

SLE coordinate changes

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ABSTRACT. The purpose of this note is to describe a framework which unifies radial, chordal and dipolar SLE. When the definition of $SLE(\kappa; \rho)$ is extended to the setting where the force points can be in the interior of the domain, radial $SLE(\kappa)$ becomes chordal $SLE(\kappa; \rho)$, with $\rho = \kappa - 6$, and vice versa. We also write down the martingales describing the Radon–Nykodim derivative of $SLE(\kappa; \rho_1, \dots, \rho_n)$ with respect to $SLE(\kappa)$.

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1. Introduction

SLE, or Stochastic Loewner Evolution [Sch00], describes random paths in the plane by specifying a differential equation satisfied by the conformal maps into the complement. The main interest in these paths stems from their relationship with scaling limits of critical models from statistical physics and two dimensional Brownian motion. For motivation and background on SLE, the reader is advised to consult the surveys [Wer04b, Law04, KN04, Car05] and the references cited there.

SLE comes in several different flavors, and the principal goal of this short paper is to observe a unification of these different variations. Basically, SLE is Loewner evolution with Brownian motion as its driving parameter. More explicitly, chordal SLE is defined as follows: fix $\kappa \geq 0$. Let B_t be one-dimensional standard Brownian motion, started as $B_0 = 0$. Set $W_t = \sqrt{\kappa} B_t$. If we fix any $z \in \mathbb{H}$, we may consider the solution of the ODE

$$(1) \quad \partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z.$$

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Let $\tau = \tau_z$ be the supremum of the set of t such that $g_t(z)$ is well-defined. Then either $\tau_z = \infty$, or $\lim_{t \nearrow \tau} g_t(z) - W_t = 0$. Set $H_t := \{z \in \mathbb{H} : \tau_z > t\}$. It is easy to check that for every $t > 0$ the map $g_t : H_t \rightarrow \mathbb{H}$ is conformal. It has been shown [RS05, LSW04] that the limit $\gamma(t) = \lim_{z \rightarrow W_t} g_t^{-1}(z)$ exists and is continuous in t . This is the path defined by SLE. It is easy to check that for every $t \geq 0$, H_t is the unbounded connected component of $\mathbb{H} \setminus \gamma[0, t]$.

Radial SLE is defined in a similar manner, except that the upper half plane \mathbb{H} is replaced by the unit disk \mathbb{D} , and the differential equation (1) is replaced by

$$(2) \quad \partial_t g_t(z) = -g_t(z) \frac{g_t(z) + W_t}{g_t(z) - W_t}, \quad g_0(z) = z,$$

where W_t is Brownian motion on the unit circle $\partial\mathbb{D}$, with time scaled by a factor of κ . The definition is presented in detail and generalized in Section 2. Yet another version of SLE, dipolar SLE, was introduced in [BB04] and further studied in [BBH05].

In [LSW01a] it was shown that radial SLE and chordal SLE have equivalent (i.e., mutually absolutely continuous) laws, up to a time change, when stopped before the disconnection time (the precise meaning of this should become clear soon). Here, we carry this calculation a step further, and describe what a radial SLE looks like when transformed to the chordal coordinate system and vice versa. We extend the definition of the $\text{SLE}(\kappa; \rho)$ processes from [LSW03, Dub05] to the setting where the force points are permitted to be in the interior of the domain. With the extended definitions, the $\text{SLE}(\kappa; \rho)$ radial processes are transformed into appropriate $\text{SLE}(\kappa; \rho)$ chordal processes, and vice versa.

We start with the definition of chordal $\text{SLE}(\kappa; \rho)$ with a force point in the interior.

Definition 1. Let $z_0 \in \mathbb{H}$, $w_0 \in \mathbb{R}$, $\kappa \geq 0$, $\rho \in \mathbb{R}$. Let B_t be standard one-dimensional Brownian motion. Define (W_t, V_t) to be the solution of the system of SDEs

$$\begin{aligned} dW_t &= \sqrt{\kappa} dB_t + \rho \Re \frac{1}{W_t - V_t} dt, \\ dV_t &= \frac{2 dt}{V_t - W_t}, \end{aligned}$$

starting at $(W_0, V_0) = (w_0, z_0)$, up to the first time $\tau > 0$ such that

$$\inf\{|W_t - V_t| : t \in [0, \tau]\} = 0.$$

(Here, $\Re z$ denotes the real part of z .) Then the solution of Loewner's chordal equation (1) with W_t as the driving term will be called chordal $\text{SLE}(\kappa; \rho)$ starting at (w_0, z_0) .

