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Comparing globular complex and flow

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ABSTRACT. A functor is constructed from the category of globular CW-complexes to that of flows. It allows the comparison of the S-homotopy equivalences (resp. the T-homotopy equivalences) of globular complexes with the S-homotopy equivalences (resp. the T-homotopy equivalences) of flows. Moreover, it is proved that this functor induces an equivalence of categories from the localization of the category of globular CW-complexes with respect to S-homotopy equivalences to the localization of the category of flows with respect to weak S-homotopy equivalences. As an application, we construct the underlying homotopy type of a flow.

CONTENTS

Part I. Outline of the paper

The category of *globular CW-complexes* **glCW** was introduced in [GG03] for modelling higher-dimensional automata and dihomotopy, the latter being an equivalence relation preserving their computer-scientific properties, like the *initial or final states*, the presence or not of *deadlocks* or of *unreachable states*, and more generally any computer-scientific property invariant by refinement of observation. More precisely, the classes of S-*homotopy equivalences* and of T-*homotopy equivalences* were defined. The category of *flows* as well as the notion of S-*homotopy equivalence* of flows are introduced in [Gau03d]. The notion of S-homotopy equivalence of flows is interpreted in [Gau03d] as the notion of homotopy arising from a model category structure. The weak equivalences of this model structure are called the *weak* S-*homotopy equivalences*.

The purpose of this paper is the comparison of the framework of globular CW-complexes with the framework of flows. More precisely, we are going to construct a functor

$$\mathrm{cat} : \mathbf{glCW} \longrightarrow \mathbf{Flow}$$

from the category of globular CW-complexes to that of flows inducing an equivalence of categories from the localization $\mathbf{glCW}[\mathcal{SH}^{-1}]$ of the category of globular CW-complexes with respect to the class $\mathcal{SH}$ of S-homotopy equivalences to the localization $\mathbf{Flow}[\mathcal{S}^{-1}]$ of the category of flows with respect to the class $\mathcal{S}$ of weak S-homotopy equivalences. Moreover, a class of T-homotopy equivalences of flows will be constructed in this paper so that there exists, up to weak S-homotopy, a T-homotopy equivalence of globular CW-complexes $f : X \longrightarrow Y$ if and only if there exists a T-homotopy equivalence of flows $g : \mathrm{cat}(X) \longrightarrow \mathrm{cat}(Y)$.

Part II introduces the category of *globular complexes* **glTop**, which is slightly larger than the category of globular CW-complexes **glCW**. Indeed, the latter category is not a big enough setting for several constructions that are going to be used. Part III builds the functor cat : **glTop** $\longrightarrow$ **Flow**. Part IV is a technical part which proves that two globular complexes X and U are S-homotopy equivalent if and only if the corresponding flows $\mathrm{cat}(X)$ and $\mathrm{cat}(U)$ are S-homotopy equivalent. Part V proves that the functor cat : **glCW** $\longrightarrow$ **Flow** from the category of globular CW-complexes to that of flows induces an equivalence of categories from the localization $\mathbf{glCW}[\mathcal{SH}^{-1}]$ of the category of globular CW-complexes with respect to the class of S-homotopy equivalences to the localization $\mathbf{Flow}[\mathcal{S}^{-1}]$ of the category of flows with respect to the class of weak S-homotopy equivalences. At last, Part VI studies and compares the notion of T-homotopy equivalence for globular complexes and flows. And Part VII applies all previous results to the construction of the *underlying homotopy type* of a flow.

Warning. This paper is the sequel of *A model category for the homotopy theory of concurrency* [Gau03d], where the category of flows was introduced. This work is focused on the relation between the category of globular CW-complexes and the category of flows. A first version of the category of globular CW-complexes was

introduced in a joint work with Eric Goubault [GG03]. A detailed abstract (in French) of [Gau03d] and of this paper can be found in [Gau03b] and [Gau03c].

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Part II. S-homotopy and globular complex

1. Introduction

The category of *globular complexes* is introduced in Section 2. This requires the introduction of several other notions, for instance the notion of *multipointed topological space*. Section 3 carefully studies the behavior of the functor $X \mapsto$ **glTOP**(X, Y) for a given Y with respect to the globular decomposition of X where **glTOP**(X, Y) is the set of morphisms of globular complexes from X to Y equipped with the Kelleyfication of the compact-open topology. At last, Section 4 defines and studies the notion of S-homotopy equivalence of globular complexes. In particular, a cylinder functor corresponding to this notion of equivalence is constructed.

2. The category of globular complexes

2.1. Compactly generated topological spaces. The category **Top** of *compactly generated topological spaces* (i.e., of weak Hausdorff k-spaces) is complete, cocomplete and cartesian closed (details for this kind of topological spaces are in [Bro88, May99], the appendix of [Lew78] and also the preliminaries of [Gau03d]). Let us denote by **TOP**$(X, -)$ the right adjoint of the functor $- \times X :$ **Top** $\longrightarrow$ **Top**. For any compactly generated topological space X and Y, the space **TOP**(X, Y) is the set of continuous maps from X to Y equipped with the Kelleyfication of the compact-open topology. For the sequel, any topological space will be supposed to be compactly generated. A *compact space* is always Hausdorff.

2.2. NDR pairs.

Definition II.2.1. Let $i : A \longrightarrow B$ and $p : X \longrightarrow Y$ be maps in a category $\mathcal{C}$. Then i has the *left lifting property* (LLP) with respect to p (or p has the *right lifting property* (RLP) with respect to i) if for any commutative square

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ i \downarrow & \nearrow^{g} & \downarrow p \\ B & \xrightarrow{\beta} & Y \end{array}$$

there exists g making both triangles commutative.

A Hurewicz fibration is a continuous map having the RLP with respect to the continuous maps $\{0\} \times M \subset [0, 1] \times M$ for any topological space M. In particular, any continuous map having a discrete codomain is a Hurewicz fibration. A Hurewicz cofibration is a continuous map having the homotopy extension property. In the category of compactly generated topological spaces, any Hurewicz cofibration is a closed inclusion of topological spaces [Lew78]. There exists a model structure on the category of compactly generated topological spaces such that the cofibrations

are the Hurewicz cofibrations, the fibrations are the Hurewicz fibrations, and the weak equivalences are the homotopy equivalences ([Str66, Str68, Str72] and also [Col99]). In this model structure, all topological spaces are fibrant and cofibrant. The class of Hurewicz cofibrations coincides with the class of NDR pairs. For any NDR pair $(Z, \partial Z)$, one has [Ste67, Whi78, FHT01, Hat02]:

1. There exists a continuous map $\mu : Z \longrightarrow [0, 1]$ such that $\mu^{-1}(\{0\}) = \partial Z$.
2. There exists a continuous map $r : Z \times [0, 1] \longrightarrow Z \times \{0\} \cup \partial Z \times [0, 1]$ which is the identity on $Z \times \{0\} \cup \partial Z \times [0, 1] \subset Z \times [0, 1]$.

These properties are used in the proofs of Theorem III.5.2 and Theorem VI.3.5.

2.3. Definition of a globular complex. A *globular complex* is a topological space together with a structure describing the sequential process of attaching *globular cells*. The class of globular complexes includes the class of *globular* CW-*complexes*. A general globular complex may require an arbitrary long transfinite construction. We must introduce this generalization because several constructions do not stay within the class of globular CW-complexes.

Definition II.2.2. A *multipointed topological space* (X, X^0) is a pair of topological spaces such that X^0 is a discrete subspace of X. A morphism of multipointed topological spaces $f : (X, X^0) \longrightarrow (Y, Y^0)$ is a continuous map $f : X \longrightarrow Y$ such that $f(X^0) \subset Y^0$. The corresponding category is denoted by $\mathbf{Top}^{\mathrm{m}}$. The set X^0 is called the 0-*skeleton* of (X, X^0). The space X is called the *underlying topological space* of (X, X^0).

A multipointed space of the form (X^0, X^0) where X^0 is a discrete topological space will be called a *discrete multipointed space* and will be frequently identified with X^0 itself.

Proposition II.2.3. *The category of multipointed topological spaces is cocomplete.*

Proof. This is due to the facts that the category of topological spaces is cocomplete and that the colimit of discrete spaces is a discrete space. □

Definition II.2.4. Let Z be a topological space. The *globe of Z*, which is denoted by $\mathrm{Glob}^{\mathrm{top}}(Z)$, is the multipointed space $(|\mathrm{Glob}^{\mathrm{top}}(Z)|, \{0, 1\})$ where the topological space $|\mathrm{Glob}^{\mathrm{top}}(Z)|$ is the quotient of $\{0, 1\} \sqcup (Z \times [0, 1])$ by the relations $(z, 0) = (z', 0) = 0$ and $(z, 1) = (z', 1) = 1$ for any $z, z' \in Z$.

In particular, $\mathrm{Glob}^{\mathrm{top}}(\varnothing)$ is the multipointed space $(\{0, 1\}, \{0, 1\})$.

Notation II.2.5. If Z is a singleton, then the globe of Z is denoted by $\overrightarrow{I}^{\mathrm{top}}$.

Any ordinal can be viewed as a small category whose objects are the elements of λ, that is the ordinals $\gamma < \lambda$, and where there exists a morphism $\gamma \longrightarrow \gamma'$ if and only if $\gamma \leqslant \gamma'$.

Definition II.2.6. Let $\mathcal{C}$ be a cocomplete category. Let λ be an ordinal. A λ-*sequence* in $\mathcal{C}$ is a colimit-preserving functor $X : \lambda \longrightarrow \mathcal{C}$. Since X preserves colimits, for all limit ordinals $\gamma < \lambda$, the induced map $\varinjlim_{\beta < \gamma} X_\beta \longrightarrow X_\gamma$ is an isomorphism. The morphism $X_0 \longrightarrow \varinjlim X$ is called the *transfinite composition* of X.

Definition II.2.7. A *relative globular precomplex* is a λ-sequence of multipointed topological spaces $X : \lambda \longrightarrow \mathbf{Top}^{\mathrm{m}}$ such that for any $\beta < \lambda$, there exists a pushout diagram of multipointed topological spaces

$$
\begin{array}{ccc}
\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta) & \xrightarrow{\ \phi_\beta\ } & X_\beta \\
\downarrow & & \downarrow \\
\mathrm{Glob}^{\mathrm{top}}(Z_\beta) & \longrightarrow & X_{\beta+1}
\end{array}
$$

where the pair $(Z_\beta, \partial Z_\beta)$ is a NDR pair of compact spaces. The morphism

$$\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta) \longrightarrow \mathrm{Glob}^{\mathrm{top}}(Z_\beta)$$

is induced by the closed inclusion $\partial Z_\beta \subset Z_\beta$.

Definition II.2.8. A *globular precomplex* is a λ-sequence of multipointed topological spaces $X : \lambda \longrightarrow \mathbf{Top}^{\mathrm{m}}$ such that X is a relative globular precomplex and such that $X_0 = (X^0, X^0)$ with X^0 a discrete space.

Let X be a globular precomplex. The 0-skeleton of $\varinjlim X$ is equal to X^0.

Definition II.2.9. A morphim of globular precomplexes $f : X \longrightarrow Y$ is a morphism of multipointed spaces still denoted by f from $\varinjlim X$ to $\varinjlim Y$.

Notation II.2.10. If X is a globular precomplex, then the underlying topological space of the multipointed space $\varinjlim X$ is denoted by $|X|$ and the 0-skeleton of the multipointed space $\varinjlim X$ is denoted by X^0.

Definition II.2.11. Let X be a globular precomplex. The space $|X|$ is called the *underlying topological space* of X. The set X^0 is called the *0-skeleton* of X. The family $(\partial Z_\beta, Z_\beta, \phi_\beta)_{\beta < \lambda}$ is called the *globular decomposition* of X.

As set, the topological space X is by construction the disjoint union of X^0 and of the $|\mathrm{Glob}^{\mathrm{top}}(Z_\beta \backslash \partial Z_\beta)| \backslash \{0, 1\}$.

Definition II.2.12. Let X be a globular precomplex. A morphism of globular precomplexes $\gamma : \overrightarrow{I}^{\mathrm{top}} \longrightarrow X$ is a *nonconstant execution path* of X if there exists $t_0 = 0 < t_1 < \cdots < t_n = 1$ such that:

1. $\gamma(t_i) \in X^0$ for any i.
2. $\gamma(]t_i, t_{i+1}[) \subset \mathrm{Glob}^{\mathrm{top}}(Z_{\beta_i} \backslash \partial Z_{\beta_i})$ for some $(\partial Z_{\beta_i}, Z_{\beta_i})$ of the globular decomposition of X.
3. For $0 \leqslant i < n$, there exists $z_\gamma^i \in Z_{\beta_i} \backslash \partial Z_{\beta_i}$ and a strictly increasing continuous map $\psi_\gamma^i : [t_i, t_{i+1}] \longrightarrow [0, 1]$ such that $\psi_\gamma^i(t_i) = 0$ and $\psi_\gamma^i(t_{i+1}) = 1$ and for any $t \in [t_i, t_{i+1}]$, $\gamma(t) = (z_\gamma^i, \psi_\gamma^i(t))$.

In particular, the restriction $\gamma\restriction_{]t_i, t_{i+1}[}$ of γ to $]t_i, t_{i+1}[$ is one-to-one. The set of nonconstant execution paths of X is denoted by $\mathbb{P}^{\mathrm{ex}}(X)$.

Definition II.2.13. A morphism of globular precomplexes $f : X \longrightarrow Y$ is *non-decreasing* if the canonical set map $\mathbf{Top}([0,1], |X|) \longrightarrow \mathbf{Top}([0,1], |Y|)$ induced by composition by f yields a set map $\mathbb{P}^{\mathrm{ex}}(X) \longrightarrow \mathbb{P}^{\mathrm{ex}}(Y)$. In other terms, one has the
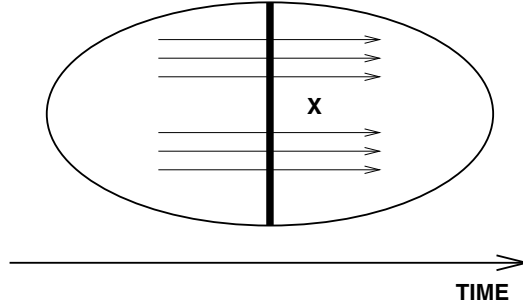
FIGURE 1. Symbolic representation of $\mathrm{Glob}^{\mathrm{top}}(X)$ for some compact topological space X.

commutative diagram of sets

$$\begin{array}{ccc} \mathbb{P}^{\mathrm{ex}}(X) & \longrightarrow & \mathbb{P}^{\mathrm{ex}}(Y) \\ \cup \downarrow & & \cup \downarrow \\ \mathbf{Top}([0,1],|X|) & \longrightarrow & \mathbf{Top}([0,1],|Y|). \end{array}$$

Definition II.2.14. A *globular complex* (resp. a *relative globular complex*) X is a globular precomplex (resp. a relative globular precomplex) such that the attaching maps ϕ_β are nondecreasing. A morphism of globular complexes is a morphism of globular precomplexes which is nondecreasing. The category of globular complexes together with the morphisms of globular complexes as defined above is denoted by **glTop**. The set **glTop**(X,Y) of morphisms of globular complexes from X to Y equipped with the Kelleyfication of the compact-open topology is denoted by **glTOP**(X,Y).

Forcing the restrictions $\gamma \upharpoonright_{]t_i,t_{i+1}[}$ to be one-to-one means that only the "stretched situation" is considered. It would be possible to build a theory of nonstretched execution paths, nonstretched globular complexes and nonstretched morphisms of globular complexes but this would be without interest regarding the complexity of the technical difficulties we would meet.

Definition II.2.15. Let X be a globular complex. A point α of X^0 such that there are no nonconstant execution paths ending at α (resp. starting from α) is called an *initial state* (resp. *final state*). More generally, a point of X^0 will be sometimes called a *state* as well.

A very simple example of globular complex is obtained by concatenating globular complexes of the form $\mathrm{Glob}^{\mathrm{top}}(Z_j)$ for $1 \leqslant i \leqslant n$ by identifying the final state 1 of $\mathrm{Glob}^{\mathrm{top}}(Z_j)$ with the initial state 0 of $\mathrm{Glob}^{\mathrm{top}}(Z_{j+1})$.

Notation II.2.16. This globular complex will be denoted by

$$\mathrm{Glob}^{\mathrm{top}}(Z_1) * \mathrm{Glob}^{\mathrm{top}}(Z_2) * \cdots * \mathrm{Glob}^{\mathrm{top}}(Z_n).$$

2.4. Globular CW-complex. Let $n \geqslant 1$. Let $\mathbf{D}^n$ be the closed n-dimensional disk defined by the set of points $(x_1, \ldots, x_n)$ of $\mathbb{R}^n$ such that $x_1^2 + \cdots + x_n^2 \leqslant 1$ endowed with the topology induced by that of $\mathbb{R}^n$. Let $\mathbf{S}^{n-1} = \partial \mathbf{D}^n$ be the boundary of $\mathbf{D}^n$ for $n \geqslant 1$, that is to say the set of $(x_1, \ldots, x_n) \in \mathbf{D}^n$ such that $x_1^2 + \cdots + x_n^2 = 1$. Notice that $\mathbf{S}^0$ is the discrete two-point topological space $\{-1, +1\}$. Let $\mathbf{D}^0$ be the one-point topological space. Let $\mathbf{S}^{-1}$ be the empty space.

Definition II.2.17. [GG03] A *globular* CW-*complex* X is a globular complex such that its globular decomposition $(\partial Z_\beta, Z_\beta, \phi_\beta)_{\beta < \lambda}$ satisfies the following property: there exists a strictly increasing sequence $(\kappa_n)_{n \geqslant 0}$ of ordinals with

$$\kappa_0 = 0, \quad \sup_{n \geqslant 0} \kappa_n = \lambda,$$

and such that for any $n \geqslant 0$, one has the following facts:

1. For any $\beta \in [\kappa_n, \kappa_{n+1}[$, $(Z_\beta, \partial Z_\beta) = (\mathbf{D}^n, \mathbf{S}^{n-1})$.
2. One has the pushout of multipointed topological spaces

$$
\begin{array}{ccc}
\bigsqcup_{i \in [\kappa_n, \kappa_{n+1}[} \mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n-1}) & \xrightarrow{\phi_n} & X_{\kappa_n} \\
\downarrow & & \downarrow \\
\bigsqcup_{i \in [\kappa_n, \kappa_{n+1}[} \mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^n) & \longrightarrow & X_{\kappa_{n+1}}
\end{array}
$$

where ϕ_n is the morphism of globular complexes induced by the ϕ_β for $\beta \in [\kappa_n, \kappa_{n+1}[$.

The full and faithful subcategory of **glTop** of globular CW-complexes is denoted by **glCW**. Notice that we necessarily have $\varinjlim_n X_{\kappa_n} = X$.

One also has:

Proposition II.2.18 ([GG03])**.** *The globe functor* $X \mapsto \mathrm{Glob}^{\mathrm{top}}(X)$ *induces a functor from* CW-*complexes to globular* CW-*complexes.*

3. Morphisms of globular complexes and colimits

The category of general topological spaces is denoted by $\mathcal{T}$.

Proposition II.3.1. *The inclusion of sets* $i : \mathbf{glTOP}(X, Y) \longrightarrow \mathbf{TOP}(|X|, |Y|)$ *is an inclusion of topological spaces, that is* $\mathbf{glTOP}(X, Y)$ *is the subset of morphisms of globular complexes of the space* $\mathbf{TOP}(|X|, |Y|)$ *equipped with the Kelleyfication of the relative topology.*

Proof. Let $\mathrm{Cop}(|X|, |Y|)$ be the set of continuous maps from $|X|$ to $|Y|$ equipped with the compact-open topology. The continuous map

$$\mathbf{glTop}(X, Y) \cap \mathrm{Cop}(|X|, |Y|) \longrightarrow \mathrm{Cop}(|X|, |Y|)$$

is an inclusion of topological spaces. Let $f : Z \to k(\mathrm{Cop}(|X|, |Y|))$ be a continuous map such that $f(Z) \subset \mathbf{glTop}(X, Y)$ where Z is an object of **Top** and where $k(-)$ is the Kelleyfication functor. Then $f : Z \longrightarrow \mathrm{Cop}(|X|, |Y|)$ is continuous since the Kelleyfication is a right adjoint and since Z is a k-space. So f induces a continuous map $Z \to \mathbf{glTop}(X, Y) \cap \mathrm{Cop}(|X|, |Y|)$, and therefore a continuous map

$$Z \longrightarrow k(\mathbf{glTop}(X, Y) \cap \mathrm{Cop}(|X|, |Y|)) \cong \mathbf{glTOP}(X, Y). \qquad \square$$

Proposition II.3.2. *Let (X_i) and (Y_i) be two diagrams of objects of $\mathcal{T}$. Let $f :$ $(X_i) \longrightarrow (Y_i)$ be a morphism of diagrams such that for any i, $f_i : X_i \longrightarrow Y_i$ is an inclusion of topological spaces, i.e., f_i is one-to-one and X_i is homeomorphic to $f(X_i)$ equipped with the relative topology coming from the set inclusion $f(X_i) \subset Y_i$. Then the continuous map $\varprojlim X_i \longrightarrow \varprojlim Y_i$ is an inclusion of topological spaces, the limits $\varprojlim X_i$ and $\varprojlim Y_i$ being calculated in $\mathcal{T}$.*

Loosely speaking, the proposition above means that the limit in $\mathcal{T}$ of the relative topology is the relative topology of the limit.

