A],$$

whereas

$$ev(\iota(v_2 \otimes \phi_2) \circ \iota(v_1 \otimes \phi_1)) = \phi_1(v_2) \cdot \phi_2(v_1) + [A, A],$$

i.e., the difference of the two elements is zero in $A/[A, A]$. Because $\mathrm{End}_A(V)$ is linearly generated (using the isomorphism ι to $\mathrm{Hom}_A(V, A) \otimes_A V$) by elements of the form $\iota(v \otimes \phi)$, Equation (2.30) follows. □

An immediate consequence of Lemma 2.29 is the following lemma.

Lemma 2.31. *Let Z be a commutative C^*-algebra (e.g., $\mathbb{C}$ or the center of A). Let $\tau\colon A \to Z$ be a trace, i.e., τ is linear and $\tau(ab) = \tau(ba)$ for each $a, b \in A$, or, in other words, τ factors through $A/[A, A]$. Then the composition of τ and ev is well-defined and is a Z-valued trace on $\mathrm{End}_A(V)$ for each finitely generated projective Hilbert A-module V, and correspondingly for a finitely generated projective Hilbert A-module bundle W on M. In the latter case it extends to a linear homomorphism*

$$\tau\colon \Omega^*(M; \mathrm{End}_A(W)) \to \Omega^*(M; Z);\ \eta \otimes \Phi \mapsto \eta \otimes \tau(ev(\Phi)).$$

3. Connections and curvature on Hilbert A-module bundles

Definition 3.1. Let V be a Hilbert A-module. Consider the trivialized Hilbert A-module bundle $M \times V$. For a smooth section $f \in \Gamma(M \times V)$, define

$$df \in \Gamma(T^*M \otimes (M \times V))$$

by the formula (locally) $df := \sum dx_i \otimes \frac{\partial f}{\partial x_i}$.

Definition 3.2. A *connection* ∇ *on a smooth Hilbert A-module bundle* W is an A-linear map $\nabla\colon \Gamma(W) \to \Gamma(T^*M \otimes W)$ which is a derivation with respect to multiplication with sections of the trivial bundle $M \times A$, i.e.,

$$\nabla(sf) = sdf + \nabla(s)f \qquad \forall s \in \Gamma(W),\ f \in C^\infty(M; A).$$

Here we use the multiplication $W \otimes T^*M \otimes (M \times A) \to W \otimes T^*M\colon s \otimes \eta \otimes f \mapsto sf \otimes \eta$. (In particular, elements of A are considered to be of degree zero.)

We say that ∇ is a *metric connection*, if

$$d\langle s_1, s_2\rangle = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle$$

for all smooth sections s_1, s_2 of W. Here, we consider $\langle s_1, s_2\rangle$ to be a section of the trivial bundle $M \times A$.

If L is only a smooth bundle of Banach spaces, a *connection* on L is a $\mathbb{C}$-linear map $\nabla\colon \Gamma(L) \to \Gamma(T^*M \otimes L)$ which is a derivation with respect to multiplication with smooth functions $f \in C^\infty(M, \mathbb{C})$.

Observe that in this sense d as defined in Definition 3.1 is a connection, the so-called *trivial connection* on the trivial bundle $M \times V$, which is actually even a metric connection with respect to the pointwise A-valued inner product $\langle s_1, s_1\rangle(x) = \langle s_1(x), s_2(x)\rangle_V$.

Lemma 3.3. *Given two connections ∇_1, ∇_2 on a smooth finitely generated projective Hilbert A-module bundle W, their difference $\omega := \nabla_1 - \nabla_2$ is a 1-form with values in the endomorphisms $\mathrm{End}_A(W)$, i.e., a section of $T^*M \otimes End_A(W)$. If both connections are metric connections, ω takes values in the skew adjoint endomorphisms of W.*

The difference being an endomorphism valued 1-form means that for each smooth section s of W and each vector field X

$$(\nabla_1)_X(s) - (\nabla_2)_X(s) = \omega(X)(s),$$

where on the right-hand side the endomorphism $\omega(X)$ is applied fiberwise to the value of the section s.

Proof. We define $\omega(X)$ by the left-hand side. The expression is $C^\infty(M)$-linear in X and A-linear in s. We have to check that it really defines an endomorphism valued 1-form, i.e., that $\omega(X)(s)_x$ depends only on s_x (for arbitrary $x \in M$), or equivalently (because of linearity), that $\omega(X)(s)$ vanishes at x if s vanishes at x.

Observe first that, from the multiplicativity formula for connections, $\omega(sf) = \omega(s)f$ for every smooth section s of W and every smooth A-valued function f on M.

Secondly, using a smooth cutoff function, we can write $s = s_1 + s_2$ such that s_1 is supported on a neighborhood U of x over which W is trivial, and s_2 vanishes in a neighborhood of x. Locally, $W|_U \subset U \times A^n$ as a direct summand. Using this trivialization, we can write $s_1 = \sum e_i f_i$ with $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, and $s(x) = 0$ if and only $f_i(x) = 0$ for all i. Extending ω (arbitrarily) to the complement of W, we can conclude that $\omega(X)(s)(x) = \sum \omega(X)(e_i)(x) f_i(x) = 0$, if $f_i(x) = 0$ for all i. In other words, $\omega(X)(s)_x = 0$ if $s_x = 0$, i.e., ω is a 1-form.

Assume now that ∇_1 and ∇_2 are metric connections. Then $0 = \langle \omega(s_1), s_2 \rangle - \langle s_1, \omega(s_2) \rangle$. Since the inner product is taken fiberwise, the operator $\omega(X)(x)$ is skew adjoint for each $x \in M$ and each vector field X. □

Definition 3.4. Let $f\colon M \to N$ be a smooth map between smooth manifolds and $W \to N$ a smooth finitely generated projective Hilbert A-module bundle with a connection ∇. Then we define on the pull back bundle f^*W a connection $f^*\nabla$ in the following way:

$$(3.5)\quad (f^*\nabla)_X((f^*s)u) := (f^*s)(du(X)) + f^*(\nabla_{f_* X}(s))u \qquad \forall u \in C^\infty(M),\ s \in \Gamma(W),\ X \in \Gamma(TN).$$

The existence of local trivializations (and the fact that the fibers are finitely generated A-module) imply that each section of f^*W is (locally) a $C^\infty(M)$-linear combination of sections of the form $(f^*s)u$ as above. By linearity, we therefore define $f^*\nabla$ for arbitrary sections of f^*W. The expression is well-defined because ∇ satisfies the Leibnitz rule.

Lemma 3.6. *Let $f\colon M \to N$ be a smooth map and $W \to N$ a smooth finitely generated projective Hilbert A-module bundle. Assume that ∇ and ∇_1 are connections on W with difference $\omega = \nabla - \nabla_1$. Then $f^*\nabla - f^*\nabla_1 = f^*\omega$.*

Proof. This follows immediately from formula (3.5) for the pullback connection and the definition of the pullback of a differential form. □

Definition 3.7. The *curvature* Ω of the connection ∇ on the finitely generated projective Hilbert A-module bundle W is the operator $\nabla \circ \nabla$.

Here, ∇ is extended to differential forms with values in W using the Leibnitz rule

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^{\deg(\omega)} \omega \nabla(s)$$

for all differential forms ω and all sections s of W.

Proposition 3.8. *The curvature is a 2-form with values in $\mathrm{End}_A(W)$. If the connection is a metric connection, then Ω takes values in skew adjoint 2-forms.*

Locally, we can trivialize $W|_U \cong U \times V$. Then on $W|_U$ the connection ∇ and a trivial connection ∇_V (depending on the trivialization) are given. They differ by the endomorphism valued 1-form ω, i.e., $\nabla = \nabla_V + \omega$.

Then $\Omega = d\omega + \omega \wedge \omega$. This implies $d\Omega = \Omega \wedge \omega - \omega \wedge \Omega$. We use the product

$$\begin{aligned}
&\Gamma(T^*M \otimes \mathrm{End}_A(W)) \otimes \Gamma(T^*M \otimes \mathrm{End}_A(W)) \\
&\quad \to \Gamma(T^*M \otimes T^*M \otimes \mathrm{End}_A(W) \otimes \mathrm{End}_A(W)) \\
&\quad \to \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}_A(W)) = \Omega^2(M; \mathrm{End}_A(W)).
\end{aligned}$$

Proof. As in the proof of Lemma 3.3, we only have to show that Ω is $C^\infty(M;A)$-linear. We compute for $s \in \Gamma(W)$ and $f \in C^\infty(M;A)$

$$\begin{aligned}\nabla(\nabla(sf)) &= \nabla(s\otimes df) + \nabla(\nabla(s)f) = s\otimes d(df) + \nabla(s)df - \nabla(s)df + \nabla(\nabla(s))f\\ &= \nabla(\nabla(s))f.\end{aligned}$$

Here we used that $d^2 = 0$ by Lemma 3.10. The minus sign arises since $\nabla(s)$ is a 1-form. From $C^\infty(M;A)$-linearity, if follows that Ω is an endomorphism valued 2-form, since W is finitely generated projective.

Next observe that by Lemma 3.9

$$\begin{aligned}\nabla\circ\nabla &= (\nabla_V+\omega)(\nabla_V+\omega)\\ &= \nabla_V\nabla_V + \omega\nabla_V + \nabla_V\circ\omega + \omega\wedge\omega = \omega\nabla_V + d\omega - \omega\nabla_V + \omega\wedge\omega\\ &= d\omega + \omega\wedge\omega.\end{aligned}$$

Here we use the fact that for each $s \in \Gamma(W)$

$$\nabla_V(\omega\wedge s) = d\omega\wedge s - \omega\wedge\nabla_V s$$

(the minus arises because ω is a 1-form, i.e., has odd degree). Moreover, $\nabla_V\nabla_V = 0$ by Lemma 3.10, since ∇_V is by definition a trivial connection.

Then

$$d\Omega = dd\omega + (d\omega)\wedge\omega - \omega\wedge d\omega = (d\omega+\omega)\wedge\omega - \omega\wedge(\omega+d\omega) = \Omega\wedge\omega - \omega\wedge\Omega.$$

If ∇ is a metric connection, then ω takes values in skew adjoint endomorphisms by Lemma 3.3 (our trivialization $W|_U \cong U\times V$ is a trivialization of Hilbert A-modules, therefore its trivial connection is a metric connection). The same is then true for $d\omega$, since the skew adjoint endomorphisms form a linear subspace of all endomorphisms. Moreover, the square $\omega\wedge\omega$ is a two form with values in skew adjoint endomorphisms because of the anti-symmetrization procedure involved in the square:

$$\omega\wedge\omega(X,Y) = \omega(X)\circ\omega(Y) - \omega(Y)\circ\omega(X),$$

whereas

$$\begin{aligned}(\omega\wedge\omega(X,Y))^* &= \omega(Y)^*\omega(X)^* - \omega(X)^*\omega(Y)^*\\ &= \omega(Y)\omega(X) - \omega(X)\omega(Y)\\ &= -\omega\wedge\omega(X,Y).\end{aligned}$$

□

In the proof of Proposition 3.8 we have used that the curvature of a trivial connection is zero, and that the difference of two connections is known even for the extension to differential forms. We prove both facts now.

Lemma 3.9. *If $\nabla_1 - \nabla_2 = \omega$ for two connections on the Hilbert A-module bundle W, as in Lemma 3.3, then the same formula holds for the extension of the connection to differential forms with values in W, i.e., the action of ω is given by the following composition:*

$$\begin{aligned}\Gamma(\Lambda^pT^*M\otimes W) &\xrightarrow{\cdot\otimes\omega\otimes\cdot} \Gamma(\Lambda^pT^*M\otimes T^*M\otimes \mathrm{End}_A(W)\otimes W)\\ &\xrightarrow{\wedge\otimes\cdot} \Gamma(\Lambda^{p+1}T^*M\otimes W).\end{aligned}$$

Proof. We only have to check that $\nabla_1 + \omega$ satisfies the Leibnitz rule. However,

$$(\nabla_1 + \omega)(\eta \otimes s) = d\eta \otimes s + (-1)^{\deg(\eta)} \eta \wedge \nabla_1 s + (-1)^{\deg(\eta)} \eta \wedge \omega s,$$

for each $s \in \Gamma(W)$ and each differential form η, since multiplication with ω is $C^\infty(M; A)$-linear and in particular $C^\infty(M)$-linear. $\square$

Lemma 3.10. *For the trivial connection d on a trivialized bundle $M \times V$, $d \circ d = 0$, i.e., the curvature is zero.*

Proof. We compute in local coordinates for a smooth section f of $M \times V$

$$d(df) = d\left(\sum dx_i \frac{\partial f}{\partial x_i}\right) = \sum dx_j dx_i \frac{\partial^2 f}{\partial x_j \partial x_i} = 0,$$

since $dx_i dx_i = 0$ and $dx_i dx_j = -dx_j dx_i$. $\square$

Definition 3.11. Connections ∇_W and ∇_{W_2} on the Hilbert A-module bundles W and W_2, respectively, induce by the Leibnitz rule a connection ∇ on the smooth bundle of Banach spaces $\mathrm{Hom}_A(W, W_2)$ with

$$\nabla_{W_2}(\Phi(s)) = (\nabla\Phi)(s) + \Phi(\nabla_W s)$$

for each smooth section Φ of $\mathrm{Hom}_A(W, W_2)$ and each smooth section s of W.

Lemma 3.12. *Assume that E is a smooth finite-dimensional complex Hermitian vector bundle and W_2 is a smooth Hilbert A-module bundle with connections ∇_E and ∇_{W_2}, respectively. The fiberwise (algebraic) tensor product over $\mathbb{C}$ is then a Hilbert A-module bundle $E \otimes W_2$, since E is finite-dimensional and W_2 is finitely generated projective. By the Leibnitz rule it carries a connection $\nabla_\otimes$ with*

$$\nabla_\otimes(\sigma \otimes s) = \nabla_E(\sigma) \otimes s + \sigma \otimes \nabla_{W_2}(s) \qquad \forall \sigma \in \Gamma(E),\ s \in \Gamma(W_2).$$

If Ω_E is the curvature of ∇_E and Ω_{W_2} the one of ∇_{W_2}, then

$$\Omega_\otimes = \Omega_E \otimes \mathrm{id}_{W_2} + \mathrm{id}_E \otimes \Omega_{W_2}$$

is the curvature of $\nabla_\otimes$.

Proof. If V is a finite-dimensional Hermitian $\mathbb{C}$-vector space and W a Hilbert A-module, then $V \otimes W \cong W^{\dim V}$ with isomorphism canonical up to the choice of an orthonormal basis of V. This implies that $E \otimes W_2$ becomes a Hilbert A-module bundle in a canonical way. It is a standard calculation that the formula defines a connection on the tensor product.

For the curvature we obtain

$$\begin{aligned}\Omega_\otimes =& \nabla_\otimes \nabla_\otimes = (\nabla_E \otimes \mathrm{id}_{W_2} + \mathrm{id}_E \otimes \nabla_{W_2})(\nabla_E \otimes \mathrm{id}_{W_2} + \mathrm{id}_E \otimes \nabla_{W_2})\\ =& (\nabla_E \nabla_E) \otimes \mathrm{id}_{W_2} + \mathrm{id}_E \otimes (\nabla_{W_2} \nabla_{W_2})\\ &+ (\nabla_E \otimes \mathrm{id}_{W_2})(\mathrm{id}_E \otimes \nabla_{W_2}) + (\mathrm{id}_E \otimes \nabla_{W_2})(\nabla_E \otimes \mathrm{id}_{W_2}).\end{aligned}$$

Observe that operators of the form $f \otimes \mathrm{id}$ commute with operators of the form $\mathrm{id} \otimes g$ on $E \otimes W_2$. Consequently, the usual sign rule when interchanging the 1-forms $\mathrm{id}_{W_2} \otimes \nabla_{W_2}$ and $\nabla_E \otimes \mathrm{id}_{W_2}$ applies to give

$$(\mathrm{id}_E \otimes \nabla_{W_2})(\nabla_E \otimes \mathrm{id}_{W_2}) = -(\nabla_E \otimes \mathrm{id}_{W_2})(\mathrm{id}_E \otimes \nabla_{W_2}).$$

This finally implies the desired formula

$$\Omega_\otimes = \Omega_E \otimes \mathrm{id}_{W_2} + \mathrm{id}_E \otimes \Omega_{W_2}.$$ $\square$

Lemma 3.13. *Let $f\colon M \to N$ be a smooth map and $W \to N$ a smooth finitely generated projective Hilbert A-module bundle with connection ∇ and curvature Ω. Then the curvature of the pullback connection $f^*\nabla$ on the pullback bundle f^*W is $f^*\Omega$.*

Proof. By Proposition 3.8, locally $\Omega = d\omega + \omega \wedge \omega$, where ω is the difference between ∇ and a trivial connection.

The pullback of a trivial connection is by Definition 3.4 trivial. By Lemma 3.6, $f^*\omega$ therefore is the difference between $f^*\nabla$ and a trivial connection. Consequently, Proposition 3.8 implies that the curvature Ω^* of $f^*\nabla$ is given by

$$\Omega^* = d(f^*\omega) + f^*\omega \wedge f^*\omega = f^*(d\omega + \omega\wedge\omega) = f^*\Omega. \qquad \square$$

4. Chern–Weil theory

The prototype of the characteristic classes we want to define is the Chern character.

Definition 4.1. Consider the formal power series $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. A differential form of degree ≥ 1 with values in a ring on a finite dimensional manifold can be substituted for x.

In particular, if W is a Hilbert A-module bundle on a manifold M with connection ∇ and curvature $\Omega \in \Omega^2(M; \mathrm{End}_A(W))$, we define

$$\exp(\Omega) := \sum_{k=0}^{\infty} \frac{\overbrace{\Omega\wedge\cdots\wedge\Omega}^{k\text{ times}}}{k!} \in \Omega^{2*}(M; \mathrm{End}_A(W)).$$

Given a commutative C^*-algebra Z and a trace $\tau\colon A \to Z$, if W is a finitely generated projective Hilbert A-module bundle, we now define

$$\mathrm{ch}_\tau(\Omega) := \tau(ev(\exp(\Omega))) \in \Omega^{2*}(M; Z),$$

using the homomorphism ev of Lemma 2.29.