One motivation for this definition comes from the fact that, as we will later see, ordinary radial $\text{SLE}(\kappa)$ starting at $W_0 = w$ is transformed by a Möbius transformation $\psi : \mathbb{D} \rightarrow \mathbb{H}$ to a time changed chordal $\text{SLE}(\kappa; \kappa - 6)$ starting at $(\psi(w), \psi(0))$ (both up to a corresponding positive stopping time).

In [Wer04a] it was shown that a chordal $\text{SLE}(\kappa; \rho)$ process can be viewed as an ordinary $\text{SLE}(\kappa)$ process weighted by a martingale. In Section 5 we extend this to SLEs with force points in the interior of the domain.

2. A unified framework for SLE($\kappa; \rho$)

In order to produce a clean and general formulation of the change of coordinate results, we need to introduce a bit of notation which will enable us to deal with the chordal and radial versions simultaneously. Let $X \in \{\mathbb{D}, \mathbb{H}\}$. We let $\Psi_X(w, z)$ denote the corresponding Loewner vector field; that is,

$$\Psi_{\mathbb{D}}(w, z) = -z \frac{z + w}{z - w}, \quad \Psi_{\mathbb{H}}(w, z) = \frac{2}{z - w}.$$

Thus, Loewner’s radial [respectively chordal] equation may be written as

$$\partial_t g_t(z) = \Psi_X(W_t, g_t(z))$$

if $X = \mathbb{D}$ [respectively $X = \mathbb{H}$]. Let $I_{\mathbb{D}}$ denote the inversion in the unit circle $\partial\mathbb{D}$, and let $I_{\mathbb{H}}$ denote the inversion in the real line $\partial\mathbb{H}$, i.e., complex conjugation. Set

$$\tilde{\Psi}_X(z, w) := \frac{\Psi_X(z, w) + \Psi_X(I_X(z), w)}{2}.$$

In radial ordinary SLE, $W_t = W_0 \exp(i\sqrt{\kappa}B_t)$, where B_t is a Brownian motion. By Itô’s formula, $dW_t = -(\kappa/2)W_t dt + i\sqrt{\kappa}W_t dB_t$. On the other hand, in ordinary chordal SLE, one simply has $dW_t = \sqrt{\kappa}dB_t$. We thus set

$$\mathcal{G}_{\mathbb{D}}(W_t, dB_t, dt) := -(\kappa/2)W_t dt + i\sqrt{\kappa}W_t dB_t, \quad \mathcal{G}_{\mathbb{H}}(W_t, dB_t, dt) := \sqrt{\kappa}dB_t.$$

Definition 2. Let $\kappa \geq 0$, $m \in \mathbb{N}$, $\rho_1, \rho_2, \dots, \rho_m \in \mathbb{R}$. Let $X \in \{\mathbb{H}, \mathbb{D}\}$ and let $V^1, V^2, \dots, V^m \in \overline{X}$. (When $X = \mathbb{H}$, we include ∞ in \overline{X} .) Let $w_0 \in \partial X \setminus \{\infty, V^1, \dots, V^m\}$ and let B_t be standard one-dimensional Brownian motion. Consider the solution of the SDE system

$$(3) \quad dW_t = \mathcal{G}_X(W_t, dB_t, dt) + \sum_{j=1}^m \frac{\rho_j}{2} \tilde{\Psi}_X(V_t^j, W_t) dt,$$

$$(4) \quad dV_t^j = \Psi_X(W_t, V_t^j) dt, \quad j = 1, 2, \dots, m$$

starting at $W_0 = w_0$ and $V_0^j = V^j$, $j = 1, \dots, m$, up to the first time τ such that for some j $\inf\{|W_t - V_t^j| : t < \tau\} = 0$. Let $g_t(z)$ be the solution of the ODE

$$\partial_t g_t(z) = \Psi_X(W_t, g_t(z))$$

starting at $g_0(z) = z$. Let $K_t = \{z \in \overline{X} : \tau_z \leq t\}$, $t < \tau$, be the corresponding hull, defined in the same way as for ordinary SLE. Then the evolution $t \mapsto K_t$ will be called X -SLE($\kappa; \rho_1, \dots, \rho_m$) starting at (w_0, V^1, \dots, V^m) . When $X = \mathbb{D}$, we refer to \mathbb{D} -SLE as radial SLE, while \mathbb{H} -SLE is chordal. The points V_t^j will be called **force points**.