Proof. Saying that $X_i \longrightarrow Y_i$ is an inclusion of topological spaces is equivalent to saying that the isomorphism of sets

$$\mathcal{T}(Z, X_i) \cong \{f \in \mathcal{T}(Z, Y_i); f(X_i) \subset Y_i\}$$

holds for any i and for any object Z of $\mathcal{T}$. But like in any category, one has the isomorphism of sets

$$\varprojlim \mathcal{T}(Z, X_i) \cong \mathcal{T}(Z, \varprojlim X_i)$$

and

$$\varprojlim \mathcal{T}(Z, Y_i) \cong \mathcal{T}(Z, \varprojlim Y_i).$$

Using the construction of limits in the category of sets, it is then obvious that the set $\mathcal{T}(Z, \varprojlim X_i)$ is isomorphic to the set

$$\{f \in \varprojlim \mathcal{T}(Z, Y_i); f_i(X_i) \in Y_i\}$$

for any object Z of $\mathcal{T}$. Hence the result. $\square$

Theorem II.3.3. *Let X be a globular complex with globular decomposition*

$$(\partial Z_\beta, Z_\beta, \phi_\beta)_{\beta < \lambda}.$$

Then for any limit ordinal $\beta \leqslant \lambda$, one has the homeomorphism

$$\mathbf{glTOP}(X_\beta, U) \cong \varprojlim{}_{\alpha < \beta}\mathbf{glTOP}(X_\alpha, U).$$

And for any $\beta < \lambda$, one has the pullback of topological spaces

$$
\begin{array}{ccc}
\mathbf{glTOP}(X_{\beta+1}, U) & \longrightarrow & \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z_\beta), U) \\
\downarrow & & \downarrow \\
\mathbf{glTOP}(X_\beta, U) & \longrightarrow & \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta), U).
\end{array}
$$

Proof. One has the isomorphism of sets

$$\mathbf{glTop}(X_\beta, U) \cong \varprojlim{}_{\alpha < \beta}\mathbf{glTop}(X_\alpha, U)$$

and the pullback of sets

$$
\begin{array}{ccc}
\mathbf{glTop}(X_{\beta+1}, U) & \longrightarrow & \mathbf{glTop}(\mathrm{Glob}^{\mathrm{top}}(Z_\beta), U) \\
\downarrow & & \downarrow \\
\mathbf{glTop}(X_\beta, U) & \longrightarrow & \mathbf{glTop}(\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta), U).
\end{array}
$$

One also has the isomorphism of topological spaces

$$\mathbf{TOP}(|X_\beta|, |U|) \cong \varprojlim{}_{\alpha < \beta}\mathbf{TOP}(|X_\alpha|, |U|)$$

and the pullback of spaces

$$\begin{array}{ccc} \mathbf{TOP}(|X_{\beta+1}|,|U|) & \longrightarrow & \mathbf{TOP}(|\mathrm{Glob}^{\mathrm{top}}(Z_\beta)|,|U|) \\ \downarrow & & \downarrow \\ \mathbf{TOP}(|X_\beta|,|U|) & \longrightarrow & \mathbf{TOP}(|\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta)|,|U|). \end{array}$$

The theorem is then a consequence of Proposition II.3.1, of Proposition II.3.2 and of the fact that the Kelleyfication functor is a right adjoint which therefore preserves all limits. $\qquad\square$

Proposition II.3.4. *Let X and U be two globular complexes. Then one has the homeomorphism*

$$\mathbf{glTOP}(X,U) \cong \bigsqcup_{\phi:X^0\longrightarrow U^0} \{f \in \mathbf{glTOP}(X,U), f^0 = \phi\}.$$

Proof. The composite set map

$$\mathbf{glTOP}(X,U) \to \mathbf{TOP}(X,U) \longrightarrow \mathbf{TOP}(X^0,U^0)$$

is continuous and $\mathbf{TOP}(X^0,U^0)$ is a discrete topological space. $\qquad\square$

Let X be a globular complex. The set $\mathbb{P}^{\mathrm{ex}}X$ of nonconstant execution paths of X can be equipped with the Kelleyfication of the compact-open topology. The mapping $\mathbb{P}^{\mathrm{ex}}$ yields a functor from $\mathbf{glTop}$ to $\mathbf{Top}$ by sending a morphism of globular complexes f to $\gamma \mapsto f \circ \gamma$.

Definition II.3.5. A *globular subcomplex* X of a globular complex Y is a globular complex X such that the underlying topological space is included in the one of Y and such that the inclusion map $X \subset Y$ is a morphism of globular complexes.

The following is immediate:

Proposition II.3.6. *Let X be a globular complex. Then there is a natural isomorphism of topological spaces* $\mathbf{glTOP}(\overrightarrow{I}^{\mathrm{top}}, X) \cong \mathbb{P}^{\mathrm{ex}}X$.

Proposition II.3.7. *Let Z be a topological space. Then one has the isomorphism of topological spaces* $\mathbb{P}^{\mathrm{ex}}(\mathrm{Glob}^{\mathrm{top}}(Z)) \cong Z \times \mathbf{glTOP}(\overrightarrow{I}^{\mathrm{top}}, \overrightarrow{I}^{\mathrm{top}})$.

Proof. There is a canonical inclusion

$$\mathbb{P}^{\mathrm{ex}}(\mathrm{Glob}^{\mathrm{top}}(Z)) \subset \mathbf{TOP}([0,1], Z \times [0,1]).$$

The image of this inclusion is exactly the subspace of

$$f = (f_1, f_2) \in \mathbf{TOP}([0,1], Z \times [0,1])$$

such that $f_1 : [0,1] \longrightarrow Z$ is a constant map and such that $f_2 : [0,1] \longrightarrow [0,1]$ is a nondecreasing continuous map with $f_2(0) = 0$ and $f_2(1) = 1$. Hence the isomorphism of topological spaces. $\qquad\square$

4. S-homotopy in glTop

4.1. S-homotopy in glTop. We now recall the notion of S-homotopy introduced in [GG03] for a particular case of globular complex.

Definition II.4.1. Two morphisms of globular complexes f and g from X to Y are said to be S-*homotopic* or S-*homotopy equivalent* if there exists a continuous map $H : [0, 1] \times X \longrightarrow Y$ such that for any $u \in [0, 1]$, $H_u = H(u, -)$ is a morphism of globular complexes from X to Y and such that $H_0 = f$ and $H_1 = g$. We denote this situation by $f \sim_S g$.

Proposition II.3.6 justifies the following definition:

Definition II.4.2. Two execution paths of a globular complex X are S-*homotopic* or S-*homotopy equivalent* if the corresponding morphisms of globular complexes from $\overrightarrow{I}^{\mathrm{top}}$ to X are S-homotopy equivalent.

Definition II.4.3. Two globular complexes X and Y are S-*homotopy equivalent* if and only if there exists two morphisms of **glTop** $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g \sim_S \mathrm{Id}_Y$ and $g \circ f \sim_S \mathrm{Id}_X$. This defines an equivalence relation on the set of morphisms between two given globular complexes called S-*homotopy*. The maps f and g are called S-*homotopy equivalences*. The mapping g is called an S-*homotopy inverse* of f.

4.2. The pairing $\boxtimes$ between a compact topological space and a globular complex. Let U be a compact topological space. Let X be a globular complex with the globular decomposition $(\partial Z_\beta, Z_\beta, \phi_\beta)_{\beta < \lambda}$. Let $(U \boxtimes X)_0 := (X^0, X^0)$. If Z is any topological space, let $U \boxtimes \mathrm{Glob}^{\mathrm{top}}(Z) := \mathrm{Glob}^{\mathrm{top}}(U \times Z)$.

If $(Z, \partial Z)$ is a NDR pair, then the continuous map $i : [0, 1] \times \partial Z \cup \{0\} \times Z \longrightarrow [0, 1] \times Z$ has a retract $r : [0, 1] \times Z \longrightarrow [0, 1] \times \partial Z \cup \{0\} \times Z$. Therefore $i \times \mathrm{Id}_U : [0, 1] \times \partial Z \times U \cup \{0\} \times Z \times U \longrightarrow [0, 1] \times Z \times U$ has a retract $r \times \mathrm{Id}_U : [0, 1] \times Z \times U \longrightarrow [0, 1] \times \partial Z \times U \cup \{0\} \times Z \times U$. Therefore $(U \times Z, U \times \partial Z)$ is a NDR pair.

Let us suppose $(U \boxtimes X)_\beta$ defined for an ordinal β such that $\beta + 1 < \lambda$ and assume that $(U \boxtimes X)_\beta$ has the globular decomposition $(U \times \partial Z_\mu, U \times Z_\mu, \psi_\mu)_{\mu < \beta}$. From the morphism of globular complexes $\phi_\beta : \mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta) \longrightarrow X_\beta$, one obtains the morphism of globular complexes $\psi_\beta : \mathrm{Glob}^{\mathrm{top}}(U \times \partial Z_\beta) \longrightarrow (U \boxtimes X)_\beta$ defined as follows: an element $\phi_\beta(z)$ belongs to a unique $Z_\mu \backslash \partial Z_\mu$. Then let $\psi_\beta(u, z) = (u, \phi_\beta(z))$. Then let us define $(U \boxtimes X)_{\beta+1}$ by the pushout of multipointed topological spaces

$$
\begin{array}{ccc}
U \boxtimes \mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta) & \xrightarrow{\psi_\beta} & U \boxtimes X_\beta \\
\downarrow & & \downarrow \\
U \boxtimes \mathrm{Glob}^{\mathrm{top}}(Z_\beta) & \longrightarrow & U \boxtimes X_{\beta+1}.
\end{array}
$$

Then the globular decomposition of $(U \boxtimes X)_{\beta+1}$ is $(U \times \partial Z_\mu, U \times Z_\mu, \psi_\mu)_{\mu < \beta+1}$. If $\beta \leqslant \lambda$ is a limit ordinal, let $(U \boxtimes X)_\beta = \varinjlim_{\mu < \beta}(U \boxtimes X)_\mu$ as multipointed topological spaces.

Proposition II.4.4. *Let U be a compact space. Let X be a globular complex. Then the underlying space $|U \boxtimes X|$ of $U \boxtimes X$ is homeomorphic to the quotient of $U \times |X|$ by*

the equivalence relation making the identification $(u, x) = (u', x)$ for any $u, u' \in U$ and for any $x \in X^0$ and equipped with the final topology.

Proof. The graph of this equivalence relation is $\Delta U \times |X| \times |X| \subset U \times U \times |X| \times |X|$ where ΔU is the diagonal of U. It is a closed subspace of $U \times U \times |X| \times |X|$. Therefore the quotient set equipped with the final topology is still weak Hausdorff, and therefore compactly generated. It then suffices to proceed by transfinite induction on the globular decomposition of X. $\square$

The underlying set of $U \boxtimes X$ is then exactly equal to $X^0 \sqcup (U \times (X \backslash X^0))$. The point (u, x) with $x \in X \backslash X^0$ will be denoted also by $u \boxtimes x$. If $x \in X^0$, then by convention $u \boxtimes x = u' \boxtimes x$ for any $u, u' \in [0, 1]$.

Proposition II.4.5. *Let U and V be two compact spaces. Let X be a globular complex. Then there exists a natural morphism of globular complexes $(U \times V) \boxtimes X \cong U \boxtimes (V \boxtimes X)$.*

Proof. Transfinite induction on the globular decomposition of X. $\square$

4.3. Cylinder functor for S-homotopy in glTop.

Proposition II.4.6. *Let f and g be two morphisms of globular complexes from X to Y. Then f and g are S-homotopic if and only if there exists a continuous map*

$$h \in \mathbf{Top}([0, 1], \mathbf{glTOP}(X, Y))$$

such that $h(0) = f$ and $h(1) = g$.

Proof. Suppose that f and g are S-homotopic. Then the S-homotopy H yields a continuous map

$$h \in \mathbf{Top}([0, 1] \times |X|, |Y|) \cong \mathbf{Top}([0, 1], \mathbf{TOP}(|X|, |Y|))$$

by construction, and h is necessarily in

$$\mathbf{Top}([0, 1], \mathbf{glTOP}(X, Y))$$

by hypothesis. Conversely, if

$$h \in \mathbf{Top}([0, 1], \mathbf{glTOP}(X, Y))$$

is such that $h(0) = f$ and $h(1) = g$, then the isomorphism

$$\mathbf{Top}([0, 1], \mathbf{TOP}(|X|, |Y|)) \cong \mathbf{Top}([0, 1] \times |X|, |Y|)$$

provides a map $H \in \mathbf{Top}([0, 1] \times |X|, |Y|)$ which is a S-homotopy from f to g. $\square$

Theorem II.4.7. *Let U be a connected nonempty topological space. Let X and Y be two globular complexes. Then there exists an isomorphism of sets*

$$\mathbf{glTop}(U \boxtimes X, Y) \cong \mathbf{Top}(U, \mathbf{glTOP}(X, Y)).$$

Proof. If X is a singleton (this implies in particular that $X = X^0$), then $U \boxtimes X = X$. So in this case, $\mathbf{glTop}(U \boxtimes X, Y) \cong \mathbf{Top}(U, \mathbf{glTOP}(X, Y)) \cong Y^0$ *since U is connected and nonempty* and by Proposition II.3.4. Now if $X = \mathrm{Glob}^{\mathrm{top}}(Z)$ for some compact space Z, then

$$\mathbf{glTop}(U \boxtimes X, Y) \cong \mathbf{glTop}(\mathrm{Glob}^{\mathrm{top}}(Z \times U), Y)$$

and it is straightforward to check that the latter space is isomorphic to

$$\mathbf{Top}(U, \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), Y)).$$

Hence the isomorphism

$$\mathbf{glTop}(U \boxtimes X, Y) \cong \mathbf{Top}(U, \mathbf{glTOP}(X, Y))$$

if X is a point or a globe.

Let $(\partial Z_\beta, Z_\beta, \phi_\beta)_{\beta < \lambda}$ be the globular decomposition of X. Then one deduces that

$$\mathbf{glTop}((U \boxtimes X)_\beta, Y) \cong \mathbf{Top}(U, \mathbf{glTOP}(X_\beta, Y)).$$

for any β by an easy transfinite induction, using the construction of $U \boxtimes X$ and Theorem II.3.3. $\qquad\square$

Definition II.4.8. Let $\mathcal{C}$ be a category. A *cylinder* is a functor $I : \mathcal{C} \longrightarrow \mathcal{C}$ together with natural transformations $i_0, i_1 : \mathrm{Id}_\mathcal{C} \longrightarrow I$ and $p : I \longrightarrow \mathrm{Id}_\mathcal{C}$ such that $p \circ i_0$ and $p \circ i_1$ are the identity natural transformation.

Corollary II.4.9. *The mapping $X \mapsto [0,1] \boxtimes X$ induces a functor from $\mathbf{glTop}$ to itself which is a cylinder functor with the natural transformations $e_i : \{i\} \boxtimes - \longrightarrow [0,1] \boxtimes -$ induced by the inclusion maps $\{i\} \subset [0,1]$ for $i \in \{0,1\}$ and with the natural transformation $p : [0,1] \boxtimes - \longrightarrow \{0\} \boxtimes -$ induced by the constant map $[0,1] \longrightarrow \{0\}$. Moreover, two morphisms of globular complexes f and g from X to Y are S-homotopic if and only if there exists a morphism of globular complexes $H : [0,1] \boxtimes X \longrightarrow Y$ such that $H \circ e_0 = f$ and $H \circ e_1 = g$. Moreover $e_0 \circ H \sim_S \mathrm{Id}$ and $e_1 \circ H \sim_S \mathrm{Id}$.*

Proof. Consequence of Proposition II.4.6 and Theorem II.4.7. $\qquad\square$

We are now ready for the construction of the functor $\mathrm{cat} : \mathbf{glTop} \longrightarrow \mathbf{Flow}$.

Part III. Associating a flow with any globular CW-complex

1. Introduction

After a short reminder about the category of flows in Section 2, the functor $\mathrm{cat} : \mathbf{glTop} \longrightarrow \mathbf{Flow}$ is constructed in Section 3. For that purpose, the notion of *quasi-flow* is introduced. Section 4 comes back to the case of flows by explicitly calculating the pushout of a morphism of flows of the form $\mathrm{Glob}(\partial Z) \longrightarrow \mathrm{Glob}(Z)$. This will be used in Section 5 and in Part V. Section 5 proves that for any globular complex X, the natural continuous map $\mathbb{P}^{\mathrm{top}} X \longrightarrow \mathbb{P} X$ has a right-hand inverse $i_X : \mathbb{P} X \longrightarrow \mathbb{P}^{\mathrm{top}} X$ (Theorem III.5.2). The latter map has no reason to be natural.

2. The category of flows

Definition III.2.1 ([Gau03d])**.** A *flow* X consists of a topological space $\mathbb{P} X$, a discrete space X^0, two continuous maps s and t from $\mathbb{P} X$ to X^0 and a continuous and associative map $* : \{(x,y) \in \mathbb{P} X \times \mathbb{P} X; t(x) = s(y)\} \longrightarrow \mathbb{P} X$ such that $s(x*y) = s(x)$ and $t(x*y) = t(y)$. A morphism of flows $f : X \longrightarrow Y$ consists of a set map $f^0 : X^0 \longrightarrow Y^0$ together with a continuous map $\mathbb{P} f : \mathbb{P} X \longrightarrow \mathbb{P} Y$ such that $f(s(x)) = s(f(x))$, $f(t(x)) = t(f(x))$ and $f(x*y) = f(x)*f(y)$. The corresponding category is denoted by $\mathbf{Flow}$.

The continuous map $s : \mathbb{P}X \longrightarrow X^0$ is called the *source map*. The continuous map $t : \mathbb{P}X \longrightarrow X^0$ is called the *target map*. One can canonically extend these two maps to the whole underlying topological space $X^0 \sqcup \mathbb{P}X$ of X by setting $s(x) = x$ and $t(x) = x$ for $x \in X^0$.

The topological space X^0 is called the 0-*skeleton of* X.[1] The 0-dimensional elements of X are called *states* or *constant execution paths*.

The elements of $\mathbb{P}X$ are called *nonconstant execution paths*. If γ_1 and γ_2 are two nonconstant execution paths, then $\gamma_1 * \gamma_2$ is called the concatenation or the composition of γ_1 and γ_2. For $\gamma \in \mathbb{P}X$, $s(\gamma)$ is called the *beginning* of γ and $t(\gamma)$ the *ending* of γ.

Notation III.2.2. For $\alpha, \beta \in X^0$, let $\mathbb{P}_{\alpha,\beta}X$ be the subspace of $\mathbb{P}X$ equipped the Kelleyfication of the relative topology consisting of the nonconstant execution paths of X with beginning α and with ending β.

Definition III.2.3 ([Gau03d])**.** Let Z be a topological space. Then the *globe* of Z is the flow $\mathrm{Glob}(Z)$ defined as follows: $\mathrm{Glob}(Z)^0 = \{0, 1\}$, $\mathbb{P}\mathrm{Glob}(Z) = Z$, $s(z) = 0$, $t(z) = 1$ for any $z \in Z$ and the composition law is trivial.

Definition III.2.4 ([Gau03d])**.** The *directed segment* $\overrightarrow{I}$ is the flow defined as follows: $\overrightarrow{I}^0 = \{0, 1\}$, $\mathbb{P}\overrightarrow{I} = \{[0, 1]\}$, $s = 0$ and $t = 1$.

Definition III.2.5. Let X be a flow. A point α of X^0 such that there are no nonconstant execution paths γ such that $t(\gamma) = \alpha$ (resp. $s(\gamma) = \alpha$) is called *initial state* (resp. *final state*).

Notation III.2.6. The space $\mathbf{FLOW}(X, Y)$ is the set $\mathbf{Flow}(X, Y)$ equipped with the Kelleyfication of the compact-open topology.

Proposition III.2.7 ([Gau03d] Proposition 4.15)**.** *Let X be a flow. Then one has the following natural isomorphism of topological spaces* $\mathbb{P}X \cong \mathbf{FLOW}(\overrightarrow{I}, X)$.

Theorem III.2.8 ([Gau03d] Theorem 4.17)**.** *The category* **Flow** *is complete and cocomplete. In particular, a terminal object is the flow* $\mathbf{1}$ *having the discrete set* $\{0, u\}$ *as underlying topological space with* 0-*skeleton* $\{0\}$ *and with path space* $\{u\}$. *And the initial object is the unique flow* $\varnothing$ *having the empty set as underlying topological space.*

Theorem III.2.9 ([Gau03d] Theorem 5.10)**.** *The mapping*

$$(X, Y) \mapsto \mathbf{FLOW}(X, Y)$$

induces a functor from **Flow** $\times$ **Flow** *to* **Top** *which is contravariant with respect to X and covariant with respect to Y. Moreover:*

1. *One has the homeomorphism*

$$\mathbf{FLOW}(\varinjlim_i X_i, Y) \cong \varprojlim_i \mathbf{FLOW}(X_i, Y)$$

for any colimit $\varinjlim_i X_i$ in **Flow**.
2. *For any finite limit $\varprojlim_i X_i$ in* **Flow**, *one has the homeomorphism*

$$\mathbf{FLOW}(X, \varprojlim_i Y_i) \cong \varprojlim_i \mathbf{FLOW}(X, Y_i).$$

---

[1] The reason of this terminology: the 0-skeleton of a flow will correspond to the 0-skeleton of a globular CW-complex by the functor cat; one could define for any $n \geqslant 1$ the n-skeleton of a globular CW-complex in an obvious way.

3. The functor cat from glTop to Flow

The purpose of this section is the proof of the following theorems:

Theorem III.3.1. *There exists a unique functor* $\mathrm{cat} : \mathbf{glTop} \longrightarrow \mathbf{Flow}$ *such that:*

1. *If* $X = X^0$ *is a discrete globular complex, then* $\mathrm{cat}(X)$ *is the achronal flow* X^0 *("achronal" meaning with an empty path space).*
2. *For any compact topological space* Z, $\mathrm{cat}(\mathrm{Glob}^{\mathrm{top}}(Z)) = \mathrm{Glob}(Z)$.
3. *For any globular complex* X *with globular decomposition* $(\partial Z_\beta, Z_\beta, \phi_\beta)_{\beta < \lambda}$, *for any limit ordinal* $\beta \leqslant \lambda$, *the canonical morphism of flows*

$$\varinjlim{}_{\alpha < \beta} \mathrm{cat}(X_\alpha) \longrightarrow \mathrm{cat}(X_\beta)$$

is an isomorphism of flows.
4. *For any globular complex* X *with globular decomposition* $(\partial Z_\beta, Z_\beta, \phi_\beta)_{\beta < \lambda}$, *for any* $\beta < \lambda$, *one has the pushout of flows*

$$
\begin{array}{ccc}
\mathrm{Glob}(\partial Z_\beta) & \xrightarrow{\mathrm{cat}(\phi_\beta)} & \mathrm{cat}(X_\beta) \\
\downarrow & & \downarrow \\
\mathrm{Glob}(Z_\beta) & \longrightarrow & \mathrm{cat}(X_{\beta+1}).
\end{array}
$$

Notation III.3.2. Let M be a topological space. Let γ_1 and γ_2 be two continuous maps from $[0, 1]$ to M with $\gamma_1(1) = \gamma_2(0)$. Let us denote by $\gamma_1 *_a \gamma_2$ (with $0 < a < 1$) the following continuous map: if $0 \leqslant t \leqslant a$, $(\gamma_1 *_a \gamma_2)(t) = \gamma_1(\frac{t}{a})$ and if $a \leqslant t \leqslant 1$, $(\gamma_1 *_a \gamma_2)(t) = \gamma_2(\frac{t-a}{1-a})$.

Let us notice that if γ_1 and γ_2 are two nonconstant execution paths of a globular complex X, then $\gamma_1 *_a \gamma_2$ is a nonconstant execution path of X as well for any $0 < a < 1$.

Notation III.3.3. If X is a globular complex, let $\mathbb{P}X := \mathbb{P}\mathrm{cat}(X)$.

Theorem III.3.4. *The functor* $\mathrm{cat} : \mathbf{glTop} \longrightarrow \mathbf{Flow}$ *induces a natural transformation* $p : \mathbb{P}^{\mathrm{ex}} \longrightarrow \mathbb{P}$ *characterized by the following facts:*

1. *If* $X = \mathrm{Glob}^{\mathrm{top}}(Z)$, *then* $p_{\mathrm{Glob}^{\mathrm{top}}(Z)}(t \mapsto (z, t)) = z$ *for any* $z \in Z$.
2. *If* $\phi \in \mathbf{glTop}(\overrightarrow{I}^{\mathrm{top}}, \overrightarrow{I}^{\mathrm{top}})$, *if* γ *is a nonconstant execution path of a globular complex* X, *then* $p_X(\gamma \circ \phi) = p_X(\gamma)$.
3. *If* γ_1 *and* γ_2 *are two nonconstant execution paths of a globular complex* X, *then* $p_X(\gamma_1 *_a \gamma_2) = p_X(\gamma_1) * p_X(\gamma_2)$ *for any* $0 < a < 1$.

Proof. See Theorem III.3.11. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.1. Quasi-flow. In order to write down in a rigorous way the construction of the functor cat, the notion of *quasi-flow* seems to be required.