Lemma 4.2. *If τ is a trace then the characteristic class $\mathrm{ch}_\tau(\Omega)$ of Definition 4.1 is closed. The cohomology class represented by $\mathrm{ch}_\tau(\Omega)$ does not depend on the connection ∇ but only on the finitely generated projective Hilbert A-module bundle W.*

Proof. Recall that by Proposition 3.8 we have locally $d\Omega = \Omega\wedge\omega - \omega\wedge\Omega$ for a suitable endomorphism valued 1-form ω. It suffices to check that for each $k \in \mathbb{N}$

$$d\tau(ev(\Omega^k)) = 0.$$

We will show that $d\tau(ev(\eta)) = \tau(ev(\nabla\eta))$ for each $\eta \in \Omega^*(M; \mathrm{End}_A(W))$. This holds for an arbitrary connection ∇, consequently we can apply it using (locally) the connection d obtained from a trivialization. Once this is established, we compute (locally) and using that $\tau \circ ev$ has the trace property and that Ω is a form of even degree,

$$\begin{aligned}
d\tau(ev(\Omega^k)) = \tau(ev(\nabla(\Omega^k))) &= \sum_{i=0}^{k-1} \tau(ev(\Omega^i \wedge (\nabla\Omega) \wedge \Omega^{k-i-1})) \\
&= \sum_{i=0}^{k-1} \tau(ev(\Omega^i \wedge (\Omega \wedge \omega - \omega \wedge \Omega) \wedge \Omega^{k-i-1})) \\
&= \sum_{i=1}^{k} \tau(ev(\Omega^i \wedge \omega \wedge \Omega^{k-i})) - \sum_{i=0}^{k-1} \tau(ev(\Omega^i \wedge \omega \wedge \Omega^{k-i})) \\
&= k\left(\tau(ev(\Omega^k \wedge \omega)) - \tau(ev(\Omega^k \wedge \omega))\right) = 0.
\end{aligned}$$

To establish the formula $d\tau(ev(\eta)) = \tau(ev(\nabla\eta))$ which we have used above, it suffices to consider $\eta = \alpha\phi \otimes v$ with $\alpha \in \Omega^*(M)$, $\phi \in \Gamma(\mathrm{Hom}_A(W, A))$ and $v \in \Gamma(W)$. This is the case because such forms locally generate $\Omega^*(M; \mathrm{End}_A(W))$, using the isomorphism of Proposition 2.26. Then, on the one hand by Definition 3.11

$$\begin{aligned}
d(\tau(ev(\eta))) = d(\tau(\alpha\phi(v))) &= \tau((d\alpha)\phi(v) + (-1)^{\deg(\alpha)}\alpha \wedge d(\phi(v))) \\
&= \tau((d\alpha)\phi(v) + (-1)^{\deg(\alpha)}\alpha \wedge ((\nabla\phi)(v) + \phi(\nabla v))).
\end{aligned}$$

Here, we used that the homomorphism $\tau\colon M \times A \to M \times Z$ is given by fiberwise application of $\tau\colon A \to Z$. It follows that $d\tau(\beta) = \tau d\beta$ for each $\beta \in \Omega^*(M; A)$, where we use $d\colon \Omega^*(M; A) \to \Omega^*(M; A)$ as defined in Definition 3.1.

On the other hand,

$$\begin{aligned}
\tau(ev(\nabla\eta)) &= \tau(ev((d\alpha)\phi \otimes v + (-1)^{\deg(\alpha)}\alpha\nabla(\phi \otimes v))) \\
&= \tau((d\alpha)\phi(v) + (-1)^{\deg(\alpha)} ev(\alpha \wedge (\nabla\phi) \otimes v + \phi \otimes \nabla(v))) \\
&= \tau((d\alpha)\phi(v) + (-1)^{\deg(\alpha)}\alpha \wedge ((\nabla\phi)(v) + \phi(\nabla v))).
\end{aligned}$$

We now have to check that the cohomology class is unchanged if we replace ∇ by a second connection ∇_1.

Consider the projection $\pi\colon M \times [0,1] \to M$ and pull the bundle W back to $M \times [0,1]$ using this projection. Using the fact that the space of connections is convex, we equip π^*W with a connection ∇_b which, when restricted (i.e., pulled back) to $M \times \{0\}$ gives ∇, and when restricted to $M \times \{1\}$ gives ∇_1.

By Lemma 3.13, if Ω_b is the curvature of ∇_b, then its restriction to $M \times \{0\}$ is the curvature Ω of ∇, and its restriction to $M \times \{1\}$ is the curvature Ω_1 of ∇_1. Application of τ, ev and exp commutes with pullback. Therefore,

$$\mathrm{ch}_\tau(W; \nabla) = i_0^*(\tau(ev(\exp(\Omega_b)))), \quad \text{and } \mathrm{ch}_\tau(W; \nabla_1) = i_1^*(\tau(ev(\exp(\Omega_b)))),$$

where $i_0, i_1\colon M \to M \times [0,1]$ are the inclusions $m \mapsto (m, 0)$ and $m \mapsto (m, 1)$ respectively. Since these maps are homotopic, the two cohomology classes represented by the two differential forms are equal. □

Remark 4.3. Recall that the Chern character determines the rational Chern classes (and of course also vice versa). Therefore, the definition of $\mathrm{ch}_\tau(W) \in H^{2*}(X; Z)$ immediately gives rise also to Chern classes $c_{i,\tau}(W) \in H^{2i}(X; Z)$. They can then be used to define all other kinds of characteristic classes. We are not going to use this in this paper and therefore refrain from any further discussion.

Theorem 4.4. *The Chern character is compatible with Bott periodicity in the following sense: given a smooth finitely generated projective Hilbert A-module bundle W on a compact manifold M and a trace $\tau\colon A \to Z$, the cohomology classes*

$$\mathrm{ch}_\tau(W) \in H^{2*}(M;Z) \qquad \text{and} \qquad \int_{\mathbb{R}^2} \mathrm{ch}_\tau(\beta(W)) \in H^{2*}(M;Z)$$

are equal.

Here, $\mathrm{ch}_\tau(\beta(W)) = \mathrm{ch}_\tau(W \otimes B_+) - \mathrm{ch}_\tau(W \otimes B_-) \in H^*_c(X \times \mathbb{R}^2; Z)$, where $[B_+] - [B_-] = B \in K^0_c(\mathbb{R}^2)$ is the Bott virtual bundle on $\mathbb{R}^2$ of Theorem 2.24. The construction of ch_τ, together with the proof of all its properties, immediately generalizes from compact base manifolds to the present case. We simply have to use on the two bundles two connections which coincide near infinity (using the given isomorphism between B_+ and B_- near infinity) to produce a compactly supported closed form on $X \times \mathbb{R}^2$ representing a well-defined element in compactly supported cohomology $H^*_c(X \times \mathbb{R}^2; Z)$.

The map $\int_{\mathbb{R}^2}\colon H^*_c(X \times \mathbb{R}^2; Z) \to H^{*-2}(X;Z)$ is the usual integration over the fiber homomorphism (tensored with the identity on Z), which in terms of de Rham cohomology is given by integration over the fibers of the product $X \times \mathbb{R}^2$.

Proof. To prove the result, on $W \otimes B_+$ and $W \otimes B_-$ we choose product connections. By Lemma 3.12 we then obtain for the curvature

$$\Omega_{W\otimes B_+} = \Omega_W \otimes \mathrm{id}_{B_+} + \mathrm{id}_W \otimes \Omega_{B_+}.$$

Since the two summands commute,

$$\begin{aligned} \exp(\Omega_{W\otimes B_+}) &= \exp(\Omega_W \otimes \mathrm{id}_{B_+}) \wedge \exp(\mathrm{id}_W \otimes \Omega_{B_+}) \\ &= (\exp(\Omega_W) \otimes \mathrm{id}_{B_+}) \wedge (\mathrm{id}_W \otimes \exp(\Omega_{B_+})). \end{aligned} \tag{4.5}$$

Consequently, we have to study

$$\tau\left(ev((a(\phi\otimes v)\otimes \mathrm{id}_{B_+}) \wedge b\,\mathrm{id}_W \otimes(\psi\otimes u))\right)$$

with $a, b \in \Omega^*(M)$, $\phi \in \Gamma(\mathrm{Hom}_A(W,A))$, $v \in \Gamma(W)$, $\psi \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(B_+,\mathbb{C}))$, $u \in \Gamma(B_+)$. We obtain

$$\begin{aligned} \tau\left(ev((a(\phi\otimes v)\otimes \mathrm{id}_{B_+}) \wedge b\,\mathrm{id}_W \otimes(\psi\otimes u))\right) &= \tau\big(ev(a\wedge b(\phi\otimes v)\otimes(\psi\otimes u))\big) \\ &= a\wedge b\tau(\phi(v)\cdot\psi(u)) \\ &= a\tau(\phi(v))\wedge b\psi(u) \end{aligned}$$

(observe that $\psi(v) \in \mathbb{C}$). Recall that $\psi(u)$ is the ordinary fiberwise trace tr of the endomorphism valued section corresponding to

$$\psi\otimes u \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(B_+,\mathbb{C}))\otimes\Gamma(B_+) \cong \Gamma(\mathrm{End}_{\mathbb{C}}(B_+)).$$

We obtain for general endomorphism valued forms of the form $(\omega\otimes\mathrm{id}_{B_+})\wedge(\mathrm{id}_W\otimes\eta)$ with $\omega \in \Omega^p(M;\mathrm{End}_A(W))$ and $\eta \in \Omega^q(M;\mathrm{End}_{\mathbb{C}}(B_+))$ (since the special ones considered above locally span the space of such sections)

$$\tau\left(ev((\omega\otimes\mathrm{id}_{B_+})\wedge(\mathrm{id}_W\otimes\eta))\right) = \tau(ev(\omega))\wedge\mathrm{tr}(\eta).$$

In particular, applying this formula to (4.5), we get

$$\begin{aligned}\mathrm{ch}_\tau(W\otimes B_+) &= \tau\left(ev(\exp(\Omega_{W\otimes B_+}))\right)\\ &= \tau\left(ev(\exp(\Omega_W))\wedge \mathrm{tr}(\exp(\Omega_{B_+}))\right)\\ &= \mathrm{ch}_\tau(W)\wedge \mathrm{ch}(B_+),\end{aligned}$$

where $\mathrm{ch}(B_+)$ is the ordinary real differential form representing the Chern character. It follows that

$$\mathrm{ch}_\tau(W\otimes B_+) - \mathrm{ch}_\tau(W\otimes B_-) = \mathrm{ch}_\tau(W)\wedge(\mathrm{ch}(B_+)-\mathrm{ch}(B_-)),$$

where the factor $\mathrm{ch}(B_+)-\mathrm{ch}(B_-)$ is a compactly supported closed 2-form on $\mathbb{R}^2$ representing the Chern character $\mathrm{ch}(B)=c_1(B)$ of the virtual bundle B (note that this is a compactly supported closed differential form of even degree on $\mathbb{R}^2$, and the 0-degree part is zero). Therefore, by Fubini's theorem

$$\int_{\mathbb{R}^2}(\mathrm{ch}_\tau(W\otimes B_+) - \mathrm{ch}_\tau(W\otimes B_-)) = \mathrm{ch}_\tau(W)\cdot\int_{\mathbb{R}^2}(\mathrm{ch}(B)).$$

A fundamental property of the Bott bundle is that $\int_{\mathbb{R}^2}(\mathrm{ch}(B))=1$, and this concludes the proof. □

An important question in the classical theory of characteristic classes is the group where the characteristic classes live in, in particular integrality results. We know, e.g., that the degree $2n$-part of the Chern character after multiplication with $n!$ belongs to the image of integral cohomology in de Rham cohomology. In our situation, the result can not be as easy as that and depends on the trace, as is evident from the fact that the degree zero-part is equal to the τ-dimension of the fiber of the Hilbert A-module bundle (a locally constant function). Only after restriction to particular choices of bundles and particular choices of traces, meaningful restriction can be expected. This will not be discussed in this paper.

5. Index and KK-theory

In this section, we give our proofs of the index theorems for operators twisted with Hilbert A-module bundles. We do this by using heavily the machinery of K-theory and KK-theory to reduce to the classical Atiyah–Singer index theorem. The main tool we will use is the Künneth theorem and associativity of the Kasparov product.

For this paper, we want to avoid all technicalities about Kasparov's bivariant KK-theory for C^*-algebras. We will just recall a few basic facts to be used here. Detailed expositions can be found in Kasparov's original paper [9], or in [2].

We consider KK to be an additive category whose objects are the C^*-algebras, and with morphism sets $KK(A,B)$. There is a functor from the category of C^*-algebras to the category KK which is the identity on objects, i.e., every C^*-algebra morphism $f\colon A\to B$ gives rise to an element $[f]\in KK(A,B)$.

We define

$$\begin{aligned}KK_0(A,B) &:= KK(A,B),\\ KK_1(A,B) &:= KK(SA,B),\end{aligned}$$

where $SA:=C_0(\mathbb{R})\otimes A$ is the suspension of A.

We have the following properties:

Proposition 5.1.

(1) *$KK(A,\mathbb{C})$ is the K-homology of the C^*-algebra A, $KK(\mathbb{C},A)$ its K-theory (defined in terms of projective finitely generated modules). In particular, if X is a finite CW-complex, then*

$$KK(C(X),\mathbb{C}) = K_0(X)$$
$$KK(\mathbb{C},C(X)) = K^0(X)$$

are the usual K-homology and K-theory of the space X.

(2) *An elliptic differential operator D on a smooth compact manifold M of dimension congruent to i modulo 2 defines an element $[D]$ in $KK_i(C(M),\mathbb{C})$. (The general idea is that the KK-groups are defined as equivalence classes of generalized elliptic operators.)*

(3) *On the other hand, every smooth complex vector bundle E on an even-dimensional manifold M defines an element $[E]$ in $KK(\mathbb{C},C(M))$. If D is a (generalized) Dirac operator, then the composition product*

$$[E]\circ[D] \in KK(\mathbb{C},\mathbb{C}) = K_0(\mathbb{C}) = \mathbb{Z}$$

equals the Fredholm index $\operatorname{ind}(D_E)$ of the operator D twisted by the bundle E.

(4) *There is an exterior product*

$$KK(A_1,B_2)\otimes KK(A_2,B_2) \to KK(A_1\otimes A_2, B_1\otimes B_2),$$

where we use the minimal (spacial) tensor product throughout.

This exterior product commutes with the composition product of the category, i.e., if we have $f_i \in KK(A_i,B_i)$, $g_i\in KK(B_i,C_i)$ for $i=1,2$ then

$$(f_1\circ g_1)\otimes(f_2\circ g_2) = (f_1\otimes f_2)\circ(g_1\otimes g_2).$$

(5) *Let Z be a commutative C^*-algebra, e.g., $Z=\mathbb{C}$. Any trace $\operatorname{tr}\colon A\to Z$, i.e., a continuous linear map with $\operatorname{tr}(ab)=\operatorname{tr}(ba)$ for each $a,b\in A$ induces a homomorphism of abelian groups, denoted with the same letter,*

$$\operatorname{tr}\colon K(A) = KK(\mathbb{C},A)\to Z.$$

Definition 5.2. Let D be an elliptic differential operator on a closed smooth manifold M, and W a smooth Hilbert A-module bundle over M. We define the index of D twisted by W

$$\operatorname{ind}_A(D_W) := \operatorname{ind}(D_W) := [W]\circ([D]\otimes[\mathrm{id}_A]) \in KK(\mathbb{C},A).$$

Here $[D]\in KK(C(M),\mathbb{C})$, $\mathrm{id}_A\in KK(A,A)$, $[D]\otimes[\mathrm{id}_A]\in KK(C(M)\otimes A, A)$ and $[W]\in KK(\mathbb{C},C(M,A))$. We also use the fact that $C(M)\otimes A = C(M,A)$.

Given a trace $\tau\colon A\to Z$, from this we can define a "numerical" index

$$\operatorname{ind}_\tau(D_W) := \tau(\operatorname{ind}(D_W)) \in Z.$$

Theorem 5.3. *Let X be a compact Hausdorff space and A a separable C^*-algebra. There is an exact sequence*

$$0\to K^0(X)\otimes K_0(A)\oplus K^1(X)\otimes K_1(A)\to K^0(X;A)$$
$$\to \operatorname{Tor}(K^0(X),K_1(A))\oplus\operatorname{Tor}(K^1(X),K_0(A))\to 0.$$

The restriction of the first map to the summand $K^0(X) \otimes K_0(A)$ sends $[E] \otimes [P]$ to $[E \otimes P]$, i.e., we tensor the complex finite-dimensional vector bundle E (over $\mathbb{C}$) with the Hilbert A-module P (considered as the trivial bundle $X \times P$).

The restriction of the first map to the summand $K^1(X) \otimes K_1(A) = K^0_c(X \times \mathbb{R}) \otimes K^0_c(\mathbb{R}; A)$ is given by the exterior tensor product as above, producing an element in $K^0_c(X \times \mathbb{R} \times \mathbb{R}; A)$, which then has to be mapped to $K^0(X; A)$ by the inverse of the Bott isomorphism of Theorem 2.24.

The sequence implies in particular that, after tensoring with $\mathbb{Q}$,

$$K^0(X; A) \otimes \mathbb{Q} \cong K^*(X) \otimes K_*(A) \otimes \mathbb{Q}. \tag{5.4}$$

Proposition 5.5. *If A is a finite von Neumann algebra, e.g., $A = \mathcal{N}\Gamma$ for a discrete group Γ, then $K_0(A)$ is torsion-free and $K_1(A) = 0$. In particular, for each compact Hausdorff space X we have an isomorphism*

$$K^0(X) \otimes K_0(A) \xrightarrow{\cong} K^0(X; A).$$

Proof. By [2, 7.1.11], $K_1(A) = \{0\}$ for an arbitrary von Neumann algebra A. For a finite von Neumann algebra, the canonical center valued trace induces an injection $\mathrm{tr}_{Z(A)}\colon K_0(A) \to Z(A)$. Since the latter is a vector space, $K_0(A)$ is torsion-free. Then we apply the exact sequence of Theorem 5.3. □

Remark 5.6. Observe that there is no explicit formula for the inverse of this isomorphism. Our work with connections and curvature in the previous sections is motivated by the attempt to overcome this difficulty.

Proposition 5.7. *Let $\tau\colon A \to Z$ be a trace on A with values in a commutative C^*-algebra Z. If A is a finite von Neumann algebra, consider the composition*

$$\psi_\tau\colon K^0(X; A) \xleftarrow{\cong} K^0(X) \otimes K_0(A) \xrightarrow{\mathrm{ch} \otimes \tau} H^{ev}(X; \mathbb{Q}) \otimes Z = H^{ev}(X; Z).$$

If Z is the center of the finite von Neumann algebra A and τ is the canonical center valued trace, then this map is rationally injective:

$$\psi_\tau\colon K_0(X; A) \otimes \mathbb{Q} \hookrightarrow H^{ev}(X; Z).$$

For arbitrary A, the map is defined at least after tensoring with $\mathbb{Q}$:

$$\psi_\tau\colon K^0(X; A) \otimes \mathbb{Q} \xleftarrow{\cong} K^0(X) \otimes K_0(A) \otimes \mathbb{Q} \oplus K^1(X) \otimes K_1(A) \otimes \mathbb{Q} \\ \xrightarrow{(\mathrm{ch} \otimes \tau) \circ \mathrm{pr}_1} H^{ev}(X; Z) \otimes \mathbb{Q}.$$

If W and W_2 are smooth finitely generated projective Hilbert A-module bundles on M with connections ∇_W and ∇_{W_2}, respectively, then

$$\mathrm{ch}_\tau(W) - \mathrm{ch}_\tau(W_2) = \psi_\tau([W] - [W_2]).$$

Proof. The map $W \mapsto \mathrm{ch}_\tau(W)$ induces a well-defined homomorphism

$$\mathrm{ch}_\tau\colon K^0(X; A) \to H^{ev}(X; Z)$$

because of the following observations:

Assume that W_1 and W_2 are finitely generated projective Hilbert A-module bundles. We can give them a (unique) smooth structure by Theorem 2.14. Equipping them with connections ∇_{W_1} and ∇_{W_2}, respectively, then, by using on $W_1 \oplus W_2$ the connection $\nabla_{W_1} \oplus \nabla_{W_2}$, we see that ch_τ is additive wit respect to direct sum. Two smooth bundles W, W_2 represent the same K-theory element if and only if they are stably isomorphic, i.e., if $W \oplus M \times V \cong W_2 \oplus M \times V$. By Theorem 2.13, we can assume this isomorphism to be a smooth isomorphism. By 4.2 ch_τ is independent of the connection chosen. Together with additivity, $\mathrm{ch}_\tau(W) = \mathrm{ch}_\tau(W_2)$.