Note that $V_t^j = g_t(V^j)$, for all $t \in [0, \tau)$.

In some situations, it is possible and worthwhile to extend the definition of SLE($\kappa; \rho_1, \dots, \rho_m$) beyond the time τ (see, e.g., [LSW03]), but we do not deal with this here.

3. Some Möbius coordinate changes

Set $o_{\mathbb{H}} := \infty$ and $o_{\mathbb{D}} := 0$ and as before $X \in \{\mathbb{H}, \mathbb{D}\}$. Observe that if $V^m = o_X$, then the value of ρ_m has no effect on the X -SLE($\kappa; \rho_1, \dots, \rho_m$), and, in fact, the SLE($\kappa; \rho_1, \dots, \rho_m$) reduces to SLE($\kappa; \rho_1, \dots, \rho_{m-1}$). Consequently, by increasing m and appending o_X to the force points, if necessary, there is no loss of generality in assuming that $\sum_{j=1}^m \rho_j = \kappa - 6$. This assumption simplifies somewhat the statement of the following theorem:

Theorem 3. *Let $X, Y \in \{\mathbb{D}, \mathbb{H}\}$, and let $\psi : X \rightarrow Y$ be a conformal homeomorphism; that is, a Möbius transformation satisfying $\psi(X) = Y$. Let $w_0 \in \partial X \setminus \{\infty\}$, $V^1, V^2, \dots, V^m \in \overline{X} \setminus \{w_0\}$. Suppose that $\rho_1, \rho_2, \dots, \rho_m \in \mathbb{R}$ satisfy $\sum_{j=1}^m \rho_j = \kappa - 6$. Then the image under ψ of the X -SLE($\kappa; \rho_1, \dots, \rho_m$) starting from (w_0, V^1, \dots, V^m) and stopped at some a.s. positive stopping time has the same law as a time change of the Y -SLE($\kappa; \rho_1, \dots, \rho_m$) starting from $(\psi(w_0), \psi(V^1), \dots, \psi(V^m))$ stopped at an a.s. positive stopping time.*

The stopping time for the X -SLE may be taken as the minimum of τ and the first time t such that $\psi^{-1}(o_Y) \in K_t$.

The case when $X = Y = \mathbb{H}$ and the force points are on the boundary appears in [Dub04].

Proof. Suppose first that $X = \mathbb{H}$ and $Y = \mathbb{D}$. Let g_t be the family of maps $g_t : \mathbb{H} \setminus K_t \rightarrow \mathbb{H}$. Let $\phi_t : \mathbb{H} \rightarrow \mathbb{D}$ be the Möbius transformation such that $F_t := \phi_t \circ g_t \circ \psi^{-1}$ satisfies the radial normalization $F_t(0) = 0$ and $F'_t(0) \in (0, \infty)$. Let $z_t = x_t + iy_t := g_t \circ \psi^{-1}(0)$, with $x_t, y_t \in \mathbb{R}$. There is some (unique) $\lambda_t \in \partial \mathbb{D}$ such that

$$\phi_t(z) = \lambda_t \frac{z - z_t}{z - \bar{z}_t}.$$

Let $s(t) := \log F'_t(0)$ and let $t(s)$ denote the inverse of the map $t \mapsto s(t)$. By the chain rule,

$$\begin{aligned} s'(t) &= \partial_t \log F'_t(0) = \partial_t \log \left(\frac{\lambda_t}{2iy_t} g'_t(\psi^{-1}(0)) (\psi^{-1})'(0) \right) \\ &= \Re(\partial_t \log g'_t(\psi^{-1}(0))) - \partial_t \log y_t. \end{aligned}$$

We have by (1)

$$\partial_t y_t = \frac{-2y_t}{|z_t - W_t|^2},$$

and

$$\partial_t g'_t(z) = \partial_z \partial_t g_t(z) = \partial_z \frac{2}{g_t(z) - W_t} = \frac{-2g'_t(z)}{(g_t(z) - W_t)^2}.$$