Definition III.3.5. A *quasi-flow* X is a set X^0 (the 0-skeleton) together with a topological space $\mathbb{P}^{\mathrm{top}}_{\alpha,\beta}X$ (which can be empty) for any $(\alpha, \beta) \in X^0 \times X^0$ and for any $\alpha, \beta, \gamma \in X^0 \times X^0 \times X^0$ a continuous map $]0, 1[\times \mathbb{P}^{\mathrm{top}}_{\alpha,\beta}X \times \mathbb{P}^{\mathrm{top}}_{\beta,\gamma}X \to \mathbb{P}^{\mathrm{top}}_{\alpha,\gamma}X$ sending (t, x, y) to $x *_t y$ and satisfying the following condition: if $ab = c$ and $(1-c)(1-d) = (1-b)$, then $(x *_a y) *_b z = x *_c (y *_d z)$ for any $(x, y, z) \in \mathbb{P}^{\mathrm{top}}_{\alpha,\beta}X \times \mathbb{P}^{\mathrm{top}}_{\beta,\gamma}X \times \mathbb{P}^{\mathrm{top}}_{\gamma,\delta}X$. A morphism of quasi-flows $f : X \longrightarrow Y$ is a set map $f^0 : X^0 \longrightarrow Y^0$ together

with for any $(\alpha, \beta) \in X^0 \times X^0$, a continuous map $\mathbb{P}^{\mathrm{top}}_{\alpha,\beta} X \to \mathbb{P}^{\mathrm{top}}_{f^0(\alpha), g^0(\beta)} Y$ such that $f(x *_t y) = f(x) *_t f(y)$ for any x, y and any $t \in]0, 1[$. The corresponding category is denoted by **qFlow**.

Theorem III.3.6 (Freyd's Adjoint Functor Theorem [Bor94, ML98])**.** *Let A and X be locally small categories. Assume that A is complete. Then a functor $G : A \longrightarrow X$ has a left adjoint if and only if it preserves all limits and satisfies the following "Solution Set Condition": for each object $x \in X$, there is a set of arrows $f_i : x \longrightarrow Ga_i$ such that for every arrow $h : x \longrightarrow Ga$ can be written as a composite $h = Gt \circ f_i$ for some i and some $t : a_i \longrightarrow a$.*

Theorem III.3.7. *The category of quasi-flows is complete and cocomplete.*

Proof. Let $X : I \longrightarrow \mathbf{qFlow}$ be a diagram of quasi-flows. Then the limit of this diagram is constructed as follows:

1. The 0-skeleton is $\varprojlim X^0$.
2. Let α and β be two elements of $\varprojlim X^0$ and let α_i and β_i be their image by the canonical continuous map $\varprojlim X^0 \longrightarrow X(i)^0$.
3. Let $\mathbb{P}^{\mathrm{top}}_{\alpha,\beta}(\varprojlim X) := \varprojlim_i \mathbb{P}^{\mathrm{top}}_{\alpha_i,\beta_i} X(i)$.

So all axioms required for the family of topological spaces $\mathbb{P}^{\mathrm{top}}_{\alpha,\beta}(\varprojlim X)$ are clearly satisfied. Hence the completeness.

The constant diagram functor Δ_I from the category of quasi-flows **qFlow** to the category of diagrams of quasi-flows $\mathbf{qFlow}^I$ over a small category I commutes with limits. It then suffices to find a set of solutions to prove the existence of a left adjoint by Theorem III.3.6. Let D be an object of $\mathbf{qFlow}^I$ and let $f : D \longrightarrow \Delta_I Y$ be a morphism in $\mathbf{qFlow}^I$. Then one can suppose that the cardinal $\mathrm{card}(Y)$ of the underlying topological space $Y^0 \sqcup (\bigsqcup_{(\alpha,\beta) \in X^0, X^0} \mathbb{P}^{\mathrm{top}}_{\alpha,\beta} Y)$ of Y is lower than the cardinal $M := \sum_{i \in I} \mathrm{card}(D(i))$ where $\mathrm{card}(D(i))$ is the cardinal of the underlying topological space of the quasi-flow $D(i)$. Then let $\{Z_i, i \in I\}$ be the set of isomorphism classes of quasi-flows whose underlying topological space is of cardinal lower than M. Then to describe $\{Z_i, i \in I\}$, one has to choose a 0-skeleton among 2^M possibilities, for each pair (α, β) of the 0-skeleton, one has to choose a topological space among $2^M \times 2^{(2^M)}$ possibilities, and maps $*_t$ among $(2^{(M \times M \times M)})^{(2^{\aleph_0})}$ possibilities. Therefore the cardinal $\mathrm{card}(I)$ of I satisfies

$$\mathrm{card}(I) \leqslant 2^M \times M \times M \times 2^M \times 2^{(2^M)} \times (2^{(M \times M \times M)})^{(2^{\aleph_0})}$$

so the class I is actually a set. Therefore the class $\bigcup_{i \in I} \mathbf{qFlow}(D, \Delta_I(Z_i))$ is a set as well. $\qquad\square$

There is a canonical embedding functor from the category of flows to that of quasi-flows by setting $*_t = *$ (the composition law of the flow).

3.2. Associating a quasi-flow with any globular complex.

Proposition III.3.8. *Let M be a topological space. Let γ_1 and γ_2 be two continuous maps from $[0, 1]$ to M with $\gamma_2(1) = \gamma_1(0)$. Let $\gamma_3 : [0, 1] \longrightarrow M$ be another continuous map with $\gamma_2(1) = \gamma_3(0)$. Assume that $a, b, c, d \in]0, 1[$ such that $ab = c$ and $(1 - c)(1 - d) = (1 - b)$. Then $(\gamma_1 *_a \gamma_2) *_b \gamma_3 = \gamma_1 *_c (\gamma_2 *_d \gamma_3)$.*

Proof. Let us calculate $((\gamma_1 *_a \gamma_2) *_b \gamma_3)(t)$. There are three possibilities:

1. $0 \leqslant t \leqslant ab$. Then $((\gamma_1 *_a \gamma_2) *_b \gamma_3)(t) = \gamma_1(\frac{t}{ab})$.

2. $ab \leqslant t \leqslant b$. Then $((\gamma_1 *_a \gamma_2) *_b \gamma_3)(t) = \gamma_2\left(\frac{\frac{t}{b}-a}{1-a}\right) = \gamma_2(\frac{t-ab}{b(1-a)})$.

3. $b \leqslant t \leqslant 1$. Then $((\gamma_1 *_a \gamma_2) *_b \gamma_3)(t) = \gamma_3(\frac{t-b}{1-b})$.

Let us now calculate $(\gamma_1 *_c (\gamma_2 *_d \gamma_3))(t)$. There are again three possibilities:

1. $0 \leqslant t \leqslant c$. Then $(\gamma_1 *_c (\gamma_2 *_d \gamma_3))(t) = \gamma_1(\frac{t}{c})$.

2. $0 \leqslant \frac{t-c}{1-c} \leqslant d$, or equivalently $c \leqslant t \leqslant c + d(1-c)$. Then

$$(\gamma_1 *_c (\gamma_2 *_d \gamma_3))(t) = \gamma_2\left(\frac{t-c}{d(1-c)}\right).$$

3. $d \leqslant \frac{t-c}{1-c} \leqslant 1$, or equivalently $c + d(1-c) \leqslant t \leqslant 1$. Then

$$(\gamma_1 *_c (\gamma_2 *_d \gamma_3))(t) = \gamma_3\left(\frac{\frac{t-c}{1-c}-d}{1-d}\right) = \gamma_3\left(\frac{t-c-d(1-c)}{(1-d)(1-c)}\right).$$

From $(1-c)(1-d) = (1-b)$, one deduces that $1 - c - (1-b) = d(1-c)$, so $d(1-c) = b-c = b-ab = b(1-a)$. Therefore $d(1-c) = b(1-a)$. So $c+d(1-c) = b$. The last two equalities complete the proof. $\square$

Proposition III.3.9. *Let X be a globular complex. Let* $\mathrm{qcat}(X) := X^0$ *and*

$$\mathbb{P}_{\alpha,\beta}^{\mathrm{top}}\mathrm{qcat}(X) := \mathbb{P}_{\alpha,\beta}^{\mathrm{ex}}X.$$

for any $(\alpha, \beta) \in X^0 \times X^0$. This defines a functor $\mathrm{qcat} : \mathbf{glTop} \longrightarrow \mathbf{qFlow}$.

Proof. Immediate consequence of Proposition III.3.8. $\square$

Proposition III.3.10. *Let X be a globular complex with globular decomposition*

$$(\partial Z_\beta, Z_\beta, \phi_\beta)_{\beta < \lambda}.$$

Then:

1. *For any $\beta < \lambda$, one has the pushout of quasi-flows*

$$
\begin{array}{ccc}
\mathrm{qcat}(\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta)) & \xrightarrow{\mathrm{qcat}(\phi_\beta)} & \mathrm{qcat}(X_\beta) \\
\downarrow & & \downarrow \\
\mathrm{qcat}(\mathrm{Glob}^{\mathrm{top}}(Z_\beta)) & \longrightarrow & \mathrm{qcat}(X_{\beta+1}).
\end{array}
$$

2. *For any limit ordinal $\beta < \lambda$, the canonical morphism of quasi-flows*

$$\varinjlim_{\alpha<\beta}\mathrm{qcat}(X_\alpha) \longrightarrow \mathrm{qcat}(X_\beta)$$

is an isomorphism of quasi-flows.

Proof. The first part is a consequence of Proposition III.3.8. For any globular complex X, the continuous map $|X_\beta| \longrightarrow |X_{\beta+1}|$ is a Hurewicz cofibration, and in particular a closed inclusion of topological spaces. Since $[0,1]$ is compact, it is $\aleph_0$-small relative to closed inclusions of topological spaces [Hov99]. Since β is a limit ordinal, then $\beta \geqslant \aleph_0$. Therefore any continuous map $[0,1] \longrightarrow X_\beta$ factors as a composite $[0,1] \longrightarrow X_\alpha \longrightarrow X_\beta$ for some $\alpha < \beta$. Hence the second part of the statement. $\square$

3.3. Construction of the functor cat on objects. Let X be a globular complex with globular decomposition $(\partial Z_\beta, Z_\beta, \phi_\beta)_{\beta < \lambda}$. We are going to construct by induction on β a flow $\mathrm{cat}(X_\beta)$ and a morphism of quasi-flows $p_{X_\beta} : \mathrm{qcat}(X_\beta) \longrightarrow \mathrm{cat}(X_\beta)$.

There is nothing to do if $X = X_0 = (X^0, X^0)$ is a discrete globular complex. If $X = \mathrm{Glob}^{\mathrm{top}}(Z)$, then $\mathrm{qcat}(X)^0 = \{0, 1\}$ and

$$\mathbb{P}_{0,1}^{\mathrm{top}}\mathrm{qcat}(X) = Z \times \mathbf{glTOP}(\overrightarrow{I}^{\mathrm{top}}, \overrightarrow{I}^{\mathrm{top}})$$

by Proposition II.3.7. The projection $\mathbb{P}_{0,1}^{\mathrm{top}}\mathrm{qcat}(X) \longrightarrow Z$ yields a morphism of quasi-flows $p_X : \mathrm{qcat}(X) \longrightarrow \mathrm{cat}(X)$.

Let us consider the pushout of multipointed spaces

$$
\begin{array}{ccc}
\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta) & \xrightarrow{\phi_\beta} & X_\beta \\
\downarrow & & \downarrow \\
\mathrm{Glob}^{\mathrm{top}}(Z_\beta) & \longrightarrow & X_{\beta+1}.
\end{array}
$$

Let us suppose $p_{X_\beta} : \mathrm{qcat}(X_\beta) \longrightarrow \mathrm{cat}(X_\beta)$ constructed. Let us consider the set map $i_Z : Z \longrightarrow \mathbb{P}^{\mathrm{ex}}\mathrm{Glob}^{\mathrm{top}}(Z)$ defined by $i_Z(z)(t) = (z, t)$. It is continuous since it corresponds, by the set map

$$\mathbf{Top}(Z, \mathbb{P}^{\mathrm{ex}}\mathrm{Glob}^{\mathrm{top}}(Z)) \longrightarrow \mathbf{Top}(Z \times [0,1], |\mathrm{Glob}^{\mathrm{top}}(Z)|),$$

to the continuous map $(z, t) \mapsto (z, t)$. The composite

$$\partial Z_\beta \xrightarrow{i_{\partial Z_\beta}} \mathbb{P}^{\mathrm{ex}}\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta) \longrightarrow \mathbb{P}^{\mathrm{top}}(X_\beta) \xrightarrow{\mathrm{qcat}(\phi_\beta)} \mathbb{P}X_\beta$$

yields a morphism of flows $\mathrm{cat}(\phi_\beta) : \mathrm{Glob}(\partial Z_\beta) \longrightarrow \mathrm{cat}(X_\beta)$. Then let $\mathrm{cat}(X_{\beta+1})$ be the flow defined by the pushout of flows

$$
\begin{array}{ccc}
\mathrm{Glob}(\partial Z_\beta) & \xrightarrow{\phi_\beta} & \mathrm{cat}(X_\beta) \\
{\scriptstyle \mathrm{cat}(\phi_\beta)}\downarrow & & \downarrow \\
\mathrm{Glob}(Z_\beta) & \longrightarrow & \mathrm{cat}(X_{\beta+1}).
\end{array}
$$

The morphisms of quasi-flows

$$\mathrm{qcat}(X_\beta) \longrightarrow \mathrm{cat}(X_\beta)$$

and

$$\mathrm{qcat}(\mathrm{Glob}^{\mathrm{top}}(Z_\beta)) \longrightarrow \mathrm{cat}(\mathrm{Glob}^{\mathrm{top}}(Z_\beta))$$

induce a commutative square of quasi-flows

$$
\begin{array}{ccc}
\mathrm{qcat}(\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta)) & \longrightarrow & \mathrm{cat}(X_\beta) \\
\downarrow & & \downarrow \\
\mathrm{qcat}(\mathrm{Glob}^{\mathrm{top}}(Z_\beta)) & \longrightarrow & \mathrm{cat}(X_{\beta+1})
\end{array}
$$

and therefore a morphism of quasi-flows $p_{X_{\beta+1}} : \mathrm{qcat}(X_{\beta+1}) \longrightarrow \mathrm{cat}(X_{\beta+1})$. If β is a limit ordinal, then $\mathrm{cat}(X_\alpha)$ and the morphism of flows $p_{X_\alpha} : \mathrm{qcat}(X_\alpha) \longrightarrow$

$\mathrm{cat}(X_\alpha)$ are defined by induction hypothesis for any $\alpha < \beta$. Then let $\mathrm{cat}(X_\beta) := \varinjlim_{\alpha<\beta}\mathrm{cat}(X_\alpha)$ and $p_{X_\beta} := \varinjlim_{\alpha<\beta}p_{X_\alpha}$.

3.4. Construction of the functor cat on arrows. Let $f : X \longrightarrow U$ be a morphism of globular complexes. The purpose of this section is the construction of $\mathrm{cat}(f) : \mathrm{cat}(X) \longrightarrow \mathrm{cat}(U)$.

If $X = X^0$, then there is nothing to do since the set map $\mathbf{glTop}(X,U) \longrightarrow \mathbf{Flow}(X,U)$ is just the identity of $\mathbf{Set}(X^0,U^0)$.

If $X = \mathrm{Glob}^{\mathrm{top}}(Z)$ for some compact space Z, let $f : \mathrm{Glob}^{\mathrm{top}}(Z) \longrightarrow U$ be a morphism of globular complexes. Let $\mathrm{cat}(f) = p_U \circ \mathrm{qcat}(f) \circ i_Z$. Then the mapping $f \mapsto \mathrm{cat}(f)$ yields a set map $\mathbf{glTop}(\mathrm{Glob}^{\mathrm{top}}(Z),U) \longrightarrow \mathbf{Flow}(\mathrm{Glob}(Z),\mathrm{cat}(U))$.

Now take a general globular complex X with globular decomposition

$$(\partial Z_\beta, Z_\beta, \phi_\beta)_{\beta<\lambda}.$$

Using Theorem II.3.3 and Theorem III.2.9, one obtains a set map

$$\mathbf{glTop}(X_\beta,U) \longrightarrow \mathbf{Flow}(\mathrm{cat}(X_\beta),\mathrm{cat}(U))$$

and by passage to the limit, a set map

$$\mathrm{cat} : \mathbf{glTop}(X,U) \longrightarrow \mathbf{Flow}(\mathrm{cat}(X),\mathrm{cat}(U)).$$

3.5. Functoriality of the functor cat.

Theorem III.3.11. *The mapping* $\mathrm{cat}(-)$ *becomes a functor from* $\mathbf{glTop}$ *to* $\mathbf{Flow}$. *The mapping* $p_X : \mathrm{qcat}(X) \longrightarrow \mathrm{cat}(X)$ *yields a natural transformation* $p : \mathrm{qcat} \longrightarrow \mathrm{cat}$. *The mapping* $p_X : \mathbb{P}^{\mathrm{top}}X \longrightarrow \mathbb{P}X$ *yields a natural transformation* $p : \mathbb{P}^{\mathrm{top}} \longrightarrow \mathbb{P}$.

Proof. Let U and V be two topological spaces. Let $h : U \longrightarrow V$ be a continuous map. Let Z be a topological space. Then the following diagram is clearly commutative:

$$\mathbf{glTop}(\mathrm{Glob}^{\mathrm{top}}(Z),\mathrm{Glob}^{\mathrm{top}}(U)) \xrightarrow{\mathrm{cat}(-)} \mathbf{Flow}(\mathrm{Glob}(Z),\mathrm{Glob}(U))$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\mathbf{glTop}(\mathrm{Glob}^{\mathrm{top}}(Z),\mathrm{Glob}^{\mathrm{top}}(V)) \xrightarrow{\mathrm{cat}(-)} \mathbf{Flow}(\mathrm{Glob}(Z),\mathrm{Glob}(V))$$

where the horizontal maps are both defined by the above construction and where the right vertical map $\mathbf{Flow}(\mathrm{Glob}(Z),\mathrm{Glob}(U)) \longrightarrow \mathbf{Flow}(\mathrm{Glob}(Z),\mathrm{Glob}(V))$ is induced by the composition by $\mathrm{Glob}(h)$.

So for any morphism $h : U \longrightarrow V$ of globular complexes and for any topological space Z, one has the following commutative diagram

$$\mathbf{glTop}(\mathrm{Glob}^{\mathrm{top}}(Z),U) \xrightarrow{\mathrm{cat}(-)} \mathbf{Flow}(\mathrm{Glob}(Z),\mathrm{cat}(U))$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\mathbf{glTop}(\mathrm{Glob}^{\mathrm{top}}(Z),V) \xrightarrow{\mathrm{cat}(-)} \mathbf{Flow}(\mathrm{Glob}(Z),\mathrm{cat}(V))$$

where both horizontal maps are defined by the above construction and where the right vertical map $\mathbf{Flow}(\mathrm{Glob}(Z),\mathrm{cat}(U)) \longrightarrow \mathbf{Flow}(\mathrm{Glob}(Z),\mathrm{cat}(V))$ is induced

by the composition by $\mathrm{cat}(h) \in \mathbf{Flow}(\mathrm{cat}(U), \mathrm{cat}(V)) \cong \varprojlim \mathbf{Flow}(\mathrm{cat}(U_\beta), \mathrm{cat}(V))$. Indeed locally, we are reduced to the situation of the first square.

Take now a general globular complex X with globular decomposition

$$(\partial Z_\beta, Z_\beta, \phi_\beta)_{\beta < \lambda}.$$

Then using Theorem II.3.3 and Theorem III.2.9, one immediately proves by transfinite induction on β that the diagram

$$
\begin{array}{ccc}
\mathbf{glTop}(X_\beta, U) & \xrightarrow{\mathrm{cat}(-)} & \mathbf{Flow}(\mathrm{cat}(X_\beta), \mathrm{cat}(U)) \\
\downarrow & & \downarrow \\
\mathbf{glTop}(X_\beta, V) & \xrightarrow{\mathrm{cat}(-)} & \mathbf{Flow}(\mathrm{cat}(X_\beta), \mathrm{cat}(V))
\end{array}
$$

is commutative for any ordinal $\beta < \lambda$. So one obtains the following commutative diagram

$$
\begin{array}{ccc}
\mathbf{glTop}(X, U) & \xrightarrow{\mathrm{cat}(-)} & \mathbf{Flow}(\mathrm{cat}(X), \mathrm{cat}(U)) \\
\downarrow & & \downarrow \\
\mathbf{glTop}(X, V) & \xrightarrow{\mathrm{cat}(-)} & \mathbf{Flow}(\mathrm{cat}(X), \mathrm{cat}(V))
\end{array}
$$

where both horizontal maps are defined by the above construction and where the right vertical map $\mathbf{Flow}(\mathrm{Glob}(Z), \mathrm{cat}(U)) \longrightarrow \mathbf{Flow}(\mathrm{Glob}(Z), \mathrm{cat}(V))$ is induced by the composition by $\mathrm{cat}(h) \in \mathbf{Flow}(\mathrm{cat}(U), \mathrm{cat}(V)) \cong \varprojlim \mathbf{Flow}(\mathrm{cat}(U_\beta), \mathrm{cat}(V))$. This is exactly the functoriality of $\mathrm{cat}(-)$.

By specializing the second square to $Z = \{*\}$ and by Proposition II.3.6 and Proposition III.2.7, one obtains the commutative square of topological spaces

$$
\begin{array}{ccc}
\mathbb{P}^{\mathrm{top}}U & \xrightarrow{p_U} & \mathbb{P}U \\
{\scriptstyle \mathbb{P}^{\mathrm{top}}h} \downarrow & & \downarrow {\scriptstyle \mathbb{P}h} \\
\mathbb{P}^{\mathrm{top}}V & \xrightarrow{p_V} & \mathbb{P}V.
\end{array}
$$

The result follows. $\square$

4. Pushout of $\mathrm{Glob}(\partial Z) \longrightarrow \mathrm{Glob}(Z)$ in Flow

Let $\partial Z \longrightarrow Z$ be a continuous map. Let us consider a diagram of flows as follows:

$$
\begin{array}{ccc}
\mathrm{Glob}(\partial Z) & \xrightarrow{\phi} & A \\
\downarrow & & \downarrow \\
\mathrm{Glob}(Z) & \longrightarrow & X.
\end{array}
$$

This short section is devoted to an explicit description of the pushout X in the category of flows.