Since for a finite von Neumann algebra A the map $K^0(X) \otimes K_0(A) \to K^0(X; A)$ is an isomorphism by Proposition 5.5, and for general A the map

$$K^0(X) \otimes K_0(A) \otimes \mathbb{Q} \oplus K^1(X) \otimes K_1(A) \otimes \mathbb{Q} \to K^0(X; A) \otimes \mathbb{Q}$$

is an isomorphism by Theorem 5.3, it suffices to consider the following two cases:

First, we consider a bundle $E \otimes V$ where E is a finite-dimensional complex vector bundle over M and V is a finitely generated projective Hilbert A-module. A connection ∇ on E and the trivial connection on $M \times V$ induce the tensor product connection on $E \otimes V$ by Lemma 3.12.

The calculations in the proof of Theorem 4.4 show that

$$\mathrm{ch}_\tau(E \otimes V) = \mathrm{ch}(E) \cdot \tau(V),$$

since V is a "bundle" on the one-point space and in this case $\mathrm{ch}_\tau(V) = \tau(V) \in Z$. This shows that ψ_τ coincides with ch_τ on $K^0(X) \otimes K_0(A)$, or on the summand $K^0(X) \otimes K_0(A) \otimes \mathbb{Q}$, respectively.

Secondly, we have to consider elements which under Bott periodicity correspond to $E \otimes V$ where $E \in K^0_c(X \times \mathbb{R})$ is a finite dimensional virtual vector bundle over $X \times \mathbb{R}$ which is zero at infinity, and $V \in K^0_c(\mathbb{R}; A)$ is a virtual finitely generated projective Hilbert A-module bundle which is zero at infinity (such virtual bundles are by definition tuples as in Proposition 2.22).

By Theorem 4.4, we have to show that $\mathrm{ch}_\tau(E \otimes V) = 0$. The proof of Theorem 4.4 shows that

$$\mathrm{ch}_\tau(E \otimes V) = \mathrm{ch}(E) \wedge \mathrm{ch}_\tau(V),$$

with $\mathrm{ch}(E) \in H^{2*}_c(X \times \mathbb{R}; \mathbb{R})$ and $\mathrm{ch}_\tau(V) \in H^{2*}_c(\mathbb{R}; Z)$, and where the product is an "exterior" wedge product (i.e., one first has to pull back to the product $X \times \mathbb{R} \times \mathbb{R}$). However, in even degrees the compactly supported cohomology of $\mathbb{R}$ vanishes, therefore the whole expression is zero as we had to show. $\square$

The importance of Proposition 5.7 lies in the explicit formula, where it is not necessary to invert the isomorphism of Proposition 5.5. We get for instance the following immediate corollary.

Corollary 5.8. *Assume that W is a flat finitely generated projective Hilbert A-module bundle over the connected manifold M with typical fiber V. Then*

$$\mathrm{ch}_\tau(W) = \psi_\tau([W]) = \psi_\tau([M \times V]) = \dim_\tau(V) \in H^0(M; Z)$$

for each trace τ on A, i.e., the K-theory class represented by W can not be distinguished from the K-theory class represented by the trivial bundle using these traces. $\dim_\tau(V)$ is the zero dimensional cohomology class represented by the (locally) constant function $\dim_\tau(V)$.

5.1. The Mishchenko–Fomenko index theorem. We are now ready to reprove the cohomological version of the Mishchenko–Fomenko index theorem. Our goal is to give a (cohomological) formula for $\mathrm{ind}_\tau(D_W)$ as defined in Definition 5.2.

Theorem 5.9. *Assume that M is a closed smooth manifold, D an elliptic differential operator defined between sections of finite-dimensional bundles over M. Let W be a finitely generated projective Hilbert A-module bundle, and $\tau\colon A \to Z$ a trace on A with values in an abelian C^*-algebra Z. Then*

$$\operatorname{ind}_\tau(D_W) = \langle \operatorname{ch}(\sigma(D)) \cup \operatorname{Td}(T_{\mathbb{C}}M) \cup \operatorname{ch}_\tau(W), [TM]\rangle. \tag{5.10}$$

Here, $\operatorname{ch}(\sigma(D))$ is the Chern character of the symbol of D, a compactly supported (real) cohomology class on the manifold TM, $\operatorname{Td}(T_{\mathbb{C}}M)$ is the Todd class of the complexified tangent bundle, pulled back to TM and $\operatorname{ch}_\tau(W)$ is the pull back of $\operatorname{ch}_\tau(W)$ to TM. $\langle\cdot,\cdot\rangle$ stands for the pairing of the compactly supported cohomology class with the locally finite fundamental homology class $[TM]$.

If M is oriented of dimension n, then integration over the fibers of $\pi\colon TM \to M$ immediately gives the following consequence:

$$\operatorname{ind}_\tau(D_W) = (-1)^{n(n+1)/2}\langle \pi_! \operatorname{ch}(\sigma(D)) \cup \operatorname{Td}(T_{\mathbb{C}}M) \cup \operatorname{ch}_\tau(W), [M]\rangle. \tag{5.11}$$

The sign compensates for the difference between the orientation of TM induced from M and its canonical orientation as a symplectic manifold.

Remark 5.12. If, in Theorem 5.9, A is a finite von Neumann algebra and $\tau\colon A \to Z$ is the canonical center valued trace, then we can recover $\operatorname{ind}(D_W)$ using the right-hand side of Equation (5.10) or (5.11), since τ induces an injection $K_0(A) \to Z$ by Proposition 5.5 applied to $X = \{*\}$.

Proof of Theorem 5.9. By definition, $\operatorname{ind}(D_W)$ and in particular $\operatorname{ind}_\tau(D_W)$ depend only on the K-theory class represented by W. The same is true for $\operatorname{ch}_\tau(W)$ and therefore for the right-hand side of Equation (5.10).

By Theorem 5.3, there is an integer $k \in \mathbb{Z}$ such that

$$k[W] = \sum_{i=1}^{n} \epsilon_i [E_i \otimes V_i] + \sum_{j=1}^{m} \beta^{-1}(F_j \otimes U_j)$$

where $n, m \in \mathbb{N}$, $\epsilon_i \in \{-1, 1\}$, E_i are finite-dimensional complex vector bundles on M and V_i are finitely generated projective Hilbert A-modules. The F_j are elements of $K^1(X) = KK_1(\mathbb{C}, C(M))$ and the U_j are elements of $K_1(A) = KK_1(\mathbb{C}, A)$. β is the Bott periodicity isomorphism of Theorem 2.24. Note that $[E_1 \otimes V_1] \in K^0(M; A) = KK_0(\mathbb{C}, C(M) \otimes A)$ is obtained as the exterior Kasparov product of $[E_1] \in KK_0(\mathbb{C}, C(M)) = K^0(M)$ and $[V_1] \in KK_0(\mathbb{C}, A) = K_0(A)$.

We now study the summands $\beta^{-1}(F_j \otimes U_j)$ and $\epsilon_i[E_i \otimes V_i]$ separately. By definition,

$$\operatorname{ind}(D_{\beta^{-1}(F_j \otimes U_j)}) = (\beta^{-1}(F_j \otimes U_j)) \circ ([D] \otimes \mathrm{id}_A).$$

Using associativity of the (exterior and interior) Kasparov product, and the fact that β^{-1} is also given by Kasparov product with a certain element, we get

$$\operatorname{ind}(D_{\beta^{-1}(F_j \otimes U_j)}) = \beta^{-1}(F_j \circ [D]) \otimes U_j) = 0,$$

since $F_j \circ [D] \in KK_1(\mathbb{C}, \mathbb{C}) = 0$.

In the same way,

$$\begin{aligned}\operatorname{ind}(D_{E_i \otimes V_i}) = [D] \circ ([E_i] \circ [V_i]) = ([D] \circ [E_i]) \circ [V_i] &= \operatorname{ind}(D_{E_i}) \circ [V_i] \\ &= \operatorname{ind}(D_{E_i}) \cdot [V_i].\end{aligned}$$

Here, for the finite-dimensional bundle $[E_i]$,

$$[D] \circ [E_i] = \mathrm{ind}(D_{E_i}) \in KK_0(\mathbb{C}, \mathbb{C}) = \mathbb{Z},$$

i.e., the Kasparov product gives the Fredholm index of the twisted operator.

Moreover, by the classical Atiyah–Singer index theorem [11, Theorem 13.8]

$$\mathrm{ind}(D_{E_i}) = \langle \mathrm{ch}(\sigma(D)) \, \mathrm{ch}(E_i) \, \mathrm{Td}(T_{\mathbb{C}}M), [TM] \rangle,$$

therefore $\mathrm{ind}(D_{E_i \otimes V_i}) = \langle \mathrm{ch}(\sigma(D)) \, \mathrm{ch}(E_i), [TM] \rangle [V_i]$, and

$$\begin{aligned} \mathrm{ind}_\tau(D_{E_i \otimes V_i}) &= \langle \mathrm{ch}(\sigma(D)) \, \mathrm{ch}(E_i) \, \mathrm{Td}(T_{\mathbb{C}}M), [TM] \rangle \tau([V_i]) \\ &= \langle \mathrm{ch}(\sigma(D)) \psi_\tau(E_i \otimes V_i) \, \mathrm{Td}(T_{\mathbb{C}}M), [TM] \rangle. \end{aligned}$$

Consequently,

$$\begin{aligned} k \, \mathrm{ind}_\tau(D_W) &= \sum_{i=1}^n \epsilon_i \, \mathrm{ind}_\tau(D_{E_i \otimes V_i}) + \sum_{j=1}^m \mathrm{ind}_\tau(D_{F_j \otimes U_j}) \\ &= \sum_{i=1}^n \epsilon_i \langle \mathrm{ch}(\sigma(D)) \psi_\tau(E_i \otimes V_i) \, \mathrm{Td}(T_{\mathbb{C}}M), [TM] \rangle \; + \; 0 \\ &= \langle \mathrm{ch}(\sigma(D)) \psi_\tau([\sum_{i=1}^n \epsilon_i E_i \otimes V_i]) \, \mathrm{Td}(T_{\mathbb{C}}M), [TM] \rangle \\ &= \langle \mathrm{ch}(\sigma(D)) \psi_\tau(k[W]) \, \mathrm{Td}(T_{\mathbb{C}}M), [TM] \rangle \\ &= k \langle \mathrm{ch}(\sigma(D)) \, \mathrm{ch}_\tau(W) \, \mathrm{Td}(T_{\mathbb{C}}M), [TM] \rangle, \end{aligned}$$

where we also use that ψ_τ vanishes on the summand $K^1(X) \otimes K_1(A) \otimes \mathbb{Q}$ of $K^0(X; A) \otimes \mathbb{Q}$. The index formula follows. □

Corollary 5.13. *Assume that, in the situation of Theorem 5.9, W is a flat Hilbert A-module bundle with typical fiber V. Then*

$$\mathrm{ind}_\tau(D_W) = \mathrm{ind}(D) \dim_\tau(W).$$

Proof. Combine Theorem 5.9 and Corollary 5.8 and use the classical Atiyah–Singer index formula for $\mathrm{ind}(D)$. □

Corollary 5.14. *If D in Theorem 5.9 is the spin Dirac operator of a spin manifold M of dimension $n = 2m$, then*

$$\mathrm{ind}_\tau(D_W) = \langle \hat{A}(M) \, \mathrm{ch}_\tau(W), [M] \rangle.$$

Proof. Under this assumption, $\pi_!(\mathrm{ch}(\sigma(D)) \, \mathrm{Td}(T_{\mathbb{C}}M)) = (-1)^m \hat{A}(M)$. Compare the proof of [11, Theorem 13.10]. □

5.2. Atiyah's L^2-index. Now we are in the situation to give a proof of one of the goals of this paper: Atiyah's L^2-index, and its center valued generalization considered by Lück in [12] can be obtained from the index which an operator defines in the K-theory of a corresponding C^*-algebra.

Assume that M is a closed manifold, Γ a discrete group and $M \to B\Gamma$ the classifying map of a Γ-covering $\widetilde{M}$ of M. Consider the corresponding flat bundles $V = \widetilde{M} \times_\Gamma C_r^*\Gamma$ and $H = \widetilde{M} \times_\Gamma l^2(\Gamma)$. Let $t = \tau$ be the canonical trace $\mathcal{N}\Gamma \to \mathbb{C}$ or the canonical center valued trace $\mathcal{N}\Gamma \to Z$. Let D be a generalized Dirac operator on M with lift $\widetilde{D}$ to $\widetilde{M}$. Using $\widetilde{D}$ and t, Atiyah [1] and Lück [12] define L^2-indices $\operatorname{ind}_{(2)}(\widetilde{D}) \in \mathbb{C}$ or $\operatorname{ind}_{(2)}(\widetilde{D}) \in Z$, respectively.

Theorem 5.15. *In the situation just described*

$$\operatorname{ind}_{(2)}(\widetilde{D}) = t(\operatorname{ind}(D_V)).$$

Proof. We have $t([C^*\Gamma]) = 1$. By Corollary 5.13 and the main result of [1] therefore

$$t(\operatorname{ind}(D_V)) = \operatorname{ind}(D) = \operatorname{ind}_{(2)}(\widetilde{D}). \qquad \square$$

Remark 5.16. The proof of Theorem 5.15 we have just given is far from elegant, since we compute two indices and then realize that the answers are equal. We will give an alternative proof in Section 7.10.

5.3. Twisted operators. In Definition 5.2 we cheated somewhat when defining the index of D_W without defining the operator D_W itself. However, it is well-known that, at least if D is a generalized Dirac operator, D_W can be defined as a differential A-operator in the sense of [14].

We want to quickly review the relevant constructions. Let $D\colon \Gamma(E) \to \Gamma(E)$ be a generalized Dirac operator on the closed Riemannian manifold (M, g), acting on the finite dimensional graded Dirac bundle E with Clifford connection ∇_E, i.e., D is the composition

$$\Gamma(E_+) \xrightarrow{\nabla_E} \Gamma(T^*M \otimes E_+) \xrightarrow{g} \Gamma(TM \otimes E_+) \xrightarrow{c} \Gamma(E_-), \tag{5.17}$$

where c denotes Clifford multiplication.

Assume that W is a finitely generated projective Hilbert A-module bundle with connection ∇_W. Then we define the twisted Dirac operator D_W in the usual way by

$$\begin{tikzcd} \Gamma(E_+ \otimes W) \arrow[r, "\nabla_E \otimes 1 + 1 \otimes \nabla_W"] & \Gamma(T^*M \otimes E_+ \otimes W) \arrow[r, "g"] & \Gamma(TM \otimes E_+ \otimes W) \arrow[d, "c"] \\ & & \Gamma(E_- \otimes W). \end{tikzcd}$$

This is an elliptic differential A-operator of order 1 in the sense of [14] with an index in $K_0(A)$ defined as follows.

Definition 5.18. Given a finitely generated smooth Hilbert A-module bundle E over a compact manifold M, Sobolev spaces $H^s(E)$ can be defined ($s \in \mathbb{R}$), compare, e.g., [14]. One way to do this is to pick a trivializing atlas (U_α) with subordinate partition of unity (ϕ_α) and then define for smooth sections u, v of E the inner product

$$(u, v)_s = \sum_\alpha \int_{U_\alpha} \langle (1 + \Delta_\alpha)^s \phi_\alpha u(x), \phi_\alpha v(x) \rangle \, dx,$$

where Δ_α is the ordinary Laplacian on $\mathbb{R}^n$ acting on the trivialized bundle (some diffeomorphisms of the trivializations are omitted to streamline the notation).

This inner product is A-valued, and the completion of $\Gamma(E)$ with respect to this inner product is $H^s(E)$.

Remark 5.19. Of course, the inner product on $H^s(E)$ depends on a number of choices, However, two different choices give rise to equivalent inner products and therefore isomorphic Sobolev spaces.

Then D_W, being a first-order differential operator, induces a bounded operator $D_W : H^s(E_+ \otimes W) \to H^{s-1}(E_- \otimes W)$ for each $s \in \mathbb{R}$.

The key point is now that the ellipticity of D allows the construction of a *parametrix* Q_W which induces bounded operators $Q_W : H^{s-1}(E_- \otimes W) \to H^s(E_+ \otimes W)$ for each $s \in \mathbb{R}$. Parametrix means that

$$D_W Q_W = 1 - S_0 \qquad Q_W D_W = 1 - S_1 \tag{5.20}$$

where S_0 and S_1 are operators of negative order, i.e., induce bounded operators $S_0 : H^s(E_- \otimes W) \to H^{s+r}(E_- \otimes W)$ and $S_1 : H^s(E_+ \otimes W) \to H^{s+r}(E_+ \otimes W)$ for some $r > 0$.

Of course, S_0 and S_1 in Equation (5.20) have to be interpreted as composition of the above operators with the inclusion $H^{s+r} \hookrightarrow H^s$.

We now can conclude that D_W indeed gives rise to A-Fredholm operators because of the appropriate version of the Rellich lemma:

Theorem 5.21. *If M is compact then the inclusion*

$$H^{s+r}(E) \to H^s(E)$$

is A-compact for each finitely generated projective Hilbert A-module bundle E, as long as $r > 0$.

Proof. If $E = M \times V$, V a finitely generated projective Hilbert A-module, then the definition of $H^s(E)$ amounts to

$$H^s(E) = H^s(M) \otimes V,$$

and $i : H^{s+r}(E) \hookrightarrow H^s(E)$ becomes $(i : H^{s+r}(M) \to H^s(M)) \otimes \mathrm{id}_V$, i.e., the tensor product of a compact operator (by the classical Rellich lemma) with id_V. Such an operator is A-compact. The general case follows from an appropriate partition of unity argument. A similar argument can be found in [18, Section 3]. $\square$

In particular, S_0 and S_1 in Equation (5.20) are A-compact as composition of the A-compact inclusion of the Rellich Lemma 5.21 with a bounded operator. Therefore, if we consider D_W as bounded operator between H^s and H^{s-1} then $\mathrm{ind}(D_W) \in K_0(A)$ is defined.

Theorem 5.22. *The index just defined is equal to* $\mathrm{ind}\, D_W$ *as defined in Definition 5.2. In particular, it does not depend on $s \in \mathbb{R}$.*

Proof. This is a well-known fact. For completeness, we want to indicate how this can be done. We do this in several steps.