Using this with $z = o$ in our previous expression for $s'(t)$ gives

$$s'(t) = \Re \frac{-2}{(z_t - W_t)^2} + \frac{2}{|z_t - W_t|^2} = \frac{4y_t^2}{|z_t - W_t|^4}.$$

Set $\hat{g}_s(z) := F_{t(s)}(z)$, $\hat{W}_s := \phi_{t(s)}(W_{t(s)})$ and $\hat{K}_s := \psi(K_{t(s)})$. We need to verify that Loewner's equation holds:

$$(5) \quad \partial_s \hat{g}_s(z) = \Psi_{\mathbb{D}}(\hat{W}_s, \hat{g}_s(z)), \quad z \in \mathbb{D} \setminus \hat{K}_s.$$

This may be verified by brute-force calculation or may be deduced from Loewner’s theorem, as follows: first, \hat{g}_s is appropriately normalized: $\hat{g}_s(0) = 0$ and $\hat{g}'_s(0) = e^s$. Second, the domain of definition of \hat{g}_s is clearly $\mathbb{D} \setminus \hat{K}_s$. Lastly, if $\varepsilon > 0$ is small, then $g_{t+\varepsilon} \circ g_t^{-1}$ is defined on the complement in \mathbb{H} of a small neighborhood of W_t . Consequently, $\hat{g}_{s+\varepsilon} \circ \hat{g}_s^{-1}$ is defined on the complement in \mathbb{D} of a small neighborhood of \hat{W}_s when $\varepsilon > 0$ is small. Therefore, Loewner’s theorem gives (5).

It remains to calculate $d\hat{W}_s$. Since $\hat{W}_s = \lambda_t \frac{W_t - z_t}{W_t - \bar{z}_t}$, the Itô differential $d \log \hat{W}_s$ satisfies

$$(6) \quad d \log \hat{W}_s = d \log \lambda_t + d \log(W_t - z_t) - d \log(W_t - \bar{z}_t).$$

Since $\hat{g}_s(\psi(\infty)) = \phi_{t(s)}(\infty) = \lambda_{t(s)}$, Equation (5) with $z = \psi(\infty)$ gives $\partial_s \lambda_{t(s)} = \Psi_{\mathbb{D}}(\hat{W}_s, \lambda_{t(s)})$, so

$$A_0 := d \log \lambda_t = \lambda_t^{-1} \Psi_{\mathbb{D}}(\hat{W}_s, \lambda_t) ds = -\hat{W}_s^{-1} \Psi_{\mathbb{D}}(\lambda_t, \hat{W}_s) ds.$$

Itô’s formula gives

$$(7) \quad \begin{aligned} d \log(W_t - z_t) &= -\frac{\kappa dt}{2(W_t - z_t)^2} + \frac{dW_t - dz_t}{W_t - z_t} \\ &= -\frac{\kappa dt}{2(W_t - z_t)^2} + \frac{dW_t}{W_t - z_t} + \frac{2 dt}{(z_t - W_t)^2} = \frac{(4 - \kappa) dt}{2(W_t - z_t)^2} + \frac{dW_t}{W_t - z_t}. \end{aligned}$$

Likewise,

$$(8) \quad -d \log(W_t - \bar{z}_t) = -\frac{(4 - \kappa) dt}{2(W_t - \bar{z}_t)^2} - \frac{dW_t}{W_t - \bar{z}_t}.$$

We now handle the sum of the dt terms in (7) and (8), using our above expression for $s'(t)$.

$$\begin{aligned} A_1 &:= \frac{(4 - \kappa) dt}{2(W_t - z_t)^2} - \frac{(4 - \kappa) dt}{2(W_t - \bar{z}_t)^2} = \frac{2i(4 - \kappa)y_t(W_t - x_t)s'(t)^{-1} ds}{|W_t - z_t|^4} \\ &= \frac{i(4 - \kappa)(W_t - x_t) ds}{2y_t} \\ &= -\frac{4 - \kappa}{2} \hat{W}_s^{-1} \Psi_{\mathbb{D}}(\lambda_t, \hat{W}_s) ds. \end{aligned}$$

