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Image partition regularity over the integers, rationals and reals

Neil Hindman and Dona Strauss

Abstract. There is only one reasonable definition of *kernel partition regularity* over any subsemigroup of the reals. On the other hand, there are several reasonable definitions of *image partition regularity*. We establish the exact relationships that can hold among these various notions for finite matrices and infinite matrices with rational entries. We also introduce some hybrid notions and describe their relationship to what is probably the major unsolved problem in kernel partition regularity, namely whether an infinite matrix which is kernel partition regular over $\mathbb{Q}$ must be kernel partition regular over $\mathbb{N}$.

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1. Introduction

Image partition regularity is one of the most important concepts of Ramsey Theory. Suppose that A is a finite or infinite matrix over $\mathbb{Q}$ in which there are only a finite number of nonzero entries in each row. A is said to be image partition regular over the set $\mathbb{N}$ of positive integers, if given any finite partition of $\mathbb{N}$, there is a vector $\vec{x}$, with entries in $\mathbb{N}$, such that $A\vec{x}$ is defined and all the entries of $A\vec{x}$ lie in the same cell of the partition.

The significance of this concept can be illustrated by considering some of the historically important theorems of Ramsey theory. For example, Schur's Theorem [16], which states that in any finite partition of $\mathbb{N}$, there is a cell containing integers x, y and $x + y$, is equivalent to the image partition regularity of the matrix

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$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$. Van der Waerden's Theorem [17], which states that, for any $l \in \mathbb{N}$ and any finite partition of $\mathbb{N}$, there is a cell containing an arithmetic progression of length l, is equivalent to the image partition regularity of the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & l-1 \end{pmatrix}$.

The Finite Sums Theorem [6], which states that, in any finite partition of $\mathbb{N}$ there is a cell which contains all the finite sums of distinct terms of some infinite sequence in $\mathbb{N}$, is equivalent to the statement that an infinite matrix is image partition regular if its entries are in $\{0, 1\}$, with only a finite number of nonzero entries in each row and no row identically zero.

See [5, Theorems 2.1 and 3.1] for proofs of van der Waerden's Theorem and Schur's Theorem. See [12, Corollary 5.10] for a proof of the Finite Sums Theorem.

In this paper we investigate image partition regularity of finite and infinite matrices over subsemigroups of $(\mathbb{R}, +)$. We represent countable infinity by the ordinal $\omega = \mathbb{N} \cup \{0\}$. For consistency of treatment between the finite and infinite cases, we shall treat $u \in \mathbb{N}$ as an ordinal, so that $u = \{0, 1, \ldots, u-1\}$. Thus, if $u, v \in \mathbb{N} \cup \{\omega\}$ and A is a $u \times v$ matrix, the rows and columns of A will be indexed by $u = \{i : i < u\}$ and $v = \{i : i < v\}$, respectively.

The concept of image partition regularity is closely related to that of kernel partition regularity. A matrix A over $\mathbb{Q}$ is said to be kernel partition regular over $\mathbb{N}$ if, in any finite partion of $\mathbb{N}$, there is a vector $\vec{x}$, whose entries all lie in the same cell of the partition, such that $A\vec{x} = \vec{0}$.

It is natural to consider the extensions of these concepts of partition regularity from $\mathbb{N}$ to more general subsemigroups of $(\mathbb{R}, +)$. As we shall explain, there is only one reasonable way to define kernel partition regularity over a subsemigroup of $\mathbb{R}$; but this statement is not true for image partition regularity.

Definition 1.1. A matrix A is *admissible* provided there exist $u, v \in \mathbb{N} \cup \{\omega\}$ such that A is a $u \times v$ matrix with entries from $\mathbb{Q}$ which has finitely many nonzero entries in each row.

Definition 1.2. Let S be a subsemigroup of $(\mathbb{R}, +)$, let T be the subgroup of $(\mathbb{R}, +)$ generated by S, let $u, v \in \mathbb{N} \cup \{\omega\}$ and let A be an admissible $u \times v$ matrix.

(a) A is *kernel partition regular over* S (KPR/S) if and only if whenever $S \setminus \{0\}$ is finitely colored there exists a monochromatic $\vec{x} \in (S \setminus \{0\})^v$ such that $A\vec{x} = \vec{0}$.
(b) A is *image partition regular over* S (IPR/S) if and only if whenever $S \setminus \{0\}$ is finitely colored, there exists $\vec{x} \in (S \setminus \{0\})^v$ such that the entries of $A\vec{x}$ are monochrome.
(c) A is *weakly image partition regular over* S (WIPR/S) if and only if whenever $S \setminus \{0\}$ is finitely colored, there exists $\vec{x} \in T^v \setminus \{\vec{0}\}$ such that the entries of $A\vec{x}$ are monochrome.

When defining kernel partition regularity of A over S, there is only one reasonable definition, namely the one given in Definition 1.1. Since the entries of $\vec{x}$ are to be monochrome, they must come from the set being colored. And if 0 were not excluded from the set being colored, one would allow the trivial solution $\vec{x} = \vec{0}$ and so all admissible matrices would be KPR$/S$. (One might argue for the requirement that S be colored and the entries of $\vec{x}$ should be monochrome and not constantly 0. But then, by assigning 0 to its own color, one sees that this is equivalent to the definition given.)

By contrast, when defining image partition regularity, there are several reasonable choices that can be made. If $0 \in S$, then one may color S or $S \setminus \{0\}$ and one may demand that one gets the entries of $A\vec{x}$ monochrome with $\vec{x} \in (S \setminus \{0\})^v$, $\vec{x} \in S^v \setminus \{\vec{0}\}$, $\vec{x} \in (T \setminus \{0\})^v$, or $\vec{x} \in T^v \setminus \{\vec{0}\}$. If $0 \notin S$ one may demand that one gets the entries of $A\vec{x}$ monochrome with $\vec{x} \in S^v$, $\vec{x} \in (T \setminus \{0\})^v$, or $\vec{x} \in T^v \setminus \{\vec{0}\}$. (We note that there is never a point in allowing $\vec{x} = \vec{0}$. If $S \setminus \{0\}$ is colored, then $\vec{x} = \vec{0}$ is impossible, and if $0 \in S$ and S is colored, then $\vec{x} = \vec{0}$ yields a trivial solution for any matrix.) Since these choices are all reasonable, it is natural to consider the relations between them.

In Section 2 we consider all of these reasonable choices for each of the subsemigroups $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}^+$, $\mathbb{Q}$, $\mathbb{R}^+$ and $\mathbb{R}$ of $\mathbb{R}$. (Here $\mathbb{Q}^+ = \{x \in \mathbb{Q} : x > 0\}$ and $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.) If S is $\mathbb{N}$, $\mathbb{Q}^+$, or $\mathbb{R}^+$, then $0 \notin S$ and $S \neq T$ so there are exactly three of these reasonable choices for S. If S is $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$, then $0 \in S$ and $S = T$ so there are exactly four of these reasonable choices for S. Thus, for these semgroups there are a total of 21 possible reasonable choices. Some of these are, however, equivalent. We show that there are a total of 15 distinct notions and establish the exact pattern of implications that hold among them.

In [15, Theorem VII], Rado established that for any subring R of $\mathbb{C}$, a finite matrix with coefficients from $\mathbb{C}$ is kernel partition regular over $R \setminus \{0\}$ if and only if it satisfies the columns condition over the field generated by R. We now give this condition.

Definition 1.3. Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, denote the columns of A by $\vec{c}_0, \vec{c}_1, \ldots, \vec{c}_{v-1}$, and let R be a subring of $(\mathbb{R}, +, \cdot)$. Then A satisfies the *columns condition over* R if and only if there is a partition $\{I_1, I_2, \ldots, I_m\}$ of $\{0, 1, \ldots, v-1\}$ such that:

(a) $\sum_{i \in I_1} \vec{c}_i = \vec{0}$.
(b) For each $t \in \{2, 3, \ldots, m\}$ (if any), $\sum_{i \in I_t} \vec{c}_i$ is a linear combination of members of $\bigcup_{i=1}^{t-1} I_i$ with coefficients from R.

It follows easily that, for a finite admissible matrix A, the statements that A is kernel partition regular over each of the subsemigroups $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}^+$, $\mathbb{Q}$, $\mathbb{R}^+$, and $\mathbb{R}$ of $\mathbb{R}$, are equivalent (see Theorem 1.4). However, this statement is not true for image partition regularity or weak image partition regularity.

Call a set $C \subseteq \mathbb{N}$ *large* provided that C contains a solution set for every partition regular finite system of homogeneous linear equations with coefficients from $\mathbb{Q}$. Rado's Theorem then easily implies that whenever $\mathbb{N}$ is partitioned into finitely many cells, one of those cells is large. Rado conjectured that whenever any large set is partitioned into finitely many cells, one of those cells must be large. This conjecture was proved by W. Deuber [2] whose proof utilized what Deuber called (m, p, c)*-sets*. These sets are images of certain image partition regular matrices. (See [11] for an algebraic proof of Deuber's result.)

Several characterizations of finite matrices that are image partition regular over $\mathbb{N}$ were found in [8], and one of these characterizations was in terms of the kernel partition regularity of a related matrix (and thus image partition regularity can be determined by means of the columns condition applied to this related matrix). Thus there is an intimate connection, in both directions, between kernel partition regular and image partition regular finite matrices.