Mishchenko and Fomenko consider the bounded operators $D_W : H^s(E_+ \otimes W) \to H^{s-1}(W_- \otimes W)$. These are genuine differential operators. We want, however, to relate the operators for different s and show that the index is equal to the index of the pseudodifferential operator $D_W / \big(\sqrt{1 + D_W^2}\big) : L^2(E_+ \otimes W) \to L^2(E_- \otimes W)$. To do this, we have to observe that $\sqrt{1 + D_W^2}$ defines bounded even invertible operators $H^s(E_\pm \otimes W) \to H^{s-1}(E_\pm \otimes W)$ which commute with the operator D as described above. Note that $D_W / \big(\sqrt{1 + D_W^2}\big)$ usually is defined in terms of unbounded normal operators on Hilbert modules, as explained in [10, Section 9]. Here, we have to relate this to the operators between Sobolev spaces. This is not quite automatic, since functional calculus for unbounded operators on Hilbert A-modules is not quite developed in the same way as for the case $A = \mathbb{C}$. A possible method of proof using integral representations (which explicitly includes some of the results needed here) can be found in [3, Section 1].

Since the index does not change if we compose with a bounded invertible operator we conclude two facts:

(1) The index of $D_W : H^s(E_+ \otimes W) \to H^{s-1}(E_- \otimes W)$ is the index of

$$D_W : H^1(E_+ \otimes W) \to L^2(E_- \otimes W),$$

since the first operator is obtained from the second by conjugation with the invertible bounded operator $(1 + D^2)^{s/2}$.

(2) The index of the bounded operator $D_W(1 + D_W^2)^{-1/2} : L^2(E_+ \otimes W) \to L^2(E_- \otimes W)$ is equal to the Mishchenko–Fomenko index, since it is obtained from $D_W : H^1(E_+ \otimes W) \to L^2(E_- \otimes W)$ by composition with the invertible bounded operator $(1 + D^2)^{-1/2}$.

We have to relate the Mishchenko–Fomenko index to the KK-index. Recall from [2, Section 17.5] that the identification of $K_0(A)$ with $KK(\mathbb{C}, A)$ identifies the index of $D_W : H^s(E_+ \otimes W) \to H^{s-1}(E_- \otimes W)$ with the KK-element represented by the Kasparov tuple

$$\left(L^2(E_+ \otimes W) \oplus L^2(E_- \otimes W), \begin{pmatrix} 0 & \frac{D_W}{\sqrt{1+D_W^2}} \\ \frac{D_W}{\sqrt{1+D_W^2}} & 0 \end{pmatrix} \right)$$

(note that on L^2, $D_W/\sqrt{1 + D_W^2}$ is a self adjoint odd operator).

We now have to compute the Kasparov product of our first definition of the twisted index, and to prove that it equals the KK-element just described. Unfortunately, the calculation of the Kasparov product is somewhat complicated. We follow here an idea due to Ulrich Bunke. Eventually, this comes down to the construction of suitable connections in the sense of Kasparov.

Recall that $\operatorname{ind}(D_W) = [W] \circ ([D] \otimes \mathrm{id}_A) \in KK(\mathbb{C}, A)$. To analyze the formula, we need explicit representatives of the ingredients. Here we have

$$\begin{aligned} [W] =& [\Gamma(W) \oplus 0, 0] \in KK(\mathbb{C}, C(M; A)) \\ [\mathrm{id}_A] =& [A \oplus 0, 0] \in KK(A, A) \\ [D] =& \left[L^2(E^+) \oplus L^2(E^-), \begin{pmatrix} 0 & D/\sqrt{1+D^2} \\ D/\sqrt{1+D^2} & 0 \end{pmatrix} \right] \in KK(C(M), \mathbb{C}); \end{aligned}$$

$$[D] \otimes [\mathrm{id}_A] = \left[L^2(E^+) \otimes A \oplus L^2(E^-) \otimes A, \begin{pmatrix} 0 & \frac{D}{\sqrt{1+D^2}} \otimes \mathrm{id}_A \\ \frac{D}{\sqrt{1+D^2}} \otimes \mathrm{id}_A & 0 \end{pmatrix} \right] \in KK(C(M; A), A).$$

If W is a graded bundle, a second summand for the negative part has to be added.

From this, $[W] \circ ([D] \otimes [\mathrm{id}_A]) = (L^2(E^+ \otimes W) \oplus L^2(E^- \otimes W), X)$ with a suitable operator X.

We claim that $X = \begin{pmatrix} 0 & \frac{D_W}{\sqrt{1+D_W^2}} \\ \frac{D_W}{\sqrt{1+D_W^2}} & 0 \end{pmatrix}$ is a possible description of this Kasparov product. Since $[W]$ is given by a Kasparov tuple with operator 0, it suffices by [2, Definition 18.4.1] to show that X is a $\begin{pmatrix} 0 & \frac{D}{\sqrt{1+D^2}} \otimes \mathrm{id}_A \\ \frac{D}{\sqrt{1+D^2}} \otimes \mathrm{id}_A & 0 \end{pmatrix}$-connection for $L^2(E \otimes W)$. Since D and D_W both are self adjoint, the connection property follows as soon as we show that for each $\gamma \in \Gamma(W)$ the operator

$$T_\gamma \circ \frac{D}{(D^2+1)^{-1/2}} \otimes \mathrm{id}_A - \frac{D_W}{(D_W^2+1)^{-1/2}} \circ T_\gamma$$

is a compact operator from $L^2(E) \otimes A$ to $L^2(E \otimes W)$, with $T_\gamma s := s \otimes \gamma$.

To do this, we use the integral representation

$$\frac{D}{(D^2+1)^{-1/2}} = \int_0^\infty D(D^2+1+\lambda^2)^{-1}\, d\lambda,$$

which by [3, Lemma 1.8] is norm convergent. By definition of the twisted Dirac operator, for each section $s \in L^2(E \otimes A)$

$$\begin{aligned} D_W(D_W^2+1+\lambda^2)^{-1}(s\otimes\gamma) &= (D_W^2+1+\lambda^2)^{-1}(Ds\otimes\gamma) \\ &\quad - \sum_i (D_W^2+1+\lambda^2)^{-1} X_i \cdot s \otimes \nabla_{X_i}\gamma, \end{aligned} \tag{5.23}$$

where $\{X_i\}$ is a local orthonormal frame and $X_i \cdot s$ denotes Clifford multiplication. $(D_W^2+1+\lambda^2)^{-1}\colon L^2(E\otimes W) \to L^2(E \otimes W)$ is compact, since it factors by [3, Lemma 1.5] as a bounded operator to H^2 composed with the compact inclusion $H^2(E\otimes W) \to L^2(E\otimes W)$ (we use here that the base manifold M is compact).

By [3, Lemma 1.5] $\left\|(D_W^2+1+\lambda^2)^{-1}\right\|_{\mathcal{B}(L^2)} \le (d+\lambda^2)^{-1}$ for a suitable constant d. For fixed $\gamma \in \Gamma(W)$, the operator

$$s \mapsto \int_0^\infty \left(\sum_i (D_W^2+1+\lambda^2)^{-1} X_i \cdot s \otimes \nabla_{X_i}\gamma \right) d\lambda$$

therefore is compact as norm convergent integral of compact operators.

Consequently, modulo compact operators,

$$\begin{aligned} &T_\gamma \circ \frac{D}{(D^2+1)^{-1/2}} \otimes \mathrm{id}_A - \frac{D_W}{(D_W^2+1)^{-1/2}} \circ T_\gamma \\ &\quad \equiv \int_0^\infty (T_\gamma \circ (D^2+1+\lambda^2)^{-1} D \otimes \mathrm{id}_A - (D_W^2+1+\lambda^2)^{-1} T_\gamma (D \otimes \mathrm{id}_A))\, d\lambda, \end{aligned}$$

using Equation (5.23) to commute D_W and T_γ. For each fixed λ, the integrand is of order -1 and therefore a compact operator on $L^2(E\otimes W)$ (the argument is the same as above).

Finally,

$$\begin{aligned}
&\big(T_\gamma \circ (D^2+1+\lambda^2)^{-1} \otimes \mathrm{id}_A - (D_W^2+1+\lambda^2)^{-1} \circ T_\gamma\big) \circ (D \otimes \mathrm{id}_A)\\
&\quad = (D_W^2+1+\lambda^2)^{-1} \big((D_W^2+1+\lambda^2) T_\gamma - T_\gamma (D^2+1+\lambda^2) \otimes \mathrm{id}_A\big) \cdot \\
&\qquad \cdot (D^2+1+\lambda^2)^{-1} D \otimes \mathrm{id}_A\\
&\quad = (D_W^2+1+\lambda^2)^{-1} \cdot \\
&\qquad \left(T_\gamma D^2 \otimes \mathrm{id}_A + \sum_i T_{\nabla_{X_i}\gamma} X_i \cdot D \otimes \mathrm{id}_A + D_W \circ \sum_i T_{\nabla_{X_i}\gamma} X_i \cdot (-T_\gamma D^2) \otimes \mathrm{id}_A \right)\\
&\qquad\qquad \cdot (D^2+1+\lambda^2)^{-1} D \otimes \mathrm{id}_A
\end{aligned}$$

For the last step, we use first that T_γ commutes with $(1+\lambda^2)$, and then we use twice Equation (5.23) to commute D_W and T_γ.

This representation shows that for each fixed λ the operator in question is actually of order -2. Moreover, by [3, Lemma 1.5] we have

$$\left|\left|(D^2+1+\lambda^2)^{-1}\right|\right|_{\mathcal{B}(L^2)} \le (d+\lambda^2)^{-1}.$$

The term in the middle braces is a bounded operator from H^1 to L^2 and is independent of λ. The operator $(D^2+1+\lambda^2)^{-1}D$ is a bounded operator from L^2 to H^1 with norm bounded independent of λ (since this is an operator on a finite-dimensional bundle, this is a classical fact, it also follows from the definition of the norm on H^1 as in [3, Equation (2)], where $|s|^2_{H^1} = |s|^2_{L^2} + |Ds|^2_{L^2}$, together with the estimates $\left|\left|D^2(D^2+1+\lambda^2)^{-1}\right|\right|_{\mathcal{B}(L^2)} \le C$ and $\left|\left|D(D^2+1+\lambda^2)^{-1}\right|\right|_{\mathcal{B}(L^2)} \le C$ with C independent of λ, as given in [3, Lemmas 1.5 and 1.6].

It follows that

$$\int_0^\infty \big(T_\gamma \circ (D^2+1+\lambda^2)^{-1} \otimes \mathrm{id}_A - (D_W^2+1+\lambda^2)^{-1} \circ T_\gamma\big) \circ (D \otimes \mathrm{id}_A)\, d\lambda$$

converges in operator norm on $L^2(E \otimes W)$. Since the integrand consists of compact operators and the ideal of compact operator is norm closed, it follows as above that the whole integral is compact.

Modulo compact operators, this is equal to $T_\gamma \circ \frac{D}{(D^2+1)^{-1/2}} \otimes \mathrm{id}_A - \frac{D_W}{(D_W^2+1)^{-1/2}} T_\gamma$, which is therefore compact as we had to show.

Putting the above arguments together, it follows that the Mishchenko–Fomenko index equals the Kasparov product, as claimed. This finishes the proof of the Theorem.

□

In [14], a "cohomological" formula for this index is derived similar to our formula 5.9. The underlying strategy uses similar ideas, namely the Künneth theorem 5.3 to reduce to the classical Atiyah–Singer index theorem. The original index theorem is less explicit, because it does not take the curvature of the twisting bundle into account. In particular, Corollary 5.13 does not follow directly. On the other hand, it is more precise because it gives K-theoretic information, whereas we neglect the part of K-theory which is not detectable by traces. Note that, if A is a finite von Neumann algebra, by Proposition 5.5 no information is lost.

6. A simplified A-index for von Neumann algebras

In this section, A is assumed to be a von Neumann algebra.

Let H_A be the Hilbert A-module which is the completion of $\oplus_{i=1}^{\infty} A$. Then $\mathrm{End}_A(H_A) \cong \mathcal{B}(H) \otimes A$, where H is a separable Hilbert space. The "compact" operators $K_A(H_A)$ in $\mathrm{End}_A(H_A)$, i.e., the C^*-algebra generated by the operators of the form $x \mapsto v\langle w, x\rangle$ for some $v, w \in H_A$ are isomorphic to $K(H) \otimes A$.

One can now define the A-Fredholm operators $F_A(H_A)$ in $\mathrm{End}_A(H_A)$ to be those operators which are invertible module $K_A(H_A)$. The generalized Atkinson theorem states that a suitably defined index induces an isomorphism between the set of path components of $F_A(H_A)$ (a group under composition) and $K_0(A)$, compare [24, Chapter 17], originally proved by Kasimov in [8]. We refer to the textbook [24] because of its easy availability and because it is rather self contained.

The problem with the definition of the index is that kernel and cokernel of a Fredholm operator as defined above are not necessarily finitely generated projective A-modules. The way around this is to compactly perturb a given Fredholm operator.

We want to show here that this is not necessary if A is a von Neumann algebra.

The main virtue of the following result is that in case A is a von Neumann algebra, the index of an A-Fredholm operator is determined using spectral calculus instead of some compact perturbation which can hardly be made explicit.

Theorem 6.1. *Assume that A is a von Neumann algebra and $f \in \mathrm{End}_A(H_A)$ is an A-Fredholm operator. Since $\mathrm{End}_A(H_A)$ is a von Neumann algebra, we can use the measurable functional calculus and define the projections $p_{\mathrm{ker}} := \chi_{\{0\}}(f^*f)$ and $p_{\mathrm{coker}} := \chi_{\{0\}}(ff^*)$, where $\chi_{\{0\}}$ is the characteristic function of the set $\{0\}$. Then $\mathrm{im}(p_{\mathrm{ker}})$ and $\mathrm{im}(p_{\mathrm{coker}})$ are finitely generated projective Hilbert A-modules and $[\mathrm{im}(p_{\mathrm{ker}})] - [\mathrm{im}(p_{\mathrm{coker}})] = \mathrm{ind}_A(f) \in K_0(A)$, with $\mathrm{ind}_A :=$ Mindex defined in* [24, Chapter 17] *as $[\ker(f+k)] - [\mathrm{coker}(f+k)]$ for a suitable A-compact perturbation of f (any k such that range, kernel and cokernel of $f+k$ are closed will do).*

Proof. Since f is invertible module A-compact operators and $fp_{\mathrm{ker}} = 0$, p_{ker} is zero module compact operators, i.e., a compact projection. The same is true for p_{coker}. By [24, Theorem 16.4.2], their images are finitely generated projective Hilbert A-modules, so that in particular $[\mathrm{im}(p_{\mathrm{ker}})] - [\mathrm{im}(p_{\mathrm{coker}})] \in K_0(A)$ is defined.

Since $\mathrm{End}_A(H_A)$ is a von Neumann algebra, each operator has a polar decomposition (for general A, this is only assured for those with closed range, compare [24, Theorem 15.3.8].) Write therefore $f = u\,|f|$ with a partial isometry u. By spectral calculus, $1 - u^*u = p_{\mathrm{ker}}$ and $1 - uu^* = p_{\mathrm{coker}}$. If $g = f + k$ is an A-compact perturbation of f, and $g = v\,|g|$ is its polar decomposition, then $u - v$ is A-compact, as follows from the proof of [24, Corollary 17.2.5] and therefore by [24, Corollary 17.2.4]

$$\begin{aligned}[p_{\mathrm{ker}}] - [p_{\mathrm{coker}}] &= [1 - u^*u] - [1 - uu^*] \\ &= [1 - v^*v] - [1 - vv^*] = [\ker(g)] - [\ker(g^*)] \in K_0(A).\end{aligned}$$

Since the latter is by definition the A-index of f, we are done. $\square$

Remark 6.2. Occasionally, we will use the notation $[p_{\ker(f)}] \in K_0(A)$ for the K-theory element represented by the image of $\ker(f)$, if we are in the situation of Theorem 6.1. Note that we have to enlarge the standard "finite projective matrices" description a little bit here, since the projection is only unitarily equivalent (with a unitary close to one) to a finite projective matrix, as is proved, e.g., in [24, Lemma 15,4.1]. We have to keep in mind that not all constructions immediately generalize to these generalized projections, e.g., when applying traces to them.

Definition 6.3. Let V and W be (topologically) countably generated Hilbert A-modules and $f \in \mathrm{Hom}_A(V, W)$. We call f *Fredholm* if $f \oplus \mathrm{id}_{H_A} : V \oplus H_A \to W \oplus H_A$ is Fredholm. If this is the case, then

$$\mathrm{ind}_A(f) := \mathrm{ind}_A(f \oplus \mathrm{id}_{H_A}) \in K_0(A).$$

Observe that this definition makes sense and reduces to the situation of Theorem 6.1 since by Kasparov's stabilization theorem [24, Theorem 15.4.6] $V \oplus H_A \cong H_A$.

Corollary 6.4. *If A is a von Neumann algebra, V and W are countably generated Hilbert A-modules and $f \in \mathrm{Hom}_A(V, W)$ is Fredholm, then*

$$\mathrm{ind}_A(f) = [\chi_{\{0\}}(f^* f)] - [\chi_{\{0\}}(f f^*)] \in K_0(A).$$

Proof. This is an immediate consequence of Definition 6.3 and of Theorem 6.1. □

We can apply this to the twisted generalized Dirac operators considered in Section 5.3

Corollary 6.5. *Let $D : \Gamma(E_+) \to \Gamma(E_-)$ be a generalized Dirac operator, acting on the sections of a finite-dimensional bundle E over the smooth compact manifold M without boundary. Let A be a von Neumann algebra and W a smooth finitely generated projective Hilbert A-module bundle. Then the A-index of the twisted operator D_W as defined in Definition 5.2 or Subsection 5.3 can be expressed as follows:*

$$\mathrm{ind}_A(D_W) = [\chi_{\{0\}}(D_W^* D_W)] - [\chi_{\{0\}}(D_W D_W^*)] \in K_0(A),$$

where we understand D_W to be the bounded operator

$$D_W : H^1(E_+ \otimes W) \to H^0(E_- \otimes D_W).$$

7. A general Atiyah L^2-index theorem

7.1. A-Hilbert spaces and bundles. Atiyah's L^2-index theorem [1] and its generalization by Lück [12] deal with indices obtained from an ordinary elliptic differential operator and a trace on a von Neumann algebra A, but this is done in a different way compared to the construction in Definition 5.2.

Atiyah is looking at coverings of a compact manifold and a lifted Dirac type operator (this corresponds to the twist with the canonical flat bundle of the covering of Example 7.11), and is proving that the L^2-index (associated to a canonical trace) coincides with the ordinary index of the operator on the compact base manifold. He is using a parametrix construction to directly show that the two numbers coincide. Lück, in the same situation, is studying all the other normal traces. He proves that they don't contain additional information. Lück is using the heat kernel on the covering manifold. A proof of Atiyah's original result using heat kernel methods is given in [15]. Lück is also giving a K-theoretic interpretation of his result: the index in question defines an element of $K_0(\mathcal{N}\Gamma)$ which is a multiple of the trivial element 1. This is an infinite-dimensional generalization of the well-known rigidity theorem which says that for a free action of a finite group, the equivariant index contains no more information than the ordinary index (compare [12, Remark after Theorem 0.4]).

Despite the different definitions and methods, there is an easy direct translation between the two aspects, which is well-known and frequently used in the literature, but seems not to be documented with proof. Therefore, our goal here is to prove this connection. This is inspired by a remark of Alain Valette who missed a citable reference for the result.