Next, we look at the dW_t terms. First, write (3) in slightly different form

$$dW_t = \sqrt{\kappa} dB_t + \sum_{j=1}^{2m} \frac{\rho_j}{4} \Psi_{\mathbb{H}}(v_t^j, W_t) dt,$$

where $v_t^j := V_t^j$, $v_t^{j+m} := I_{\mathbb{H}}(v_t^j)$ and $\rho_{j+m} = \rho_j$ for $j = 1, 2, \dots, m$. The sum of the dW_t terms in (7) and (8) is

$$\begin{aligned} A_2 &:= \frac{dW_t}{W_t - z_t} - \frac{dW_t}{W_t - \bar{z}_t} = \frac{2iy_t}{|W_t - z_t|^2} \left(\sqrt{\kappa} dB_t + \sum_{j=1}^{2m} \frac{\rho_j}{4} \Psi_{\mathbb{H}}(v_t^j, W_t) dt \right) \\ &= i\sqrt{\kappa s'(t)} dB_t + i s'(t)^{-1/2} \sum_{j=1}^{2m} \frac{\rho_j}{4} \Psi_{\mathbb{H}}(v_t^j, W_t) ds. \end{aligned}$$

Now define

$$\hat{B}_s := \int_0^{t(s)} \sqrt{s'(t)} dB_t.$$

Then $\langle \hat{B} \rangle_{s_0} = \int_0^{t(s_0)} s'(t) dt = s_0$, and therefore $s \mapsto \hat{B}_s$ is standard Brownian motion. Also set $\hat{v}_t^j := \phi_t(v_t^j)$, and note that $\phi_t(I_{\mathbb{H}}(z)) = I_{\mathbb{D}}(\phi_t(z))$, since $\phi_t : \mathbb{H} \rightarrow \mathbb{D}$ is a Möbius transformation. To further translate our expression for A_2 to the \mathbb{D} coordinate system, we calculate

$$i s'(t)^{-1/2} \Psi_{\mathbb{H}}(v_t^j, W_t) = \frac{i |z_t - W_t|^2}{y_t(W_t - v_t^j)} = \hat{W}_s^{-1} \Psi_{\mathbb{D}}(\hat{v}_t^j, \hat{W}_s) - \hat{W}_s^{-1} \Psi_{\mathbb{D}}(\lambda_t, \hat{W}_s).$$

Consequently, since $\sum_{j=1}^m \rho_j = \kappa - 6$,

$$A_2 = i \sqrt{\kappa} d\hat{B}_s - \frac{\kappa - 6}{2} \hat{W}_s^{-1} \Psi_{\mathbb{D}}(\lambda_t, \hat{W}_s) ds + \hat{W}_s^{-1} \sum_{j=1}^m \frac{\rho_j}{2} \tilde{\Psi}_{\mathbb{D}}(\hat{v}_t^j, \hat{W}_s) ds.$$

Setting $\hat{w}_s := \log \hat{W}_s$, we get

$$d\hat{w}_s = A_0 + A_1 + A_2 = i \sqrt{\kappa} d\hat{B}_s + \hat{W}_s^{-1} \sum_{j=1}^m \frac{\rho_j}{2} \tilde{\Psi}_{\mathbb{D}}(\hat{v}_t^j, \hat{W}_s) ds.$$

Now, Itô's formula gives

$$\begin{aligned} d\hat{W}_s &= d \exp(\hat{w}_s) = \exp(\hat{w}_s) d\hat{w}_s + \frac{1}{2} \exp(\hat{w}_s) d\langle \hat{w}_s \rangle \\ &= i \sqrt{\kappa} \hat{W}_s d\hat{B}_s + \sum_{j=1}^m \frac{\rho_j}{2} \tilde{\Psi}_{\mathbb{D}}(\hat{v}_t^j, \hat{W}_s) ds - \frac{\kappa \hat{W}_s}{2} ds. \end{aligned}$$

This completes the proof in the case $X = \mathbb{H}$ and $Y = \mathbb{D}$.