The question of which infinite matrices are image partition regular or kernel partition regular is a difficult open problem, which we have addressed in previous papers. (See, for example, [3] and [13].) We shall not be specifically concerned with this question in this paper.

In Section 3 we investigate the relationship between kernel and image partition regularity for infinite matrices. We also introduce some additional "hybrid" notions of partition regularity. (For example "very weakly image partition regular" refers to coloring $\mathbb{N}$ and asking for $\vec{x} \in \mathbb{Q}^v \setminus \{\vec{0}\}$ with the entries of $A\vec{x}$ monochrome.) In these cases, the exact pattern of implications is not known, and the unanswered questions about them turn out to be intimately related to the main open problem about kernel partition regularity. That is, does KPR/$\mathbb{Q}$ imply KPR/$\mathbb{N}$?

We shall have need of the following result, which is well-known among afficianados.

Theorem 1.4. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent:*

(a) *A is* KPR/$\mathbb{N}$.
(b) *A is* KPR/$\mathbb{Z}$.
(c) *A is* KPR/$\mathbb{Q}^+$.
(d) *A is* KPR/$\mathbb{Q}$.
(e) *A is* KPR/$\mathbb{R}^+$.
(f) *A is* KPR/$\mathbb{R}$.

Proof. The implications in the following diagram are all trivial:

$$\begin{tikzcd}
 & & \mathrm{KPR}/\mathbb{N} \arrow[dl] \arrow[dr] & & \\
 & \mathrm{KPR}/\mathbb{Z} \arrow[dr] & & \mathrm{KPR}/\mathbb{Q}^+ \arrow[dl] \arrow[dr] & \\
 & & \mathrm{KPR}/\mathbb{Q} \arrow[dr] & & \mathrm{KPR}/\mathbb{R}^+ \arrow[dl] \\
 & & & \mathrm{KPR}/\mathbb{R}\,. &
\end{tikzcd}$$

We shall show that KPR/$\mathbb{R} \Rightarrow$ KPR/$\mathbb{N}$. So assume that A is KPR/$\mathbb{R}$. Then by [15, Theorem VII] A satisfies the columns condition over $\mathbb{R}$. But since a rational vector is in the linear span over $\mathbb{R}$ of a set of rational vectors if and only if it is in the linear span over $\mathbb{Q}$ of those same vectors, this tells us that A satisfies the columns condition over $\mathbb{Q}$. But then, by the original version of Rado's Theorem ([14, Satz IV], or see [5, Theorem 3.5] or [12, Theorem 15.20]) A is KPR/$\mathbb{N}$. $\square$

We also shall need the following deep result of Baumgartner and Hajnal. We denote by $[A]^k$ the set of k-element subsets of A.

Theorem 1.5. *Let A be a linearly ordered set with the property that whenever $\varphi : A \to \omega$, there is an infinite increasing sequence in A on which φ is constant. Then for any $k < \omega$, any countable ordinal α, and any $\psi : [A]^2 \to \{0, 1, \ldots, k\}$ there is a subset B of A which has order type α such that ψ is constant on $[B]^2$.*

Proof. This is [1, Theorem 1], where it was proved using Martin's Axiom and then shown by absoluteness considerations to be a theorem of ZFC. A direct combinatorial proof was obtained by Galvin [4, Theorem 4]. □

We shall only need the following very special case. It is an indication of the strength of Theorem 1.5 that even this special case does not seem to be easy to prove.

Corollary 1.6. *Let $[\mathbb{R}^+]^2$ be finitely colored. There is a set $B \subseteq \mathbb{R}^+$ of order type $\omega + 1$ such that $[B]^2$ is monochrome.*

Proof. To see that $\mathbb{R}^+$ satisfies the hypotheses of Theorem 1.5, note that by the Baire Category Theorem, when $\mathbb{R}^+$ is colored with countably many colors, the closure of one of the color classes has nonempty interior. □

We mention two conventions that we will use throughout. The entries of a matrix will be denoted by lower case letters corresponding to the upper case letter which denotes the matrix. Also, we shall use the notation $\vec{x}$ for both column and row vectors, expecting the reader to rely on the context to determine which is intended.

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2. Image partition regularity over $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}^+$, $\mathbb{Q}$, $\mathbb{R}^+$ and $\mathbb{R}$

Let S be a subsemigroup of $(\mathbb{R}, +)$ and let T be the group generated by S. Let $u, v \in \mathbb{N} \cup \{\omega\}$, and let A be an admissible $u \times v$ matrix. As we observed in the introduction, when defining image partition regularity, there are several reasonable choices that can be made. One may color S or $S \setminus \{0\}$ and one may demand that one gets the entries of $A\vec{x}$ monochrome with $\vec{x} \in (S \setminus \{0\})^v$, $\vec{x} \in S^v \setminus \{\vec{0}\}$, $\vec{x} \in (T \setminus \{0\})^v$, or $\vec{x} \in T^v \setminus \{\vec{0}\}$. We show in this section that there are exactly fifteen distinct nontrivial notions arising from these choices for the semigroups $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}^+$, $\mathbb{Q}$, $\mathbb{R}^+$ and $\mathbb{R}$. We also establish the exact patterns of implications among these notions. We first need to define two additional notions.

Definition 2.1. Let $u, v \in \mathbb{N} \cup \{\omega\}$ and let A be an admissible $u \times v$ matrix.

(a) A satisfies the statement θ_S if, whenever S is finitely colored, there exists $\vec{x} \in (S \setminus \{0\})^v$ such that $A\vec{x}$ is monochrome.
(b) A satisfies the statement ψ_S if, whenever S is finitely colored, there exists $\vec{x} \in S^v \setminus \{\vec{0}\}$ such that $A\vec{x}$ is monochrome.

The fifteen notions that we shall investigate are the notions IPR/S for $S \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Q}^+, \mathbb{Q}, \mathbb{R}^+, \mathbb{R}\}$, and the notions WIPR/$S$, θ_S and ψ_S for $S \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$.

Theorem 2.2. *Theorem. Let S be one of $\mathbb{N}$, $\mathbb{Q}^+$, $\mathbb{R}^+$, $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$ and let T be the subgroup of $(\mathbb{R}, +)$ generated by S. Let $u, v \in \mathbb{N} \cup \{\omega\}$ and let A be an admissible $u \times v$ matrix. Let $B \in \big\{S, S \setminus \{0\}\big\}$ and let $C \in \big\{(S \setminus \{0\})^v, (T \setminus \{0\})^v, S^v \setminus \{\vec{0}\}, T^v \setminus \{\vec{0}\}\big\}$. Define the property $(*)$ by:*

(∗) *Whenever B is finitely colored, there exists $\vec{x} \in C$ such that the entries of $A\vec{x}$ are monochrome.*

Then (∗) *is equivalent to one of the fifteen notions described above. In particular,* WIPR/$\mathbb{Z} \Leftrightarrow$ WIPR/$\mathbb{N}$*,* WIPR/$\mathbb{Q} \Leftrightarrow$ WIPR/$\mathbb{Q}^+$ *and* WIPR/$\mathbb{R} \Leftrightarrow$ WIPR/$\mathbb{R}^+$*.*

Proof. Notice that if $S = \mathbb{N}$, $S = \mathbb{Q}^+$, or $S = \mathbb{R}^+$, then $S = S \setminus \{0\}$ and $(S \setminus \{0\})^v = S^v \setminus \{\vec{0}\}$. Also if $S = \mathbb{Z}$, $S = \mathbb{Q}$, or $S = \mathbb{R}$, then $S = T$. Thus, in addition to our fifteen notions, we have the following possibilities to consider:

	S	B	C
(16)	$\mathbb{N}$	$\mathbb{N}$	$(\mathbb{Z} \setminus \{0\})^v$
(17)	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{Z}^v \setminus \{\vec{0}\}$
(18)	$\mathbb{Q}^+$	$\mathbb{Q}^+$	$(\mathbb{Q} \setminus \{0\})^v$
(19)	$\mathbb{Q}^+$	$\mathbb{Q}^+$	$\mathbb{Q}^v \setminus \{\vec{0}\}$
(20)	$\mathbb{R}^+$	$\mathbb{R}^+$	$(\mathbb{R} \setminus \{0\})^v$
(21)	$\mathbb{R}^+$	$\mathbb{R}^+$	$\mathbb{R}^v \setminus \{\vec{0}\}$.

Notice that (17), (19) and (21) are WIPR/$\mathbb{N}$, WIPR/$\mathbb{Q}^+$ and $WIPR/\mathbb{R}^+$, respectively. We claim that (16) $\Leftrightarrow$ IPR/$\mathbb{Z}$, (17) $\Leftrightarrow$ WIPR/$\mathbb{Z}$, (18) $\Leftrightarrow$ IPR/$\mathbb{Q}$, (19) $\Leftrightarrow$ WIPR/$\mathbb{Q}$, (20) $\Leftrightarrow$ IPR/$\mathbb{R}$ and (21) $\Leftrightarrow$ WIPR/$\mathbb{R}$. The proofs of these equivalences are essentially identical. We shall write out the proof that (16) $\Leftrightarrow$ IPR/$\mathbb{Z}$.