Let us consider the set $\mathcal{M}$ of finite sequences $\alpha_0 \ldots \alpha_p$ of elements of $A^0 = X^0$ with $p \geqslant 1$ and such that, for any i with $0 \leqslant i \leqslant p-2$, at least one of the two pairs

(α_i, α_{i+1}) and $(\alpha_{i+1}, \alpha_{i+2})$ is equal to $(\phi(0), \phi(1))$. Let us consider the pushout diagram of topological spaces

$$
\begin{array}{ccc}
\partial Z & \xrightarrow{\ \phi\ } & \mathbb{P}_{\phi(0),\phi(1)}A \\
\downarrow & & \downarrow \\
Z & \longrightarrow & T.
\end{array}
$$

Let $Z_{\alpha,\beta} = \mathbb{P}_{\alpha,\beta}A$ if $(\alpha,\beta) \neq (\phi(0),\phi(1))$ and let $Z_{\phi(0),\phi(1)} = T$. At last, for any $\alpha_0 \ldots \alpha_p \in \mathcal{M}$, let $[\alpha_0 \ldots \alpha_p] = Z_{\alpha_0,\alpha_1} \times Z_{\alpha_1,\alpha_2} \times \ldots \times Z_{\alpha_{p-1},\alpha_p}$. And $[\alpha_0 \ldots \alpha_p]_i$ denotes the same product as $[\alpha_0 \ldots \alpha_p]$ except that $(\alpha_i, \alpha_{i+1}) = (\phi(0),\phi(1))$ and that the factor $Z_{\alpha_i,\alpha_{i+1}} = T$ is replaced by $\mathbb{P}_{\phi(0),\phi(1)}A$. That means that in the product $[\alpha_0 \ldots \alpha_p]_i$, the factor $\mathbb{P}_{\phi(0),\phi(1)}A$ appears exactly once. For instance, one has (with $\phi(0) \neq \phi(1)$)

$$
\begin{aligned}
[\alpha\phi(0)\phi(1)\phi(0)\phi(1)] \ &= \mathbb{P}_{\alpha,\phi(0)}A \times T \times \mathbb{P}_{\phi(1),\phi(0)}A \times T \\
[\alpha\phi(0)\phi(1)\phi(0)\phi(1)]_1 &= \mathbb{P}_{\alpha,\phi(0)}A \times \mathbb{P}_{\phi(0),\phi(1)}A \times \mathbb{P}_{\phi(1),\phi(0)}A \times T \\
[\alpha\phi(0)\phi(1)\phi(0)\phi(1)]_3 &= \mathbb{P}_{\alpha,\phi(0)}A \times T \times \mathbb{P}_{\phi(1),\phi(0)}A \times \mathbb{P}_{\phi(0),\phi(1)}A.
\end{aligned}
$$

The idea is that in the products $[\alpha_0 \ldots \alpha_p]$, there are no possible simplifications using the composition law of A. On the contrary, exactly one simplification is possible using the composition law of A in the products $[\alpha_0 \ldots \alpha_p]_i$. For instance, with the examples above, there exist continuous maps

$$[\alpha\phi(0)\phi(1)\phi(0)\phi(1)]_1 \longrightarrow [\alpha\phi(0)\phi(1)]$$

and

$$[\alpha\phi(0)\phi(1)\phi(0)\phi(1)]_3 \longrightarrow [\alpha\phi(0)\phi(1)\phi(1)]$$

induced by the composition law of A and there exist continuous maps

$$[\alpha\phi(0)\phi(1)\phi(0)\phi(1)]_1 \longrightarrow [\alpha\phi(0)\phi(1)\phi(0)\phi(1)]$$

and

$$[\alpha\phi(0)\phi(1)\phi(0)\phi(1)]_3 \longrightarrow [\alpha\phi(0)\phi(1)\phi(0)\phi(1)]$$

induced by the continuous map $\mathbb{P}_{\phi(0),\phi(1)}A \longrightarrow T$.

Let $\mathbb{P}_{\alpha,\beta}M$ be the colimit of the diagram of topological spaces consisting of the topological spaces $[\alpha_0 \ldots \alpha_p]$ and $[\alpha_0 \ldots \alpha_p]_i$ with $\alpha_0 = \alpha$ and $\alpha_p = \beta$ and with the two kinds of maps above defined. The composition law of A and the free concatenation obviously gives a continuous associative map $\mathbb{P}_{\alpha,\beta}M \times \mathbb{P}_{\beta,\gamma}M \longrightarrow \mathbb{P}_{\alpha,\gamma}M$.

Proposition III.4.1 ([Gau03d] Proposition 15.1). *One has the pushout diagram of flows*

$$
\begin{array}{ccc}
\mathrm{Glob}(\partial Z) & \xrightarrow{\ \phi\ } & A \\
\downarrow & & \downarrow \\
\mathrm{Glob}(Z) & \longrightarrow & M.
\end{array}
$$

5. Geometric realization of execution paths

Proposition III.5.1. *Let Z be a compact topological space. Let f and g be two morphisms of globular complexes from $\mathrm{Glob}^{\mathrm{top}}(Z)$ to a globular complex U such that the continuous maps $\mathbb{P}f$ and $\mathbb{P}g$ from Z to $\mathbb{P}U$ are equal. Then there exists one and only one map $\phi : |\mathrm{Glob}^{\mathrm{top}}(Z)| \longrightarrow [0,1]$ such that*

$$f((z,t)) = \mathbb{P}^{\mathrm{top}}g(t \mapsto (z,t))(\phi(z,t)).$$

Moreover this map ϕ is necessarily continuous.

Notice that the map $\phi : \mathrm{Glob}^{\mathrm{top}}(Z) \longrightarrow [0,1]$ induces a morphism of globular complexes from $\mathrm{Glob}^{\mathrm{top}}(Z)$ to $\overrightarrow{I}^{\mathrm{top}}$.

Proof. By hypothesis, the equality $\mathbb{P}^{\mathrm{top}}f([0,1]) = \mathbb{P}^{\mathrm{top}}g([0,1])$ holds. For a given $z_0 \in Z$, if

$$\{0 = t_0 < \cdots < t_p = 1\} = \mathbb{P}^{\mathrm{top}}f(t \mapsto (z_0,t))([0,1]) \cap U^0$$

and

$$\{0 = t'_0 < \cdots < t'_p = 1\} = \mathbb{P}^{\mathrm{top}}g(t \mapsto (z_0,t))([0,1]) \cap U^0$$

then necessarily $\phi(z_0, t_i) = t'_i$ for $0 \leqslant i \leqslant p$. For $t \in \,]t_i, t_{i+1}[$, the map

$$\mathbb{P}^{\mathrm{top}}g(t \mapsto (z_0,t)) \upharpoonright_{]t'_i, t'_{i+1}[}$$

is one-to-one by hypothesis. Therefore for $t \in \,]t_i, t_{i+1}[$, $\mathbb{P}^{\mathrm{top}}f(t \mapsto (z_0,t))(t)$ is equal to

$$(\mathbb{P}^{\mathrm{top}}g(t \mapsto (z_0,t)) \upharpoonright_{]t'_i, t'_{i+1}[})(\mathbb{P}^{\mathrm{top}}g(t \mapsto (z_0,t)) \upharpoonright_{]t'_i, t'_{i+1}[})^{-1}\mathbb{P}^{\mathrm{top}}f(t \mapsto (z_0,t))(t)$$

so necessarily one has

$$\phi(z_0, t) = (\mathbb{P}^{\mathrm{top}}g(t \mapsto (z_0,t)) \upharpoonright_{]t'_i, t'_{i+1}[})^{-1}\mathbb{P}^{\mathrm{top}}f(t \mapsto (z_0,t))(t).$$

Now suppose that ϕ is not continuous at (z_∞, t_∞). Then there exists an open neighborhood U of $\phi(z_\infty, t_\infty)$ such that for any open V containing (z_∞, t_∞), for any $(z,t) \in V \backslash \{(z_\infty, t_\infty)\}$, $\phi(z,t) \notin U$. Take a sequence $(z_n, t_n)_{n \geqslant 0}$ of V tending to (z_∞, t_∞). Then there exists a subsequence of $(\phi(z_n, t_n))_{n \geqslant 0}$ tending to some $t' \in [0,1]$ since $[0,1]$ is compact: by hypothesis t' is in the topological closure of the complement of U; this latter being closed, $t' \notin U$. So we can take $(z_n, t_n)_{n \geqslant 0}$ such that $(\phi(z_n, t_n))_{n \geqslant 0}$ converges. Then $f((z_n, t_n))$ tends to $f((z_\infty, t_\infty))$ because f is continuous, $\mathbb{P}^{\mathrm{top}}g(t \mapsto (z_n, t))$ tends to $\mathbb{P}^{\mathrm{top}}g(t \mapsto (z_\infty, t))$ for the Kelleyfication of the compact-open topology so $f((z_\infty, t_\infty)) = \mathbb{P}^{\mathrm{top}}g(t \mapsto (z_\infty, t))(t')$ with $t' \notin U$ and $t' = \phi(z_\infty, t_\infty) \in U$: contradiction. $\qquad \square$

Theorem III.5.2. *For any globular complex X, there exists a continuous map $i_X : \mathbb{P}X \longrightarrow \mathbb{P}^{\mathrm{top}}X$ such that $p_X \circ i_X = \mathrm{Id}_{\mathbb{P}X}$.*

Notice that i_X cannot be obtained from a morphism of quasi-flows. Otherwise one would have $(x *_a y) *_a z = x *_a (y *_a z)$ in $\mathbb{P}^{\mathrm{top}}X$ for some fixed $a \in \,]0,1[$, and this is impossible.

Proof. First of all, notice that there is an inclusion of sets $\mathbf{Top}(\mathbb{P}X, \mathbb{P}^{\mathrm{top}}X) \subset \mathbf{Top}(\mathbb{P}X \times [0,1], X)$. So constructing a continuous map from $\mathbb{P}X$ to $\mathbb{P}^{\mathrm{top}}X$ is equivalent to constructing a continuous map from $\mathbb{P}X \times [0,1]$ to X satisfying some obvious properties, since the category $\mathbf{Top}$ of compactly generated topological spaces is cartesian closed.

Let X be a globular complex with globular decomposition $(\partial Z_\beta, Z_\beta, \phi_\beta)_{\beta < \lambda}$. We are going to construct a continuous map $i_{X_\beta} : \mathbb{P}X_\beta \longrightarrow \mathbb{P}^{\mathrm{top}}X_\beta$. For $\beta = 0$, there is nothing to do since the topological spaces are both discrete. Assume that $i_{X_\beta} : \mathbb{P}X_\beta \longrightarrow \mathbb{P}^{\mathrm{top}}X_\beta$ is constructed for some $\beta \geqslant 0$ such that $p_{X_\beta} \circ i_{X_\beta} = \mathrm{Id}_{\mathbb{P}X_\beta}$. Let us consider the pushout of multipointed spaces

$$
\begin{array}{ccc}
\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta) & \xrightarrow{\phi_\beta} & X_\beta \\
\downarrow & & \downarrow \\
\mathrm{Glob}^{\mathrm{top}}(Z_\beta) & \xrightarrow{\overline{\phi_\beta}} & X_{\beta+1}.
\end{array}
$$

Proposition III.4.1 provides an explicit method for the calculation of $\mathbb{P}X_{\beta+1}$ as the colimit of a diagram of topological spaces. Let us consider the pushout diagram of topological spaces

$$
\begin{array}{ccc}
\partial Z_\beta & \xrightarrow{\phi_\beta} & \mathbb{P}_{\phi_\beta(0),\phi_\beta(1)}X_\beta \\
\downarrow & & \downarrow \\
Z_\beta & \longrightarrow & T.
\end{array}
$$

Constructing a continuous map $\mathbb{P}X_{\beta+1} \longrightarrow \mathbb{P}^{\mathrm{top}}X_{\beta+1}$ is then equivalent to constructing continuous maps $[\alpha_0 \ldots \alpha_p] \longrightarrow \mathbb{P}^{\mathrm{top}}X_{\beta+1}$ and $[\alpha_0 \ldots \alpha_p]_i \longrightarrow \mathbb{P}^{\mathrm{top}}X_{\beta+1}$ for any finite sequence $\alpha_0 \ldots \alpha_p$ of $\mathcal{M}$ such that any diagram like

$$
\begin{array}{ccc}
[\alpha_0 \ldots \alpha_p]_i & \longrightarrow & \mathbb{P}^{\mathrm{top}}X_{\beta+1} \\
\downarrow & \nearrow & \\
[\alpha_0 \ldots \alpha_p] & &
\end{array}
\qquad
\begin{array}{ccc}
[\alpha_0 \ldots \alpha_p]_i & \longrightarrow & \mathbb{P}^{\mathrm{top}}X_{\beta+1} \\
\downarrow & \nearrow & \\
[\alpha_0 \ldots \widehat{\phi(0)\phi(1)} \ldots \alpha_p] & &
\end{array}
$$

is commutative.

We are going to proceed by induction on p. If $p = 1$, then $[\alpha_0\alpha_1]$ is equal to $\mathbb{P}_{\alpha_0,\alpha_1}X_\beta$ if $(\alpha_0, \alpha_1) \neq (\phi_\beta(0), \phi_\beta(1))$ and is equal to T if $(\alpha_0, \alpha_1) = (\phi_\beta(0), \phi_\beta(1))$. For $p = 1$, the only thing we then have to prove is that the continuous map $p_{X_{\beta+1}} : \mathbb{P}^{\mathrm{top}}X_{\beta+1} \longrightarrow \mathbb{P}X_{\beta+1}$ has the right lifting property with respect to the continuous map $\mathbb{P}_{\phi_\beta(0),\phi_\beta(1)}X_\beta \longrightarrow T$, in other terms that there exists a continuous map $k : T \longrightarrow \mathbb{P}^{\mathrm{top}}X_{\beta+1}$ making commutative the diagram of topological spaces

$$
\begin{array}{ccc}
\mathbb{P}_{\phi_\beta(0),\phi_\beta(1)}X_\beta & \xrightarrow{i_{X_\beta}} & \mathbb{P}^{\mathrm{top}}X_{\beta+1} \\
\downarrow & \overset{k}{\nearrow} & \downarrow {\scriptstyle p_{X_{\beta+1}}} \\
T & \longrightarrow & \mathbb{P}X_{\beta+1}.
\end{array}
$$

Since $\mathbb{P}_{\phi_\beta(0),\phi_\beta(1)}X_\beta \longrightarrow T$ is a pushout of a NDR pair of spaces, then the pair of spaces $(T, \mathbb{P}_{\phi_\beta(0),\phi_\beta(1)}X_\beta)$ is a NDR pair as well. If $z \in Z_\beta$, let $[z](t) = (z, t)$ for $t \in [0, 1]$. This defines an execution path of $\mathrm{Glob}^{\mathrm{top}}(Z_\beta)$. Then $\overline{\phi_\beta} \circ [z]$ is still an execution path. Since $\mathbb{P}_{\phi_\beta(0),\phi_\beta(1)}X_\beta \longrightarrow T$ is a (closed) inclusion of topological spaces, then for any $z \in \partial Z_\beta$, $\overline{\phi_\beta} \circ [z]$ is an execution path of X_β. By Proposition III.5.1 and since ∂Z_β is compact, there exists a continuous map $\psi : \partial Z_\beta \times [0, 1] \longrightarrow [0, 1]$ such that

$$i_{X_\beta}(z)(t) = (\overline{\phi_\beta} \circ [z])(\psi(z, t)).$$

Then define k by: $k(x) = i_{X_\beta}(x)$ if $x \in \mathbb{P}_{\psi(0),\psi(1)}X_\beta$ and

$$k(x)(t) = (\overline{\phi_\beta} \circ [x])(\mu(x)t + (1 - \mu(x))\psi(x, t))$$

if $x \in Z_\beta \backslash \partial Z_\beta$. The case $p = 1$ is complete.

We now have to construct $[\alpha_0 \ldots \alpha_p]_i \longrightarrow \mathbb{P}^{\mathrm{top}}X_{\beta+1}$ and $[\alpha_0 \ldots \alpha_p] \longrightarrow \mathbb{P}^{\mathrm{top}}X_{\beta+1}$ by induction on $p \geqslant 1$. The product $[\alpha_0 \ldots \widehat{\phi(0)\phi(1)} \ldots \alpha_p]$ is of length strictly lower than p. Therefore the continuous map $[\alpha_0 \ldots \widehat{\phi(0)\phi(1)} \ldots \alpha_p] \longrightarrow \mathbb{P}^{\mathrm{top}}X_{\beta+1}$ is already constructed. Then the commutativity of the diagram

$$
\begin{array}{ccc}
[\alpha_0 \ldots \alpha_p]_i & \longrightarrow & \mathbb{P}^{\mathrm{top}}X_{\beta+1} \\
\downarrow & \nearrow & \\
[\alpha_0 \ldots \widehat{\phi(0)\phi(1)} \ldots \alpha_p] & &
\end{array}
$$

entails the definition of $[\alpha_0 \ldots \alpha_p]_i \longrightarrow \mathbb{P}^{\mathrm{top}}X_{\beta+1}$. It remains to prove that there exists k making the following diagram commutative:

$$
\begin{array}{ccc}
[\alpha_0 \ldots \alpha_p]_i & \longrightarrow & \mathbb{P}^{\mathrm{top}}X_{\beta+1} \\
\downarrow & \overset{k}{\nearrow} & \downarrow{\scriptstyle p_{X_{\beta+1}}} \\
[\alpha_0 \ldots \alpha_p] & \longrightarrow & \mathbb{P}X_{\beta+1}.
\end{array}
$$

Once again the closed inclusion $[\alpha_0 \ldots \alpha_p]_i \longrightarrow [\alpha_0 \ldots \alpha_p]$ is a Hurewicz cofibration. There are three mutually exclusive possible cases:

1. $[\alpha_0 \ldots \alpha_p]_i = P \times \mathbb{P}_{\phi_\beta(0),\phi_\beta(1)}X_\beta \times Q$ and $[\alpha_0 \ldots \alpha_p] = P \times T \times Q$ where P and Q are objects of the diagram of topological spaces calculating $\mathbb{P}X_{\beta+1}$.
2. $[\alpha_0 \ldots \alpha_p]_i = P \times \mathbb{P}_{\phi_\beta(0),\phi_\beta(1)}X_\beta$ and $[\alpha_0 \ldots \alpha_p] = P \times T$ where P is an object of the diagram of topological spaces calculating $\mathbb{P}X_{\beta+1}$.
3. $[\alpha_0 \ldots \alpha_p]_i = \mathbb{P}_{\phi_\beta(0),\phi_\beta(1)}X_\beta \times Q$ and $[\alpha_0 \ldots \alpha_p] = T \times Q$ where Q is an object of the diagram of topological spaces calculating $\mathbb{P}X_{\beta+1}$.

Let us treat for instance the first case. The products P and Q are of length strictly lower than p. So by induction hypothesis, $i_{X_{\beta+1}} : P \longrightarrow \mathbb{P}^{\mathrm{top}}X_{\beta+1}$ and $i_{X_{\beta+1}} : Q \longrightarrow \mathbb{P}^{\mathrm{top}}X_{\beta+1}$ are already constructed. For any $z \in Z_\beta$ and any $(p, q) \in P \times Q$, consider the execution path

$$\Gamma(p, z, q) := (i_{X_{\beta+1}}(p) *_{1/2} (\overline{\phi_\beta} \circ [z])) *_{1/2} i_{X_{\beta+1}}(q).$$

By Proposition III.5.1 and since ∂Z_β is compact, there exists a continuous map $\psi : \partial P \times Z_\beta \times Q \times [0, 1] \longrightarrow [0, 1]$ such that $\Gamma(p, z, q)(\psi(p, z, q, t)) = i_{X_\beta}(p, z, q)(t)$. Then define k by:

1. $k(p, x, q) = i_{X_\beta}(p, x, q)(t)$ if $x \in \mathbb{P}_{\psi(0), \psi(1)} X_\beta$.
2. $k(p, x, q) = \Gamma(p, z, q)(\mu(x)t + (1 - \mu(x))\psi(p, x, q, t))$ if $x \in Z_\beta \backslash \partial Z_\beta$.

The induction is complete. $\qquad\square$

Since the functor cat : **glTop** $\longrightarrow$ **Flow** is constructed, we are now ready to compare the S-homotopy equivalences in the two frameworks.

Part IV. S-homotopy and flow

1. Introduction

Section 2 studies the notion of S-*homotopy extension property* for morphisms of globular complexes. This is the analogue in our framework of the notion of Hurewicz cofibration. This section, as short as possible, studies some analogues of well-known theorems in homotopy theory of topological spaces. The goal of Section 3 is the comparison of the space of morphisms of globular complexes from a globular complex X to a globular complex U with the space of morphisms of flows from the flow cat(X) to the flow cat(U). It turns out that these two spaces are homotopy equivalent. The proof requires the careful study of two transfinite towers of topological spaces and needs the introduction of a model category of topological spaces which is not the usual one, but another one whose weak equivalences are the homotopy equivalences [Str66, Str68, Str72]. At last, Section 4 makes the comparison between the two notions of S-homotopy equivalences using all previous results.

2. S-homotopy extension property

We first need to develop some of the theory of morphisms of globular complexes satisfying the S-homotopy extension property in order to obtain Corollary IV.2.8.

Definition IV.2.1. Let $i : A \longrightarrow X$ be a morphism of globular complexes and let Y be a globular complex. The morphism $i : A \longrightarrow X$ satisfies the S-*homotopy extension property* for Y if for any morphism $f : X \longrightarrow Y$ and any S-homotopy $h : [0, 1] \boxtimes A \longrightarrow Y$ such that for any $a \in A$, $h(0 \boxtimes a) = f(i(a))$, there exists an S-homotopy $H : [0, 1] \boxtimes X \longrightarrow Y$ such that for any $x \in X$, $H(0 \boxtimes x) = f(x)$ and for any $(t, a) \in [0, 1] \times A$, $H(t \boxtimes i(a)) = h(t \boxtimes a)$.

Definition IV.2.2. A morphism of globular complexes $i : A \longrightarrow X$ satisfies the S-*homotopy extension property* if $i : A \longrightarrow X$ satisfies the S-homotopy extension property for any globular complex Y.

Proposition IV.2.3. *Let $i : A \longrightarrow X$ be a morphism of globular complexes. Let us consider the cocartesian diagram of multipointed topological spaces*

$$\begin{array}{ccc} \{0\} \boxtimes A & \longrightarrow & [0, 1] \boxtimes A \\ {\scriptstyle i}\downarrow & & \downarrow \\ \{0\} \boxtimes X & \longrightarrow & Mi. \end{array}$$

Then Mi inherits a globular decomposition from those of A and X. This makes the multipointed topological space Mi into a globular complex. Moreover both morphisms $X \longrightarrow Mi$ and $[0, 1] \boxtimes A \longrightarrow Mi$ are morphisms of globular complexes. One

even has $(Mi)_\beta = X$ for some ordinal β and $X \longrightarrow Mi$ is the canonical morphism induced by the globular decomposition of Mi.

Proof. Let $(\partial Z_\beta, Z_\beta, \phi_\beta)_{\beta < \lambda}$ be the globular decomposition of A. The morphism of multipointed spaces $\{0\} \boxtimes A \longrightarrow [0,1] \boxtimes A$ can be viewed as a composite

$$\{0\} \boxtimes A \longrightarrow \{0,1\} \boxtimes A \longrightarrow [0,1] \boxtimes A.$$

The morphism of globular complexes $\{0\} \boxtimes A \longrightarrow \{0,1\} \boxtimes A$ is the transfinite composition of pushouts of the morphisms $\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta) \longrightarrow \mathrm{Glob}^{\mathrm{top}}(Z_\beta)$ for $\beta < \lambda$. The morphism of globular complexes $\{0,1\} \boxtimes A \longrightarrow [0,1] \boxtimes A$ is the transfinite composition of pushouts of the $\mathrm{Glob}^{\mathrm{top}}(Z_\beta \sqcup Z_\beta) \longrightarrow \mathrm{Glob}^{\mathrm{top}}([0,1] \times Z_\beta)$. Therefore the morphism of multipointed spaces $X \longrightarrow Mi$ is a relative globular complex. So Mi has a canonical structure of globular complexes and both morphisms $X \longrightarrow Mi$ and $[0,1] \boxtimes A \longrightarrow Mi$ are morphisms of globular complexes. $\square$

The commutative diagram of globular complexes

$$\begin{array}{ccc} \{0\} \boxtimes A & \longrightarrow & [0,1] \boxtimes A \\ \downarrow{\scriptstyle i} & & \downarrow \\ \{0\} \boxtimes X & \longrightarrow & [0,1] \boxtimes X \end{array}$$

gives rise to a morphism of multipointed spaces $\psi(i) : Mi \longrightarrow [0,1] \boxtimes X$. Since by Proposition III.3.10, one also has the cocartesian diagram of quasi-flows

$$\begin{array}{ccc} \mathrm{qcat}(\{0\} \boxtimes A) & \longrightarrow & \mathrm{qcat}([0,1] \boxtimes A) \\ \downarrow{\scriptstyle i} & & \downarrow \\ \mathrm{qcat}(\{0\} \boxtimes X) & \longrightarrow & \mathrm{qcat}(Mi) \end{array}$$

then there exists a morphism of quasi-flows $\mathrm{qcat}(Mi) \longrightarrow \mathrm{qcat}([0,1] \boxtimes X)$. Therefore the morphism of multipointed spaces $\psi(i) : Mi \longrightarrow [0,1] \boxtimes X$ satisfies

$$\psi(i)(\mathbb{P}^{\mathrm{top}} Mi) \subset \mathbb{P}^{\mathrm{top}}([0,1] \boxtimes X).$$

So $\psi(i)$ is a morphism of globular complexes.