In the present subsection, we will introduce the notation and concepts necessary to give the definition of Atiyah's (and Lück's) L^2-index. We do this in a more general setting, making transparent some of the connections to the previous parts of this paper.

We have to introduce some further notation. Unfortunately, the term "(finitely generated projective) Hilbert A-module" is used in the literature for two different things: the objects we have introduced so far, but also the objects on which Atiyah's definition of the L^2-index is based. The latter are honest Hilbert spaces with an action of the C^*-algebra A. To distinguish them from the objects introduced above, we use the term "A-Hilbert space" (deviating from the literature at this point). We will see in Section 7.6 how to translate between these two concepts.

For our construction, we use a trace on A with particular properties. This will exist in our main example, the von Neumann algebra of a discrete group. For the following, we recall the construction of $l^2(A)$ which is used to pass from the algebra A to an A-Hilbert space.

Definition 7.1. Let A be a C^*-algebra and Z a commutative C^*-algebra (most important is the example $Z = \mathbb{C}$). A *trace* $\tau\colon A \to Z$ is a linear map such that:

(1) $\tau(ab) = \tau(ba)$ for each $a, b \in A$.
(2) It is called *positive* if $\tau(a^*a) \geq 0$ for each $a \in A$.
(3) It is called *faithful* if $\tau(a^*a) = 0$ only for $a = 0$.
(4) It is called *normalized* if $\tau(1) = 1$.
(5) If A and Z are von Neumann algebras, a positive trace τ is called *normal* if it is ultraweakly continuous.

Notation 7.2. From now on, we assume the existence and fix a positive faithful normalized trace $\tau\colon A \to \mathbb{C}$.

Lemma 7.3. *Given a trace τ as in 7.2, we have the following inequality:*

$$\tau(a^*xa) \leq |x|\,\tau(a^*a) \qquad \text{if } x \in A \text{ is positive,} a \in A,$$

with $|x|$ the $\mathbb{R}$-valued norm of $x \in A$.

In particular, with $a = 1$, the map $\tau\colon A \to \mathbb{C}$ is norm continuous.

Proof. In A, we have $x \leq |x|$ and therefore $a^*xa \leq a^*\,|x|\,a = |x|\,a^*a$. Positivity and linearity of the trace implies the inequality. □

Definition 7.4. Given the positive faithful normalized trace τ on the C^*-algebra A as in 7.2, define a sesquilinear inner product on a Hilbert A-module V by

$$\langle v, w\rangle_2 = \tau(\langle v, w\rangle)$$

(linear in the second entry), i.e., we compose the A-valued inner product with τ.

Lemma 7.5. *In the situation of Definition 7.4, V with the constructed inner product becomes a pre Hilbert space. Its completion is denoted $l^2(V)$. Right multiplication of A on V induces a C^*-homomorphism from A to the bounded operators on $l^2(V)$.*

When $V = A^n$, left and right multiplication both induce C^*-embeddings of A into the bounded operators on $l^2(A)^n = l^2(A^n)$.

Proof. Since τ is faithful and positive and the same is true for $\langle\cdot,\cdot\rangle$, $\langle\cdot,\cdot\rangle_2$ induces a norm $\|\cdot\|$. If $a, x \in A$, $v \in V$ then by Lemma 7.3

$$\begin{aligned}\|va\| &= \tau(\langle va, va\rangle)^{1/2} = \tau(a^*\langle v,v\rangle a)^{1/2} = \tau(\sqrt{\langle v,v\rangle}aa^*\sqrt{\langle v,v\rangle})^{1/2}\\ &\le |a^*a|^{1/2}\,\tau(\langle v,v\rangle)^{1/2} = |a|\cdot\|v\|\,.\end{aligned}$$

For left multiplication of A on A

$$\|ax\| = \tau(x^*a^*ax)^{1/2} \le |a|\,\|x\|\,.$$

We conclude that right multiplication and for $V = A^n$ also left multiplication by a give rise to bounded operators with operator norm $\le |a|$. The corresponding maps are $*$-homomorphisms since

$$\begin{aligned}\langle va, w\rangle_2 &= \tau(a^*\langle v,w\rangle) = \tau(\langle v,w\rangle a^*) = \langle v, wa^*\rangle_2 \qquad &&\forall a\in A,\ v,w\in V.\\ \langle ax, y\rangle_2 &= \tau(x^*a^*y) = \langle x, a^*y\rangle_2 &&\forall a,x,y\in A.\end{aligned}$$

Left or right multiplication by a on A^n is the zero map only if $a = 0$. □

Remark 7.6.
(1) In Lemma 7.5, $l^2(A)$ and $l^2(V)$ depend of course on the chosen trace τ. We will not indicate this in the notation since we adopt the convention that the trace τ is fixed throughout. Moreover, we will see in Section 7.6 that one can recover V from $l^2(V)$, such that the particular choice of τ does not play too much of a role.
(2) Lemma 7.5 contains the easy case of the representation theorem for C^*-algebras: if A has a trace as in Definition 7.1 then A can be isometrically embedded into the algebra of bounded operators on the Hilbert space $l^2(A)$.

Definition 7.7. A finitely generated projective A-Hilbert space V is a Hilbert space together with a right action of A such that V embeds isometrically preserving the A-module structure as a direct summand into $l^2(A)^n$ for some n, and such that the orthogonal projection $l^2(A)^n \twoheadrightarrow V$ is given by left multiplication with an element of $M_n(A)$.

A (general) *A-Hilbert space* V satisfies the same conditions a finitely generated projective A-Hilbert space does, with the exception that $l^2(A)^n$ is replaced by $H \otimes l^2(A)$ for some Hilbert space H (the tensor product has to be completed), and $M_n(A)$ by $\mathcal{B}(H)\otimes A$ (where A is here understood to act by right multiplication). Observe that, if H is separable, then $H\otimes l^2(A) \cong l^2(H_A)$, and $\mathcal{B}(H)\otimes A \cong \mathrm{End}_A(H_A)$.

Remark 7.8. Assume that, in Definition 7.7, A is a von Neumann algebra. Then the condition that the projection $H\otimes l^2(A) \twoheadrightarrow V$ belongs to $\mathcal{B}(H)\otimes A$ is automatically satisfied, since the commutant of the right multiplication of A on $H\otimes l^2(A)$ is $\mathcal{B}(H)\otimes A$ (and on $l^2(A^n)$ is $M_n(A)$), and the projection by definition commutes with the right multiplication of A.

7.2. A-Hilbert space bundles.

Definition 7.9. An A-Hilbert space morphism is a bounded A-linear map between two A-Hilbert spaces. If it is an isometry for the Hilbert space structure, it is called an A-Hilbert space isometry.

An A-Hilbert space bundle H on a space X is a locally trivial bundle of A-Hilbert spaces, the transition functions being A-Hilbert space isometries. A smooth structure is given by a trivializing atlas where all the transition functions are smooth.

If the fibers are finitely generated projective A-Hilbert space, the bundle is called a *finitely generated projective A-Hilbert space bundle.*

Lemma 7.10. *The L^2-sections of an A-Hilbert space bundle W on a Riemannian manifold X form themselves an A-Hilbert space.*

Proof. The action of A is given by pointwise multiplication. We want to show that $L^2(W) \cong L^2(M) \otimes V$, where V is a typical fiber of M (we assume for simplicity that M is connected). Since V embeds into $H \otimes l^2(A)$, the same is then true of $L^2(W)$.

To prove that $L^2(W) \cong L^2(M) \otimes V$, choose a subset $U \subset M$ such that $M \setminus U$ has measure zero, and such that $W|_U$ is trivial (U could, e.g., consist of the interiors of the top cells of a smooth triangulation of M). Then

$$L^2(W) \cong L^2(W|_U) \cong L^2(U) \otimes V \cong L^2(M) \otimes V,$$

since U and M differ only by a set of measure zero, and since $W|_U \cong U \times V$. $\square$

As an example, we now want to give the most important A-modules, A-Hilbert spaces and corresponding bundles. To do this, we have in particular to specify the von Neumann algebra A. This is the A-Hilbert space bundle featuring in Atiyah's L^2-index theorem and its generalization by Lück.

Example 7.11. Let M be a smooth compact manifold and Γ its fundamental group. Let $\pi\colon \widetilde{M} \to M$ be a universal covering of M, with Γ-action from the right by deck transformations.

The Hilbert space $l^2(\Gamma)$ is the space of complex valued square summable functions on the discrete group Γ. $\mathbb{C}\Gamma$ acts through bounded operators on $l^2(\Gamma)$ by left as well as right convolution multiplication. By definition, the reduced C^*-algebra $C^*_r\Gamma$ of Γ is the norm closure in $\mathcal{B}(l^2(\Gamma))$ of $\mathbb{C}\Gamma$ acting from the right, and $\mathcal{N}\Gamma$ is the weak closure of the same algebra. By the double commutant theorem, this is the set of all operators which commute with left convolution of $\mathbb{C}\Gamma$.

On $\mathcal{N}\Gamma$ and therefore also on its subalgebra $C^*_r\Gamma$ we have the canonical faithful positive trace τ with $\tau(f) = \langle f(1), 1\rangle_{l^2\Gamma}$, where $1 \in l^2(\Gamma)$ is by definition the characteristic function of the unit element.

The construction of $l^2(C^*_r\Gamma)$ and of $l^2(\mathcal{N}\Gamma)$ with respect to this trace yields precisely $l^2(\Gamma)$.

Since the left Γ-action and the right $C^*_r\Gamma$ or $\mathcal{N}\Gamma$-action, respectively, on $l^2(\Gamma)$ and $C^*_r\Gamma$ or $\mathcal{N}\Gamma$, respectively, commute, the bundles $\widetilde{M} \times_\Gamma C^*_r\Gamma$ and $\widetilde{M} \times_\Gamma \mathcal{N}\Gamma$ are smooth finitely generated projective Hilbert $C^*_r\Gamma$ and Hilbert $\mathcal{N}\Gamma$ module bundle, and $\widetilde{M} \times_\Gamma l^2(\Gamma)$ is a finitely generated projective $C^*_r\Gamma$-Hilbert space or $\mathcal{N}\Gamma$-Hilbert space bundle, all on M. Moreover, $\widetilde{M} \times_\Gamma l^2(\Gamma)$ can be considered as the A-Hilbert space completion of the former bundles with respect to the canonical trace.

To see that the bundles are smooth, observe that the canonical trivializations are obtained by choosing lifts to $\widetilde{M}$, and the transition functions are then given by left multiplication with fixed elements $\gamma \in \Gamma$. Since these maps do not depend on the basepoint in M they are smooth (the argument shows that these bundles are actually flat).

The same construction works if Γ is some homomorphic image of the fundamental group of M, and $\widetilde{M}$ the corresponding normal covering space of M.

The trivial connection on $\widetilde{M} \times C_r^*\Gamma$ and $\widetilde{M} \times \mathcal{N}\Gamma$ descents to a canonical flat connection on $\widetilde{M} \times_\Gamma C_r^*\Gamma$ and $\widetilde{M} \times_\Gamma \mathcal{N}\Gamma$, since left (as well as right) multiplication with an element $\gamma \in \Gamma$ is parallel.

7.3. Connections on A-Hilbert space bundles.

Definition 7.12. Let A be a von Neumann algebra with a trace τ as in 7.2. Assume that M is a smooth manifold and X is a smooth finitely generated projective A-Hilbert space bundle on M. A *connection* ∇ on X is an A-linear map $\nabla\colon \Gamma(X) \to \Gamma(T^*M \otimes X)$ which is a derivation with respect to multiplication with sections of the trivial bundle $M \times A$, i.e.,

$$\nabla(sf) = sdf + \nabla(s)f \qquad \forall s \in \Gamma(X),\ f \in C^\infty(M; A).$$

Here we use the multiplication $X \otimes T^*M \otimes (M \times A) \to X \otimes T^*M\colon s \otimes \eta \otimes f \mapsto sf \otimes \eta$. (In particular, elements of A are considered to be of degree zero.)

We say that ∇ is a *metric connection* if

$$d\langle s_1, s_2\rangle = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle$$

for all smooth sections s_1, s_2 of X. Here, we consider $\langle s_1, s_2\rangle$ to be a section of the trivial bundle $M \times \mathbb{C}$.

Example 7.13. In the situation of Example 7.11, $\widetilde{M} \times_\Gamma l^2(\Gamma)$ inherits a canonical flat connection, descending from $\widetilde{M} \times l^2(\Gamma)$, which extends the corresponding flat connection on the subbundle $\widetilde{M} \times_\Gamma \mathcal{N}\Gamma$.

7.4. Operators twisted by A-Hilbert space bundles. In this paper, we will only twist ordinary Dirac type differential operators with A-Hilbert space bundles. For a more complete theory of (pseudo)differential operators on such bundles compare, e.g., [4, Section 2].

Definition 7.14. Let $D\colon \Gamma(E^+) \to \Gamma(E^-)$ be a generalized Dirac operator between sections of finite-dimensional bundles on the Riemannian manifold (M, g).

Let H be a smooth A-Hilbert space bundle with connection ∇_H. Then we define (as usual) the twisted Dirac operator, D_H, to be the composite

$$\begin{tikzcd}
\Gamma(E_+ \otimes H) \arrow[r, "\nabla\otimes 1 + 1\otimes\nabla_H"] & \Gamma(T^*M \otimes E_+ \otimes H) \arrow[r, "g"] & \Gamma(TM \otimes E_+ \otimes H) \arrow[d, "c"] \\
& & \Gamma(E_- \otimes H),
\end{tikzcd}$$

where c stands for Clifford multiplication.

This is an elliptic differential operator of order 1 on A-Hilbert space bundles in the sense of [4]. In particular, it extends to an unbounded operator on $L^2(E \otimes H)$.

If A is a von Neumann algebra, then the kernel as well as the orthogonal complement of the image are A-Hilbert spaces. The A-action is evident. The assertion about the projections follows from the fact that by measurable functional calculus, the projection onto the kernel of A is given by $\chi_{\{0\}}(D_H^* D_H)$ ($\chi_{\{0\}}$ being the characteristic function of $\{0\}$), and similarly for the cokernel.

Remark 7.15. If A is not a von Neumann algebra, kernel and cokernel are not necessarily A-Hilbert modules.

Definition 7.16. Assume that A is a von Neumann algebra with a trace τ as in 7.2. Let $t\colon A \to Z$ be a second trace which is required to be positive and normal (but not necessarily faithful or normalized), with values in a commutative von Neumann algebra Z ($t = \tau$ is permitted). Given an A-Hilbert module V, we define

$$\dim_t(V) := t(\mathrm{pr}_V),$$

where $\mathrm{pr}_V : l^2(A) \otimes H \to l^2(A) \otimes H$ is the orthogonal projection onto V, and t here also stands for the extension of the trace to $A \otimes \mathcal{B}(H)$ (to do this, the fact that the trace t is normal has to be used). We will discuss the definition and properties of these traces in Section 7.8.

Definition 7.17. Let A be a von Neumann algebra with traces t and τ as in Definition 7.16.

Let D_H be a generalized Dirac operator twisted by a finitely generated projective A-Hilbert space bundle H as in Definition 7.14. Assume that M is compact without boundary. Ellipticity implies that $\chi_{\{0\}}(D_H^* D_H)$ and $\chi_{\{0\}}(D_H D_H^*)$ are of t-trace class (compare Section 7.8 for the definition and Section 7.10 for a proof of this fact). Then define

$$\mathrm{ind}_t(D_H) := t(\chi_{\{0\}}(D_H^* D_H)) - t(\chi_{\{0\}}(D_H D_H^*)).$$

Our goal now is to prove an index formula for $\mathrm{ind}_t(D_H)$ in the general situation of Definition 7.14. One way to do this would be the following:

(1) Develop a theory of connections and curvature for A-Hilbert space bundles similar to what we have done for Hilbert A-module bundles. This is possible in exactly the same way as done above.
(2) Show that ind_t is unchanged by lower order perturbations of D_H (in particular if the connection on H is changed). One way to do this would be to prove that ind_t can be calculated from the remainder terms S_0 and S_1 in $D_H Q = 1 - S_0$ and $Q D_H = 1 - S_1$, where Q is a suitable parametrix (such that the remainder terms are of t-trace class), namely

$$\mathrm{ind}_t(D_H) = t(S_1) - t(S_0).$$

This step is already done by Atiyah [1] (in his special situation), and his proof does only use a few general properties of the trace, in particular that it is normal, a trace, and that operators of order $-k$, for k sufficiently big, are of trace class. Since all these properties are satisfied here, the proof goes through. A more formal discussion of this prove can be found in [18]. For a lower order perturbation $D_H - a$ of D_H, we can then use the parametrix

$Q' = Q + QaQ + QaQaQ + \cdots + QaQ \cdots aQ$. Then

$$(D_H - a)Q' = 1 - S_0 - aQ \cdots aQ, \quad \text{and}$$
$$Q'(A_H - a) = 1 - S_1 - Qa \cdots Qa,$$

and the trace property implies immediately that

$$t(S_1') - t(S_0') = t(S_1) - t(S_0).$$

(3) Follow the proof of Theorem 5.9 to get a very similar formula for ind_t.

Although all this can be done, Step (2) is rather lengthy. Therefore, we prefer to show in Section 7.10 that the "new" situation can be reduced to the index theorem 5.9 by directly showing that

$$\operatorname{ind}_t(D_H) = t(\operatorname{ind}(D_V)) \tag{7.18}$$

for a finitely generated projective Hilbert A-module bundle V canonically associated to H (in particular, $\operatorname{ind}(D_V) \in K_0(A)$).

7.5. Flat A-Hilbert space bundles and coverings. Assume that $A = \mathcal{N}\Gamma$ is the von Neumann algebra of the discrete group Γ and $t = \tau$ is the canonical trace of Example 7.11. Let $H = \widetilde{M} \times_\Gamma l^2(\Gamma)$ be the canonical flat $l^2(\Gamma)$-bundle of Example 7.11, and let $D\colon \Gamma(E^+) \to \Gamma(E^-)$ be a generalized Dirac operator on M. In this situation, we have defined $\operatorname{ind}_t(D_H) \in \mathbb{R}$. Fix, more generally, an element $g \in \Gamma$ which has only finitely many conjugates, and let $[g]$ be this finite conjugacy class. Then it is well-known that $\sum_{\gamma\in[g]} f(\gamma)$ for $f \in \mathbb{C}[\Gamma]$ extends to a finite normal trace t_g on $\mathcal{N}\Gamma$, a so-called delocalized trace. The indices generated by these traces are studied by Lück in[12].

In [1], Atiyah is working with the lifted operator to $\widetilde{M}$: lift the differential (and hence local) operator D to $\widetilde{D}\colon \Gamma(\widetilde{E}^+) \to \Gamma(\widetilde{E}^-)$, where $\widetilde{E}^\pm$ are the pullbacks of $E^\pm$ to the universal covering $\widetilde{M}$.

In this situation, there is a literal translation between spaces of sections and operators on them for $\widetilde{E}^\pm$ on the one hand, and for $E^\pm \otimes H$ on the other hand. This is rather straightforward (and well-known). For the sake of completeness we indicate the constructions. Other accounts (with more details) can be found, e.g., in [19, Section 3.1] and [20, Example 3.39].