Trivially (16) implies IPR/$\mathbb{Z}$. To see that IPR/$\mathbb{Z}$ implies (16), let $r \in \mathbb{N}$ and let $\varphi : \mathbb{N} \to \{1, 2, \ldots, r\}$. Define $\psi : \mathbb{Z} \setminus \{0\} \to \{1, 2, \ldots, 2r\}$ by

$$\psi(x) = \begin{cases} \varphi(x) & \text{if } x > 0 \\ r + \varphi(-x) & \text{if } x < 0\,. \end{cases}$$

Pick $\vec{x} \in (\mathbb{Z} \setminus \{0\})^v$ and $j \in \{1, 2, \ldots, 2r\}$ such that $A\vec{x} \in (\psi^{-1}[\{j\}])^u$. If $j \leq r$, let $\vec{y} = \vec{x}$ and let $i = j$. If $j > r$, let $\vec{y} = -\vec{x}$ and let $i = j - r$. Then $A\vec{y} \in (\varphi^{-1}[\{i\}])^u$. □

We show in the following lemma that, for $S \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, the properties θ_S and ψ_S are simply described in terms of the properties IPR/S and WIPR/S.

Lemma 2.3. *Let $S \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. Let $u, v \in \mathbb{N} \cup \{\omega\}$ and let A be a $u \times v$ admissible matrix.*

(a) *A satisfies property θ_S of Definition* 2.1 *if and only if either A is* IPR/S *or there exists $\vec{x} \in (S \setminus \{0\})^v$ such that $A\vec{x} = \vec{0}$.*

(b) *A satisfies property ψ_S of Definition* 2.1 *if and only if either A is* WIPR/S *or there exists $\vec{x} \in S^v \setminus \{\vec{0}\}$ such that $A\vec{x} = \vec{0}$.*

Proof. In each case the sufficiency is trivial. For the necessity, given an r-coloring of $S \setminus \{0\}$ define an $(r+1)$-coloring of S by assigning 0 to its own color. □

If the matrix is finite, the fifteen properties collapse to four, as we shall see in the following theorem. The proof that (I)(c) $\Rightarrow$ (I)(a) uses the algebraic structure of the Stone–Čech compactification of a discrete semigroup. (By $\mathbb{R}_d^+$ we mean $\mathbb{R}^+$ with the discrete topology.) The reader is referred to [12] for background material on this structure.

Theorem 2.4. *Let $u, v \in \mathbb{N}$ and let A be an admissible $u \times v$ matrix.*

(I) *The following are equivalent:*
  (a) *A is* IPR/$\mathbb{N}$.
  (b) *A is* IPR/$\mathbb{Q}^+$.
  (c) *A is* IPR/$\mathbb{R}^+$.

(II) *The following are equivalent:*
  (a) *A is* IPR/$\mathbb{Z}$.
  (b) *A is* IPR/$\mathbb{Q}$.
  (c) *A is* IPR/$\mathbb{R}$.
  (d) *A is* WIPR/$\mathbb{Z}$.
  (e) *A is* WIPR/$\mathbb{Q}$.
  (f) *A is* WIPR/$\mathbb{R}$.

(III) *The following are equivalent:*
  (a) *A satisfies property $\theta_{\mathbb{Z}}$ of Definition* 2.1.
  (b) *A satisfies property $\theta_{\mathbb{Q}}$ of Definition* 2.1.
  (c) *A satisfies property $\theta_{\mathbb{R}}$ of Definition* 2.1.

(IV) *The following are equivalent:*
  (a) *A satisfies property $\psi_{\mathbb{Z}}$ of Definition* 2.1.
  (b) *A satisfies property $\psi_{\mathbb{Q}}$ of Definition* 2.1.
  (c) *A satisfies property $\psi_{\mathbb{R}}$ of Definition* 2.1.

Proof. (I) We show that IPR/$\mathbb{R}^+ \Rightarrow$ IPR/$\mathbb{N}$. Assume that A is IPR/$\mathbb{R}^+$. If k is a common multiple of the denominators of entries of A, then kA is also IPR/$\mathbb{R}^+$ and, if kA is IPR/$\mathbb{N}$ then A is IPR/$\mathbb{N}$. Thus we may assume that the entries of A are integers.

Define $\varphi : (\mathbb{R}^+)^v \to \mathbb{R}^u$ by $\varphi(\vec{x}) = A\vec{x}$ and let $\widetilde{\varphi} : \beta\big((\mathbb{R}_d^+)^v\big) \to (\beta\mathbb{R}_d)^u$ be its continuous extension. Let p be an idempotent in the smallest ideal $K(\beta\mathbb{R}_d^+)$ of $\beta\mathbb{R}_d^+$ and let

$$\overline{p} = \begin{pmatrix} p \\ p \\ \vdots \\ p \end{pmatrix} \in (\beta\mathbb{R}_d)^u \,.$$

Pick by [7, Lemma 2.8 and Theorem 4.1] an idempotent $q \in K\big(\beta\big((\mathbb{R}^+)^v\big)\big)$ such that $\widetilde{\varphi}(q) = \overline{p}$. Notice that $[1,\infty)^v$ is an ideal of $\big((\mathbb{R}^+)^v,+\big)$ so by [12, Theorem 4.17] $c\ell\big([1,\infty)^v\big)$ is an ideal of $\beta\big((\mathbb{R}_d^+)^v\big)$ and so $[1,\infty)^v \in q$.

Let $r \in \mathbb{N}$ and let $\psi : \mathbb{N} \to \{1, 2, \ldots, r\}$. Define $g : \mathbb{R}^+ \to \mathbb{N}$ by

$$g(x) = \begin{cases} \lfloor x + \frac{1}{2} \rfloor & \text{if } x \geq \frac{1}{2} \\ 1 & \text{if } 0 < x < \frac{1}{2} \,. \end{cases}$$

Then $\psi \circ g : \mathbb{R}^+ \to \{1, 2, \ldots, r\}$ so pick $l \in \{1, 2, \ldots, r\}$ such that $(\psi \circ g)^{-1}[\{l\}] \in p$. Let $B = [1,\infty) \cap (\psi \circ g)^{-1}[\{l\}]$. Then $B \in p$ and so $\varphi^{-1}[B^u] \in q$.

Define $\tau : (\mathbb{R}^+)^v \to (\mathbb{R}/\mathbb{Z})^v$ by $\tau(\vec{x})_j = \mathbb{Z} + x_j$ and let $\widetilde{\tau} : \beta\big((\mathbb{R}^+)^v\big) \to (\mathbb{R}/\mathbb{Z})^v$ be its continuous extension. By [12, Corollary 4.22] $\widetilde{\tau}$ is a homomorphism so $\widetilde{\tau}(q)$ is an idempotent, and thus $\widetilde{\tau}(q)_j = \mathbb{Z} + 0$ for each $j \in \{0, 1, \ldots, v-1\}$. There exists $\delta > 0$ such that the entries of $A\vec{x}$ are contained in $(-\frac{1}{2}, \frac{1}{2})$ whenever the entries of $\vec{x}$ are contained in $(-\delta, \delta)$. Let $U = \times_{j=0}^{v-1}\{\mathbb{Z} + x : -\delta < x < \delta\}$. Then U is a neighborhood of $\widetilde{\tau}(q)$ so $\tau^{-1}[U] \in q$. Pick $\vec{x} \in \tau^{-1}[U] \cap \varphi^{-1}[B^u] \cap [1,\infty)^v$.

Let $y_j = g(x_j)$ for each $j \in \{0, 1, \ldots, v-1\}$. Then $\vec{y} \in \mathbb{N}^v$ and for each j, $y_j = \lfloor x_j + \frac{1}{2} \rfloor$. Let $\vec{w} = A\vec{y}$. We claim that $\vec{w} \in (\psi^{-1}[\{l\}])^u$. Let $\vec{z} = \varphi(\vec{x}) = A\vec{x}$. Then $\vec{z} \in B^u \subseteq \big((\psi \circ g)^{-1}[\{l\}]\big)^u$. Thus it suffices to show that for each $i \in \{0, 1, \ldots, u-1\}$, $w_i = g(z_i)$, so let $i \in \{0, 1, \ldots, u-1\}$. Since $\vec{x} \in \tau^{-1}[U]$, for each $j \in \{0, 1, \ldots, v-1\}$, $x_j = g(x_j) + \gamma_j$ for some $\gamma_j \in (-\delta, \delta)$. So $z_i = \sum_{j=0}^{v-1} a_{i,j} \cdot x_j = \sum_{j=0}^{v-1} a_{i,j} \cdot y_j + \sum_{j=0}^{v-1} a_{i,j} \cdot \gamma_j = w_i + \sum_{j=0}^{v-1} a_{i,j} \cdot \gamma_j$. Since $|\sum_{j=0}^{v-1} a_{i,j} \cdot \gamma_j| < \frac{1}{2}$, we have that $g(z_i) = w_i$ as required.