Theorem IV.2.4. *Let $i : A \longrightarrow X$ be a morphism of globular complexes. Then the following assertions are equivalent:*

1. *The morphism i satisfies the S-homotopy extension property.*
2. *The morphism of globular complexes $\psi(i)$ has a retract r, that is to say there exists a morphism of globular complexes*

$$r : [0,1] \boxtimes X \longrightarrow ([0,1] \boxtimes A) \sqcup_{\{0\} \boxtimes A} (\{0\} \boxtimes X)$$

such that $r \circ \psi(i) = \mathrm{Id}_{([0,1] \boxtimes A) \sqcup_{\{0\} \boxtimes A}(\{0\} \boxtimes X)}$.

The proof is exactly the same as the one of [Gau03d] Theorem 9.4. The main point is that the multipointed space Mi is a globular complex.

Proof. Giving two morphisms of globular complexes $f : X \longrightarrow Y$ and $h : [0,1] \boxtimes A \longrightarrow Y$ such that $h(0 \boxtimes a) = f(i(a))$ for any $a \in A$ is equivalent to giving a morphism of globular complexes still denoted by h from $([0,1] \boxtimes A) \sqcup_{\{0\} \boxtimes A}(\{0\} \boxtimes X)$ to Y. The S-homotopy extension problem for i has then always a solution if and only

for any morphism of globular complexes $h : ([0,1] \boxtimes A) \sqcup_{\{0\} \boxtimes A} (\{0\} \boxtimes X) \longrightarrow Y$, there exists a morphism of globular complexes $H : [0,1] \boxtimes X \longrightarrow Y$ such that $H \circ \psi(i) = h$. Take $Y = ([0,1] \boxtimes A) \sqcup_{\{0\} \boxtimes A} (\{0\} \boxtimes X)$ and let h be the identity map of Y. This yields the retract r. Conversely, let r be a retract of i. Then $H := h \circ r$ is always a solution of the S-homotopy extension problem. $\qquad\square$

Theorem IV.2.5. *Let $(Z, \partial Z)$ be a* NDR *pair of compact spaces. Then the inclusion of globular complexes $i : \mathrm{Glob}^{\mathrm{top}}(\partial Z) \longrightarrow \mathrm{Glob}^{\mathrm{top}}(Z)$ satisfies the S-homotopy extension property.*

Proof. Since $(Z, \partial Z)$ is a NDR pair, then the closed inclusion $[0,1] \times \partial Z \cup \{0\} \times Z \longrightarrow [0,1] \times Z$ has a retract $[0,1] \times Z \longrightarrow [0,1] \times \partial Z \cup \{0\} \times Z$. Then the morphism of globular complexes $\mathrm{Glob}^{\mathrm{top}}([0,1] \times \partial Z \cup \{0\} \times Z) \longrightarrow \mathrm{Glob}^{\mathrm{top}}([0,1] \times Z)$ has a retract $\mathrm{Glob}^{\mathrm{top}}([0,1] \times Z) \longrightarrow \mathrm{Glob}^{\mathrm{top}}([0,1] \times \partial Z \cup \{0\} \times Z)$. Hence the result by Theorem IV.2.4. $\qquad\square$

Theorem IV.2.6. *Let U be a compact connected nonempty space. Let X and Y be two globular complexes. Then there exists a natural homeomorphism*

$$\mathbf{TOP}(U, \mathbf{glTOP}(X,Y)) \cong \mathbf{glTOP}(U \boxtimes X, Y).$$

Proof. We already know by Theorem II.4.7 that there exists a natural bijection

$$\mathbf{Top}(U, \mathbf{glTOP}(X,Y)) \cong \mathbf{glTop}(U \boxtimes X, Y).$$

Let $(\partial Z_\beta, Z_\beta, \phi_\beta)_{\beta < \lambda}$ be the globular decomposition of X. We are going to prove that

$$\mathbf{TOP}(U, \mathbf{glTOP}(X_\beta, Y)) \cong \mathbf{glTOP}(U \boxtimes X_\beta, Y).$$

Using the construction of $\boxtimes$ and Theorem II.3.3, it suffices to prove the homeomorphism for $X = X_0$ and $X = \mathrm{Glob}^{\mathrm{top}}(Z)$. The space $\mathbf{glTOP}(X_0, Y)$ is the discrete space of set maps $\mathbf{Set}(X^0, Y^0)$ from X^0 to Y^0. Since U is connected and nonempty, one has the homeomorphism $\mathbf{TOP}(U, \mathbf{glTOP}(X_0, Y)) \cong \mathbf{Set}(X^0, Y^0)$. On the other hand, $\mathbf{glTOP}(U \boxtimes X_0, Y) \cong \mathbf{glTOP}(X_0, Y) \cong \mathbf{Set}(X^0, Y^0)$, hence the result for X_0. At last,

$$\mathbf{Top}(W, \mathbf{TOP}(U, \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), Y))) \cong \mathbf{Top}(W \times U, \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), Y))$$
$$\cong \mathbf{glTop}((W \times U) \boxtimes \mathrm{Glob}^{\mathrm{top}}(Z), Y)$$
$$\cong \mathbf{glTop}(\mathrm{Glob}^{\mathrm{top}}(W \times U \times Z), Y)$$

and $\mathbf{Top}(W, \mathbf{glTOP}(U \boxtimes \mathrm{Glob}^{\mathrm{top}}(Z), Y)) \cong \mathbf{Top}(W, \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(U \times Z), Y))$. It is then easy to see that both sets $\mathbf{glTop}(\mathrm{Glob}^{\mathrm{top}}(W \times U \times Z), Y)$ and $\mathbf{Top}(W, \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(U \times Z), Y))$ can be identified with the same subset of $\mathbf{Top}([0,1] \times W \times U \times Z, Y)$. Hence the result by Yoneda. $\qquad\square$

Theorem IV.2.7. *A morphism of globular complexes $i : A \longrightarrow X$ satisfies the S-homotopy extension property if and only if for any globular complex Y, the continuous map $i^* : \mathbf{glTOP}(X,Y) \longrightarrow \mathbf{glTOP}(A,Y)$ is a Hurewicz fibration.*

Proof. For any topological space M, one has

$$\mathbf{Top}([0,1] \times M, \mathbf{glTOP}(A,Y)) \cong \mathbf{Top}(M, \mathbf{TOP}([0,1], \mathbf{glTOP}(A,Y)))$$

since **Top** is cartesian closed. One also has

$$\mathbf{Top}(M, \mathbf{TOP}([0,1], \mathbf{glTOP}(A,Y))) \cong \mathbf{Top}(M, \mathbf{glTOP}([0,1] \boxtimes A, Y))$$

by Theorem IV.2.6. Considering a commutative diagram like

$$\begin{array}{ccc}
\{0\} \times M & \xrightarrow{\phi} & \mathbf{glTOP}(X,Y) \\
\downarrow & \overset{k}{\nearrow} & \downarrow i^* \\
[0,1] \times M & \xrightarrow{\psi} & \mathbf{glTOP}(A,Y)
\end{array}$$

is then equivalent to considering a commutative diagram of topological spaces

$$\begin{array}{ccc}
M & \longrightarrow & \mathbf{glTOP}(\{0\} \boxtimes X, Y) \\
\downarrow & & \downarrow \\
\mathbf{glTOP}([0,1] \boxtimes A, Y) & \longrightarrow & \mathbf{glTOP}(\{0\} \boxtimes A, Y)
\end{array}$$

Since $\{0\} \boxtimes A \longrightarrow [0,1] \boxtimes A$ is a relative globular complex, using again Theorem II.3.3, considering such a commutative diagram is equivalent to considering a continuous map $M \longrightarrow \mathbf{glTOP}(Mi, Y)$. Finding a continuous map k making both triangles commutative is equivalent to finding a commutative diagram of the form

$$\begin{array}{ccc}
M & \xrightarrow{\overline{\phi}} & \mathbf{glTOP}(Mi,Y) \\
=\downarrow & & \uparrow \psi(i)^* \\
M & \xrightarrow{\ell} & \mathbf{glTOP}([0,1] \boxtimes X, Y).
\end{array}$$

If $i : A \longrightarrow X$ satisfies the S-homotopy extension property, then $\psi(i) : Mi \longrightarrow [0,1] \boxtimes X$ has a retract $r : [0,1] \boxtimes X \longrightarrow Mi$. Then take $\ell = \overline{\phi} \circ r$. Conversely, if ℓ exists for any M and any Y, take $M = \{0\}$ and $Y = Mi$ and $\overline{\phi}(0) = \mathrm{Id}_{Mi}$. Then $\ell(0)$ is a retract of $\psi(i)$. Therefore $i : A \longrightarrow X$ satisfies the S-homotopy extension property. $\qquad\square$

Corollary IV.2.8. *Let Z be a compact space and let $\partial Z \subset Z$ be a compact subspace such that the canonical inclusion is a NDR pair. Let U be a globular complex. Then the canonical restriction map*

$$\mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), U) \to \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(\partial Z), U)$$

is a Hurewicz fibration.

3. Comparing execution paths of globular complexes and of flows

3.1. Morphisms of globular complexes and morphisms of flows.

Proposition IV.3.1. *Let Z be a compact topological space. Let U be a globular complex. Consider the set map*

$$\mathrm{cat} : \mathbf{glTop}(\mathrm{Glob}^{\mathrm{top}}(Z), U) \longrightarrow \mathbf{Flow}(\mathrm{Glob}(Z), \mathrm{cat}(U)).$$

1. *The mapping*

$$\mathrm{cat} : \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), U) \longrightarrow \mathbf{FLOW}(\mathrm{Glob}(Z), \mathrm{cat}(U))$$

   is continuous.
2. *There exists a continuous map*

$$r : \mathbf{FLOW}(\mathrm{Glob}(Z), \mathrm{cat}(U)) \longrightarrow \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), U)$$

   such that $\mathrm{cat} \circ r = \mathrm{Id}_{\mathbf{FLOW}(\mathrm{Glob}(Z), \mathrm{cat}(U))}$. *In particular, this means that* cat *is onto.*
3. *The map* $r \circ \mathrm{cat}$ *is homotopic to* $\mathrm{Id}_{\mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), U)}$. *In particular, this means that* $\mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), U)$ *and* $\mathbf{FLOW}(\mathrm{Glob}(Z), \mathrm{cat}(U))$ *are homotopy equivalent.*

Proof. One has

$$\mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), U) \subset \mathbf{TOP}(Z \times [0,1], U) \cong \mathbf{TOP}(Z, \mathbf{TOP}([0,1], U))$$

therefore

$$\mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), U) \cong \bigsqcup_{(\alpha,\beta) \in U^0 \times U^0} \mathbf{TOP}(Z, \mathbb{P}^{\mathrm{top}}_{\alpha,\beta} U).$$

On the other hand,

$$\mathbf{FLOW}(\mathrm{Glob}(Z), \mathrm{cat}(U)) \cong \bigsqcup_{(\alpha,\beta) \in U^0 \times U^0} \mathbf{TOP}(Z, \mathbb{P}_{\alpha,\beta} U).$$

So the set map

$$\mathrm{cat} : \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), U) \to \mathbf{FLOW}(\mathrm{Glob}(Z), \mathrm{cat}(U))$$

is induced by p_U which is continuous. Hence

$$\mathrm{cat} : \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), U) \to \mathbf{FLOW}(\mathrm{Glob}(Z), \mathrm{cat}(U))$$

is continuous. Choose a map i_U like in Theorem III.5.2. Let

$$r(\phi) \in \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), U)$$

defined by $r(\phi)((z,t)) := (i_U \phi(z))(t)$. Then

$$
\begin{aligned}
\mathrm{cat}(r(\phi))(z) &= p_U \circ \mathbb{P}^{\mathrm{top}}(r(\phi))(t \mapsto (z,t)) && \text{by definition of } \mathrm{cat}(-) \\
&= p_U \circ \mathbb{P}^{\mathrm{top}}(r(\phi)) \circ i_{\mathrm{Glob}^{\mathrm{top}}(Z)}(z) && \text{by definition of } i_{\mathrm{Glob}^{\mathrm{top}}(Z)} \\
&= p_U(r(\phi) \circ i_{\mathrm{Glob}^{\mathrm{top}}(Z)}(z)) && \text{by definition of } \mathbb{P}^{\mathrm{top}} \\
&= p_U(t \mapsto r(\phi)((z,t))) && \text{by definition of } i_{\mathrm{Glob}^{\mathrm{top}}(Z)} \\
&= p_U(i_U \phi(z)) && \text{by definition of } r(\phi) \\
&= \phi(z) && \text{since } p_U \circ i_U = \mathrm{Id}.
\end{aligned}
$$

Therefore $\mathrm{cat}(r(\phi)) = \phi$. So the second assertion holds. One has

$$
\begin{aligned}
(r \circ \mathrm{cat}(f))(z,t) &= (i_U \circ \mathrm{cat}(f)(z))(t) && \text{by definition of } r \\
&= (i_U \circ p_U \circ \mathbb{P}^{\mathrm{top}}(f)(t \mapsto (z,t)))(t) && \text{by definition of } \mathrm{cat}.
\end{aligned}
$$

Since $(i_U \circ p_U \circ \mathbb{P}^{\mathrm{top}}(f)(t \mapsto (z,t)))$ is an execution path of U by Theorem III.5.2, since

$$p_U \circ (i_U \circ p_U \circ \mathbb{P}^{\mathrm{top}}(f)(t \mapsto (z,t))) = p_U \circ \mathbb{P}^{\mathrm{top}}(f)(t \mapsto (z,t)) = \mathbb{P}f(z),$$

then by Proposition III.5.1, there exists a continuous map $\phi : Z \times [0,1] \longrightarrow [0,1]$ such that

$$f((z,t)) = \mathbb{P}^{\mathrm{top}}(f)(t \mapsto (z,t)) = (r \circ \mathrm{cat}(f))(z, \phi(z,t)).$$

Notice that for a given $z \in Z$, the mapping $t \mapsto \phi(z,t)$ is necessarily nondecreasing. Hence the third assertion by considering the homotopy

$$H(f,u)((z,t)) = (r \circ \mathrm{cat}(f))(z, u\phi(z,t) + (1-u)t). \qquad \square$$

Proposition IV.3.2 ([Gau03d] Corollary 9.9)**.** *Let Z be a compact space and let $\partial Z \subset Z$ be a compact subspace such that the canonical inclusion is a* NDR *pair. Let U be a flow. Then the canonical restriction map*

$$\mathbf{FLOW}(\mathrm{Glob}(Z), U) \to \mathbf{FLOW}(\mathrm{Glob}(\partial Z), U)$$

is a Hurewicz fibration.

3.2. Homotopy limit of a transfinite tower and homotopy pullback. The corollaries IV.3.6 and IV.3.9 below are of course not new. But the author does not know where the proofs of these two facts can be found. So a short argument involving Strøm's model structure is presented.

Let λ be an ordinal. Any ordinal can be viewed as a small category whose objects are the elements of λ, that is the ordinal $\gamma < \lambda$, and where there exists a morphism $\gamma \longrightarrow \gamma'$ if and only if $\gamma \leqslant \gamma'$. The notation λ^{op} will then denote the opposite category. Let us then denote by $\mathcal{C}^{\lambda^{\mathrm{op}}}$ the category of functors from λ^{op} to $\mathcal{C}$ where $\mathcal{C}$ is a category. An object of $\mathcal{C}^{\lambda^{\mathrm{op}}}$ is called a *tower*.

Proposition and Definition IV.3.3 ([Hir03, Hov99])**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be two model categories. A Quillen adjunction is a pair of adjoint functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ between the model categories $\mathcal{C}$ and $\mathcal{D}$ such that one of the following equivalent properties holds:*

1. *If f is a cofibration* (*resp. a trivial cofibration*), *then so does $F(f)$.*
2. *If g is a fibration* (*resp. a trivial fibration*), *then so does $G(g)$.*

One says that F is a left Quillen functor. *One says that G is a* right Quillen functor.

Definition IV.3.4 ([Hir03, Hov99])**.** An object X of a model category $\mathcal{C}$ is *cofibrant* if the canonical morphism $\varnothing \longrightarrow X$ from the initial object of $\mathcal{C}$ to X is a cofibration, and is *fibrant* if the canonical morphism $X \longrightarrow \mathbf{1}$ from X to the final object $\mathbf{1}$ is a fibration.

Proposition IV.3.5. *Let $\mathcal{C}$ be a model category. There exists a model structure on $\mathcal{C}^{\lambda^{\mathrm{op}}}$ such that the limit functor $\varprojlim : \mathcal{C}^{\lambda^{\mathrm{op}}} \longrightarrow \mathcal{C}$ is a right Quillen functor and such that the fibrant towers T are exactly the towers $T : \lambda^{\mathrm{op}} \longrightarrow \mathcal{C}$ such that T_0 is fibrant and such that for any ordinal γ with $0 \leqslant \gamma < \lambda$, the canonical morphism $T_\gamma \longrightarrow \varprojlim_{\beta < \gamma} T_\beta$ is a fibration of $\mathcal{C}$.*

This proposition is proved for $\lambda = \aleph_0$ in [GJ99].

Sketch of proof. For a reminder about the Reedy model structure, see [Hir03] and [Hov99]. With the Reedy structure corresponding to the indexing, let us calculate the latching space functors $L_\gamma T$ and the matching space functors $M_\gamma T$ of a tower T:

1. If $\gamma + 1 < \lambda$, then $L_\gamma T = T_{\gamma+1}$.

2. If $\gamma + 1 = \lambda$, then $L_\gamma T = \varnothing$ (the initial object of $\mathcal{C}$).

3. For any $\gamma < \lambda$, $M_\gamma T = \varprojlim_{\beta<\gamma} T_\beta$.

So a morphism of towers $T \longrightarrow T'$ is a cofibration for the Reedy model structure if and only if:

1. For $\gamma + 1 < \lambda$, the morphism $T_\gamma \sqcup_{T_{\gamma+1}} T'_{\gamma+1} \longrightarrow T'_\gamma$ is a cofibration of $\mathcal{C}$.

2. For $\gamma + 1 = \lambda$, $T_\gamma \longrightarrow T'_\gamma$ is a cofibration of $\mathcal{C}$.

The limit functor $\varprojlim : \mathcal{C}^{\lambda^{\mathrm{op}}} \longrightarrow \mathcal{C}$ is a right Quillen functor if and only if its left adjoint, the constant diagram functor $\Delta : \mathcal{C} \longrightarrow \mathcal{C}^{\lambda^{\mathrm{op}}}$ is a left Quillen functor. Consider a cofibration $X \longrightarrow Y$ of $\mathcal{C}$. Then the morphism of towers $\Delta(X) \longrightarrow \Delta(Y)$ is a cofibration if and only if either $\gamma + 1 < \lambda$ and $\Delta(Y)_{\gamma+1} \longrightarrow \Delta(Y)_\gamma$ is a cofibration or $\gamma + 1 = \lambda$ and $\Delta(X)_\gamma \longrightarrow \Delta(Y)_\gamma$ is a cofibration. This holds indeed. Therefore the limit functor is a right Quillen functor.

But a morphism of towers $T \longrightarrow T'$ is a fibration for the Reedy model structure if and only if for any $\gamma < \lambda$, $T_\gamma \longrightarrow T'_\gamma \times_{(\varprojlim_{\beta<\gamma} T'_\beta)} (\varprojlim_{\beta<\gamma} T_\beta)$ is a fibration. Hence the result. $\qquad\square$

Corollary IV.3.6. *Let T and T' be two objects of $\mathbf{Top}^{\lambda^{\mathrm{op}}}$ such that:*

1. *For any $\gamma < \lambda$ such that $\gamma + 1 < \lambda$, the morphism $T_{\gamma+1} \longrightarrow T_\gamma$ is a Hurewicz fibration of topological spaces.*

2. *For any $\gamma < \lambda$ such that γ is a limit ordinal, the canonical morphism $T_\gamma \longrightarrow \varprojlim_{\beta<\gamma} T_\beta$ is an homeomorphism.*

If $f : T \longrightarrow T'$ is an objectwise homotopy equivalence, then $\varprojlim f : \varprojlim T \longrightarrow \varprojlim T'$ is a homotopy equivalence.

Proof. There exists a model structure on the category of topological spaces $\mathbf{Top}$ where the cofibrations are the Hurewicz cofibrations, the fibrations the Hurewicz fibrations and the weak homotopy equivalences the homotopy equivalences [Str66, Str68, Str72]. All topological spaces are fibrant and cofibrant for this model structure. The corollary is then due to the fact that a right Quillen functor preserves weak homotopy equivalences between fibrant objects and to the fact that any topological space is fibrant for this model structure. $\qquad\square$

Lemma IV.3.7 (Cube lemma [Hir03, Hov99]). *Let $\mathcal{C}$ be model category. Let*

$$
\begin{array}{ccc}
A_i & \longrightarrow & B_i \\
\downarrow & & \\
C_i & &
\end{array}
$$

be two diagrams D_i with $i = 1, 2$ of cofibrant objects of $\mathcal{C}$ such that both morphisms $A_i \longrightarrow B_i$ with $i = 1, 2$ are cofibrations of the model structure. Then any morphism of diagrams $D_1 \longrightarrow D_2$ which is an objectwise weak equivalence induces a weak equivalence $\varinjlim D_1 \longrightarrow \varinjlim D_2$.

The dual version is as follows:

Lemma IV.3.8. *Let $\mathcal{C}$ be a model category. Let*

$$
\begin{array}{ccc}
 & & B_i \\
 & & \downarrow \\
A_i & \longrightarrow & C_i
\end{array}
$$

be two diagrams D_i with $i = 1, 2$ of fibrant objects of $\mathcal{C}$ such that both morphisms $B_i \longrightarrow C_i$ with $i = 1, 2$ are fibrations of the model structure. Then any morphism of diagrams $D_1 \longrightarrow D_2$ which is an objectwise weak equivalence induces a weak equivalence $\varprojlim D_1 \longrightarrow \varprojlim D_2$.

Corollary IV.3.9. *Let*

$$
\begin{array}{ccc}
 & & B_i \\
 & & \downarrow \\
A_i & \longrightarrow & C_i
\end{array}
$$

be two diagrams D_i with $i = 1, 2$ of topological spaces such that both morphisms $B_i \longrightarrow C_i$ with $i = 1, 2$ are Hurewicz fibrations. Then any morphism of diagrams $D_1 \longrightarrow D_2$ which is an objectwise homotopy equivalence induces a homotopy equivalence $\varprojlim D_1 \longrightarrow \varprojlim D_2$.