If $X = \mathbb{D}$ and $Y = \mathbb{H}$, we may just reverse the above equivalence. To handle the case $X = Y = \mathbb{H}$, we may just write the Möbius transformation $\psi : X \rightarrow Y$ as a composition $\psi = \psi_2 \circ \psi_1$, where $\psi_1 : \mathbb{H} \rightarrow \mathbb{D}$ and $\psi_2 : \mathbb{D} \rightarrow \mathbb{H}$, and appeal to the above situations. The case $X = Y = \mathbb{D}$ is similar. \square

Remark 4. It is possible to come up with a somewhat more conceptual version of some parts of this proof. First, the time change $s'(t)$ is the rate at which the radial capacity changes with respect to the chordal capacity. This is known to be $\phi_t'(W_t)^2$. Second, we have $\hat{W}_s = \phi_t(W_t)$. Itô's formula gives

$$d\hat{W}_s = (\partial_t \phi_t)(W_t) dt + \phi_t'(W_t) dW_t + (\kappa/2) \phi_t''(W_t) dt.$$

The terms $\phi_t'(W_t) \Psi_{\mathbb{H}}(V_t^j, W_t)$ that arise by expanding dW_t should be thought of as the image under ϕ_t of the vector field $z \mapsto \Psi_{\mathbb{H}}(V_t^j, z)$, evaluated at \hat{W}_s . The vector field $z \mapsto \Psi_{\mathbb{H}}(V_t^j, z)$ is the Loewner vector field in \mathbb{H} , and it should be mapped under ϕ_t to a multiple of the Loewner vector field in \mathbb{D} plus some vector field which preserves \mathbb{D} .

4. Strip SLE

For comparison and illustration, we now mention another type of SLE. It generalizes dipolar SLE as introduced in [BB04] and studied in [BBH05] as well as a version of dipolar SLE with force points, which was used in [LSW01b, §3]

Definition 5 (Dipolar SLE). Let $S = \{x + iy : x \in \mathbb{R}, y \in (0, \pi/2)\}$. Set

$$\begin{aligned} \Psi_S(w, z) &= 2 \coth(z - w), & I_S &= I_{\mathbb{H}}, \\ \tilde{\Psi}_S(z, w) &= \frac{\Psi_S(z, w) + \Psi_S(I_S(z), w)}{2}, & \mathcal{G}_S &= \mathcal{G}_{\mathbb{H}}. \end{aligned}$$

Then strip-SLE($\kappa; \rho_1, \rho_2, \dots, \rho_m$) is defined using Definition 2 with $X = S$ and $w_0 \in \mathbb{R}$. In the case where $m = 0$, this coincides with dipolar SLE, as defined in [BBH05].

Note that in strip-SLE($\kappa; \rho_1, \dots, \rho_m$), possible force points at $+\infty$ and at $-\infty$ do exert an effect on the motion of W_t , since $\Psi_S(\pm\infty, w) = \mp 2$. However, if $V^{m-1} = +\infty$ and $V^m = -\infty$, then adding a constant to both ρ_{m-1} and ρ_m has no effect, since the resulting forces cancel. Therefore, we may again assume with no loss of generality that $\sum_{j=1}^m \rho_j = \kappa - 6$. In that case, the image of this process under any conformal map $\psi : S \rightarrow \mathbb{H}$ that satisfies $\psi(w_0) \neq \infty$ is chordal-SLE($\kappa; \rho_1, \rho_2, \dots, \rho_m$). The proof is left to the dedicated reader.

5. Associated martingales

In [Wer04a] it was shown that a chordal SLE($\kappa; \rho$) process can be viewed as an ordinary SLE(κ) process weighted by a martingale. In the following, we extend this to most of the SLE-like processes discussed in the previous sections.

Suppose that $(Y_t, t \geq 0)$ is some random process taking values in some space X . If $h : X \rightarrow [0, \infty)$ is some function such that $h(Y_1)$ is measurable and $0 < \mathbb{E}[h(Y_1)] < \infty$, then we may weight our given probability measure by $h(Y_1)$; that is, we may consider the probability measure $\tilde{\mathbf{P}}$ whose Radon–Nykodim derivative with respect to \mathbf{P} is $h(Y_1)/\mathbb{E}[h(Y_1)]$. In some cases, the new law of Y_t is called the Doob-transform of the unweighted law.