(II) We show that WIPR/$\mathbb{R}$ $\Rightarrow$ IPR/$\mathbb{Z}$. Assume that A is WIPR/$\mathbb{R}$. Let $l = \text{rank}(A)$. Rearrange the rows of A so that the first l rows are linearly independent over $\mathbb{Q}$ (and therefore are linearly independent over $\mathbb{R}$ because finding $\alpha_0, \alpha_1, \ldots, \alpha_{l-1}$ such that $\sum_{i=0}^{l-1} \alpha_i \vec{r_i} = \vec{0}$ amounts to solving linear equations with rational coefficients). Let $\vec{r_0}, \vec{r_1}, \ldots, \vec{r_{u-1}}$ be the rows of A. For each $t \in \{l, l+1, \ldots, u-1\}$, if any, let $\gamma_{t,0}, \gamma_{t,1}, \ldots, \gamma_{t,l-1} \in \mathbb{Q}$ be determined by $\vec{r_t} = \sum_{i=0}^{l-1} \gamma_{t,i} \cdot \vec{r_i}$. If $u > l$, let D be the $(u-l) \times v$ matrix such that, for $t \in \{0, 1, \ldots, u-l-1\}$ and $i \in \{0, 1, \ldots, u-1\}$,

$$d_{t,i} = \begin{cases} \gamma_{l+t,i} & \text{if } i < l \\ -1 & \text{if } i = l+t \\ 0 & \text{otherwise.} \end{cases}$$

Then by [7, Theorem 3.1], $l = u$ or D is KPR/$\mathbb{R}$. Thus by Theorem 1.4 either $l = u$ or D is KPR/$\mathbb{Q}$ and thus by [8, Theorem 2.2] we may pick $b_0, b_1, \ldots, b_{v-1} \in \mathbb{Q} \setminus \{0\}$ such that the matrix

$$B = \begin{pmatrix} b_0 & 0 & \ldots & 0 \\ 0 & b_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_{v-1} \\ & & & \\ & & A & \end{pmatrix}$$

is WIPR/$\mathbb{Z}$. Now let $\mathbb{Z} \setminus \{0\}$ be finitely colored and pick $\vec{x} \in \mathbb{Z}^v \setminus \{\vec{0}\}$ such that the entries of $B\vec{x}$ are monochrome (and in particular the entries of $A\vec{x}$ are monochrome). Since each $b_i x_i \neq 0$ we have that $\vec{x} \in (\mathbb{Z} \setminus \{0\})^v$.

The equivalence of the statements in (III) and the equivalence of the statements in (IV) now follow from Lemma 2.3 and the fact that, if $A\vec{x} = \vec{0}$ for some $\vec{x} \in \mathbb{R}^v$, then $A\vec{r} = \vec{0}$ for some $\vec{r} \in \mathbb{Q}^v$, with the property that, for each $i \in \{0, 1, 2, \cdots, v-1\}$, $x_i = 0$ if and only if $r_i = 0$. See [9, Lemma 2.5] for a proof of this elementary fact. $\square$

Theorem 2.5. *The collections* (I), (II), (III) *and* (IV) *of equivalent properties in Theorem* 2.4 *are listed in strictly decreasing order of strength.*

Proof. It is trivial that collections (I), (II) and (III) imply collections (II), (III) and (IV) respectively.

The matrix $\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 4 & 6 \end{pmatrix}$ was shown in [8, pages 461–462] to be WIPR/$\mathbb{Z}$ but not IPR/$\mathbb{N}$, so (II) $\not\Rightarrow$ (I).

To see that (III) $\not\Rightarrow$ (II), consider the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. Then $A\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so A satisfies $\theta_{\mathbb{Z}}$. Any 2-coloring of $\mathbb{Z}$ for which one never has a nonzero x with x and $2x$ the same color establishes that A is not IPR/$\mathbb{Z}$.

To see that (IV) $\not\Rightarrow$ (III), consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 6 & 4 \\ 2 & 4 & 2 \end{pmatrix}. \text{ Then } A\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so A satsifies $\psi_{\mathbb{Z}}$. If $\vec{x} \in \mathbb{Z}^3$ and $A\vec{x} = \vec{0}$, then $x_2 = 0$. Thus by Lemma 2.3(a), to see that A does not satisfy $\theta_{\mathbb{Z}}$, it suffices to show that A is not IPR/$\mathbb{Z}$. To this end, color $\mathbb{Z} \setminus \{0\}$ with two colors so that there is no $x \in \mathbb{Z} \setminus \{0\}$ such that x and $2x$ have the same color. Given any $\vec{x} \in \mathbb{Z}^3$, if $\vec{y} = A\vec{x}$, then $y_2 = 2y_0$. $\square$

The situation is more complicated when infinite matrices are allowed.

Theorem 2.6. *Consider the following diagram among the properties of Definition* 2.1:

$$\begin{tikzcd}
 & & \mathrm{IPR}/\mathbb{N} \arrow[dl] \arrow[dr] & & \\
 & \mathrm{IPR}/\mathbb{Z} \arrow[dl] \arrow[d] \arrow[dr] & & \mathrm{IPR}/\mathbb{Q}^+ \arrow[dl] \arrow[dr] & \\
\mathrm{WIPR}/\mathbb{Z} \arrow[d] \arrow[dr] & \theta_{\mathbb{Z}} \arrow[dl] \arrow[dr] & \mathrm{IPR}/\mathbb{Q} \arrow[dl] \arrow[d] \arrow[dr] & & \mathrm{IPR}/\mathbb{R}^+ \arrow[dl] \\
\psi_{\mathbb{Z}} \arrow[dr] & \mathrm{WIPR}/\mathbb{Q} \arrow[d] \arrow[dr] & \theta_{\mathbb{Q}} \arrow[dl] \arrow[dr] & \mathrm{IPR}/\mathbb{R} \arrow[dl] \arrow[d] & \\
 & \psi_{\mathbb{Q}} \arrow[dr] & \mathrm{WIPR}/\mathbb{R} \arrow[d] & \theta_{\mathbb{R}} \arrow[dl] & \\
 & & \psi_{\mathbb{R}}. & &
\end{tikzcd}$$

Each of the diagramed implications is valid, and no implication among these notions holds in general unless it is forced to hold by the diagramed implications and transitivity.

Proof. Each of the listed implications is trivial. To establish that none of the missing implications is valid, it suffices to show that:

(A) IPR/$\mathbb{Z} \not\Rightarrow$ IPR/$\mathbb{R}^+$.
(B) IPR/$\mathbb{Q}^+ \not\Rightarrow \psi_{\mathbb{Z}}$.
(C) WIPR/$\mathbb{Z} \not\Rightarrow \theta_{\mathbb{R}}$.
(D) IPR/$\mathbb{R}^+ \not\Rightarrow \psi_{\mathbb{Q}}$.
(E) $\theta_{\mathbb{Z}} \not\Rightarrow$ WIPR/$\mathbb{R}$.

By Theorem 2.5 there are finite matrices establishing (A) and (E).

To see that IPR/$\mathbb{Q}^+ \not\Rightarrow \psi_{\mathbb{Z}}$, consider

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1/2 & 1 & 0 & 0 & \dots \\ 1/3 & 0 & 1 & 0 & \dots \\ 1/4 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and let } \vec{z} = \begin{pmatrix} 1 \\ 1/2 \\ 2/3 \\ 3/4 \\ \vdots \end{pmatrix}.$$

Then $A\vec{z} = \vec{1}$ so A is IPR/$\mathbb{Q}^+$.

To see that A does not satisfy $\psi_{\mathbb{Z}}$, observe first that if $\vec{x} \neq \vec{0}$, then $A\vec{x} \neq \vec{0}$. (If $x_0 \neq 0$, the first entry of $A\vec{x}$ is x_0 while if $x_0 = 0$ and $x_n \neq 0$, then entry n of $A\vec{x}$ is x_n.) Thus by Lemma 2.3(b) it suffices to show that A is not WIPR/$\mathbb{Z}$. To this end, let $\vec{x} \in \mathbb{Z}^\omega$. If $x_0 = 0$, then the first entry of $A\vec{x}$ is 0. Otherwise there is some $n \in \mathbb{N}$ such that $x_0/n \notin \mathbb{Z}$ so the entry n of $A\vec{x}$, namely $x_0/n + x_n$, is not in $\mathbb{Z}$. Therefor e $A\vec{x} \notin (\mathbb{Z} \cup \{0\})^\omega$.

To see that WIPR/$\mathbb{Z} \not\Rightarrow \theta_{\mathbb{R}}$, let

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ \vdots & \vdots \end{pmatrix}.$$

Then $B\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{1}$ so B is WIPR/$\mathbb{Z}$. To see that B does not satisfy $\theta_{\mathbb{R}}$, two color $\mathbb{R}$ so that there are no monochrome infinite arithmetic progressions.

Finally we show that IPR/$\mathbb{R}^+ \not\Rightarrow \psi_{\mathbb{Q}}$. Let $D = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & 0 & \dots \\ 0 & 0 & 1 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ and let $C = \begin{pmatrix} I \\ D \end{pmatrix}$, where I is the $\omega \times \omega$ identity matrix. So for $\vec{x} \in \mathbb{R}^\omega$, the entries of $C\vec{x}$ consist of $\{x_n : n < \omega\} \cup \{x_n - x_{n+1} : n < \omega\}$.

To see that C is IPR/$\mathbb{R}^+$, we choose a finite coloring of $\mathbb{R}^+$ and color $[\mathbb{R}^+]^2$ by giving $\{x, y\}$ the color of $|x - y|$. By Corollary 1.6, there is an increasing $\omega + 1$ sequence $\langle y_i \rangle_{i<\omega+1}$ in $\mathbb{R}^+$ such that $\{y_j - y_i : i < j < \omega + 1\}$ is monochrome. If $x_i = y_\omega - y_i$ for every $i < \omega$, $A\vec{x}$ is monochrome.