3.3. The end of the proof.

Theorem IV.3.10. *Let X and U be two globular complexes. The set map*

$$
\mathrm{cat} : \mathbf{glTOP}(X, U) \longrightarrow \mathbf{FLOW}(\mathrm{cat}(X), \mathrm{cat}(U))
$$

is continuous and moreover is a homotopy equivalence.

Proof. The globular decomposition of X enables to view the canonical continuous map $\varnothing \longrightarrow X$ as a transfinite composition of $X_\beta \longrightarrow X_{\beta+1}$ for $\beta < \lambda$ such that for any ordinal $\beta < \lambda$, one has the pushout of topological spaces

$$
\begin{array}{ccc}
\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta) & \longrightarrow & X_\beta \\
\downarrow & & \downarrow \\
\mathrm{Glob}^{\mathrm{top}}(Z_\beta) & \longrightarrow & X_{\beta+1}
\end{array}
$$

where the pair $(Z_\beta, \partial Z_\beta)$ is a NDR pair. And by construction of the functor cat : **glTop** $\longrightarrow$ **Flow**, one also has for any ordinal $\beta < \lambda$ the pushout of flows

$$
\begin{array}{ccc}
\mathrm{Glob}(\partial Z_\beta) & \longrightarrow & \mathrm{cat}(X_\beta) \\
\downarrow & & \downarrow \\
\mathrm{Glob}(Z_\beta) & \longrightarrow & \mathrm{cat}(X_{\beta+1}).
\end{array}
$$

By Theorem II.3.3, one obtains the pullback of topological spaces

$$\begin{array}{ccc}
\mathbf{glTOP}(X_{\beta+1}, U) & \longrightarrow & \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z_\beta), U) \\
\downarrow & & \downarrow \\
\mathbf{glTOP}(X_\beta, U) & \longrightarrow & \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta), U).
\end{array}$$

By Theorem III.2.9, one obtains the pullback of topological spaces

$$\begin{array}{ccc}
\mathbf{FLOW}(\mathrm{cat}(X_{\beta+1}), \mathrm{cat}(U)) & \longrightarrow & \mathbf{FLOW}(\mathrm{Glob}(Z_\beta), \mathrm{cat}(U)) \\
\downarrow & & \downarrow \\
\mathbf{FLOW}(\mathrm{cat}(X_\beta), \mathrm{cat}(U)) & \longrightarrow & \mathbf{FLOW}(\mathrm{Glob}(\partial Z_\beta), \mathrm{cat}(U)).
\end{array}$$

For a given β, let us suppose that the space ∂Z_β is empty. Then the topological spaces $\mathbf{FLOW}(\mathrm{Glob}(\partial Z_\beta), \mathrm{cat}(U))$ and $\mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta), U)$ are both discrete. So both continuous maps

$$\mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z_\beta), U) \longrightarrow \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta), U)$$

and

$$\mathbf{FLOW}(\mathrm{Glob}(Z_\beta), \mathrm{cat}(U)) \longrightarrow \mathbf{FLOW}(\mathrm{Glob}(\partial Z_\beta), \mathrm{cat}(U))$$

are Hurewicz fibrations. Otherwise, if the space ∂Z_β is not empty, then the pair $(Z_\beta, \partial Z_\beta)$ is a NDR pair. Then by Corollary IV.2.8, the continuous map

$$\mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z_\beta), U) \longrightarrow \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta), U)$$

is a Hurewicz fibration. And by Proposition IV.3.2, the continuous map

$$\mathbf{FLOW}(\mathrm{Glob}(Z_\beta), \mathrm{cat}(U)) \longrightarrow \mathbf{FLOW}(\mathrm{Glob}(\partial Z_\beta), \mathrm{cat}(U))$$

is a Hurewicz fibration as well. One obtains for a given ordinal $\beta < \lambda$ the following commutative diagram of topological spaces:

where the symbol $\longrightarrow\!\!\!\!\!\rightarrow$ means Hurewicz fibration. One can now apply Corollary IV.3.9. Therefore, if $\mathbf{glTOP}(X_\beta, U) \longrightarrow \mathbf{FLOW}(\mathrm{cat}(X_\beta), \mathrm{cat}(U))$ is a homotopy equivalence of topological spaces, then the same holds by replacing β by $\beta + 1$. By transfinite induction, we want to prove that for any ordinal $\beta < \lambda$, one has the homotopy equivalence of topological spaces $\mathbf{glTOP}(X_\beta, U) \longrightarrow \mathbf{FLOW}(\mathrm{cat}(X_\beta), \mathrm{cat}(U))$. The initialization is trivial: if X^0 is a discrete globular

complex, then $\operatorname{cat}(X^0) = X^0$. The passage from β to $\beta+1$ is ensured by the proof above. It remains to treat the case where β is a limit ordinal. Since the pullback of a Hurewicz fibration is a Hurewicz fibration, then all continuous maps

$$\mathbf{glTOP}(X_{\beta+1}, U) \longrightarrow \mathbf{glTOP}(X_\beta, U)$$

and

$$\mathbf{FLOW}(\operatorname{cat}(X_{\beta+1}), \operatorname{cat}(U)) \longrightarrow \mathbf{FLOW}(\operatorname{cat}(X_\beta), \operatorname{cat}(U))$$

are actually Hurewicz fibrations. By Theorem II.3.3, for any limit ordinal β, one has

$$\mathbf{glTOP}(\varinjlim_{\alpha<\beta} X_\beta, U) \cong \varprojlim_{\alpha<\beta} \mathbf{glTOP}(X_\beta, U).$$

By Theorem III.2.9, for any limit ordinal β, one has

$$\mathbf{FLOW}(\varinjlim_{\alpha<\beta} \operatorname{cat}(X_\beta), \operatorname{cat}(U)) \cong \varprojlim_{\alpha<\beta} \mathbf{FLOW}(\operatorname{cat}(X_\beta), \operatorname{cat}(U)).$$

The proof is then complete with Corollary IV.3.6. $\qquad\square$

The preceding result can be slightly improved. The homotopy equivalence above is actually a Hurewicz fibration. Three preliminary propositions are necessary to establish this fact.

Proposition IV.3.11. *Let Z be a compact space. Let U be a globular complex. Then the canonical continuous map*

$$\operatorname{cat} : \mathbf{glTOP}(\operatorname{Glob}^{\mathrm{top}}(Z), U) \longrightarrow \mathbf{FLOW}(\operatorname{Glob}(Z), \operatorname{cat}(U))$$

is a Hurewicz fibration.

Proof. Let M be a topological space. Consider the following commutative diagram:

$$
\begin{array}{ccc}
M \times \{0\} & \xrightarrow{\ f\ } & \mathbf{glTOP}(\operatorname{Glob}^{\mathrm{top}}(Z), U) \\
{\scriptstyle i}\downarrow & {\overset{h}{\nearrow}} & \downarrow{\scriptstyle cat} \\
M \times [0,1] & \xrightarrow{\ g\ } & \mathbf{FLOW}(\operatorname{Glob}(Z), \operatorname{cat}(U)).
\end{array}
$$

One has to find h making the two triangles commutative where $i : M \times \{0\} \subset M \times [0,1]$ is the canonical inclusion. Let $h(m, u) \in \mathbf{glTOP}(\operatorname{Glob}^{\mathrm{top}}(Z), U)$ of the form

$$h(m, u)(z, t) = r(g(m, u))(z, \phi(m, z)(t))$$

where ϕ is a continuous map from $M \times Z$ to $\mathbf{glTOP}(\overrightarrow{I}^{\mathrm{top}}, \overrightarrow{I}^{\mathrm{top}})$. Then $\operatorname{cat} \circ h = g$ for any map ϕ. It then suffices to take ϕ such that

$$f(m, 0)(z, t) = r(g(m, u))(z, \phi(m, z)(t)).$$

Such a map ϕ is unique by the second assertion of Proposition III.5.1. The continuity of ϕ comes from its uniqueness and from the continuity of the other components, similarly to Proposition III.5.1. $\qquad\square$

Proposition IV.3.12. *Let Z be a compact space. Let U be a globular complex. Then one has the homeomorphism*

$$\mathbf{glTOP}(\operatorname{Glob}^{\mathrm{top}}(Z), U) \cong \mathbf{glTOP}(\operatorname{Glob}^{\mathrm{top}}(Z), \overrightarrow{I}^{\mathrm{top}}) \times \mathbf{FLOW}(\operatorname{Glob}(Z), \operatorname{cat}(U)).$$

Proof. Let $f \in \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), U)$. Then there exists a unique continuous map $\phi_f : |\mathrm{Glob}^{\mathrm{top}}(Z)| \longrightarrow [0, 1]$ such that for any $(z, t) \in \mathrm{Glob}^{\mathrm{top}}(Z)$, $f(z, t) = i_U(\mathrm{cat}(f)(z))(\phi_f(z, t))$ by Proposition III.5.1. The continuous map ϕ_f is actually a morphism of globular complexes from $\mathrm{Glob}^{\mathrm{top}}(Z)$ to $\overrightarrow{I}^{\mathrm{top}}$. The mapping $f \mapsto (\phi_f, \mathrm{cat}(f))$ defines a set map from $\mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), U)$ to

$$\mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), \overrightarrow{I}^{\mathrm{top}}) \times \mathbf{FLOW}(\mathrm{Glob}(Z), \mathrm{cat}(U))$$

which is obviously an isomorphism of sets. One obtains the isomorphism of sets

$$\mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), U) \cong \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), \overrightarrow{I}^{\mathrm{top}}) \times \mathbf{FLOW}(\mathrm{Glob}(Z), \mathrm{cat}(U)).$$

The set map

$$\mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), \overrightarrow{I}^{\mathrm{top}}) \times \mathbf{FLOW}(\mathrm{Glob}(Z), \mathrm{cat}(U)) \to \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), U)$$

is clearly continuous for the Kelleyfication of the compact-open topology. It remains to prove that the mapping $f \mapsto (\phi_f, \mathrm{cat}(f))$ is continuous. It suffices to prove that the mapping $f \mapsto \phi_f$ is continuous since we already know that $\mathrm{cat}(-)$ is continuous. The latter fact comes from the continuity of the mapping $(f, z, t) \mapsto f(z, t)$ which implies the continuity of $(f, z, t) \mapsto \phi_f(z, t)$. $\qquad \square$

Proposition IV.3.13. *Let Z be a compact space. Let U be a globular complex. Let $(Z, \partial Z)$ be a* NDR *pair. Then the canonical continuous map*

$$\mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), U) \longrightarrow$$
$$\mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(\partial Z), U) \times_{\mathbf{FLOW}(\mathrm{Glob}(\partial Z), \mathrm{cat}(U))} \mathbf{FLOW}(\mathrm{Glob}(Z), \mathrm{cat}(U))$$

is a Hurewicz fibration.

Proof. One has

$$\mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), U) \cong \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), \overrightarrow{I}^{\mathrm{top}}) \times \mathbf{FLOW}(\mathrm{Glob}(Z), \mathrm{cat}(U))$$

and

$$\mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(\partial Z), U) \times_{\mathbf{FLOW}(\mathrm{Glob}(\partial Z), \mathrm{cat}(U))} \mathbf{FLOW}(\mathrm{Glob}(Z), \mathrm{cat}(U))$$
$$\cong \left(\mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(\partial Z), \overrightarrow{I}^{\mathrm{top}}) \times \mathbf{FLOW}(\mathrm{Glob}(\partial Z), \mathrm{cat}(U)) \right)$$
$$\times_{\mathbf{FLOW}(\mathrm{Glob}(\partial Z), \mathrm{cat}(U))} \mathbf{FLOW}(\mathrm{Glob}(Z), \mathrm{cat}(U))$$
$$\cong \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(\partial Z), \overrightarrow{I}^{\mathrm{top}}) \times \mathbf{FLOW}(\mathrm{Glob}(Z), \mathrm{cat}(U)).$$

So the continuous map we are studying is the cartesian product of the Hurewicz fibration

$$\mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z), \overrightarrow{I}^{\mathrm{top}}) \longrightarrow \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(\partial Z), \overrightarrow{I}^{\mathrm{top}})$$

by the identity of $\mathbf{FLOW}(\mathrm{Glob}(Z), \mathrm{cat}(U))$, and hence is a Hurewicz fibration. $\quad \square$

Theorem IV.3.14. *Let X and U be two globular complexes. The set map*

$$\mathrm{cat} : \mathbf{glTOP}(X, U) \longrightarrow \mathbf{FLOW}(\mathrm{cat}(X), \mathrm{cat}(U))$$

is a Hurewicz fibration.

Sketch of proof. We use the notations of the proof of Theorem IV.3.10. We are going to prove by transfinite induction on β that the canonical continuous map

$$\mathbf{glTOP}(X_\beta, U) \longrightarrow \mathbf{FLOW}(\mathrm{cat}(X_\beta), \mathrm{cat}(U))$$

is a Hurewicz fibration. For $\beta = 0$, X_β is the discrete globular complex (X^0, X^0). Therefore $\mathbf{glTOP}(X_0, U) = \mathbf{FLOW}(\mathrm{cat}(X_0), \mathrm{cat}(U)) = U^0$. Let us suppose the fact proved for $\beta \geqslant 0$. Then one has the following diagram of topological spaces:

$$
\begin{array}{ccc}
& & \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(Z_\beta), U) \\
& & \downarrow \\
\mathbf{glTOP}(X_\beta, U) \longrightarrow & \mathbf{glTOP}(\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta), U) \\
& & \mathbf{FLOW}(\mathrm{Glob}(Z_\beta), \mathrm{cat}(U)) \\
& & \downarrow \\
\mathbf{FLOW}(\mathrm{cat}(X_\beta), \mathrm{cat}(U)) \longrightarrow & \mathbf{FLOW}(\mathrm{Glob}(\partial Z_\beta), \mathrm{cat}(U))
\end{array}
$$

where the symbol $\longrightarrow\!\!\!\!\!\rightarrow$ means Hurewicz fibration. We then consider the Reedy category

$$
\begin{array}{ccc}
& & 2 \\
& & \downarrow \\
0 & \longrightarrow & 1
\end{array}
$$

and the Reedy model category of diagrams of topological spaces over this small category. In this model category, the fibrant diagrams D are the diagrams such that D_0, D_1 and D_2 are fibrant and such that $D_2 \longrightarrow D_1$ is a fibration. And a morphism of diagrams $D \longrightarrow D'$ is fibrant if and only if both $D_0 \longrightarrow D'_0$ and $D_1 \longrightarrow D'_1$ are fibrant and if $D_2 \longrightarrow D_1 \times_{D'_1} D'_2$ is fibrant. So it remains to check that the inverse limit functor is a right Quillen functor to complete the proof. It then suffices to prove that the constant diagram functor is a left Quillen functor. For this Reedy model structure, a morphism of diagrams $D \longrightarrow D'$ is cofibrant if both $D_0 \longrightarrow D'_0$ and $D_2 \longrightarrow D'_2$ are cofibrant and if $D'_0 \sqcup_{D_0} D_1 \longrightarrow D'_1$ is cofibrant. So a diagram D is cofibrant if and only if D_0, D_1 and D_2 are cofibrant and if $D_0 \longrightarrow D_1$ is a cofibration. Hence the result. $\qquad \square$

Corollary IV.3.15. *Let X and U be two globular complexes. The set map*

$$\mathrm{cat} : \mathbf{glTOP}(X, U) \longrightarrow \mathbf{FLOW}(\mathrm{cat}(X), \mathrm{cat}(U))$$

is onto.

Proof. Any Hurewicz fibration which is a homotopy equivalence is onto since it satisfies the right lifting property with respect to $\varnothing \longrightarrow \{0\}$. $\qquad \square$

4. Comparison of S-homotopy in glTop and in Flow

4.1. The pairing $\boxtimes$ between a topological space and a flow.

Definition IV.4.1 ([Gau03d]). Let U be a topological space. Let X be a flow. The flow $\{U, X\}_S$ is defined as follows:

1. The 0-skeleton of $\{U, X\}_S$ is X^0.
2. For $\alpha, \beta \in X^0$, the topological space $\mathbb{P}_{\alpha,\beta}\{U, X\}_S$ is $\mathbf{TOP}(U, \mathbb{P}_{\alpha,\beta}X)$ with an obvious definition of the composition law.

Theorem IV.4.2 ([Gau03d] Theorem 7.8). *Let U be a topological space. The functor $\{U, -\}_S$ has a left adjoint which will be denoted by $U \boxtimes -$. Moreover:*

1. *One has the natural isomorphism of flows*

$$U \boxtimes (\varinjlim_i X_i) \cong \varinjlim_i (U \boxtimes X_i).$$

2. *There is a natural isomorphism of flows $\{*\} \boxtimes Y \cong Y$.*
3. *If Z is another topological space, one has the natural isomorphism of flows*

$$U \boxtimes \mathrm{Glob}(Z) \cong \mathrm{Glob}(U \times Z).$$

4. *For any flow X and any topological space U, one has the natural bijection of sets*

$$(U \boxtimes X)^0 \cong X^0.$$

5. *If U and V are two topological spaces, then $(U \times V) \boxtimes Y \cong U \boxtimes (V \boxtimes Y)$ as flows.*
6. *For any flow X, $\varnothing \boxtimes X \cong X^0$.*

4.2. S-homotopy of flows.

Definition IV.4.3 ([Gau03d]). A morphism of flows $f : X \longrightarrow Y$ is said *synchronized* if and only if it induces a bijection of sets between the 0-skeleton of X and the 0-skeleton of Y.

Definition IV.4.4 ([Gau03d]). Two morphisms of flows f and g from X to Y are S-*homotopy equivalent* if and only if there exists

$$H \in \mathbf{Top}([0, 1], \mathbf{FLOW}(X, Y))$$

such that $H(0) = f$ and $H(1) = g$. We denote this situation by $f \sim_S g$.

Definition IV.4.5 ([Gau03d]). Two flows are S-*homotopy equivalent* or S-*homotopic* if and only if there exist morphisms of flows $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g \sim_S \mathrm{Id}_Y$ and $g \circ f \sim_S \mathrm{Id}_X$.

Proposition IV.4.6 ([Gau03d] Proposition 7.5). *Let f and g be two morphisms of flows from X to Y. Then f and g are S-homotopy equivalent if and only if there exists a continuous map*

$$h \in \mathbf{Top}([0, 1], \mathbf{FLOW}(X, Y))$$

such that $h(0) = f$ and $h(1) = g$.

Proposition IV.4.7 (Cylinder functor. [Gau03d] Corollary 7.11)**.** *The mapping* $X \mapsto [0,1] \boxtimes X$ *induces a functor from* **Flow** *to itself which is a cylinder functor with the natural transformations* $e_i : \{i\} \boxtimes - \to [0,1] \boxtimes -$ *induced by the inclusion maps* $\{i\} \subset [0,1]$ *for* $i \in \{0,1\}$ *and with the natural transformation* $p : [0,1] \boxtimes - \longrightarrow \{0\} \boxtimes -$ *induced by the constant map* $[0,1] \longrightarrow \{0\}$. *Moreover, two morphisms of flows* f *and* g *from* X *to* Y *are* S-*homotopic if and only if there exists a morphism of flows* $H : [0,1] \boxtimes X \to Y$ *such that* $H \circ e_0 = f$ *and* $H \circ e_1 = g$. *Moreover* $e_0 \circ H \sim_S \mathrm{Id}$ *and* $e_1 \circ H \sim_S \mathrm{Id}$.

4.3. The pairing $\boxtimes$ and S-homotopy.

Proposition IV.4.8. *Let* U *be a compact space. Let* X *be a globular complex. Then one has the isomorphism of flows* $\mathrm{cat}(U \boxtimes X) \cong U \boxtimes \mathrm{cat}(X)$.

Proof. Let $(\partial Z_\beta, Z_\beta, \phi_\beta)_{\beta < \lambda}$ be the globular decomposition of X. This is clear if $X = X_0 = (X^0, X^0)$ and if $X = \mathrm{Glob}^{\mathrm{top}}(Z)$ where Z is compact. It then suffices to make a transfinite induction on β to prove $\mathrm{cat}(U \boxtimes X_\beta) \cong U \boxtimes \mathrm{cat}(X_\beta)$. $\qquad\square$

Theorem IV.4.9. *The set map* $\mathrm{cat} : \mathbf{glTop}(X, U) \to \mathbf{Flow}(\mathrm{cat}(X), \mathrm{cat}(U))$ *induces a bijection of sets* $\mathbf{glTop}(X, U)/\sim_S \cong \mathbf{Flow}(\mathrm{cat}(X), \mathrm{cat}(U))/\sim_S$.

Proof. Let f and g be two S-homotopy equivalent morphisms of globular complexes from X to Y. Then there exists a morphism of globular complexes $H : [0,1] \boxtimes X \longrightarrow Y$ such that the composite $H \circ e_0$ is equal to f and the composite $H \circ e_1$ is equal to g. Then $\mathrm{cat}(H) : [0,1] \boxtimes X \longrightarrow Y$ induces by Proposition IV.4.8 a S-homotopy between $\mathrm{cat}(f)$ and $\mathrm{cat}(g)$. So the mapping cat induces a set map $\mathbf{glTop}(X, U)/\sim_S \to \mathbf{Flow}(\mathrm{cat}(X), \mathrm{cat}(U))/\sim_S$. By Proposition II.4.6, the set $\mathbf{glTop}(X, U)/\sim_S$ is exactly the set of path-connected components of $\mathbf{glTOP}(X, U)$. By Proposition IV.4.6, the set $\mathbf{Flow}(\mathrm{cat}(X), \mathrm{cat}(U))/\sim_S$ is exactly the set of path-connected components of $\mathbf{FLOW}(\mathrm{cat}(X), \mathrm{cat}(U))$. But the set map cat : $\mathbf{glTOP}(X, U) \to \mathbf{FLOW}(\mathrm{cat}(X), \mathrm{cat}(U))$ induces a homotopy equivalence by Theorem IV.3.10. So the two topological spaces have the same path-connected components. $\qquad\square$

Corollary IV.4.10. *Two globular complexes are* S-*homotopy equivalent if and only if the corresponding flows are* S-*homotopy equivalent.*

Corollary IV.4.11. *The localization of the category of globular complexes with respect to the class of* S-*homotopy equivalences is equivalent to the localization of the full and faithful subcategory of flows of the form* $\mathrm{cat}(X)$ *with respect to the* S-*homotopy equivalences.*

Proof. This is due to the existence of the cylinder functor both for the S-homotopy of globular complexes and for the S-homotopy of flows. $\qquad\square$

5. Conclusion

This part shows that the category of flows is an appropriate framework for the study of S-homotopy equivalences. The category **Flow** has nicer categorical properties than **glTop**, for example because it is both complete and cocomplete.

Part V. Flow up to weak S-homotopy

1. Introduction

We prove that the functor from the category of globular CW-complexes to the category of flows induces an equivalence of categories from the localization of the category of globular CW-complexes with respect to the class of the S-homotopy equivalences to the localization of the category of flows with respect to the class of weak S-homotopy equivalences.

2. The model structure of Flow

Some useful references for the notion of model category are [Hov99, GJ99]. See also [DHK97, Hir03].