The translation is summarized in the following table:

$\widetilde{M}$	$\cdot \otimes H$
$L^2(\widetilde{E}^\pm)$	$L^2(E^\pm \otimes H)$
$\{s \in \Gamma(E^\pm) \mid \sum_{\gamma\in\Gamma} \lvert s(\gamma x)\rvert^2 < \infty \; \forall x \in \widetilde{M}\}$	$\Gamma(E^\pm \otimes H)$
$\widetilde{D}$	D_H
$\widetilde{D}/(1+\widetilde{D}^2)^{1/2}$	$D_H/(1+D_H^2)^{1/2}$
$\phi(\widetilde{D})$	$\phi(D_H)$
$\int_{\widetilde{M}/\Gamma} \operatorname{tr}_x k(x,x)\, dx$	t
$\sum_{\gamma\in[g]} \int_{\widetilde{M}/\Gamma} \operatorname{tr}_x k(x,\gamma x)\, dx$	t_g
$\operatorname{ind}_t(\widetilde{D})$	$\operatorname{ind}_t(D_H)$
$\operatorname{ind}_{t_g}(\widetilde{D})$	$\operatorname{ind}_{t_g}(D_H)$.

Some explanations are in order:

(1) A section s of $\widetilde{E}$ corresponds to the section $\hat{s}$ of $E \otimes H$ with

$$\hat{s}(x) = \sum_{\gamma \in \Gamma} s(\gamma \widetilde{x}) \otimes (\widetilde{x}, \gamma),$$

where $\widetilde{x}$ is an arbitrary lift of x. Of course we identify the fibers E_x and $\widetilde{E}_{\gamma \widetilde{x}}$, and $H_x = \Gamma \widetilde{x} \times_\Gamma l^2(\Gamma)$. This construction is well-defined by the definition of the twisted bundle H, with fiber identified with $l^2(\Gamma)$ using the chosen lift $\widetilde{x}$.

(2) This identification defines an isometry of the spaces of L^2-sections. Moreover, it is compatible with the Γ-action, therefore an isometry of A-Hilbert spaces. In addition, it preserves smoothness and continuity, where the condition as given in the table is used to really get a section of $E \otimes H$.

(3) The operators $\widetilde{D}$ and D_H are conjugated to each other under the isomorphism of the section spaces. This follows from their local definition. Here we use that for a small connected neighborhood U of $x \in M$ we can choose a lift $\widetilde{U}$, a connected neighborhood of a lift $\widetilde{x}$, such that there is a unique section $U \to \widetilde{U}$ of the restriction of the covering $\widetilde{M} \to M$ to U, and then $y \mapsto (\widetilde{y}, \gamma)$ is a flat section of $H|_U$ for each $\gamma \in \Gamma$.

(4) Since the self-adjoint unbounded operators $\widetilde{D}$ and D_H are unitarily equivalent, the same is true for all bounded measurable functions of them, using functional calculus. In particular, this is the case for $\widetilde{D}/(1+\widetilde{D}^2)^{1/2}$, but also for any other bounded measurable function $\phi \colon \mathbb{R} \to \mathbb{R}$. As a particular example we will have to study the projections onto the kernels of the operators.

(5) Appropriate functions of $\widetilde{D}$, e.g., the projection onto the kernel, have by elliptic regularity a smooth integral kernel $k(x,y)$ on $\widetilde{M} \times \widetilde{M}$. This kernel is invariant (in the appropriate sense) under the diagonal Γ-action, in particular, its restriction to the diagonal descends to the quotient by this action. On the diagonal, $k(x,x)$ is an endomorphism of the fiber $\widetilde{E}_x$ and therefore has a finite-dimensional trace $\operatorname{tr} k(x,x)$. Since the fiber $\widetilde{E}_{\gamma x}$ can for each $\gamma \in \Gamma$ be canonically identified with $\widetilde{E}_x$ (since they are both identified with $E_{p(x)}$, $p \colon \widetilde{M} \to M$), we can also take the finite-dimensional trace $\operatorname{tr} k(x, \gamma x)$.

The integrals in the tables define then certain traces which are the ones used by Atiyah and by Lück.

(6) Choose a subset $U \subset M$ such that $M \setminus U$ has measure zero and such that the restriction of the covering $\widetilde{M} \to M$ to U is trivial. If we choose an appropriate lift of U then $\widetilde{M}|U \cong U \times \Gamma$. This induces a trivialization $H|_U \cong U \times l^2(\Gamma)$. Using this, we identified in Lemma 7.10 $L^2(E \otimes H) = L^2(E|_U) \otimes l^2(\Gamma)$, and this in turn was used to define t and t_g on (trace class) operators acting on $L^2(E \otimes H)$, e.g., the projection onto the kernel of D_H.

On the other hand, using the corresponding trivialization of the covering $\widetilde{M}|U \cong U \times \Gamma$ we get the identification $L^2(\widetilde{E}|U) \cong L^2(E) \otimes l^2(\Gamma)$, and our unitary identification defined above becomes the identity under these identifications.

It was proved by Atiyah in [1] that the formula of the integral computes the tensor product of the ordinary Hilbert space trace on $L^2(E)$ with the trace t on $l^2(\Gamma)$ under the last identification. This proof extends to the second integral, which corresponds to the tensor product of the Hilbert space trace on $L^2(E)$ with the delocalized trace t_g on $l^2(\Gamma)$.

On the other hand, we defined t (or t_g, respectively) for operators on $L^2(E \otimes H)$ as tensor product of t (or t_g) on $l^2(\Gamma)$ with the usual trace on $L^2(E)$, using the identification $L^2(E\otimes H) \cong L^2(E|_U)\otimes l^2(\Gamma)$. Since all these identifications coincide with each other, the traces also do so.

(7) From the discussion so far, it follows in particular that the unitary isomorphism described above induces A-Hilbert space isometries between $\ker(\widetilde{D}^{\pm})$ and $\ker(D_H^{\pm})$, such that the traces of the projectors onto these kernels coincide, defined either using the integral over the diagonal in $\widetilde{M} \times \widetilde{M}/\Gamma$ for the integral kernel, or using the recipe of Definition 7.16 with the Hilbert A-module structure given by Lemma 7.10 on $L^2(E \otimes H)$.

In particular $\operatorname{ind}_t(D_H) = \operatorname{ind}_t(\widetilde{D})$, and $\operatorname{ind}_{t_g}(D_H) = \operatorname{ind}_{t_g}(\widetilde{D})$, where the left-hand side is defined in Definition 7.17, and the right-hand side is defined with the integrals of the table evaluated for the projection operators $k^{\pm}$ onto the kernels of $\widetilde{D}^+$ and $\widetilde{D}^-$:

$$\operatorname{ind}_t(\widetilde{D}) = \int_{\widetilde{M}/\Gamma} \operatorname{tr}_x k^+(x,x)\,dx - \int_{\widetilde{M}/\Gamma} \operatorname{tr}_x k^-(x,x)\,dx,$$

$$\operatorname{ind}_{t_g}(\widetilde{D}) = \sum_{\gamma\in[g]} \left(\int_{\widetilde{M}/\Gamma} \operatorname{tr}_x k^+(x,\gamma x)\,dx - \int_{\widetilde{M}/\Gamma} \operatorname{tr}_x k^-(x,\gamma x)\,dx \right).$$

In particular, we have proved:

Theorem 7.19. *The L^2-index defined in terms of a covering equals the L^2-index using the corresponding flat A-Hilbert space twisting bundle.*

Therefore, we will have proved Theorem 5.15 and then recovered Atiyah's L^2-index theorem as soon as we prove the index formula for $\operatorname{ind}_t(D_H)$, which we will reduce to Theorem 5.9 by proving Equation (7.18).

Note that Atiyah defines the L^2-index for arbitrary elliptic differential operators on M, not necessarily of Dirac type. This is possible since $\widetilde{M} \times_\Gamma l^2(\Gamma)$ is a flat bundle, and arbitrary differential operators can be twisted with every flat bundle. A corresponding construction is possible in our more general setting. Since all geometrically important operators are generalized Dirac operators, and since only those can be twisted with bundles with nonflat connections, we will stick to the latter more restricted class.

7.6. Equivalences of categories. In this section we show how one can go back and forth between Hilbert A-modules and A-Hilbert spaces, and the corresponding bundles.

Lemma 7.20. *If V is a finitely generated projective Hilbert A-module, then $l^2(V)$ is a finitely generated projective A-Hilbert space.*

Proof. Let $V \oplus W \cong A^n$ be a decomposition into V and an orthogonal complement W. Then V and W are orthogonal also with respect to the inner product $\langle\cdot,\cdot\rangle_2$, and therefore their completions add up to the completion $l^2(A)^n$ of A^n. Moreover, the projection $A^n \to A^n$ with image V is given (as is any right A-linear map from A^n to itself) by multiplication from the left with a matrix with entries in A. This same matrix will act on $l^2(A)^n$ (by Lemma 7.5) with kernel containing W (i.e., also its closure $l^2(W)$) and image containing V and —since the matrix is a projection— also its closure $l^2(V)$). This shows that the orthogonal projection is given by multiplication with the matrix. This completes the proof that $l^2(V)$ is a finitely generated projective A-Hilbert space. $\square$

Lemma 7.21. *Assume that $f\colon V \to W$ is an adjointable A-module homomorphism between Hilbert A-modules V and W. Then f extends to a bounded A-linear operator $f\colon l^2(V) \to l^2(W)$ with adjoint the extension of f^*.*

If $f\colon V \to W$ is a Hilbert A-module isometry, then f extends to an isometry $f\colon l^2(V) \to l^2(W)$.

Proof. By [10, Proposition 1.2], $\langle f(x), f(x)\rangle \le \|f\|^2 \langle x, x\rangle$ in A. Therefore, because of positivity and linearity of τ

$$\langle f(x), f(x)\rangle_2 = \tau(\langle f(x), f(x)\rangle) \le \|f\|^2 \tau(\langle x, x\rangle) = \|f\|^2 \langle x, x\rangle_2, \qquad \forall x \in V.$$

This shows that f is l^2-bounded.

For the adjoint observe that

$$\langle f(x), y\rangle_2 = \tau(\langle f(x), y\rangle) = \tau(\langle x, f^*(y)\rangle) = \langle x, f^*(y)\rangle_2 \qquad \forall x \in V.$$

If $f\colon V \to W$ is an isometry, then in particular

$$\langle f(v), f(v')\rangle_2 = \tau(\langle f(v), f(v')\rangle) = \tau(\langle v, v'\rangle) = \langle v, v'\rangle_2 \qquad \forall v, v' \in V.$$ $\square$

Definition 7.22. Let W be a Hilbert A-module bundle on a space X. Fiberwise application of the construction of Lemma 7.5 produces an A-Hilbert space bundle on X which we call $l^2(W)$. The transition functions are obtained as extensions of Hilbert A-module isometries to A-Hilbert space isometries as described in Lemma 7.5. In particular, we define an induced smooth structure on $l^2(W)$ from a smooth structure on W.

Lemma 7.23. *Assume that W is a smooth Hilbert A-module bundle on a smooth manifold M. Let ∇ be a connection on W which is locally given by the* $\mathrm{End}(W)$*-valued* 1*-form ω as in Proposition 3.8, with curvature* 2*-form Ω. Then the connection extends to $l^2(W)$, locally given by ω and with curvature Ω, where we extend the endomorphisms of W to endomorphisms of $l^2(W)$ using Lemma 7.21.*

This extension still satisfies the Leibniz rule for the right A-action. If ∇ is a metric connection, the same is true for its extension (*now with respect to the l^2-inner product*)*.*

Proof. Recall that, if a trivialization $W|_U \cong V \times U$ is given, then $\nabla = \nabla_0 + \omega$, where ∇_0 is the trivial connection given by the trivialization. The latter one extends to the trivialized bundle $l^2(V) \times U$ as the trivial connection. By Lemma 7.21 ω extends to a 1-form with values in A-Hilbert space endomorphisms of $l^2(V)$. Consequently, $\nabla_0 + \omega$ defines the desired extension of ∇. From the local formula for the curvature of Proposition 3.8, its curvature is the extensions of Ω.

The Leibniz rule holds for the trivial connection on $l^2(V) \times U$ by the usual calculus proof of the Leibniz rule (which only uses distributivity in both variables), and since ω is compatible with the A-module structure also for the extension of ∇.

If ∇ is a metric connection of W, then ω has values in skew adjoint A-module endomorphisms. By Lemma 7.21 the extension has values in skew adjoint Hilbert space endomorphism and therefore the extension of ∇ is a metric connection for the l^2-inner product. $\square$

Definition 7.24. Assume that A is a von Neumann algebra. Let X be an A-Hilbert space. Choose an embedding $X \hookrightarrow H_X \otimes l^2(A)$ for an appropriate Hilbert space H_X (finite dimensional if X is finitely generated projective), as in Definition 7.7. Let $p \in \mathcal{B}(H_X) \otimes A$ be the corresponding orthogonal projection onto X. Set

$$A(X) := p(H_X \otimes A) \subset X,$$

where $H_X \otimes A \subset H_X \otimes l^2(A)$ is the canonical Hilbert A-module contained in $H_X \otimes l^2(A)$ (isomorphic to H_A is H_X is separable). Since p is a projection in $\mathcal{B}(H_X) \otimes A = \mathcal{B}_A(H_X \otimes A)$, the image $p(H_X \otimes A)$ is itself a Hilbert A-module with the induced structure from the ambient space $H_X \otimes A$.

If X is a finitely generated projective A-Hilbert space, H_X can be chosen finite-dimensional, say $H_X = \mathbb{C}^n$. Then $A(X)$ is a finitely generated projective Hilbert A-module, the image of the projection $p \in \mathcal{B}_A(\mathbb{C}^n \otimes A) = M_n(A)$.

Of course, the construction of $A(X)$ a priori depends on the choice of the embedding and of the resulting projection p. In the next lemma, we will see that this is not the case.

Lemma 7.25. *A bounded A-linear operator $f\colon X \to Y$ between two A-Hilbert spaces induces by restriction an adjointable A-linear map $A(f)\colon A(X) \to A(Y)$, for every choice of projection $p_X \in \mathcal{B}(H_X) \otimes A$ and $p_Y \in \mathcal{B}(H_Y) \otimes A$ with image X and Y, respectively. Moreover, $A(f)^* = A(f^*)$ and $A(\cdot)$ is a functor. If f is a Hilbert space isometry, then $A(f)$ is an isometry of Hilbert A-modules.*

In particular, if we apply this to $\mathrm{id}_X\colon X \to X$, with $A(X)$ defined using two different projections, we see that id_X restricts to the identity map on $A(X)$, therefore $A(X)$ (with its structure as Hilbert A-module) is well-defined.

Proof. If $i_Y\colon Y \to H_Y \otimes l^2(A)$ is the inclusion, then

$$i_Y \circ f \circ p_X\colon H_X \otimes l^2(A) \to H_Y \otimes l^2(A)$$

is a bounded operator which commutes with right multiplication by A. Since A is a von Neumann algebra, by Lemma 7.26 the composition belongs to $\mathcal{B}(H_X, H_Y) \otimes A$, where A acts by right multiplication on $l^2(A)$. In particular, the subspace $H_X \otimes A$ is mapped to the subspace $H_Y \otimes A$, and since $A(X)$ is the intersection $X \cap (H_X \otimes A)$, and similarly $A(Y) = Y \cap (H_Y \otimes A)$, f maps these subspaces to each other.

Moreover, $\mathcal{B}(H_X, H_Y) \otimes A$ is exactly the space of adjointable operators from $H_X \otimes A$ to $H_Y \otimes A$. Since $A(f) = p_Y \circ (i_Y f p_X) \circ i_X$, and p_Y, i_X are also adjointable, the same follows for $A(f)$.

$A(f)$ is functorial by construction, since it is just given by restriction to the subspace $A(X)$. Since the representations of A on $l^2(A)$ by left and right multiplication are both C^*-homomorphisms, $\mathcal{B}(H_X, H_Y) \otimes A \to \mathcal{B}(H_X \otimes l^2(A), H_Y \otimes l^2(A))$ is also adjoint preserving. It follows that $A(f)^* = A(f^*)$.

Finally, f is an isometry $\iff$ $ff^* = 1 = f^*f$ $\iff$ $A(f)A(f)^* = 1 = A(f)^*A(f)$ $\iff$ $A(f)$ is an isometry. $\square$

Note that for Lemma 7.25 it is crucial that A is a von Neumann algebra, the corresponding result does not necessarily hold for arbitrary C^*-algebras.

We needed the following lemma.

Lemma 7.26. *Let A be a von Neumann algebra with a trace τ as in 7.2. Then A acts by left and right multiplication on $l^2(A)$. The corresponding subalgebras of $\mathcal{B}(l^2(A))$ are mutually commutants of each other, i.e., the operators given by right multiplication with elements of A are exactly those operators commuting with left multiplication by A.*

Let H_1 and H_2 be two Hilbert spaces. Then

$$\mathcal{B}(H_1 \otimes l^2(A), H_2 \otimes l^2(A))^A = \mathcal{B}(H_1, H_2) \otimes A,$$

where $\mathcal{B}(H_1 \otimes l^2(A), H_2 \otimes l^2(A))^A$ is defined as those operators commuting with left multiplication by A, and the factor A in $\mathcal{B}(H_1, H_2) \otimes A$ acts by right multiplication on $l^2(A)$.

Proof. The first assertion follows from Tomita modular theory. The vector $1 \in l^2(A)$ is a separating and generating vector for left as well as right multiplication of A on $l^2(A)$ since the trace is faithful, and since, by definition, $l^2(A)$ is the closure of the subspace A. The map

$$J = S = F\colon A \to A;\ a \mapsto a^*$$

is a conjugate linear isometry of order 2, in particular extends to all of $l^2(A)$.

By [7, Theorem 9.2.9] the elements of the commutant of right multiplication R_a with elements $a \in A$ are given as operators $JR_aJ = L_{a^*}$, $a \in A$ (where L_a denotes left multiplication with A). The first statement follows.

The second assertion follows since the commutant of $A_1 \otimes A_2$ acting on $H_1 \otimes H_2$ is $A_1' \otimes A_2'$ (here $A_1 = \mathbb{C}$, $A_1' = \mathcal{B}(H_1, H_2)$). □

Theorem 7.27. *Let A be a von Neumann algebra with a trace τ as in 7.2. The category of finitely generated projective A-Hilbert spaces is equivalent to the category of finitely generated projective Hilbert A-modules, and the category of A-Hilbert spaces is equivalent to the category of projective Hilbert A-modules. The equivalence is given by $V \mapsto l^2(V)$ and $X \mapsto A(X)$ for any Hilbert A-module V and A-Hilbert space X.*

Proof. This follows from Lemma 7.25 and Lemmas 7.21 and 7.20. □

Proposition 7.28. *Assume that A is a von Neumann algebra with a trace τ as in 7.2. The naturality of the construction of $A(X)$ for an A-Hilbert space X implies that we get a corresponding functor which assigns to each finitely generated projective (smooth) A-Hilbert space bundle a finitely generated projective (smooth) Hilbert A-module bundle. Here we also use that the transition functions (in both cases isometries) are preserved since the functors map isometries to isometries. Together with the construction of Definition 7.22 this gives rise to an equivalence between finitely generated projective (smooth) A-Hilbert space bundles and finitely generated projective Hilbert A-module bundles.*

A connection on a smooth finitely generated projective A-Hilbert space bundle preserves the Hilbert A-module subbundle and therefore gives rise to a connection on the latter. In view of Lemma 7.23, we also get an equivalence between smooth

Hilbert A-module bundles with connection and smooth A-Hilbert space bundles with connection.