To see that C does not satisfy $\psi_{\mathbb{Q}}$, we use an argument due to I. Leader. Suppose that C does satisfy $\psi_{\mathbb{Q}}$. There is no $\vec{x} \in \mathbb{Q}^\omega \setminus \{\vec{0}\}$ such that $C\vec{x} = \vec{0}$ so by Lemma 2.3, C is WIPR/$\mathbb{Q}$ and thus, by Theorem 2.2, C is WIPR/$\mathbb{Q}^+$. However, if $\vec{x} \in \mathbb{Q}^\omega$ and the entries of $C\vec{x}$ are in $\mathbb{Q}^+$, then in fact $\vec{x} \in (\mathbb{Q}^+)^\omega$, so C is IPR/$\mathbb{Q}^+$.

Given a positive rational x, write x as

$$x = \sum_{t=1}^{n(x)} b(x,t)/t!$$

where for each t, $b(x,t) \in \omega$, $b(x,t) < t$ if $t > 1$ and $b\big(x, n(x)\big) \neq 0$. Color $\mathbb{Q}^+$ according to the value of $n(x)$ mod 3, the parity of $b\big(x, n(x)\big)$ and the parity of $b(x, n(x) - 2)$.

Choose $\vec{x} \in (\mathbb{Q}^+)^\omega$ for which $C\vec{x}$ is monochrome. This implies that $\langle x_i \rangle_{i<\omega}$ is a strictly decreasing sequence in $\mathbb{Q}^+$. We can therefore choose $i \in \omega$ for which $n(x_{i+1}) > n(x_i)$. We have $x_i = x_{i+1} + y$, where $\{x_i, x_{i+1}, y\}$ is monochrome. In particular, $n(x_{i+1}) \geq n(x_i) + 3$ consequently, one must have carrying in the rightmost three positions when x_{i+1} and y are added so that $n = n(y)$ and

$$\begin{aligned} b(x_{i+1}, n) + b(y, n) &= n \\ 1 + b(x_{i+1}, n-1) + b(y, n-1) &= n - 1, \text{ and} \\ 1 + b(x_{i+1}, n-2) + b(y, n-2) &= n - 2\,. \end{aligned}$$

The first of these equations implies that that n is even and the third implies that n is odd, a contradiction. □

Notice that in particular we have the following simple pattern of implications among the named notions:

$$\begin{tikzcd}
 & & \mathrm{IPR}/\mathbb{N} \arrow[dl] \arrow[dr] & & \\
 & \mathrm{IPR}/\mathbb{Z} \arrow[dl] \arrow[dr] & & \mathrm{IPR}/\mathbb{Q}^+ \arrow[dl] \arrow[dr] & \\
\mathrm{WIPR}/\mathbb{Z} \arrow[dr] & & \mathrm{IPR}/\mathbb{Q} \arrow[dl] \arrow[dr] & & \mathrm{IPR}/\mathbb{R}^+ \arrow[dl] \\
 & \mathrm{WIPR}/\mathbb{Q} \arrow[dr] & & \mathrm{IPR}/\mathbb{R} \arrow[dl] & \\
 & & \mathrm{WIPR}/\mathbb{R}. & &
\end{tikzcd}$$

3. Connections between image and kernel partition regularity

As we remarked in the introduction, there is an intimate relationship between the notions of image and kernel partition regularity for finite matrices. We shall see in this section that some of that relationship carries over to infinite matrices. The following auxiliary notions provide a connection between image and kernel partition regularity.

Definition 3.1. Let $u, v \in \mathbb{N} \cup \{\omega\}$, let A be an admissible $u \times v$ matrix, and denote the rows of A by $\{\vec{r}_i : i < u\}$. Choose a subset $I(A)$ of u which is maximal with respect to the property that $\{\vec{r}_i : i \in I(A)\}$ is linearly independent (and $\vec{r}_i \neq \vec{r}_j$ for $i \neq j$ in $I(A)$). Let $J(A) = u \setminus I(A)$. For each $i \in J(A)$, if any, let $\langle \gamma_{i,t} \rangle_{t \in I(A)}$ be the members of $\mathbb{Q}$ for which $\vec{r}_i = \sum_{t \in I(A)} \gamma_{i,t} \cdot \vec{r}_t$. If $J(A) \neq \emptyset$, let $B(A)$ be the matrix with rows indexed by $J(A)$ and columns indexed by u such that for $i \in J(A)$ and $t \in u$,

$$b_{i,t} = \begin{cases} \gamma_{i,t} & \text{if } t \in I(A) \\ -1 & \text{if } i = t \\ 0 & \text{if } t \in J(A) \setminus \{i\}\,. \end{cases}$$

To be definite, in our examples we shall suppose that $I(A)$ is chosen inductively by always taking the first row which is not a linear combination of those previously chosen.

Lemma 3.2. *Let S be any subsemigroup of $(\mathbb{R},+)$, let $u,v\in\mathbb{N}\cup\{\omega\}$, and let A be an admissible $u\times v$ matrix. If A is* WIPR/S*, then either $J(A)=\emptyset$ or $B(A)$ is* KPR/S*.*

Proof. Let T be the subgroup of $(\mathbb{R},+)$ generated by S. Assume that $J(A)\neq\emptyset$ and let $S\setminus\{0\}$ be finitely colored. Pick $\vec{x}\in T^v\setminus\{\vec{0}\}$ such that the entries of $\vec{w}=A\vec{x}$ are monochrome. Let $B=B(A)$. We show that $B\vec{w}=\vec{0}$. Since for each $i\in J(A)$, $\vec{r}_i=\sum_{t\in I(A)}\gamma_{i,t}\cdot\vec{r}_t$, one has that for each $i\in J(A)$ and each $j<v$, $a_{i,j}=\sum_{t\in I(A)}\gamma_{i,t}\cdot a_{t,j}$. Also for each $t<u$, $w_t=\sum_{j<v}a_{t,j}\cdot x_j$. Now let $i\in J(A)$. Then

$$\begin{aligned}\sum_{t<u}b_{i,t}\cdot w_t&=\sum_{t\in I(A)}\gamma_{i,t}\cdot w_t-w_i\\&=\sum_{t\in I(A)}\gamma_{i,t}\cdot\sum_{j<v}a_{t,j}\cdot x_j-\sum_{j<v}a_{i,j}\cdot x_j\\&=\sum_{j<v}x_j\cdot\left(\sum_{t\in I(A)}\gamma_{i,t}\cdot a_{t,j}-a_{i,j}\right)=0.\end{aligned}$$ □

For finite matrices we obtain characterizations of the properties in group (II) of Theorem 2.4.

Theorem 3.3. *Let $u,v\in\mathbb{N}$ and let A be a $u\times v$ admissible matrix. The following statements are equivalent:*

(a) *A is* IPR/$\mathbb{Z}$.
(b) *A is* WIPR/$\mathbb{Z}$.
(c) *A is* IPR/$\mathbb{Q}$.
(d) *A is* WIPR/$\mathbb{Q}$.
(e) *A is* IPR/$\mathbb{R}$.
(f) *A is* WIPR/$\mathbb{R}$.
(g) *$J(A)=\emptyset$ or $B(A)$ is* KPR/$\mathbb{N}$.
(h) *$J(A)=\emptyset$ or $B(A)$ is* KPR/$\mathbb{Z}$.
(i) *$J(A)=\emptyset$ or $B(A)$ is* KPR/$\mathbb{Q}^+$.
(j) *$J(A)=\emptyset$ or $B(A)$ is* KPR/$\mathbb{Q}$.
(k) *$J(A)=\emptyset$ or $B(A)$ is* KPR/$\mathbb{R}^+$.
(l) *$J(A)=\emptyset$ or $B(A)$ is* KPR/$\mathbb{R}$.

Proof. Statements (a) through (f) are equivalent by Theorem 2.4, statements (g) through (l) are equivalent by Theorem 1.4, and by [8, Theorem 2.2], statement (g) is equivalent to the assertion that A is WIPR/$\mathbb{Z}$. □

When infinite matrices are allowed, the correspondence between IPR/S and KPR/S given in Theorem 3.3 completely disappears.

Theorem 3.4. *There is an admissible $\omega\times 2$ matrix A such that $B(A)$ is* KPR/$\mathbb{N}$ *(and thus* KPR/$\mathbb{Z}$, KPR/$\mathbb{Q}^+$, KPR/$\mathbb{Q}$, KPR/$\mathbb{R}^+$ *and* KPR/$\mathbb{R}$*) but A is not* IPR/$\mathbb{R}$ *(and thus not* IPR/$\mathbb{R}^+$, IPR/$\mathbb{Q}$, IPR/$\mathbb{Q}^+$, IPR/$\mathbb{Z}$, *or* IPR/$\mathbb{N}$*).*

Proof. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ \vdots & \vdots \end{pmatrix}.$$

We saw in the proof of Theorem 2.6 that A does not satisfy $\theta_{\mathbb{R}}$ and so is not IPR/$\mathbb{R}$. We have $I(A) = \{0, 1\}$ and for $j \geq 2$, $\vec{r}_j = (1-j) \cdot \vec{r}_0 + j \cdot \vec{r}_1$ so

$$B(A) = \begin{pmatrix} -1 & 2 & -1 & 0 & 0 & \dots \\ -2 & 3 & 0 & -1 & 0 & \dots \\ -3 & 4 & 0 & 0 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since $B(A) \cdot \vec{1} = \vec{0}$, we have that $B(A)$ is KPR/$\mathbb{N}$. □

On the other hand, we shall see in Theorem 3.6 that the correspondence between WIPR/$\mathbb{Q}$ and KPR/$\mathbb{Q}$ given in Theorem 3.3 remains.