Theorem V.2.1 ([Gau03d] Theorem 19.7). *The category of flows can be given a model structure such that:*

1. *The weak equivalences are the* weak S-homotopy equivalences, *that is a morphism of flows $f : X \longrightarrow Y$ such that $f : X^0 \longrightarrow Y^0$ is an isomorphism of sets and $f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ a weak homotopy equivalence of topological spaces.*
2. *The fibrations are the continuous maps satisfying the RLP with respect to the morphisms $\mathrm{Glob}(\mathbf{D}^n) \longrightarrow \mathrm{Glob}([0,1] \times \mathbf{D}^n)$ for $n \geqslant 0$. The fibrations are exactly the morphisms of flows $f : X \longrightarrow Y$ such that $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a Serre fibration of* **Top**.
3. *The cofibrations are the morphisms satisfying the LLP with respect to any map satisfying the RLP with respect to the morphisms $\mathrm{Glob}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n)$ for $n \geqslant 0$ and with respect to the morphisms $\varnothing \longrightarrow \{0\}$ and $\{0,1\} \longrightarrow \{0\}$.*
4. *Any flow is fibrant.*

Notation V.2.2. Let $\mathcal{S}$ be the subcategory of weak S-homotopy equivalences. Let I^{gl} be the set of morphisms of flows $\mathrm{Glob}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n)$ for $n \geqslant 0$. Let J^{gl} be the set of morphisms of flows $\mathrm{Glob}(\mathbf{D}^n) \longrightarrow \mathrm{Glob}([0,1] \times \mathbf{D}^n)$. Notice that all arrows of $\mathcal{S}$, I^{gl} and J^{gl} are synchronized. At last, denote by I^{gl}_+ be the union of I^{gl} with the two morphisms of flows $R : \{0,1\} \longrightarrow \{0\}$ and $C : \varnothing \subset \{0\}$.

3. Strongly cofibrant replacement of a flow

Definition V.3.1. Let X be a flow. Let $n \geqslant 0$. Let $f_i : \mathrm{Glob}(\mathbf{S}^{n-1}) \longrightarrow X$ be a family of morphisms of flows with $i \in I$ where I is some set. Then the pushout Y of the diagram

$$
\begin{array}{ccc}
\bigsqcup_{i \in I} \mathrm{Glob}(\mathbf{S}^{n-1}) & \xrightarrow{\ \bigsqcup_{i \in I} f_i\ } & X \\
\ \ \downarrow{\scriptstyle \subset} & & \\
\bigsqcup_{i \in I} \mathrm{Glob}(\mathbf{D}^n) & &
\end{array}
$$

is called a *n-globular extension* of X. The family of $f_i : \mathrm{Glob}(\mathbf{S}^{n-1}) \longrightarrow X$ is called the *globular decomposition of the extension*.

Definition V.3.2. Let $i : A \longrightarrow X$ be a morphism of flows. Then the morphism i is a *relative globular extension* if the flow X is isomorphic to a flow $X_\omega = \varinjlim X_n$ such that for any integer $n \geqslant 0$, X_n is a n-globular extension of X_{n-1} (by convention, let $X_{-1} = A$). One says that $\dim(X, A) = p$ if $X_\omega = X_p = X_{p+1} = \dots$ and if $X_{p-1} \neq X_p$. The flow X_n is called the n-skeleton of (X, A) and the family of $(X_n)_{n \geqslant 0}$ the skeleton.

Definition V.3.3. A flow X is said *strongly cofibrant* if and only if the pair (X, X^0), where X^0 is the 0-skeleton, is a relative globular extension. Let

$$\dim(X) = \dim(X, X^0).$$

Notice that any strongly cofibrant flow is cofibrant for the model structure of **Flow**. Using Theorem V.2.1, we already know that any flow is weakly S-homotopy equivalent to a cofibrant flow and that this cofibrant flow is unique up to S-homotopy. Such a cofibrant flow is usually called a *cofibrant replacement*. With the standard construction of the cofibrant replacement involving the "Small Object Argument", we can only say that the cofibrant replacement of a flow can be taken in the I_+^{gl}-cell complexes.

We want to prove in this section that the cofibrant replacement can be supposed strongly cofibrant. This is therefore a stronger statement than the usual one.

Theorem V.3.4 ([Gau03d] Theorem 15.2)**.** *Suppose that one has the pushout of flows*

$$
\begin{array}{ccc}
\mathrm{Glob}(\mathbf{S}^n) & \longrightarrow & A \\
\downarrow & & \downarrow \\
\mathrm{Glob}(\mathbf{D}^{n+1}) & \longrightarrow & X
\end{array}
$$

for some $n \geqslant 1$. Then the continuous map $f : \mathbb{P}A \longrightarrow \mathbb{P}X$ is a closed n-connected inclusion.

Theorem V.3.5. *Any flow is weakly S-homotopy equivalent to a strongly cofibrant flow. This "strongly cofibrant replacement" is unique up to S-homotopy.*

Proof. As usual in this kind of proof, two kinds of processes are involved; the first is that of attaching cells like $\mathrm{Glob}(\mathbf{S}^n)$ so as to create new generators; the second, of attaching cells like $\mathrm{Glob}(\mathbf{D}^n)$ to create new relations.

Let X be an object of **Flow**. Let $T_{-1} = X^0$ (so $\mathbb{P}T_{-1} = \varnothing$). Then the canonical morphism $f_{-1} : T_{-1} \longrightarrow X$ is synchronized. If $\mathbb{P}X = \varnothing$, then the proof is ended. Otherwise, for any $\gamma \in \mathbb{P}X$, let us attach a copy of $\overrightarrow{I}$ such that $[0, 1] \in \mathbb{P}\overrightarrow{I}$ is mapped to γ. Then the canonical morphism of flows $f_0 : T_0 \longrightarrow X$ induces an onto map $\pi_0(f_0) : \pi_0(\mathbb{P}T_0) \longrightarrow \pi_0(\mathbb{P}X)$ (where $\pi_i(U)$ is the i-th homotopy group of U). In other terms, T_0 is the flow having X^0 as 0-skeleton and the set $\mathbb{P}X$ equipped with the discrete topology as path space.

We are going to introduce by induction on $n \geqslant 0$ a n-globular extension T_n of T_{n-1} such that the canonical morphism of flows $f_n : T_n \longrightarrow X$ satisfies the following conditions:

1. The morphism of flows f_n is synchronized.
2. For any base-point γ, $\pi_n(f_n) : \pi_n(\mathbb{P}T_n, \gamma) \longrightarrow \pi_n(\mathbb{P}X, \gamma)$ is onto.

3. For any base-point γ, and for any $0 \leqslant i < n$, $\pi_i(f_n) : \pi_i(\mathbb{P}T_n, \gamma) \longrightarrow \pi_i(\mathbb{P}X, \gamma)$ is an isomorphism.

The passage from T_0 to T_1 is fairly different from the rest of the induction. To obtain a bijection $\pi_0(f_1) : \pi_0(\mathbb{P}T_1) \longrightarrow \pi_0(\mathbb{P}X)$, it suffices to have a bijection $\pi_0(f_1) : \pi_0(\mathbb{P}_{\alpha,\beta}T_1) \longrightarrow \pi_0(\mathbb{P}_{\alpha,\beta}X)$ for any $\alpha, \beta \in X^0$. Let x and y be two distinct elements of $\pi_0(\mathbb{P}_{\alpha,\beta}T_0)$ having the same image in $\pi_0(\mathbb{P}_{\alpha,\beta}X)$. Then x and y correspond to two nonconstant execution paths γ_x and γ_y from α to β. Consider the morphism of flows $\mathrm{Glob}(\mathbf{S}^0) \longrightarrow X$ such that $-1 \mapsto \gamma_x$ and $1 \mapsto \gamma_y$. Then let us attach a cell $\mathrm{Glob}(\mathbf{D}^1)$ by the pushout
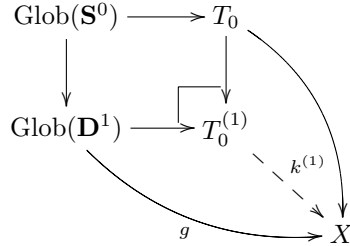
By construction, the equality $x = y$ holds in $T_0^{(1)}$. By transfinite induction, one obtains a flow U_0 and a morphism of flows $U_0 \longrightarrow X$ inducing a bijection $\pi_0(U_0) \cong \pi_0(X)$. We now have to make $\pi_1(U_0) \longrightarrow \pi_1(X)$ onto. The passage from U_0 to T_1 is analogous to the passage from U_n to T_{n+1} for $n \geqslant 1$, as explained below.

Let us suppose T_n constructed for $n \geqslant 1$. We are going to construct the morphism $T_n \longrightarrow T_{n+1}$ as a transfinite composition of pushouts of the morphism of flows $\mathrm{Glob}(\mathbf{S}^n) \longrightarrow \mathrm{Glob}(\mathbf{D}^{n+1})$. By Theorem V.3.4, the pair $(\mathbb{P}T_{n+1}, \mathbb{P}T_n)$ will be n-connected, and so the canonical maps $\pi_i(\mathbb{P}T_n) \longrightarrow \pi_i(\mathbb{P}T_{n+1})$ will be bijective for $i < n$. So the canonical map $\pi_i(\mathbb{P}T_{n+1}) \longrightarrow \pi_i(\mathbb{P}X)$ will remain bijective for $i < n$. By induction hypothesis, the map $\pi_n(f_n) : \pi_n(\mathbb{P}T_n, \gamma) \longrightarrow \pi_n(\mathbb{P}X, \gamma)$ is onto. To each element of $\pi_n(\mathbb{P}T_n, \gamma)$ with trivial image in $\pi_n(\mathbb{P}X, \gamma)$ corresponds a continuous map $\mathbf{S}^n \longrightarrow \mathbb{P}T_n$. Since $\mathbf{S}^n$ is connected, it can be associated to a morphism of flows $\mathrm{Glob}(\mathbf{S}^n) \longrightarrow T_n$. Let us attach to T_n a cell $\mathrm{Glob}(\mathbf{D}^{n+1})$ using the latter morphism. And repeat the process transfinitely. Then one obtains a relative $(n+1)$-globular extension U_n of T_n such that $\pi_i(\mathbb{P}U_n) \longrightarrow \pi_i(\mathbb{P}X)$ is still bijective for $i < n$ and such that $\pi_n(\mathbb{P}U_n) \longrightarrow \pi_n(\mathbb{P}X)$ becomes bijective. Now we have to make $\pi_{n+1}(\mathbb{P}U_n, \gamma) \longrightarrow \pi_{n+1}(\mathbb{P}X, \gamma)$ onto for any base-point γ. Let $g : (\mathbf{D}^{n+1}, \mathbf{S}^n) \longrightarrow (\mathbb{P}X, \gamma)$ be a relative continuous map which corresponds to an element of $\pi_{n+1}(\mathbb{P}X)$. Let us consider the following commutative diagram:
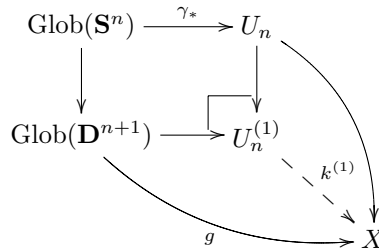
where $\gamma_*(0) = s(\gamma)$, $\gamma_*(1) = t(\gamma)$ and for any $z \in \mathbf{S}^n$, $\gamma_*(z) = \gamma$. Then because of the universal property satisfied by the pushout, there exists a morphism of flows

$k^{(1)} : U_n^{(1)} \longrightarrow X$ and by construction, the canonical morphism $\mathbf{D}^{n+1} \longrightarrow \mathbb{P}U_n^{(1)}$ is an inverse image of g by the canonical map $\pi_{n+1}(\mathbb{P}U_n^{(1)}, \gamma) \longrightarrow \pi_{n+1}(\mathbb{P}X, \gamma)$. By transfinite induction, one then obtains for some ordinal λ a flow $U_n^{(\lambda)}$ such that $\pi_{n+1}(\mathbb{P}U_n^{(\lambda)}, \gamma) \longrightarrow \pi_{n+1}(\mathbb{P}X, \gamma)$ is onto. It then suffices to set $T_{n+1} := U_n^{(\lambda)}$. The colimit $\varinjlim T_n$ is then a strongly cofibrant replacement of X and $\varinjlim f_n : \varinjlim T_n \longrightarrow X$ is then a weak S-homotopy equivalence by construction. The uniqueness of this strongly cofibrant replacement up to S-homotopy is a consequence of Theorem V.2.1. $\hfill\square$

4. The category of S-homotopy types

Theorem V.4.1. *The functor* cat *from* **glTop** *to* **Flow** *induces an equivalence between the localization* $\mathbf{glCW}[\mathcal{SH}^{-1}]$ *of globular* CW*-complexes with respect to the class* $\mathcal{SH}$ *of* S*-homotopy equivalences and the localization of the full and faithful subcategory of* **Flow** *consisting of the strongly cofibrant flows by the* S*-homotopy equivalences.*

Proof. Let X be a strongly cofibrant flow. Let $(X_n)_{n \geqslant 0}$ be the skeleton of the relative globular extension (X, X^0) (with the convention $X_{-1} = X^0$). Let $P(n)$ be the statement: "there exists a globular CW-complex Y of dimension n such that $\mathrm{cat}(Y) = X_n$ (by convention a globular CW-complex of dimension -1 will be a discrete space)". Suppose $P(n)$ proved for $n \geqslant -1$. Using Theorem III.5.2, choose a continuous map $i_Y : \mathbb{P}Y \longrightarrow \mathbb{P}^{\mathrm{top}}Y$. Let

$$
\begin{array}{ccc}
\bigsqcup_{i \in I} \mathrm{Glob}(\mathbf{S}^n) & \xrightarrow{\ \bigsqcup_{i \in I} f_i\ } & X_n \\
{\scriptstyle\subset}\big\downarrow & & \big\downarrow \\
\bigsqcup_{i \in I} \mathrm{Glob}(\mathbf{D}^{n+1}) & \longrightarrow & X_{n+1}
\end{array}
$$

be the pushout defining X_{n+1}. Then the pushout of multipointed spaces

$$
\begin{array}{ccc}
\bigsqcup_{i \in I} \mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^n) & \xrightarrow{\ \bigsqcup_{i \in I}(z,t) \mapsto i_Y(f_i(z))(t)\ } & Y \\
{\scriptstyle\subset}\big\downarrow & & \big\downarrow \\
\bigsqcup_{i \in I} \mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^{n+1}) & \longrightarrow & Y'
\end{array}
$$

gives the solution. It remains to prove that the functor is both full and faithful. Since S-homotopy in **glCW** is characterized by a cylinder functor (cf. [GG03] or Corollary II.4.9), one has the natural bijection of sets

$$
\mathbf{glCW}[\mathcal{SH}^{-1}](X, Y) \cong \mathbf{glTop}(X, Y)/\sim_S
$$

for any globular CW-complexes X and Y. Since S-homotopy in **Flow** is also characterized by a cylinder functor (cf. Proposition IV.4.7), one also has the natural bijection of sets

$$
\mathbf{Flow}(\mathrm{cat}(X), \mathrm{cat}(Y))/\sim_S \cong \mathbf{Flow}[\mathcal{SH}^{-1}](\mathrm{cat}(X), \mathrm{cat}(Y)).
$$

The theorem is then a consequence of Theorem IV.4.9. $\hfill\square$

Theorem V.4.2. *The localization* **Flow**$[\mathcal{S}^{-1}]$ *of* **Flow** *with respect to the class* $\mathcal{S}$ *of weak* S-*homotopy equivalences exists (i.e., is locally small). The functor* cat : **glCW** $\longrightarrow$ **Flow** *induces an equivalence of categories* **glCW**$[\mathcal{SH}^{-1}] \cong$ **Flow**$[\mathcal{S}^{-1}]$.

Proof. Let X be an object of **Flow**. By Theorem V.3.5, there exists a strongly cofibrant flow X' weakly S-homotopy equivalent to X. By Theorem V.4.1, there exists a globular CW-complex Y with cat$(Y) \cong X'$. So cat(Y) is isomorphic to X in **Flow**$[\mathcal{S}^{-1}]$. So the functor cat : **glCW**$[\mathcal{SH}^{-1}] \longrightarrow$ **Flow**$[\mathcal{S}^{-1}]$ is essentially surjective.

Let Y_1 and Y_2 be two globular CW-complexes. Then

$$\mathbf{glCW}[\mathcal{SH}^{-1}](Y_1, Y_2) \cong \mathbf{Flow}[\mathcal{SH}^{-1}](\mathrm{cat}(Y_1), \mathrm{cat}(Y_2))$$
$$\cong \mathbf{Flow}[\mathcal{S}^{-1}](\mathrm{cat}(Y_1), \mathrm{cat}(Y_2)),$$

the last isomorphism being due to the facts that cat(Y_1) is cofibrant and that cat(Y_2) is fibrant for the model structure of **Flow**. Therefore cat : **glCW**$[\mathcal{SH}^{-1}] \longrightarrow$ **Flow**$[\mathcal{S}^{-1}]$ is full and faithful. $\qquad\square$

Corollary V.4.3. *Let* **CW** *be the category of* CW-*complexes. Let* **Top** *be the category of compactly generated topological spaces. Let* **Ho**(**CW**) *be the localization of* **CW** *with respect to homotopy equivalences and* **Ho**(**Top**) *be the localization of* **Top** *with respect to weak homotopy equivalences. Then the commutative diagram*

$$
\begin{array}{ccc}
\mathbf{CW} & \longrightarrow & \mathbf{Top} \\
{\scriptstyle \mathrm{Glob}^{\mathrm{top}}(-)}\downarrow & & \downarrow{\scriptstyle \mathrm{Glob}(-)} \\
\mathbf{glCW} & \longrightarrow & \mathbf{Flow}
\end{array}
$$

gives rise to the commutative diagram

$$
\begin{array}{ccc}
\mathbf{Ho}(\mathbf{CW}) & \overset{\cong}{\longrightarrow} & \mathbf{Ho}(\mathbf{Top}) \\
{\scriptstyle \mathrm{Glob}^{\mathrm{top}}(-)}\big\uparrow\big\downarrow & & \big\uparrow\big\downarrow{\scriptstyle \mathrm{Glob}(-)} \\
\mathbf{glCW}[\mathcal{SH}^{-1}] & \overset{\cong}{\longrightarrow} & \mathbf{Flow}[\mathcal{S}^{-1}].
\end{array}
$$

5. Conclusion

The model structure of [Gau03d] on the category of flows provides a new interpretation of the notion of S-homotopy equivalence. It allowed us to prove in Part V that the functor from the category of globular CW-complexes to the category of flows induces an equivalence of categories from the localization of the category of globular CW-complexes with respect to the class of the S-homotopy equivalences to the localization of the category of flows with respect to the class of weak S-homotopy equivalences.

Part VI. T-homotopy and flow

1. Introduction

The purpose of this part is the construction of a class of morphisms of flows, the T-homotopy equivalences, so that the following theorem holds:
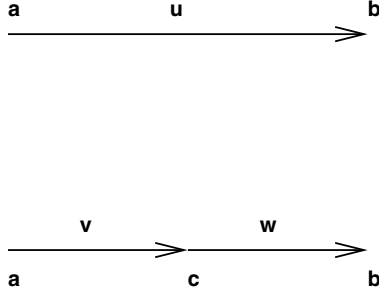
FIGURE 2. Concatenation of v and w.

Theorem VI.1.1. *Let X and U be globular complexes. If $f : X \longrightarrow U$ is a T-homotopy equivalence of globular complexes, then $\operatorname{cat}(f) : \operatorname{cat}(X) \longrightarrow \operatorname{cat}(U)$ is a T-homotopy equivalence of flows. Conversely, if $g : \operatorname{cat}(X) \longrightarrow \operatorname{cat}(U)$ is a T-homotopy equivalence of flows, then $g = \operatorname{cat}(f)$ for some T-homotopy equivalence of globular complexes $f : X \longrightarrow U$.*

This makes use of the following:

Definition VI.1.2. A T-*homotopy equivalence* is a morphism $f : X \longrightarrow Y$ of globular complexes inducing an homeomorphism between the two underlying topological spaces.

Section 2 defines the class of T-homotopy equivalences in the category of flows. Section 3 is devoted to proving the theorem above.

2. T-homotopy in Flow

The idea of T-homotopy is to change nothing globally except that new states may appear in the middle of full globes. In particular, the additional states appearing in the 0-skeleton must not create any new branching or merging areas of execution paths. For example, the unique morphism of flows F such that $F(u) = v * w$ in Figure 2 is a T-homotopy.

We need again the notion of quasi-flow introduced in Part III Section 3.1. Recall that a flow can be viewed as a particular case of quasi-flow.

Definition VI.2.1. Let X be a quasi-flow. Let Y be a subset of X^0. Then the *restriction* $X \restriction_Y$ of X over Y is the unique quasi-flow such that $(X \restriction_Y)^0 = Y$ and such that

$$\mathbb{P}^{\mathrm{top}}(X \restriction_Y) = \bigsqcup_{(\alpha,\beta) \in Y \times Y} \mathbb{P}^{\mathrm{top}}_{\alpha,\beta} X$$

equipped with the topology induced by the one of $\mathbb{P}^{\mathrm{top}} X$.

Let X be a flow. As in [Gau03a], let $\mathcal{R}^-$ be the smallest closed equivalence relation on $\mathbb{P}X$ identifying γ_1 and $\gamma_1 * \gamma_2$ whenever γ_1 and $\gamma_1 * \gamma_2$ are defined in $\mathbb{P}X$, and let

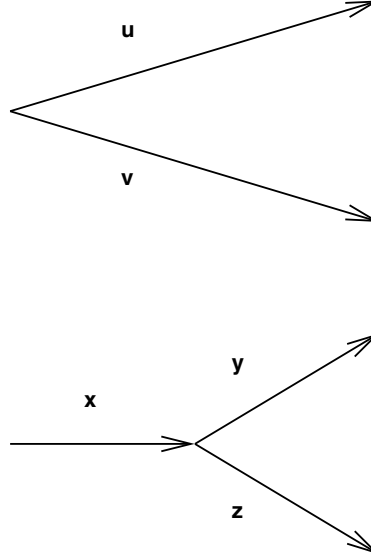$$\mathbb{P}^- X = \mathbb{P}X / \mathcal{R}^-.$$

FIGURE 3. Two 1-dimensional automata not T-homotopy equivalent.

Symmetrically, let us consider the smallest closed equivalence relation $\mathcal{R}^+$ identifying γ_2 and $\gamma_1 * \gamma_2$ if γ_2 and $\gamma_1 * \gamma_2$ belong to $\mathbb{P}X$. Then let

$$\mathbb{P}^+ X = \mathbb{P}X/\mathcal{R}^+.$$

Definition VI.2.2. A morphism of flows $f : X \longrightarrow Y$ is a T-*homotopy equivalence* if and only if the following conditions are satisfied:

1. The morphism of flows $f : X \to Y \upharpoonright_{f(X^0)}$ is an isomorphism of flows. In particular, the set map $f^0 : X^0 \longrightarrow Y^0$ is one-to-one.
2. For any $\alpha \in Y^0 \backslash f(X^0)$, the topological spaces $\mathbb{P}_\alpha^- Y$ and $\mathbb{P}_\alpha^+ Y$ are singletons.
3. For any $\alpha \in Y^0 \backslash f(X^0)$, there are execution paths u and v in Y such that $s(u) \in f^0(X^0)$, $t(u) = y$, $s(v) = y$ and $t(v) \in f^0(X^0)$.

The first condition alone does not suffice for a characterization of T-homotopy, since the unique morphisms of flows F' such that $F'(u) = v$ satisfies this condition as well. The additional state (i.e., $b = tw$) creates a new final state.

Now consider Figure 3. In the globular complex setting, there are no T-homotopy equivalences between them because the underlying topological spaces are not homeomorphic because of the calculation x before the branching. However the unique morphisms of flows F such that $F(u) = x * y$ and $F(v) = x * z$ satisfies the first and third conditions of Definition VI.2.2, but not the second one.