In many situations one can explicitly determine the Doob-transform. Consider, for example, a diffusion process Y_t adapted to the filtration \mathcal{F}_t taking values in some domain in \mathbb{R}^n . If h is as above, then $M_t := \mathbb{E}[h(Y_1) \mid \mathcal{F}_t]$ is a martingale. It turns out to be worthwhile to forget about h and consider weighting by any positive martingale. Indeed, for every event $A \in \mathcal{F}_t$ we have $\tilde{\mathbf{P}}[A] = \mathbb{E}[1_A M_t]/M_0$. Girsanov’s theorem (see, e.g. [RY99]) describes the behavior of continuous martingales N_t weighted by a positive continuous martingale M_t adapted to the same filtration \mathcal{F}_t . It states that

$$N_t - \int^t \frac{d\langle N, M \rangle_{t'}}{M_{t'}}$$

is a local martingale under the weighted measure. In particular, if B_t is Brownian motion under \mathbf{P} and $dM_t = a_t dB_t$, for some adapted process a_t , then

$$\tilde{B}_t := B_t - \int^t (a_{t'}/M_{t'}) dt'$$

is a \tilde{P} -Brownian motion (note that $\langle \tilde{B} \rangle_t = \langle B \rangle_t$). Thus, with respect to \tilde{P} , B_t has the drift term $(a_t/M_t) dt$.

Theorem 6. *Consider standard chordal SLE(κ). Let z^1, \dots, z^n be some collection of distinct points in \mathbb{H} . Let $\rho_1, \dots, \rho_n \in \mathbb{R}$ be arbitrary. Define $z_t^j = x_t^j + i y_t^j := g_t(z^j)$. Set*

$$M_t := \prod_{j=1}^n \left(|g'_t(z^j)|^{(8-2\kappa+\rho_j)\rho_j/(8\kappa)} (y_t^j)^{\rho_j^2/(8\kappa)} |W_t - z_t^j|^{\rho_j/\kappa} \right) \times \prod_{1 \leq j < j' \leq n} \left(|z_t^j - z_t^{j'}| \left| \overline{z_t^j} - z_t^{j'} \right| \right)^{\rho_j \rho_{j'}/(4\kappa)}.$$

Then M_t is a local martingale. Moreover, under the measure weighted by M_t (with an appropriate stopping time) we have chordal SLE($\kappa; \rho_1, \dots, \rho_n$) with force points z^1, \dots, z^n .

The local martingale M_t was independently discovered by Marek Biskup [Bis04].

Proof. Using Itô’s formula, we may calculate

$$dM_t = M_t \frac{1}{\kappa} \Re \left(\sum_{j=1}^n \frac{\rho_j}{W_t - z_t^j} \right) dW_t = M_t \frac{1}{\sqrt{\kappa}} \Re \left(\sum_{j=1}^n \frac{\rho_j}{W_t - z_t^j} \right) dB_t.$$

This is laborious, but straightforward. Thus, M_t is a local martingale. The drift term

$$\frac{d\langle W, M \rangle_t}{M_t}$$

from Girsanov’s theorem is thus precisely the drift one has for SLE($\kappa; \rho_1, \dots, \rho_n$) (since it is the product of the coefficient of dB_t in dM_t and in dW_t divided by M_t). \square

Remark 7. A similar variation of the theorem holds when some or all of the points z^j are on the real axis. One only needs to replace the corresponding y_t^j by $g'_t(z^j)$ in the definition of M_t .

We now discuss the radial version of the theorem. Let z^1, \dots, z^n be distinct points in the unit disk. We may extend the radial maps g_t to the complement of the unit disk by Schwarz reflection, $g_t(I_{\mathbb{D}}(z)) = I_{\mathbb{D}}(g_t(z))$. Set $z^{n+j} := I_{\mathbb{D}}(z^j)$, $\rho_{n+j} := \rho_j$ and $z_t^j = g_t(z^j)$ for $j = 1, \dots, 2n$. Then the local martingale takes the form

$$(9) \quad M_t = g'_t(0)^{q_0} \prod_{j=1}^{2n} |W_t - z_t^j|^{\rho_j/(2\kappa)} |g'_t(z^j)|^{q_j} \prod_{1 \leq j < k \leq 2n} |z_t^j - z_t^k|^{\rho_j \rho_k/(8\kappa)},$$

where $q_0 = (4 + \bar{\rho}) \bar{\rho}/(8\kappa)$, $\bar{\rho} = \sum_{j=1}^n \rho_j$ and $q_j = (8 - 2\kappa + \rho_j) \rho_j/(16\kappa)$ for $j > 0$. Of course $g'_t(0) = e^t$. Likewise, if one or more of the special points z_j , $j \leq n$, lies on the unit circle, then the corresponding vanishing term $|z_t^j - z_t^{n+j}|^{\rho_j^2/(8\kappa)}$ is replaced by $|g'_t(z^j)|^{\rho_j^2/(8\kappa)}$.