Lemma 3.5. *Let $u, v \in \mathbb{N} \cup \{\omega\}$ and let A be a $u \times v$ admissible matrix whose rows are linearly independent. For each $\vec{w} \in \mathbb{Q}^u$ there exists $\vec{x} \in \mathbb{Q}^v$ such that $A\vec{x} = \vec{w}$.*

Proof. Note that $u \leq v$ and, if u is finite, then there are only finitely many nonzero columns in A so one may presume that v is finite. If A is finite, the conclusion is part of any introductory linear algebra course. So we may presume that $u = v = \omega$.

We carry out Gaussian row operations on A, using the last nonzero entry in each row as a pivot and subtracting multiples of each row in turn from succeeding rows, so as to reduce to zero all the entries which lie below the pivot entry in the same column. We then carry out Gaussian column operations, again using the last nonzero entry in each row as a pivot and subtracting multiples of the column in which it occurs from preceding columns, so as to obtain a matrix B which has exactly one nonzero entry in each row, with the nonzero entries of different rows occurring in different columns. Then B clearly satisfies the conclusion of the lemma.

We have $B = EAF$, where E is obtained from the identity $\omega \times \omega$ matrix by carrying out the row operations applied, and F is obtained from the identity matrix by carrying out the column operations applied. We observe that E and F are admissible matrices, being lower triangular, with all diagonal entries equal to 1.

We claim that, for any $\omega \times \omega$ admissible matrix C and any $\vec{x} \in \mathbb{Q}^v$, if $EC\vec{x} = \vec{0}$, then $C\vec{x} = \vec{0}$. To see this, let $\vec{r_i}$ denote the i^{th} row of C and let $\vec{s_i}$ denote the i^{th} row of EC. Then $\vec{s}_0 = \vec{r}_0$ and, for each $i > 0$, $\vec{s_i} = \vec{r_i} + \sum_{k=0}^{i-1} e_{i,k} \vec{r_k}$. Assume that $\vec{x} \in \mathbb{Q}^v$ and $EC\vec{x} = \vec{0}$. Then $\vec{r_0}\vec{x} = \vec{s_0}\vec{x} = 0$ and by induction on i we have that for each i $\vec{r_i}\vec{x} = 0$.

We can choose $\vec{y} \in \mathbb{Q}^v$ for which $B\vec{y} = E\vec{w}$. Let $\vec{x} = F\vec{y}$. Then $EA\vec{x} = EAF\vec{y} = B\vec{y} = E\vec{w}$ so $E(A\vec{x} - \vec{w}) = \vec{0}$ and thus $A\vec{x} = \vec{w}$. □

Theorem 3.6. *Let $u, v \in \mathbb{N} \cup \{\omega\}$ and let A be a $u \times v$ admissible matrix. Then A is* WIPR/$\mathbb{Q}$ *if and only if either $J(A) = \emptyset$ or $B(A)$ is* KPR/$\mathbb{Q}$.

Proof. The necessity follows from Lemma 3.2.

For the sufficiency, if $J(A) = \emptyset$, then the rows of A are linearly independent and one can apply Lemma 3.5 to find $\vec{x} \in \mathbb{Q}^\omega$ such that $A\vec{x} = \vec{1}$. So assume that $J(A) \neq \emptyset$ and $B = B(A)$ is KPR/$\mathbb{Q}$.

Let $\langle \vec{r}_i \rangle_{i<u}$ be the rows of A and let C be the matrix with rows $\langle \vec{r}_i \rangle_{i \in I(A)}$. Let $\mathbb{Q} \setminus \{0\}$ be finitely colored and pick monochrome $\vec{w} \in (\mathbb{Q} \setminus \{0\})^u$ such that $B\vec{w} = \vec{0}$. Let $\vec{z}$ be the restriction of $\vec{w}$ to $I(A)$. Pick by Lemma 3.5 some $\vec{x} \in \mathbb{Q}^v$ such that $C\vec{x} = \vec{z}$.

We claim that $A\vec{x} = \vec{w}$. So let $i < u$. We show that $\sum_{j<v} a_{i,j} \cdot x_j = w_i$. If $i \in I(A)$, this is immediate, so assume that $i \in J(A)$. Then for each $j < v$, $a_{i,j} = \sum_{t \in I(A)} b_{i,t} \cdot a_{t,j}$. Then

$$\begin{aligned}\sum_{j<v} a_{i,j} \cdot x_j &= \sum_{j<v} \sum_{t \in I(A)} b_{i,t} \cdot a_{t,j} \cdot x_j \\ &= \sum_{t \in I(A)} b_{i,t} \cdot \sum_{j<v} a_{t,j} \cdot x_j \\ &= \sum_{t \in I(A)} b_{i,t} \cdot z_t = \sum_{t \in I(A)} b_{i,t} \cdot w_t.\end{aligned}$$

Now

$$0 = \sum_{t<u} b_{i,t} \cdot w_t = \sum_{t \in I(A)} b_{i,t} \cdot w_t - w_i$$

so

$$w_i = \sum_{t \in I(A)} b_{i,t} \cdot w_t = \sum_{j<v} a_{i,j} \cdot x_j. \qquad \square$$

However, the correspondence between WIPR/$\mathbb{Z}$ and KPR/$\mathbb{Z}$ given in Theorem 3.3 may fail for infinite matrices.

Theorem 3.7. *There is an admissible $\omega \times \omega$ matrix A such that $J(A) = \emptyset$ but A is not* WIPR/$\mathbb{Z}$.

Proof. Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ \frac{1}{2} & 1 & 0 & 0 & \ldots \\ \frac{1}{3} & 0 & 1 & 0 & \ldots \\ \frac{1}{4} & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Trivially $J(A) = \emptyset$. Also there is no $\vec{x} \in \mathbb{Z}^\omega$ with all entries of $A\vec{x}$ in $\mathbb{Z} \setminus \{0\}$. (The first row would force $x_0 \in \mathbb{Z} \setminus \{0\}$ and then for some n, $x_0/n \notin \mathbb{Z}$.) $\square$

We now introduce two other notions of image partition regularity. The reader is certainly justified in asking whether we don't have quite enough notions already. We introduce them because of the connection they provide with a major unsolved problem, namely whether KPR/$\mathbb{N}$ and KPR/$\mathbb{Q}$ are equivalent for infinite admissible matrices. (This connection will be presented in Theorem 3.14.)

Definition 3.8. Let $u, v \in \mathbb{N} \cup \{\omega\}$ and let A be an admissible $u \times v$ matrix.

(a) A is *very weakly image partition regular over* $\mathbb{N}$ (VWIPR/$\mathbb{N}$) if and only if whenever $\mathbb{N}$ is finitely colored, there exists $\vec{x} \in \mathbb{Q}^v \setminus \{\vec{0}\}$ such that the entries of $A\vec{x}$ are monochrome.
(b) Let S be a subsemigroup of $(\mathbb{R}, +)$. Then A is *specially image partition regular over* S (SIPR/S) if and only if whenever $S \setminus \{0\}$ is finitely colored there exists $\vec{x} \in S^v$ such that, if $\vec{y} = A\vec{x}$, then $\{x_j : j < v\} \cup \{y_i : i < u\}$ is monochrome.

Notice that VWIPR/$\mathbb{N}$ is a hybrid notion where the entries of $\vec{x}$ are not taken from the subgroup of $(\mathbb{R}, +)$ generated by $\mathbb{N}$.

Theorem 3.9. *Let $u, v \in \mathbb{N} \cup \{\omega\}$ and let A be a $u \times v$ admissible matrix. Then A is* VWIPR/$\mathbb{N}$ *if and only if either $J(A) = \emptyset$ or $B(A)$ is* KPR/$\mathbb{N}$.

Proof. The proof of the necessity is nearly identical to the proof of Lemma 3.2 and the proof of the sufficiency is nearly identical to the proof of the sufficiency of Theorem 3.6. □

As a consequence of Theorems 3.9 and 3.3 we see that for finite matrices, the notion of VWIPR/$\mathbb{N}$ is equivalent to the notions of group (II) of Theorem 2.4.

There are three trivial characterizations of SIPR/S.

Observation 3.10. *Let $u, v \in \mathbb{N} \cup \{\omega\}$, let A be a $u \times v$ admissible matrix, and let S be a subsemigroup of $(\mathbb{R}, +)$. Let I_u and I_v be the $u \times u$ and $v \times v$ identity matrices respectively. The following statements are equivalent:*

(a) *A is* SIPR/S.
(b) $\begin{pmatrix} A & -I_u \end{pmatrix}$ *is* KPR/S.
(c) $\begin{pmatrix} I_v \\ A \end{pmatrix}$ *is* IPR/S.
(d) $\begin{pmatrix} I_v \\ A \end{pmatrix}$ *is* WIPR/S.

Among the semigroups $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}^+$, $\mathbb{Q}$, $\mathbb{R}^+$, and $\mathbb{R}$, there are only three notions of special image partition regularity.

Theorem 3.11. *Let $u, v \in \mathbb{N} \cup \{\omega\}$ and let A be a $u \times v$ admissible matrix.*

(a) *A is* SIPR/$\mathbb{N} \Leftrightarrow A$ *is* SIPR/$\mathbb{Z}$.
(b) *A is* SIPR/$\mathbb{Q}^+ \Leftrightarrow A$ *is* SIPR/$\mathbb{Q}$.
(c) *A is* SIPR/$\mathbb{R}^+ \Leftrightarrow A$ *is* SIPR/$\mathbb{R}$.