Proof. We only have to check that a connection on an A-Hilbert space bundle indeed preserve the Hilbert A-module subbundle. This is clear for the trivial connection on a trivial bundle $U\times X$. Locally, an arbitrary connection differs from such a trivial connection by a one form with values in endomorphisms which commute with the right A-multiplication. Using Lemma 7.26 in the same way as in the proof of Lemma 7.25, such endomorphisms preserve the Hilbert A-module subbundle, and therefore the same is true for the connection. □

Corollary 7.29. *Given any smooth finitely generated projective A-Hilbert space bundle X with connection, we can assume that $X = l^2(V)$ for an appropriate smooth finitely generated projective Hilbert A-module bundle V with connection, where the connection on $l^2(V)$ is obtained as described in Lemma 7.23.*

7.7. The general version of Atiyah's L^2-index theorem. In view of Corollary 7.29 we can now formulate our general version of the L^2-index theorem.

Theorem 7.30. *Let M be a closed manifold, and $D\colon \Gamma(E^+)\to\Gamma(E^-)$ a generalized Dirac operator on M. Let A be a von Neumann algebra with a normal trace t and a faithful trace τ as in Definition 7.16. Let X be a smooth finitely generated projective A-Hilbert space bundle on M, obtained (by Corollary 7.29) as $X = l^2(V)$ for a smooth finitely generated projective Hilbert A-module bundle V. Assume that X has a connection which is extended from V as in Lemma 7.23 and Proposition 7.28. Then*

$$\operatorname{ind}_t(D_X) = t(\operatorname{ind}(D_V)),$$

where $\operatorname{ind}_t(D_X)$ is defined in Definition 7.17, and $\operatorname{ind}(D_V)\in K_0(A)$ is defined in Definition 5.2. In particular, by Theorem 5.9

$$\operatorname{ind}_t(D_X) = \langle \operatorname{ch}(\sigma(D))\cup \operatorname{Td}(T_{\mathbb{C}}M)\cup \operatorname{ch}_t(V), [TM]\rangle.$$

We might as well define $\operatorname{ch}_t(X) := \operatorname{ch}_t(V)$ and observe that it can be obtained from the connection on X (which gives rise to the connection on V simply by restriction). In particular, if X (or equivalently V) are flat, then

$$\operatorname{ind}_t(D_X) = \langle \operatorname{ch}(\sigma(D))\cup \operatorname{Td}(T_{\mathbb{C}}M), [TM]\rangle\cdot \dim_t(X_p),$$

where $\dim_t(X_p)$ is the locally constant function (with values in Z) which assigns to $p\in M$ the value $\dim_t(X_p) = \dim_t(V_p)$, where X_p and V_p are the fibers over p of X and V, respectively.

Corollary 7.31. *If A in Theorem 7.30 is a finite von Neumann algebra with center valued trace $t\colon A\to Z$, then $\operatorname{ind}_t(D_X)$ and $\operatorname{ind}(D_V)$ can be obtained from each other.*

Proof. One direction follows from $t(\operatorname{ind}(D_V)) = \operatorname{ind}_t(D_X)$. The converse is true because the center valued trace induces an *injection* $K_0(A)\xrightarrow{t} Z$ by 5.7, applied to $X = \{*\}$. □

Theorem 7.30 is a consequence of Corollary 6.5 and of properties of the trace t established in Section 7.8. Therefore, we first establish these properties of t, before completing the proof of Theorem 7.30.

7.8. Properties of traces. In Definition 7.16 we used the extension of the trace t from A to $\mathcal{B}(H)\otimes A$. Here, we want to recall the definition and the main properties (we are following [5, I 6, Exercise 7]). Similar considerations can be found in [18, Section 2].

Definition 7.32. Let A be a von Neumann algebra with a trace τ as in 7.2 and with a normal trace $t\colon A\to Z$, where Z is a commutative von Neumann algebra (e.g., $Z=\mathbb{C}$). Let H be a Hilbert space with orthonormal basis $\{e_i \mid i\in I\}$. For a positive operator $a\in\mathcal{B}(H)\otimes A$ (acting on $H\otimes l^2(A)$) define

$$t(a) := \begin{cases} \sum_{i\in I} t(U_i^* a U_i)\in Z & \text{if the sum is ultraweakly convergent}\\ \infty & \text{otherwise}\end{cases}$$

where $U_i\colon l^2(A)\to H\otimes l^2(A)$ is given by the decomposition of H according to the orthonormal basis $\{e_i\}$. Note that $U_i^*aU_i\in A$, since the map $a\mapsto U_i^*aU_i$ is norm continuous from $\mathcal{B}(H\otimes l^2(A))\to\mathcal{B}(l^2(A))$ and maps elementary tensors $T\otimes x\in\mathcal{B}(H)\otimes A$ to elements of A. Note that $\sum_{i\in I} t(U_i^*aU_i)$ is an infinite sum of *nonnegative* elements. It is convergent if and only if the corresponding collection of finite sums has an upper bound in Z, in which case the least upper bound is the limit. In particular, convergence is independent of the ordering in the sum.

The linear span of all positive operators a with $t(a)<\infty$ is an ideal in $\mathcal{B}(H)\otimes A$, and t extends by linearity to this ideal.

In the above definition, we must check that $t(a)$ does not depend on the chosen orthonormal basis $\{e_i\}$. If f_j is a second orthonormal basis with induced unitary inclusions $V_j\colon l^2(A)\to H$, then this follows from the following calculation

$$\begin{aligned}\sum_{i\in I} t(U_i^*aU_i) &= \sum_{i\in I} t(U_i^*\sum_{j\in J} V_jV_j^*aU_i)\\ &= \sum_{i\in I}\sum_{j\in J} t(U_i^*V_jV_j^*aU_i)\\ &= \sum_{i\in I, j\in J} t(V_j^*aU_iU_i^*V_j)\\ &= \sum_{j\in J} t(V_j^*a\sum_{i\in I} U_iU_i^*V_j) = \sum_{j\in J} t(V_j^*aV_j).\end{aligned}$$

Here we used the fact that $\sum_{i\in I} U_iU_i^* = \sum_{j\in J} V_jV_j^* = \mathrm{id}_{H\otimes l^2(A)}$, where the convergence is in the ultraweak sense, and that t is normal and a trace.

Moreover, we use that the linear map $a\mapsto U_i^*aV_j\colon \mathcal{B}(H)\otimes A\to\mathcal{B}(l^2(A))$ is norm continuous and maps elementary tensors $T\otimes x\in\mathcal{B}(H)\otimes A$ to elements of A, such that the image is contained in A. In particular $U_i^*V_j = U_i^*1V_j\in A$ and $V_j^*aU_i\in A$, such that $t((U_i^*V_j)(V_j^*aU_i)) = t((V_j^*aU_i)(U_i^*V_j))$ by the trace property for operators in A.

Again, since all the summands in the above infinite sums are positive elements of Z, the ordering is not an issue, and the limit (if it exists) is the least upper bound.

Definition 7.33. Let A be a von Neumann algebra with traces τ and t as above. Assume that V_1 and V_2 are A-Hilbert spaces and $f\colon V_1\to V_2$ is an A-linear bounded operator. Let $i_1\colon V_1\to H_1\otimes l^2(A)$ and $i_2\colon V_2\to H_2\otimes l^2(A)$ be inclusions as in Definition 7.7, and p_1, p_2 the corresponding orthogonal projections. We say that f

is a *t-Hilbert Schmidt* operator, if $i_1 f^* f p_1$ is of t-trace class. We say that f is of t-trace class, if there are $f_1\colon V_1 \to V_3$ and $f_2\colon V_3 \to V_1$ t-Hilbert Schmidt operators (V_3 an additional A-Hilbert space) such that $f = f_2 f_1$.

If $V_1 = V_2$ and f is of t-trace class, set $t(f) := f(i_1 f p_1)$.

If id_{V_1} is of t-trace class, define $\dim_t(V_1) := t(\mathrm{id}_{V_1})$, else set $\dim_t(V_1) := \infty$.

Again, it is necessary to check that the definitions in 7.33 are independent of the choices made. Moreover, we have to check that the trace so defined has the usual properties (which we are going to use later on). This is the content of the following theorem. Essentially the same theorem, with t complex valued, is stated in [18, Theorem 2.3] and [17, 9.13]. The proof given there also applies to the more general situation here.

Theorem 7.34. *Assume that A is a von Neumann algebra with traces τ and t as above. Let V_0, V_1, V_2 and V_3 be A-Hilbert spaces and $f\colon V_1 \to V_2$, $g\colon V_2 \to V_3$, $e\colon V_0 \to V_1$ be bounded A-linear operators. Then:*

(1) *f is of t-trace class $\iff$ f^* is of t-trace class $\iff$ $|f|$ if of t-trace class.*
(2) *f is a t-Hilbert–Schmidt operator $\iff$ f^* is a t-Hilbert–Schmidt operator.*
(3) *If f is a t-Hilbert–Schmidt operator then gf and fe are t-Hilbert–Schmidt operators.*
(4) *If f is a t-trace class operator, then gf and fe are t-trace class operators.*
(5) *If f is of t-trace class and $V_1 = V_3$ then $g \mapsto t(gf)$ is ultra-weakly continuous.*
(6) *If $V_1 = V_3$ and either f if of t-trace class or f and g are t-Hilbert–Schmidt operators then $t(gf) = t(fg)$.*
(7) *If $V_{1,2} = H \otimes l^2(A)$ for a Hilbert space H, a is a t-Hilbert–Schmidt operator and $B \in \mathcal{B}(H)$ is a Hilbert–Schmidt operator, then $f = a \otimes B$ is a t-Hilbert–Schmidt operator. If a is of t-trace class and B is of trace class, then f is of t-trace class with $t(f) = t(a)\,\mathrm{Sp}(B)$, where Sp is the ordinary trace on the trace class ideal of $\mathcal{B}(H)$.*
(8) *Assume that $u\colon V_1 \to V_2$ is bounded A-linear with a bounded (necessarily A-equivariant) inverse u^{-1}. Then $\dim_t(V_1) = \dim_t(V_2)$, i.e., $\dim_t$ does not depend on the Hilbert space structure.*

Proof. (8) We have

$$\dim_t(V_1) = \mathrm{tr}_t(\mathrm{id}_{V_1}) = \mathrm{tr}_t(u^{-1}u\,\mathrm{id}_{V_1}) = \mathrm{tr}_t(u\,\mathrm{id}_{V_1}\,u^{-1}) = \mathrm{tr}_t(\mathrm{id}_{V_2})$$

if either id_{V_1} or id_{V_2} are of t-trace class, and the calculation shows that then the other one also is of t-trace class. Here we used (6). $\square$

7.9. Trace class operators.

Definition 7.35. Assume that $f \in \mathrm{End}_A(H_A)$ is a self adjoint positive endomorphism of the standard countably generated Hilbert A-module H_A. We call f of *τ-trace class* if $\tau(f) := \sum_{n\in\mathbb{N}} \tau(\langle f(e_n), e_n\rangle_A) < \infty$. An arbitrary $f \in \mathrm{End}_A(H_A)$ is called a τ-trace class operator if it is a (finite) linear combination of self adjoint positive τ-trace class operators. Then $\tau(f)$ is defined as the corresponding linear combination.

Let V, W be countably generated Hilbert A-modules, $f \in \mathrm{Hom}_A(V, W)$. We call f of *τ-trace class*, if $f \oplus 0\colon V \oplus H_A \to W \oplus H_A$ is of τ-trace class. Recall that by Kasparov's stabilization theorem [24, Theorem 15.4.6] $V \oplus H_A \cong H_A \cong W \oplus H_A$

such that being of τ-trace class is already defined for $f \oplus 0$. The normality of τ is used to prove that this concept and the extension of τ we get this way is well-defined and that we can define traces with the usual properties in Proposition 7.36.

Proposition 7.36. *If $f \in \mathrm{Hom}_A(V, W)$ is of τ-trace class and $g \in \mathrm{Hom}_A(W, V)$ then fg and gf are both of τ-trace class and $\tau(fg) = \tau(gf)$.*

If $g\colon l^2(\mathbb{N}) \to l^2(\mathbb{N})$ is of trace class with trace $\mathrm{Sp}(g)$ (in the sense of endomorphisms of the Hilbert space $l^2(\mathbb{N})$), and $f \in \mathrm{End}_A(A)$ then $f \otimes g \in \mathrm{End}_A(H_A)$ is of τ-trace class and

$$\tau(f \otimes g) = \tau(f) \cdot \mathrm{Sp}(g).$$

Proof. The trace on $\mathrm{End}_A(H_A)$ is the tensor product of τ on A and the standard trace on $l^2(\mathbb{N})$ which both have the trace property. For a more detailed treatment of such results compare, e.g., [18, Section 2].

Recall that we define $f \otimes g(ae_n) := f(a)g(e_n)$, which extends by linearity and continuity to an element of $\mathrm{End}_A(H_A)$. □

Definition 7.37. Exactly the same kind of definition was made for A-Hilbert space morphisms. Observe that the two constructions are compatible in the sense that if $f \in \mathrm{End}_A(V)$ is of τ-trace class then the same is true for its extension to $l^2(V)$ as in Lemma 7.21 with unchanged trace $\tau(f)$.

7.10. Proof of Theorem 7.30. Note first that, by definition,

$$\mathrm{ind}_t(D_X) = t\big(\mathrm{pr}_{ker(D_X)}\big) - t\big(\mathrm{pr}_{coker(D_X)}\big),$$

where $\mathrm{pr}_{\ker(D_X)}$ is the orthogonal projection onto the kernel of D_X inside the space of L^2-section $L^2(E^+ \otimes X)$, and $\mathrm{pr}_{\mathrm{coker}(D_X)}$ is the projection onto the orthogonal complement of the image of D_X in $L^2(E^- \otimes X)$. Here, we consider

$$D_X\colon L^2(E^+ \otimes X) \to L^2(E^- \otimes X)$$

as unbounded operator.

D_X also gives rise to a bounded operator between Sobolev spaces. The following definition should be compared with Definition 5.18.

Definition 7.38. Given a finitely generated smooth A-Hilbert space bundle X over a compact smooth manifold M, Sobolev spaces $H^s(X)$ can be defined ($s \in \mathbb{R}$), compare, e.g., [4]. One way to do this is to pick a trivializing atlas (U_α) with subordinate partition of unity (ϕ_α) and then define for smooth sections u, v of X the inner product

$$(u, v)_s = \sum_\alpha \int_{U_\alpha} \langle (1 + \Delta_\alpha)^s \phi_\alpha u(x), \phi_\alpha v(x) \rangle \, dx,$$

where Δ_α is the ordinary Laplacian on $\mathbb{R}^n$ acting on the trivialized bundle (in the notation, some diffeomorphisms are omitted).

The inner product is $\mathbb{C}$-valued and the completion is an A-Hilbert space.

Theorem 7.39. *Assume that W is a smooth finitely generated projective Hilbert A-module bundle over a compact manifold M, For each $\epsilon > 0$, the natural inclusion $H^s(W) \to H^{s-\epsilon}(W)$ is A-compact.*

If $r > \dim(M)/2$, then the natural inclusion $H^s(W) \to H^{s-r}(W)$ is of τ-trace class.

The second assertion holds also if W is a finitely generated projective A-Hilbert space bundle.

Proof. Using charts and a partition of unity, it suffices to prove the statement for the trivial bundle $A \times T^n$ on the n-torus T^n. In the latter case, one obtains isomorphisms $H^s(A \times T^n) \cong H^s(T^n) \otimes A$. In particular, the inclusion

$$H^s(A \times T^n) \to H^{s-r}(A \times T^n)$$

is the tensor product of the inclusion of $H^s(T^n) \to H^{s-r}(T^n)$ with the identity on A. By Proposition 7.36, the trace class property follows, and compactness is handled in a similar way.

The same argument applies to A-Hilbert space bundles. □

A twisted Dirac operator D_H as in Definition 7.14 extends to a bounded operator between Sobolev spaces $D_H \colon H^1(W^+ \otimes X) \to L^2(W^- \otimes X)$.

Of course, the inner product on $H^s(W)$ depends on a number of choices, However, two different choices give rise to equivalent inner products and therefore isomorphic Sobolev spaces.

Observe that if V is a finitely generated projective Hilbert A-module bundle with corresponding A-Hilbert module completion $X = l^2(V)$, the A-Hilbert space completion $l^2(H^s(V))$ and $H^s(l^2(V))$ are isomorphic. This follows since the trace τ used to define $l^2(V)$ is continuous by Lemma 7.3. $l^2(H^s(V))$ is the completion of $\Gamma(V)$ with respect to the inner product $\sum \tau \int_{U_\alpha} \langle (1+\Delta)^\alpha \cdot, \cdot \rangle$, whereas $H^s(l^2(V))$ is the completion of $\Gamma(V)$ with respect to the inner product $\sum \int_{U_\alpha} \tau(\langle (1+\Delta)^s \cdot, \cdot \rangle)$ and by continuity, τ commutes with integration so that the two inner products coincide.

Moreover,

$$D_X = l^2(D_V) \colon H^1(E^+ \otimes X) \to L^2(E^- \otimes X)$$

under this identification (and is in particular a bounded operator). We can now look at $\chi_{\{0\}}(D_X^* D_X)$ and $\chi_{\{0\}}(D_X D_X^*)$. These are the projections onto the kernel of D_X in $H^1(E^+ \otimes X)$ and onto the orthogonal complement of the image of D_X in $L^2(E^- \otimes X)$. Note that the second space is exactly the same one showing up in the definition of $\dim_t(D_X)$, since H^1 is exactly the domain of the closure of the unbounded operator D_X on L^2.

However, the kernels in H^1 and in L^2 strictly speaking are different. The inclusion $H^1(E^+ \otimes X) \to L^2(E^+ \otimes X)$ maps the kernels bijectively onto each other (by elliptic regularity), but the topologies are different. Note, however, that $\ker(D_X) \subset L^2(E^+ \otimes X)$ is a closed subset, therefore complete. By the open mapping theorem, the bijection between the kernels has a bounded inverse (which is of course also A-linear). It follows from Theorem 7.34 (8) that $\dim_t(\ker(D_X))$ does not depend on the question whether we consider D_x as unbounded operator on L^2 or as bounded operator from H^1 to L^2. In particular,

$$\operatorname{ind}_t(D_X) = t\big(\chi_{\{0\}}(D_X^* D_X)\big) - t\big(\chi_{\{0\}}(D_X D_X^*)\big),$$

where D_X is considered as bounded operator from H^1 to L^2.