Remark 8. In some cases these martingales are relevant to estimating the probability of rare events in discrete models. We mention two examples, the first pertaining to the probability that a site is pivotal in critical percolation, and the second pertaining to “triple points” in uniform spanning trees. In this remark we indicate, without proof, why the martingales are relevant.

Consider critical percolation in a bounded planar domain D containing the origin where ∂D is a simple closed curve. Let z^1, \dots, z^4 be four distinct points in clockwise order on ∂D . For every sufficiently small $\varepsilon > 0$, we may consider the triangular lattice of mesh ε with a vertex at the origin and the event $\mathcal{A} = \mathcal{A}(D, \varepsilon, z^1, z^2, z^3, z^4)$ that the site at the origin is pivotal for an open crossing in critical site percolation between two opposite arcs on ∂D determined by the four marked points in the union of the triangles of the grid that meet D . One could also consider the probability that the interfaces starting at an even number n of points on ∂D reach close to the origin. Essentially, the only difference is that the event \mathcal{A} has to be defined a bit differently, since there can be at most 3 distinct interfaces that meet a hexagon.

Observe that the event \mathcal{A} is equivalent to the event that the two (or $n/2$) percolation interfaces containing the points z^j 's all visit the boundary of the hexagon dual to the site at the origin. As the percolation interfaces are explored, the conditional probability of the event \mathcal{A} is a positive martingale that depends only on the domain cut by the explored segments of the interfaces. This quantity should be (nearly) conformally invariant, modulo scaling by a power of the conformal radius of D with respect to the origin. In particular, the martingale will not depend on $g'_t(z^j)$ for the boundary points z^j . Referring to (9), we see that when $\kappa = 6$ [Smi01], by selecting each $\rho_j = 2$ the $g'_t(z^j)$ terms drop out, so that the martingale M_t simplifies to

$$M_t = g'_t(0)^{(n^2-1)/12} \prod_{1 \leq j < k \leq n} |z_t^j - z_t^k|^{1/3},$$

where W_t in (9) is here written as z_t^n . Letting $f : D \rightarrow \mathbb{D}$ denote a conformal bijection taking the domain D to the unit disk with $f(0) = 0$, it is thus reasonable to conjecture that the probability that the $n/2$ interfaces each reach (or “approach” when $n > 6$) the hexagon centered at the origin will be

$$(10) \quad (\text{const} + o(1)) \times (\varepsilon f'(0))^{(n^2-1)/12} \prod_{1 \leq j < k \leq n} |f(z^j) - f(z^k)|^{1/3}.$$

The exponent of $(n^2 - 1)/12$ was shown in [SW01].

Consider next the uniform spanning tree with wired boundary conditions in D , and let us consider the unlikely event that the tree paths leading to ∂D from three different fixed neighbors of the origin stay disjoint and hit ∂D at three specific points z^1, z^2, z^3 . After one path from a neighbor of v is generated, the probability that the next path will manage to hit its desired target is given by the harmonic measure of that target. In this case, rather than looking for a martingale in which the g'_t terms at the boundary points drop out, we want a martingale in which the g'_t terms at the boundary points have an exponent of 1. To get this exponent of 1 in (9) when $\kappa = 2$ [LSW04], we again need to pick the ρ_j 's to be 2, giving us

$$M_t = g'_t(0)^{(n^2-1)/4} \prod_{1 \leq j < n} g'_t(z^j) \prod_{1 \leq j < k \leq n} |z_t^j - z_t^k|.$$

Again letting $f : D \rightarrow \mathbb{D}$ denote a conformal bijection taking the domain D to the unit disk with $f(0) = 0$, this martingale suggests the probability of this triple point (or more generally n -tuple point) event is

$$(11) \quad (\text{const} + o(1)) \times (\varepsilon f'(0))^{(n^2-1)/4} \prod_{1 \leq j \leq n} \varepsilon f'(z^j) \prod_{1 \leq j < k \leq n} |f(z^j) - f(z^k)|.$$

The exponent of $(n^2 - 1)/4$ agrees with the value computed in [Dup87, Ken00].

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