If $u, v \in \mathbb{N}$, then all six notions are equivalent, and strictly stronger than the notions of group (I) *of Theorem* 2.4.

Proof. Conclusions (a), (b), and (c) are immediate consequences of Observation 3.10(b) and the fact that, for example, the notions of KPR/$\mathbb{N}$ and KPR/$\mathbb{Z}$ are equivalent.

The fact that the notions are equivalent for finite matrices follows from Observation 3.10(b) and Theorem 1.4. To see that they are strictly stronger than the notions of group (I) of Theorem 2.4, consider the matrix (2). Any coloring of $\mathbb{N}$ for which one never has x and $2x$ the same color establishes that (2) is not SIPR/$\mathbb{N}$. □

For finite matrices, we see that the notion of SIPR/$\mathbb{N}$ provides characterizations of matrices that are IPR/S.

Lemma 3.12. *Let $u, v \in \mathbb{N}$ and let A be an admissible $u \times v$ matrix.*

(a) *Let S be any of $\mathbb{N}$, $\mathbb{Q}^+$, or $\mathbb{R}^+$. Then A is* IPR/S *if and only if there exist $s_0, s_1, \ldots, s_{v-1} \in \mathbb{Q}^+$ such that*

$$B = A \cdot \begin{pmatrix} s_0 & 0 & \ldots & 0 \\ 0 & s_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & s_{v-1} \end{pmatrix}$$

is SIPR/$\mathbb{N}$.

(b) *Let S be any of $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$. Then A is* IPR/S *if and only if there exist $s_0, s_1, \ldots, s_{v-1} \in \mathbb{Q} \setminus \{0\}$ such that*

$$C = A \cdot \begin{pmatrix} s_0 & 0 & \ldots & 0 \\ 0 & s_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & s_{v-1} \end{pmatrix}$$

is SIPR/$\mathbb{N}$.

Proof. (a) By Theorem 2.4 it suffices to establish the equivalence for $S = \mathbb{N}$. By [8, Theorem 3.1] A is IPR/$\mathbb{N}$ if and only if there exist $s_0, s_1, \ldots, s_{v-1} \in \mathbb{Q}^+$ such that $\begin{pmatrix} B & -I_u \end{pmatrix}$ is KPR/$\mathbb{N}$, so the conclusion follows from Observation 3.10.

(b) By Theorems 2.2 and 2.4 it suffices to show that A is WIPR/$\mathbb{N}$ if and only if there exist $s_0, s_1, \ldots, s_{v-1} \in \mathbb{Q} \setminus \{0\}$ such that C is SIPR/$\mathbb{N}$. By [8, Theorem 2.2] A is WIPR/$\mathbb{N}$ if and only if there exist $s_0, s_1, \ldots, s_{v-1} \in \mathbb{Q} \setminus \{0\}$ such that $\begin{pmatrix} C & -I_u \end{pmatrix}$ is KPR/$\mathbb{N}$, so the conclusion follows from Observation 3.10. $\square$

Consider now the following extension of the diagram of Theorem 2.6:

$$\begin{tikzcd}[column sep=tiny]
 & & & & & & \text{SIPR}/\mathbb{N} \arrow[dll] \arrow[drr] & & & & \\
 & & & & \text{IPR}/\mathbb{N} \arrow[dll] \arrow[drr] & & & & \text{SIPR}/\mathbb{Q}^+ \arrow[dll] \arrow[drr] & & \\
 & & \text{IPR}/\mathbb{Z} \arrow[dll] \arrow[d] \arrow[drr] & & & & \text{IPR}/\mathbb{Q}^+ \arrow[dll] \arrow[drr] & & & & \text{SIPR}/\mathbb{R}^+ \arrow[dll] \\
\text{WIPR}/\mathbb{Z} \arrow[d] \arrow[drr] & & \theta_{\mathbb{Z}} \arrow[dll] \arrow[drr] & & \text{IPR}/\mathbb{Q} \arrow[dll] \arrow[d] \arrow[drr] & & & & \text{IPR}/\mathbb{R}^+ \arrow[dll] & & \\
\psi_{\mathbb{Z}} \arrow[drr] & & \text{WIPR}/\mathbb{Q} \arrow[d] \arrow[drr] & & \theta_{\mathbb{Q}} \arrow[dll] \arrow[drr] & & \text{IPR}/\mathbb{R} \arrow[dll] \arrow[d] & & & & \\
 & & \psi_{\mathbb{Q}} \arrow[drr] & & \text{WIPR}/\mathbb{R} \arrow[d] & & \theta_{\mathbb{R}} \arrow[dll] & & & & \\
 & & & & \psi_{\mathbb{R}}. & & & & & &
\end{tikzcd}$$

The matrix C in the proof of Theorem 2.6 is SIPR/$\mathbb{R}^+$ (because $\{x_n : n < \omega\}$ is contained in the set of entries of $C\vec{x}$) so SIPR/$\mathbb{R}^+$ does not imply any of the notions in the diagram besides those indicated. However, we do not know whether the notion of SIPR/$\mathbb{Q}^+$ implies SIPR/$\mathbb{N}$, or indeed even whether it implies $\psi_{\mathbb{Z}}$.

Similarly, we do not know whether WIPR/$\mathbb{Q}$ implies VWIPR/$\mathbb{N}$, or even whether IPR/$\mathbb{Q}^+$ implies VWIPR/$\mathbb{N}$.

The only unanswered question about the relationships of the various notions of kernel partition regularity is whether KPR/$\mathbb{Q}$ implies KPR/$\mathbb{N}$. (See the discussion surrounding [10, Question 6].) We shall see in Theorem 3.14 that these unanswered questions are in fact equivalent.

Lemma 3.13. *Let A be an $\omega \times \omega$ admissible matrix whose rows are linearly independent. There exists an $\omega \times \omega$ admissible matrix B such that:*

(a) *For all $\vec{x} \in \mathbb{Q}^\omega$, $A\vec{x} = \vec{0} \Leftrightarrow B\vec{x} = \vec{0}$.*

(b) *For each $i < \omega$, there exists $j < \omega$ such that $b_{i,j} = -1$ and $b_{k,j} = 0$ for every $k \in \omega \setminus \{i\}$.*

Proof. For each $i \in \omega$, let $l(i) = \max\{j \in \omega : a_{i,j} \neq 0\}$. For each $n \in \omega$, $\{i \in \omega : l(i) < n\}$ is finite. So we can assume that $l(i)$ is nondecreasing, because this could be ensured by rearranging the rows of A.

We carry out ordinary Gaussian row operations, always choosing as pivot the last nonzero entry in the current row. We make that entry -1 and make all other entries in the same column zero. Let B denote the matrix obtained in this way. Since any row is operated on only finitely often, B is admissible. B clearly satisfies (b), and we shall show that B satisfies (a).

We define a sequence $\langle n_i \rangle_{i<\omega}$ inductively by putting $n_0 = 0$ and $n_i = \min\{n \in \omega : l(n) > l(n_{i-1})\}$ if $i > 0$. For each $i \in \omega$, let A_i and B_i denote the matrices formed by the first $n_{i+1} - 1$ rows of A and B respectively. Suppose that $B\vec{x} = \vec{0}$. For each $i \in \omega$, B_i is obtained from A_i by elementary row operations, which are invertible. Since $B_i\vec{x} = \vec{0}$, $A_i\vec{x} = \vec{0}$ and so $A\vec{x} = \vec{0}$.

Conversely, $A\vec{x} = \vec{0}$ clearly implies that $B\vec{x} = \vec{0}$, because each row of B is a linear combination of rows of A. $\square$

Note that the choice of the last nonzero entry as pivot in the above proof is crucial. To see this, consider the matrix

$$\begin{pmatrix} -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If the algorithm described in the proof of Lemma 3.13 is modified so that the first nonzero entry in each row is chosen as pivot, then the resulting matrix is

$$\begin{pmatrix} -1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & \dots \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & \dots \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

each row of which has infinitely many nonzero entries.

Theorem 3.14. *The following statements are equivalent:*

(a) *Every admissible matrix which is* SIPR/$\mathbb{Q}^+$ *is also* SIPR/$\mathbb{N}$.

(b) *Every admissible matrix which is* KPR/$\mathbb{Q}$ *is also* KPR/$\mathbb{N}$.
(c) *Every admissible matrix which is* WIPR/$\mathbb{Q}$ *is also* VWIPR/$\mathbb{N}$.

Proof. (a) implies (b). Let A be an admissible matrix which is KPR/$\mathbb{Q}$. Let $I(A)$ be defined as in Definition 3.1, let $\vec{r_i}$ denote the i^{th} row of A and let A' denote the matrix whose rows are $\{\vec{r_i} : i \in I(A)\}$. For any $\vec{x} \in \mathbb{Q}^\omega$, $A\vec{x} = \vec{0} \Leftrightarrow A'\vec{x} = \vec{0}$, because each row of A is a linear combination of rows of A'. If A' is finite, we have by Theorem 1.4 that A' is KPR/$\mathbb{N}$ and hence that A is KPR/$\mathbb{N}$. So we may assume that A' is infinite.