Note that, on the level of operators, the functor l^2 embeds for each Hilbert A-module U the C^*-algebra $\operatorname{Hom}_A(U,U)$ into the C^*-algebra $\mathcal{B}(l^2(U))$. Embeddings of C^*-algebras commute with functional calculus. In particular, $\chi_{\{0\}}(D_X^* D_X) = l^2(\chi_{\{0\}}(D_V^* D_V))$ and $\chi_{\{0\}}(D_X D_X^*) = l^2(\chi_{\{0\}}(D_V D_V^*))$.

Next, we must look at $t(\mathrm{ind}(D_V))$. This is defined as follows: after stabilization, $L^2(E^+ \otimes V) \oplus H_A \cong H_A$. Then, there is a unitary $u \in \mathcal{B}_A(H_A)$ such that

$$p := u^*(\chi_{\{0\}}(D_V^* D_V) \oplus 0_{H_A})u \in \mathcal{B}_A(H_A)$$

(using the above isomorphism) is a projection which is represented by a matrix with finitely many nonzero entries, where we understand $M_n(A) \subset \mathcal{B}_A(H_A)$ using an orthonormal basis of H in $H_A = H \otimes A$. Similarly, $\chi_{\{0\}}(D_V D_V^*)$ gives rise to a projection q in $M_n(A) \subset \mathcal{B}_A(H_A)$. We can apply the functor l^2 to the whole construction, and therefore get elements $l^2(p)$ and $l^2(q)$, represented exactly by the same finite matrices p and q in $M_n(A)$ which are unitarily equivalent (by A-linear operators $l^2(u)$) to

$$\chi_{\{0\}}(D_X^* D_X) \oplus 0_{H\otimes l^2(A)} \quad \text{and} \quad \chi_{\{0\}}(D_X D_X^*) \oplus 0_{H\otimes l^2(A)}.$$

Then $\mathrm{ind}_t(D_V) = t(p) - t(q)$. Because t is normal, we have Theorem 7.34(6) which is valid for nonfinitely generated A-Hilbert spaces and therefore

$$t(p) = t(l^2(p)) = t(l^2(\chi_{\{0\}}(D_V^* D_V))), \quad t(q) = t(l^2(q)) = t(l^2(\chi_{\{0\}}(D_V D_V^*))).$$

For the first equal sign in both equations note that $t(p)$ and $t(l^2(p))$ are by their very definitions exactly the same thing.

This finally implies the assertion of Theorem 7.30.

8. Trace class subalgebras

Throughout this paper, we have been dealing with a C^*-algebra A with a trace $\tau\colon A \to Z$, Z being a commutative C^*-algebra. We were able to derive rather explicit index theorems for $\tau(\mathrm{ind}(D_W))$, where D is a Dirac type operator on a closed manifold M and W is a Hilbert A-module bundle on M (with finitely generated projective fibers). Here $\tau\colon K_0(A) \to Z$ is the induced map on K-theory and $\mathrm{ind}(D_W) \in K_0(A)$ is defined, e.g., by the Mishchenko–Fomenko construction.

However, there are many other situations where trace-like maps on $K_0(A)$ exist which do not come from a trace on A. One of the most prominent is if $\mathcal{B} \subset A$ is a dense subalgebra which is closed under holomorphic functional calculus in A and which has a trace $\tau\colon \mathcal{B} \to Z$. Since the inclusion $\mathcal{B} \to A$ induces an isomorphism $K_0(\mathcal{B}) \to K_0(A)$, we still get an induced map $\tau\colon K_0(A) \to Z$. The most prominent example is the trace class ideal inside the compact operators on a separable Hilbert space. In the notation of [2], $\mathcal{B}$ is a local C^*-algebra with completion A.

In this section we describe how to generalize the results obtained in the rest of the paper to this situation. In particular we will get an explicit index theorem.

Our goal is to describe how the contents of Sections 2 to 5.1 remain true almost entirely in the more general situation.

We note that the following concepts are relevant:

(1) Finitely generated projective modules make sense in exactly the same way for the local C^*-algebra $\mathcal{B}$ as for the C^*-algebra A.
(2) Finitely generated projective Hilbert $\mathcal{B}$-modules also make sense. Of course we can not assume that such a module is complete in any way. But being finitely generated projective, those modules are up to isomorphism described as an orthogonal summand of $\mathcal{B}^n$, and this makes sense for each $*$-algebra.

Restricting the $\mathcal{B}$-valued inner product of $\mathcal{B}^n$ induces a $\mathcal{B}$-valued inner product on the summands.

(3) $\mathcal{B}$ is a dense subalgebra of A, and in a canonical way is each finitely generated projective (Hilbert)$\mathcal{B}$-module $\mathcal{V}$ a dense subspace of the finitely generated projective (Hilbert) A-module $V := \mathcal{V} \otimes_{\mathcal{B}} A$. In the same way, for any two such modules $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathrm{Hom}_{\mathcal{B}}(\mathcal{V}_1, \mathcal{V}_2)$ is a dense subspace of $\mathrm{Hom}_A(V_1, V_2)$. If $V_2 = V_1$ and therefore these spaces are algebras, then the smaller one is holomorphically closed in the bigger one. This follows since $\mathrm{Hom}_{\mathcal{B}}(\mathcal{B}^n, \mathcal{B}^n) = M(n, \mathcal{B})$ is holomorphically closed in $\mathrm{Hom}_A(A^n, A^n) = M(n, A)$ by definition of being holomorphically closed.

(4) Now the definition of a bundle with finitely generated projective (Hilbert) $\mathcal{B}$-module fibers makes perfectly sense. This is also true for smooth bundles, there a map to $\mathrm{Hom}_{\mathcal{B}}(\mathcal{V}_1, \mathcal{V}_2)$ is smooth if and only if it is smooth when composed with the inclusion into $\mathrm{Hom}_A(V_1, V_2)$ as above.

In particular, we can consider each such bundle as included in an induced finitely generated projective (Hilbert) A-module bundle, and a smooth structure on the $\mathcal{B}$-bundle induces a smooth structure on the A-bundle.

(5) The results in Section 2, in particular in Section 2.1 remain true for finitely generated projective (Hilbert) $\mathcal{B}$-module bundles. This follows by carrying out the constructions for the induced A-module bundle and then observing that all the constructions, which only involve the $*$-operator, the algebra structure, and taking holomorphic functions of elements in $\mathrm{Hom}_{\mathcal{B}}(\mathcal{V}, \mathcal{V})$ remain inside this subset of $\mathrm{Hom}_A(V, V)$ by the very fact that $\mathcal{B}$ is holomorphically closed in A.

The most important such function takes the inverse of an invertible element. One should note that the set of invertible elements of $\mathcal{B}$, being the intersection of $\mathcal{B}$ with the corresponding open subset of A is itself open in $\mathcal{B}$. This property is also used occasionally.

(6) It is now possible to define K-theory groups $K^0_c(X; \mathcal{B})$ of finitely generated projective $\mathcal{B}$-modules in the same way as we define such K-groups for A-module bundles in Section 2.2. We can also consider the normed $*$-algebra $C_0(X; \mathcal{B})$ of continuous $\mathcal{B}$-valued functions on a locally compact space X which vanish at infinity., and the same proofs as for A-bundles implies that we get a commutative diagram

$$\begin{tikzcd} K^0_c(X;\mathcal{B}) \arrow[r, "\cong"] \arrow[d, "\cong"'] & K_0(C_0(X;\mathcal{B})) \arrow[d, "\cong"'] \\ K^0_c(X;A) \arrow[r, "\cong"] & K_0(C_0(X;A)) \end{tikzcd} \tag{8.1}$$

for any locally compact space X. Moreover, $C_0(X; \mathcal{B})$ is a $*$-subalgebra of $C_0(X; A)$ which is closed under holomorphic functional calculus. This is true since $f(\phi)$ for a holomorphic function f and $\phi\colon X \to A$ is given by $(f(\phi))(x) = f(\phi(x))$ whenever it is defined, so that the statement reduces to the fact that $\mathcal{B}$ is closed under holomorphic functional calculus in A. This implies that the vertical maps in (8.1) are isomorphisms.

(7) Note that we do not use $C(X) \otimes \mathcal{B}$ here, which will in general not be isomorphic to $C(X; \mathcal{B})$.

(8) We can then also define $K_c^1(X;\mathcal{B}) := K_c^0(X\times\mathbb{R};\mathcal{B})$, such that $K_c^1(X;\mathcal{B}) \xrightarrow{\cong} K_c^1(X;A)$ is an immediate consequence of the corresponding result for K^0. Moreover, we get a commutative Bott periodicity diagram

$$\begin{array}{ccc} K^0(X;\mathcal{B}) & \xrightarrow[\cong]{\beta} & K_c^0(X\times\mathbb{R}^2;\mathcal{B}) \\ \downarrow{\cong} & & \downarrow{\cong} \\ K^0(X;A) & \xrightarrow[\cong]{\beta} & K_c^0(X\times\mathbb{R}^2;A), \end{array}$$

where we might define the upper horizontal map such that the diagram commutes.

(9) Given a trace $\tau\colon \mathcal{B}\to Z$ on $\mathcal{B}$, the constructions of Section 2.3 immediately generalize. In particular, for each finitely generated projective Hilbert $\mathcal{B}$-module bundle $\mathcal{W}$ on a manifold M we get an induced map

$$\tau\colon \Omega^*(M;\mathrm{End}_{\mathcal{B}}(\mathcal{W})) \to \Omega^*(M;Z).$$

(10) Connections on smooth Hilbert $\mathcal{B}$-module bundles are defined in exactly the same way as they are defined on A-bundles, and the properties proved in Section 3 generalize immediately to Hilbert $\mathcal{B}$-module bundles. Moreover, each $\mathcal{B}$-module connection induces a connection on the induced Hilbert A-module bundle which restricts to the given one on the $\mathcal{B}$-subbundle.

(11) If ∇ is a connection on a finitely generated projective Hilbert $\mathcal{B}$-module bundle over M and Ω is its curvature, and if $\tau\colon\mathcal{B}\to Z$ is a trace, then the Chern character

$$\mathrm{ch}_\tau(\Omega)\in\Omega^{2*}(M;Z)$$

with its corresponding de Rham cohomology class are defined as in Section 4. Moreover, all the properties proved there generalize to this situation.

(12) Let M is a closed manifold, D a Dirac type operator on M and $\mathcal{W}$ a smooth finitely generated projective Hilbert $\mathcal{B}$-module bundle with connection and with curvature Ω on M with induced Hilbert A-module bundle W. Let $\tau\colon\mathcal{B}\to Z$ be a trace with induced homomorphism $\tau\colon K_0(\mathcal{B})\cong K_0(A)\to Z$. Commutativity in the Künneth diagram (compare (5.4))

$$\begin{array}{ccc} K^*(\mathcal{B})\otimes K_*(M)\otimes\mathbb{Q} & \longrightarrow & K_0(M;\mathcal{B})\otimes\mathbb{Q} \\ \downarrow{\cong} & & \downarrow{\cong} \\ K^*(A)\otimes K_*(M)\otimes\mathbb{Q} & \xrightarrow{\cong} & K_0(M;A) \end{array}$$

implies that the upper horizontal map is an isomorphism. Recall that the map sends $[\mathcal{P}]\otimes[E]$, where $\mathcal{P}$ is a finitely generated projective $\mathcal{B}$-module and E a finite dimensional vector bundle to the fiberwise tensor product bundle $[\mathcal{P}\otimes E]$, and $[\mathcal{W},\mathcal{W}_\in,\phi_{\mathcal{W}},\phi_{\mathcal{W}_2}]\otimes[E,E_2,\phi_E,\phi_{E_2}]$ to

$$\beta^{-1}[\mathcal{W}\otimes E\oplus\mathcal{W}_\in\otimes E_2,\mathcal{W}\otimes E_2\oplus\mathcal{W}_\in\otimes E,$$
$$\phi_{\mathcal{W}}\otimes\phi_E\oplus\phi_{\mathcal{W}_2}\otimes\phi_{E_2},\phi_{\mathcal{W}}\otimes\phi_{E_2}\oplus\phi_{\mathcal{W}_2}\otimes\phi_E],$$

where the objects are now corresponding tuples of bundles on $\mathbb{R}$, $M\times\mathbb{R}$ and $M\times\mathbb{R}^2$ as in the description of compactly supported K-theory of Proposition 2.22.

Given this, the proof of Theorem 5.9 now goes through for Hilbert $\mathcal{B}$-module bundles as it does for A-bundles, and we arrive, in the situation and with the notation of (12), at the formula we want to prove:

Theorem 8.2.
$$\operatorname{ind}_\tau(D_W) = \langle \operatorname{ch}(\sigma(D)) \cup \operatorname{Td}(T_{\mathbb{C}}M) \cup \operatorname{ch}_\tau(\mathcal{W}), [TM] \rangle,$$
and if M is oriented of dimension n, we get similarly
$$\operatorname{ind}_\tau(D_W) = (-1)^{n(n+1)/2} \langle \pi_! \operatorname{ch}(\sigma(D)) \cup \operatorname{Td}(T_{\mathbb{C}}M) \cup \operatorname{ch}_\tau(\mathcal{W}), [M] \rangle.$$
(*Compare also Theorem* 5.9.)

Note that $\operatorname{ch}_\tau(\mathcal{W}) = \operatorname{ch}_\tau(\Omega)$ is only defined in terms of the curvature of the $\mathcal{B}$-bundle, because the trace is only defined on $\mathcal{B}$. Only after passage to K-theory can it also be used for $K_0(A)$.

The extensions of the index theory to $\mathcal{B}$-module bundles were inspired by conversations with John Roe and his proof of the relative index theorem in [16], which also uses traces on densely defined subalgebras.

References

[1] M. F. Atiyah, *Elliptic operators, discrete groups and von Neumann algebras*, Colloque "analyse et topologie" en l'honneur de Henri Cartan (orsay, 1974), Astérisque, **32–33**, Soc. Math. France, Paris, 1976, 43–72, MR0420729 (54 #8741), Zbl 0323.58015.

[2] Bruce Blackadar, *K-theory for operator algebras* (Second Edition), Mathematical Sciences Research Institute Publications, **5**, Cambridge University Press, Cambridge, 1998, MR1656031 (99g:46104), Zbl 0913.46054.

[3] Ulrich Bunke, *A K-theoretic relative index theorem and Callias-type Dirac operators*, Math. Ann. **303** (1995), 241–279, MR1348799 (96e:58148), Zbl 0835.58035.

[4] D. Burghelea, L. Friedlander, T. Kappeler and P. McDonald, *Analytic and Reidemeister torsion for representations in finite type Hilbert modules*, Geom. Funct. Anal. **6** (1996), 751–859, MR1415762 (97i:58177), Zbl 0874.57025.

[5] Jacques Dixmier, Les algèbres d'opérateurs dans l'espace hilbertien (algèbres de von Neumann), (Deuxième édition, revue et augmentée, Cahiers Scientifiques, Fasc. XXV), Gauthier-Villars Éditeur, Paris, 1969, MR0352996 (50 #5482), Zbl 0175.43801.

[6] Bernhard Hanke and Thomas Schick, *Enlargeability and index theory*, to appear in Journal of Differential Geometry, math.GT/0403257.

[7] Richard V. Kadison and John R. Ringrose, *Fundamentals of the theory of operator algebras*. II. *Advanced theory*, Graduate Studies in Mathematics, **16**, (Corrected reprint of the 1986 original), American Mathematical Society, Providence, RI, 1997, MR1468230 (98f:46001b), Zbl 0991.46031.

[8] V. A. Kasimov, *A property of Hilbert modules and Fredholm operators over C^*-algebras*, Akad. Nauk Azerbaĭdzhan. SSR Dokl. **38** (1982) 10–14, MR0688514 (84j:46099), Zbl 0507.46043.

[9] G. G. Kasparov, *The operator K-functor and extensions of C^*-algebras*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), 571–636, 719, MR0582160 (81m:58075), Zbl 0464.46054.

[10] E. C. Lance, *Hilbert C^*-modules, A toolkit for operator algebraists*, Cambridge University Press, Cambridge, 1995, MR1325694 (96k:46100), Zbl 0822.46080.

[11] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn, Spin geometry, Princeton University Press, Princeton, NJ, 1989, MR1031992 (91g:53001), Zbl 0688.57001.

[12] Wolfgang Lück, *The relation between the Baum–Connes conjecture and the trace conjecture*, Invent. Math. **149** (2002), 123–152, MR1914619 (2003f:19005), Zbl 1035.19003.

[13] Varghese Mathai, *The Novikov conjecture for low degree cohomology classes*, Geom. Dedicata **99** (2003), 1–15, MR1998926 (2004e:19002), Zbl 1029.19006,

[14] A. S. Mishchenko and A. T. Fomenko, *The index of elliptic operators over C^*-algebras*, Math. USSR, Izv. **15** (1980), 87–112, MR0548506 (81i:46075), Zbl 0448.46039.

[15] John Roe, *Elliptic operators, topology and asymptotic methods*, Pitman Research Notes in Mathematics Series, **179**, Longman Scientific & Technical, Harlow, 1988, MR0960889 (89j:58126), Zbl 0654.58031.

[16] John Roe, *A note on the relative index theorem*, Quart. J. Math. Oxford Ser. (2) **42** (1991), no. 167, 365–373, MR1120996 (92g:58125), Zbl 0746.19009.

[17] Thomas Schick, *Analysis on boundary manifolds of bounded geometry, Hodge–de Rham isomorphism and L^2-index theorem*, Ph.D. Thesis, Johannes Gutenberg Universität, Mainz, 1996, http://www.uni-math.gwdg.de/schick/publ/dissschick.htm, printed by Shaker Verlag, ISBN=3-8265-1969-8.

[18] Thomas Schick, *L^2-index theorem for elliptic differential boundary problems*, Pacific J. Math. **197** (2001), 423–439, MR1815264 (2001m:58053), Zbl 1058.58012.

[19] Thomas Schick, *Operator algebras and topology*, Topology of high-dimensional manifolds, no. 1, 2 (Trieste, 2001), ICTP Lect. Notes, **9**, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002, 571–660, MR1937025 (2004a:46070).

[20] Thomas Schick, *Index theory and the Baum–Connes conjecture*, Geometry seminars. 2001–2004 (Italian), Univ. Stud. Bologna, Bologna, 2004, 231–280, MR2115817 (2005j:58031).

[21] Yu. P. Solovyov and E. V. Troitskiĭ, *C^*-algebras and elliptic operators in differential topology*, Translated from the 1996 Russian original by Troitsky, Translation edited by A. B. Sossinskii, Translations of Mathematical Monographs, **192**, American Mathematical Society, Providence, RI, 2001, MR1787114 (2001f:58050), Zbl 0958.46038.

[22] Evgenij V. Troitskiĭ, *An exact formula for the index of an equivariant C^*-elliptic operator*, Trudy Mat. Inst. Steklov., **193** (1992), 178–182, MR1266008 (95c:58170), Zbl 0812.46067.

[23] Evgenij V. Troitskiĭ, *"Twice" equivariant C^*-index theorem and the index theorem for families*, Noncommutative geometry and operator K-theory, Acta Appl. Math., **68** (2001), 39–70, MR1865952 (2003h:58036), Zbl 1013.46059.

[24] N. E. Wegge-Olsen, *K-theory and C^*-algebras, A friendly approach*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993, MR1222415 (95c:46116), Zbl 0780.46038.

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