Choose B as guaranteed by Lemma 3.13, with A' in place of A. Then B is KPR/$\mathbb{Q}$. By rearranging the columns of B, we may write it in the form $\begin{pmatrix} C & -I \end{pmatrix}$. Then by Observation 3.10 C is SIPR/$\mathbb{Q}$ and thus SIPR/$\mathbb{Q}^+$. By assumption C is SIPR/$\mathbb{N}$ and therefore by Observation 3.10 $\begin{pmatrix} C & -I \end{pmatrix}$ is KPR/$\mathbb{N}$ and consequently A' is KPR/$\mathbb{N}$. So A is KPR/$\mathbb{N}$.

(b) implies (c). Let A be an admissible matrix which is WIPR/$\mathbb{Q}$. By Lemma 3.2, either $J(A) = \emptyset$ or $B(A)$ is KPR/$\mathbb{Q}$. If $J(A) = \emptyset$, then by Theorem 3.9 A is VWIPR/$\mathbb{N}$. If $B(A)$ is KPR/$\mathbb{Q}$, then by assumption $B(A)$ is KPR/$\mathbb{N}$ so by Theorem 3.9 A is VWIPR/$\mathbb{N}$.

(c) implies (a). Let $u, v \in \mathbb{N} \cup \{\omega\}$ and let B be an admissible $u \times v$ matrix which is SIPR/$\mathbb{Q}^+$. Then by Observation 3.10 $\begin{pmatrix} B & -I \end{pmatrix}$ is KPR/$\mathbb{Q}$. Define the $(v + u) \times v$ matrix A by

$$a_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i < v \text{ and } i \neq j \\ b_{i-v,j} & \text{if } v \leq i < v + u \end{cases}$$

where $v + u$ is the ordinal sum. Then $B(A) = \begin{pmatrix} B & -I \end{pmatrix}$ so $B(A)$ is KPR/$\mathbb{Q}$ and hence A is WIPR/$\mathbb{Q}$ by Theorem 3.6. Thus by assumption A is VWIPR/$\mathbb{N}$ and thus by Theorem 3.9 $B(A)$ is KPR/$\mathbb{N}$ and so by Observation 3.10 B is SIPR/$\mathbb{N}$. □

Observe that any admissible matrix whose row sums are constantly equal to 1 is trivially SIPR/$\mathbb{N}$.

Definition 3.15. Let $u, v \in \mathbb{N} \cup \{\omega\}$, let A be an admissible $u \times v$ matrix, and let S be a subsemigroup of $(\mathbb{R}, +)$. Then A is *nontrivially* SIPR/S if and only if whenever $S \setminus \{0\}$ is finitely colored there exists $\vec{x} \in S^v$ such that, if $\vec{y} = A\vec{x}$, then $\{x_j : j < v\} \cup \{y_i : i < u\}$ is monochrome and the entries of $\vec{x}$ are all distinct.

We have already observed that there is a matrix which is SIPR/$\mathbb{R}^+$, but not SIPR/$\mathbb{Q}^+$, and the proof that it is SIPR/$\mathbb{R}^+$ shows that it is nontrivially so. We now show that there is a matrix which is trivially SIPR/$\mathbb{N}$, nontrivially SIPR/$\mathbb{R}^+$, and not nontrivially SIPR/$\mathbb{Q}^+$.

Theorem 3.16. *Let*

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & \dots \\ 1 & 1 & 0 & -1 & 0 & \dots \\ 1 & 1 & 0 & 0 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then A is nontrivially SIPR/$\mathbb{R}^+$ *but is not nontrivially* SIPR/$\mathbb{Q}^+$.

Proof. Let $r \in \mathbb{N}$ and let $\varphi : \mathbb{R}^+ \to \{1, 2, \ldots, r\}$. Define $\psi : [\mathbb{R}^+]^2 \to \{1, 2, \ldots, r\}$ by $\psi(\{x, y\}) = \varphi(|x - y|)$. By Corollary 1.6 choose a sequence $y_0 < y_1 < y_2 < \cdots < y_\omega$ in $\mathbb{R}^+$ such that ψ is constant on $\{y_m - y_n : n < m \leq \omega\}$. Choose $k \in \mathbb{N}$ such that $y_k - y_0 \notin \{y_\omega - y_n : k \leq n < \omega\}$. Let $x_0 = y_k - y_0$ and for $n \in \mathbb{N}$, let $x_n = y_\omega - y_{k+n-1}$. Then by the choice of k, the sequence $\langle x_n \rangle_{n<\omega}$ is injective. And for any $n \geq 2$, $x_0 + x_1 - x_n = y_{k+n-1} - y_0$.

To see that A is not nontrivially SIPR/$\mathbb{Q}$, color $\mathbb{Q}^+$ as in the proof of (D) of Theorem 2.6. That is, given a positive rational x, write x as

$$x = \textstyle\sum_{t=1}^{n(x)} b(x, t)/t!$$

where for each t, $b(x, t) \in \omega$, $b(x, t) < t$ if $t > 1$ and $b\big(x, n(x)\big) \neq 0$. Color $\mathbb{Q}^+$ according to the parity of $b\big(x, n(x)\big)$ and the parity of $b(x, n(x) - 2)$. Suppose one has an injective sequence $\langle x_i \rangle_{i<\omega}$ such that the entries of $\begin{pmatrix} I \\ A \end{pmatrix} \vec{x}$ are monochrome. We have for each $i \geq 2$ that $x_0 + x_1 - x_i > 0$ so $\{x_i : 2 \leq i < \omega\}$ is a bounded set of positive numbers, so we may pick $i \geq 2$ such that $n = n(x_i) > n(x_0 + x_1) + 2$. Let $y = x_0 + x_1 - x_i$. Then y and x_i are the same color and $n(y) = n(x_i)$. Also

$$\begin{aligned} b(x_i, n) + b(y, n) &= n \\ 1 + b(x_i, n-1) + b(y, n-1) &= n - 1, \text{ and} \\ 1 + b(x_i, n-2) + b(y, n-2) &= n - 2\,. \end{aligned}$$

The first of these equations implies that that n is even and the third implies that n is odd, a contradiction. □

We conclude by showing that there is a class of admissible matrices for which being KPR/$[1, \infty)$ is equivalent to being KPR/$\mathbb{N}$.

Theorem 3.17. *Let A be an $\omega \times \omega$ admissible matrix with entries in $\mathbb{Z}$ and the property that $\{\sum_{j<\omega} |a_{ij}| : i < \omega\}$ is bounded. If A is* KPR/$[1, \infty)$*, then A is* KPR/$\mathbb{N}$.

Proof. Assume that A is KPR/$[1, \infty)$. Let $C = \{p \in \beta([1, \infty)_d) : (\forall B \in p)(\exists \vec{x} \in B^\omega)(A\vec{x} = \vec{0})\}$. Then C is closed and nonempty. Furthermore, for every $n \in \mathbb{N}$, $n \cdot C \subseteq C$ so $q \cdot C \subseteq C$ for every $q \in \beta\mathbb{N}$.

Let $p \in C$ and let $h : \mathbb{R} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the natural homomorphism. Let $\widetilde{h} : \beta\mathbb{R} \to \mathbb{T}$ denote its continuous extension. For each $m \in \mathbb{N}$, there exists $n_m \in \mathbb{N}$ such that the distance in $\mathbb{T}$ from $n_m\widetilde{h}(p)$ to $\mathbb{Z} + 0$ is less than $\frac{1}{m}$. If q is a limit point of $\langle n_m \rangle_{m=1}^\infty$ in $\beta\mathbb{N}$, then $\widetilde{h}(q \cdot p) = 0$, so we can replace p by $q \cdot p$ and assume that $\widetilde{h}(p) = 0$.

We define $g : \mathbb{R} \to \mathbb{Z}$ by $g(x) = \lfloor x + \frac{1}{2} \rfloor$ and let $\widetilde{g} : \beta\mathbb{R}_d \to \beta\mathbb{Z}$ be its continuous extension. Notice that $\widetilde{g}(p) \in \beta\mathbb{N}$. We define $k : \mathbb{R}^\omega \to \mathbb{Z}^\omega$ by

$$k(\vec{x}) = \begin{pmatrix} g(x_0) \\ g(x_1) \\ g(x_2) \\ \vdots \end{pmatrix}.$$

We observe that because of the boundedness assumption on the rows of A, there exists $\epsilon > 0$ such that the entries of $A\vec{x}$ are contained in $(-\frac{1}{2}, \frac{1}{2})$ if the entries of $\vec{x}$

are contained in $(-\epsilon, \epsilon)$. If $P = \{x \in [1, \infty)\colon$ the distance from x to a member of $\mathbb{Z}$ is less than $\epsilon\}$, then $P \in p$.

Now let $\mathbb{N}$ be finitely colored and pick a color class D such that $D \in \widetilde{g}(p)$. Pick $\vec{x} \in (P \cap g^{-1}[D])^\omega$ such that $A\vec{x} = \vec{0}$. Then $k(\vec{x}) \in D^\omega$ and $Ak(\vec{x}) = k(A\vec{x}) = \vec{0}$. □

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DEPARTMENT OF MATHEMATICS, HOWARD UNIVERSITY, WASHINGTON, DC 20059, USA
nhindman@howard.edu, nhindman@aol.com, http://members.aol.com/nhindman/

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF HULL, HULL HU6 7RX, UK
d.strauss@hull.ac.uk

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