Requiring that $\mathbb{P}_\alpha^- Y$ and $\mathbb{P}_\alpha^+ Y$ are only contractible for $\alpha \in Y^0 \backslash f(X^0)$ is not sufficient either. Indeed, consider two contractible topological spaces X and Y and the morphism of globular complexes $f : \mathrm{Glob}^{\mathrm{top}}(X \times Y) \to \mathrm{Glob}^{\mathrm{top}}(X) * \mathrm{Glob}^{\mathrm{top}}(Y)$ such that $f((x,y),t) = (x,2t)$ for $0 \leqslant t \leqslant 1/2$ and $f((x,y),t) = (y,2t-1)$ for $1/2 \leqslant t \leqslant 1$. The morphism of flows f would be a T-homotopy equivalence.

The third condition is also necessary because otherwise, the directed segment $\overrightarrow{I}$ would be T-homotopy equivalent to the disjoint sum of $\overrightarrow{I}$ with the concatenation of an infinite number of copies of $\overrightarrow{I}$.

3. Comparison of T-homotopy in glTop and in Flow

3.1. Properties of T-homotopy. Some useful properties of T-homotopy equivalences of flows are proved in this section.

Theorem VI.3.1. *Let f be a morphism of flows from X to Y. Assume that f is the pushout of a morphism of flows of the form $\mathrm{cat}(g) : \mathrm{cat}(U) \longrightarrow \mathrm{cat}(V)$ where $g : U \longrightarrow V$ is a T-homotopy equivalence of globular complexes. Then the morphism of flows $X \longrightarrow Y \upharpoonright_{f(X^0)}$ is an isomorphism of flows. In particular, the continuous map $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is one-to-one.*

Proof. First of all, assume that $X = \mathrm{cat}(U)$, $Y = \mathrm{cat}(V)$ and $f = \mathrm{cat}(g)$ for some T-homotopy equivalence $g : U \longrightarrow V$. The morphism of quasi-flows $\mathrm{qcat}(g) : \mathrm{qcat}(U) \longrightarrow \mathrm{qcat}(V) \upharpoonright_{g(X^0)}$ has an obvious inverse from $\mathrm{qcat}(V) \upharpoonright_{g(X^0)}$ to $\mathrm{qcat}(U)$ denoted by $\mathrm{qcat}(g)^{-1}$ sending $\gamma \in \mathbb{P}^{\mathrm{top}}\mathrm{qcat}(V) \upharpoonright_{g(X^0)}$ to $g^{-1} \circ \gamma \in \mathbb{P}^{\mathrm{top}}\mathrm{qcat}(U)$. Using the natural transformation $p : \mathrm{qcat} \longrightarrow \mathrm{cat}$, one obtains that

$$p\left(\mathrm{qcat}(g)^{-1}\right) : \mathbb{P}V \upharpoonright_{g(X^0)} \longrightarrow \mathbb{P}U$$

is an inverse continuous map of $\mathbb{P}g : \mathbb{P}U \longrightarrow \mathbb{P}V \upharpoonright_{g(X^0)}$.

Now take a general T-homotopy equivalence of flows f from X to Y. By hypothesis, there exists a cocartesian diagram of flows

$$
\begin{array}{ccc}
\mathrm{cat}(U) & \longrightarrow & X \\
{\scriptstyle \mathrm{cat}(g)} \downarrow & & \downarrow \\
\mathrm{cat}(V) & \longrightarrow & Y
\end{array}
$$

for some T-homotopy equivalence of globular complexes $f : U \longrightarrow V$. Consider the following commutative diagram of flows

$$
\begin{array}{ccc}
\mathrm{cat}(U) & \longrightarrow & X \\
{\scriptstyle \mathrm{cat}(g)} \downarrow & & \downarrow \quad {\scriptstyle \phi_1} \\
\mathrm{cat}(V) \upharpoonright_{g(X^0)} & \longrightarrow & Y \upharpoonright_{f(X^0)} \\
& {\scriptstyle \phi_2} & \quad {\scriptstyle h} \\
& & Z.
\end{array}
$$

One wants to prove the existence of h making the diagram commutative. So consider the following diagram (where a new flow $\overline{Z}$ is defined as a pushout)

$$\begin{array}{c}\\ \phi_1 \end{array}\quad\begin{array}{ccccc} & & \mathrm{cat}(U) & \longrightarrow & X \\ & \nearrow^{\mathrm{cat}(g)} & \downarrow^{\mathrm{cat}(g)} & & \downarrow \\ \mathrm{cat}(V)\!\restriction_{g(X^0)} & \hookrightarrow & \mathrm{cat}(V) & \longrightarrow & Y \\ \downarrow^{\phi_2} & & \downarrow^{\overline{\phi_2}} & & \\ Z & \xrightarrow{\ u\ } & \overline{Z}. & & \end{array}$$
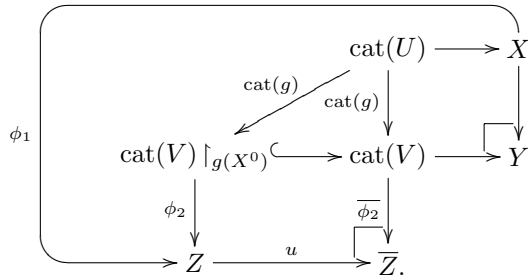
Every part of this diagram is commutative. Therefore one obtains the commutative diagram

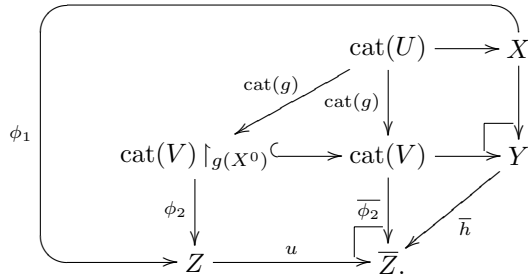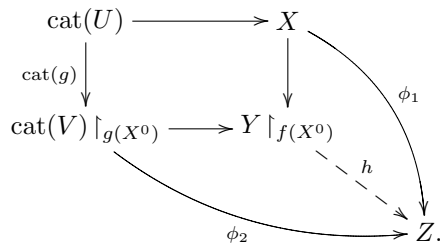$$\begin{array}{ccc} \mathrm{cat}(U) & \longrightarrow & X \\ \downarrow^{\mathrm{cat}(g)} & & \downarrow \\ \mathrm{cat}(V) & \longrightarrow & Y \\ & \searrow_{\overline{\phi_2}} & \downarrow^{\overline{h}} \\ & & \overline{Z}. \end{array}\qquad u\circ\phi_1$$

One obtains the commutative diagram

$$\begin{array}{c}\\ \phi_1 \end{array}\quad\begin{array}{ccccc} & & \mathrm{cat}(U) & \longrightarrow & X \\ & \nearrow^{\mathrm{cat}(g)} & \downarrow^{\mathrm{cat}(g)} & & \downarrow \\ \mathrm{cat}(V)\!\restriction_{g(X^0)} & \hookrightarrow & \mathrm{cat}(V) & \longrightarrow & Y \\ \downarrow^{\phi_2} & & \downarrow^{\overline{\phi_2}} & \swarrow^{\overline{h}} & \\ Z & \xrightarrow{\ u\ } & \overline{Z}. & & \end{array}$$
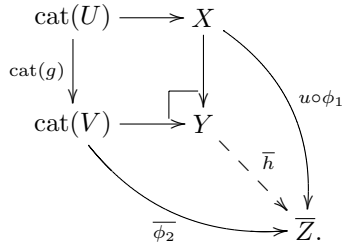
So $h = \overline{h}\!\restriction_{f(X^0)}$ makes the following diagram commutative:

$$\begin{array}{ccc} \mathrm{cat}(U) & \longrightarrow & X \\ \downarrow^{\mathrm{cat}(g)} & & \downarrow \\ \mathrm{cat}(V)\!\restriction_{g(X^0)} & \longrightarrow & Y\!\restriction_{f(X^0)} \\ & \searrow_{\phi_2} & \downarrow^{h} \quad \phi_1 \\ & & Z. \end{array}$$

Therefore the following square of flows is cocartesian:

$$
\begin{array}{ccc}
\mathrm{cat}(U) & \longrightarrow & X \\
{\scriptstyle \mathrm{cat}(g)}\downarrow & & \downarrow \\
\mathrm{cat}(V)\restriction_{g(X^0)} & \longrightarrow & Y\restriction_{f(X^0)} \ .
\end{array}
$$

Since $\mathrm{cat}(U) \cong \mathrm{cat}(V)\restriction_{g(X^0)}$, first part of the proof gives $X \cong Y\restriction_{f(X^0)}$. $\qquad\square$

Theorem VI.3.2. *Let f be a morphism of flows from X to Y. Assume that f is the pushout of a morphism of flows of the form $\mathrm{cat}(g) : \mathrm{cat}(U) \longrightarrow \mathrm{cat}(V)$ where $g : U \longrightarrow V$ is a T-homotopy equivalence of globular complexes. For any $\alpha \in Y^0\backslash f(X^0)$, the topological spaces $\mathbb{P}_\alpha^- Y$ and $\mathbb{P}_\alpha^+ Y$ are singletons.*

Proof. Let us suppose first that $X = \mathrm{cat}(U)$, $Y = \mathrm{cat}(V)$ and $f = \mathrm{cat}(g)$ for some T-homotopy equivalence of globular complexes $g : U \longrightarrow V$. Let $\alpha \in Y^0\backslash f(X^0)$. One sees by induction on the globular decomposition of U that the topological spaces $\mathbb{P}_{g^{-1}(\alpha)}^\pm X$ are singletons. Since one has $\mathbb{P}_\alpha^\pm Y \cong \mathbb{P}_{g^{-1}(\alpha)}^\pm X$ as topological spaces, the proof is complete in that case.

Let us take now a general T-homotopy equivalence of flows h from X to Y. By hypothesis, there exists a cocartesian diagram of flows

$$
\begin{array}{ccc}
\mathrm{cat}(U) & \longrightarrow & X \\
{\scriptstyle \mathrm{cat}(g)}\downarrow & {\scriptstyle h}& \downarrow \\
\mathrm{cat}(V) & \overset{\phi}{\longrightarrow} & Y
\end{array}
$$

for some T-homotopy equivalence of globular complexes $g : U \longrightarrow V$. Let $\alpha \in Y^0\backslash h(X^0)$. Since one has the cocartesian diagram of sets

$$
\begin{array}{ccc}
U^0 & \longrightarrow & X^0 \\
{\scriptstyle g^0}\downarrow & {\scriptstyle h^0}& \downarrow \\
V^0 & \overset{\phi}{\longrightarrow} & Y^0
\end{array}
$$

then there exists a unique $\beta \in V^0\backslash U^0$ such that $\phi(\beta) = \alpha$. By the first part of this proof, both topological spaces $\mathbb{P}_\beta^\pm V$ are singletons. Let $\gamma \in \mathbb{P}Y$ with $s(\gamma) = \alpha$. Then one has $\gamma = \gamma_1 * \cdots * \gamma_n$ where the γ_i are either execution paths of $\mathbb{P}V$ or execution paths of $\mathbb{P}X$. Since $\alpha = s(\gamma_1)$ and since $\alpha \in Y^0\backslash h(X^0)$, one deduces that $\gamma_1 \in \mathbb{P}V$. But since γ is $\mathcal{R}^-$-equivalent to γ_1, one deduces that $\mathbb{P}_\alpha^- Y$ is a singleton. In the same way, one can check that $\mathbb{P}_\alpha^+ Y$ is a singleton as well. $\qquad\square$

Theorem VI.3.3. *Let f be a morphism of flows from X to Y. Assume that f is the pushout of a morphism of flows of the form $\mathrm{cat}(g) : \mathrm{cat}(U) \longrightarrow \mathrm{cat}(V)$ where $g : U \longrightarrow V$ is a T-homotopy equivalence of globular complexes. For any $\alpha \in Y^0\backslash f(X^0)$, there are execution paths u and v in Y such that $s(u) \in f^0(X^0)$, $t(u) = y$, $s(v) = y$ and $t(v) \in f^0(X^0)$.*

Proof. First suppose that $X = \mathrm{cat}(U)$, $Y = \mathrm{cat}(V)$ and $f = \mathrm{cat}(g)$ for some T-homotopy equivalence of globular complexes $g : U \longrightarrow V$. Let $\alpha \in Y^0\backslash g(X^0)$. Then $g^{-1}(\alpha)$ is in the middle of a globe of the globular decomposition of X. In other terms, there exists $\gamma \in \mathbb{P}^{\mathrm{top}}X$ such that $\alpha \in \gamma(]0,1[)$. So there exists $\gamma_1 \in \mathbb{P}Y$

and $\gamma_2 \in \mathbb{P}Y$ such that $s(\gamma_1) \in g(X^0)$, $t(\gamma_2) \in g(X^0)$ and $t(\gamma_1) = s(\gamma_2) = \alpha$. Hence the conclusion in that case.

Take now a general T-homotopy equivalence of flows h from X to Y. By hypothesis, there exists a cocartesian diagram of flows

$$
\begin{array}{ccc}
\mathrm{cat}(U) & \longrightarrow & X \\
\scriptstyle\mathrm{cat}(g) \downarrow & & \downarrow \scriptstyle h \\
\mathrm{cat}(V) & \xrightarrow{\ \phi\ } & Y
\end{array}
$$

for some T-homotopy equivalence of globular complexes $g : U \longrightarrow V$. Let $\alpha \in Y^0 \backslash h(X^0)$. Like in Theorem VI.3.2, there exists a unique $\beta \in V^0 \backslash U^0$ such that $\phi(\beta) = \alpha$. Then using the first part of this proof, there exist $\gamma_1 \in \mathbb{P}V$ and $\gamma_2 \in \mathbb{P}V$ such that $s(\gamma_1) \in f(U^0)$, $t(\gamma_2) \in f(U^0)$ and $t(\gamma_1) = s(\gamma_2) = \beta$. Then $s(\phi(\gamma_1)) \in h(X^0)$, $t(\phi(\gamma_2)) \in h(X^0)$ and $t(\phi(\gamma_1)) = s(\phi(\gamma_2)) = \alpha$. Hence the conclusion in the general case. □

Corollary VI.3.4. *Let f be a morphism of flows from X to Y. Assume that f is the pushout of a morphism of flows of the form $\mathrm{cat}(g) : \mathrm{cat}(U) \longrightarrow \mathrm{cat}(V)$ where $g : U \longrightarrow V$ is a T-homotopy equivalence of globular complexes. Then f is a T-homotopy equivalence of flows.*

Proof. This is an immediate consequence of Theorem VI.3.1, Theorem VI.3.2 and Theorem VI.3.3. □

3.2. Comparison with T-homotopy of globular complexes.

Theorem VI.3.5. *Let X and U be globular complexes. Let $f : X \longrightarrow U$ be a T-homotopy equivalence of globular complexes. Then $\mathrm{cat}(f) : \mathrm{cat}(X) \longrightarrow \mathrm{cat}(U)$ is a T-homotopy equivalence of flows. Conversely, if $g : \mathrm{cat}(X) \longrightarrow \mathrm{cat}(U)$ is a T-homotopy equivalence of flows, then $g = \mathrm{cat}(f)$ for some T-homotopy equivalence $f : X \longrightarrow U$ of globular complexes.*

Proof. Let $f : X \longrightarrow U$ be a T-homotopy equivalence of globular complexes. Then $\mathrm{cat}(f) : \mathrm{cat}(X) \longrightarrow \mathrm{cat}(U)$ is a T-homotopy equivalence of flows by Corollary VI.3.4.

Conversely, let X and U be two globular complexes. Let $g : \mathrm{cat}(X) \longrightarrow \mathrm{cat}(U)$ be a T-homotopy equivalence of flows. Let $(\partial Z_\beta, Z_\beta, \phi_\beta)_{\beta < \lambda}$ be the globular decomposition of X. The morphism g gives rise to a one-to-one set map g^0 from $\mathrm{cat}(X)^0$ to $\mathrm{cat}(U)^0$ and to an homeomorphism $\mathbb{P}g : \mathbb{P}\mathrm{cat}(X) \longrightarrow \mathbb{P}\mathrm{cat}(U) \restriction_{g^0(\mathrm{cat}(X)^0)}$. Let $i_U : \mathbb{P}U \to \mathbb{P}^{\mathrm{top}}U$ given by Theorem III.5.2.

Let us suppose by induction on β that there exists a one-to-one morphism of globular complexes $f_\beta : X_\beta \longrightarrow U$ such that $\mathbb{P}f_\beta : \mathbb{P}X_\beta \longrightarrow \mathbb{P}U$ coincides with the restriction of $\mathbb{P}g$ to $\mathbb{P}X_\beta$. One has to prove that the same thing holds for $\beta + 1$. There is a cocartesian diagram of multipointed topological spaces

$$
\begin{array}{ccc}
\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta) & \xrightarrow{\ \phi_\beta\ } & X_\beta \\
\downarrow & & \downarrow \\
\mathrm{Glob}^{\mathrm{top}}(Z_\beta) & \xrightarrow{\ \overline{\phi_\beta}\ } & X_{\beta+1}.
\end{array}
$$

Let

$$k(z,t) = i_U(g \circ \overline{\phi_\beta}(z))(t)$$

for $z \in Z_\beta$ and $t \in [0,1]$. For any $z \in Z_\beta$, $k(z,-)$ is an execution path of U. The composite of morphisms of globular complexes

$$\mathrm{Glob}^{\mathrm{top}}(\partial Z_\beta) \xrightarrow{\phi_\beta} X_\beta \xrightarrow{f_\beta} U$$

gives rise to an execution path $\ell(z,t)$ for any $z \in \partial Z_\beta$. Since ∂Z_β is compact, then there exists a continuous map $\psi : \partial Z_\beta \times [0,1] \longrightarrow [0,1]$ such that $\ell(z,t) = k(z, \psi(z,t))$ for any $z \in \partial Z_\beta$ and any $t \in [0,1]$ by Proposition III.5.1. Therefore the mapping

$$\overline{k} : (z,t) \mapsto k(z, \mu(z)t + (1 - \mu(z))\psi(z,t))$$

induces a morphism of globular complexes $f_{\beta+1} : X_{\beta+1} \longrightarrow U$ which is an extension of $f_\beta : X_\beta \longrightarrow U$.

One now wants to prove that the restriction of $f_{\beta+1}$ to $\mathrm{Glob}^{\mathrm{top}}(Z_\beta \backslash \partial Z_\beta)$ is one-to-one. Suppose that there exists two points (z,t) and (z',t') of $\mathrm{Glob}^{\mathrm{top}}(Z_\beta \backslash \partial Z_\beta)$ such that $f_{\beta+1}(z,t) = f_{\beta+1}(z',t')$. If $z \neq z'$, then $g \circ \overline{\phi_\beta}(z) \neq g \circ \overline{\phi_\beta}(z')$ since g is one-to-one. So the two execution paths $\overline{k}(z,-)$ and $\overline{k}(z',-)$ are two distinct execution paths intersecting at $\overline{k}(z,t) = \overline{k}(z',t')$. The latter point necessarily belongs to U^0. Since the topological spaces $\mathbb{P}_\alpha^- U$ and $\mathbb{P}_\alpha^+ U$ are both singletons for $\alpha \in U^0 \backslash \phi(X^0)$, then

$$\overline{k}(z,t) = \overline{k}(z',t') \in \phi(X^0).$$

There are two possibilities: $\overline{k}(z,t) = \overline{k}(z',t') = g \circ \phi_\beta(0)$ and $\overline{k}(z,t) = \overline{k}(z',t') = g \circ \phi_\beta(1)$ (notice that $\phi_\beta(0)$ and $\phi_\beta(1)$ may be equal). The equality $\overline{k}(z,t) = \overline{k}(z',t') = g \circ \phi_\beta(0)$ implies $t = t' = 0$ and the equality $\overline{k}(z,t) = \overline{k}(z',t') = g \circ \phi_\beta(1)$ implies $t = t' = 1$. In both cases, one has $(z,t) = (z',t')$: contradiction. So $f_{\beta+1}$ is one-to-one.

If $\beta < \lambda$ is a limit ordinal, then let $f_\beta = \varinjlim_{\alpha < \beta} f_\alpha$. The latter map is still a one-to-one continuous map and a morphism of globular complexes. So one obtains a one-to-one morphism of globular complexes $f : X \longrightarrow U$ such that $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}U$ coincides with $\mathbb{P}g$.

Now let us prove that f is surjective. Let $x \in U$. First case: $x \in U^0$. If $x \notin g(X^0) = f(X^0)$, then by hypothesis, $\mathbb{P}_x^- \mathrm{cat}(U)$ and $\mathbb{P}_x^+ \mathrm{cat}(U)$ are singletons. So x necessarily belongs to an execution path between two points of $g(X^0)$. Since g is a bijection from $\mathbb{P}X$ to $\mathbb{P}U \restriction_{g(X^0)}$, this execution path necessarily belongs to $g(X)$. Therefore $x \in g(X)$. Second case: $x \in U \backslash U^0$. Then there exists an execution path γ of U passing by x. If $\gamma(0) \notin g(X^0)$ (resp. $\gamma(1) \notin g(X^0)$), then there exists an execution path going from a point of $g(X^0)$ to $\gamma(0)$ (resp. going from $\gamma(1)$ to a point of $g(X^0)$) because $\gamma(0)$ is not an initial state (resp. a final state) of $g(X)$. Therefore one can suppose that $\gamma(0)$ and $\gamma(1)$ belong to $g(X^0)$. Once again we recall that g is a bijection from $\mathbb{P}X$ to $\mathbb{P}U \restriction_{g(X^0)}$, so $x \in f(X)$. Therefore $U \subset f(X)$. So f is bijective.

At last, one has to check that $f^{-1} : U \longrightarrow X$ is continuous. Let T be a compact of the globular decomposition of U (not of X !), let q be the corresponding attaching

map, and consider the composite

$$\mathrm{Glob}^{\mathrm{top}}(T) \xrightarrow{\ q\ } U \xrightarrow{\ f^{-1}\ } X.$$

There are then two possibilities.

First of all, assume that $T = \{x\}$ for some $x \in U^0$. Then there exist $\gamma, \gamma' \in U$ such that $\gamma * (\mathbb{P}q)(x) * \gamma' \in g(\mathbb{P}X)$. Let $\gamma * \mathbb{P}q(x) * \gamma' = g(\gamma'')$. Let $i_X : \mathbb{P}X \longrightarrow \mathbb{P}^{\mathrm{top}}X$ given by Theorem III.5.2. Then the execution path $i_X(\gamma'')$ of X becomes an execution path $f \circ i_X(\gamma'')$ of U since f is one-to-one. Let us consider the execution path $i_U(\gamma * \mathbb{P}q(x) * \gamma')$ of U. By Proposition III.5.1, there exists a continuous nondecreasing map $\omega : [0,1] \longrightarrow [0,1]$ such that $\omega(0) = 0$, $\omega(1) = 1$ and such that

$$i_U(\gamma * \mathbb{P}q(x) * \gamma') = f \circ i_X(\gamma'') \circ \omega.$$

Then ω is necessarily bijective, and so an homemorphism since $[0,1]$ is compact. Therefore $f^{-1} \circ i_U(\gamma * \mathbb{P}q(x) * \gamma') = i_X(\gamma'') \circ \omega$. So $f^{-1} \circ q(\mathrm{Glob}^{\mathrm{top}}(T))$ is a compact of X.

Now suppose that T contains more than one element. Then $\mathbb{P}^-_{q(0)}U$ and $\mathbb{P}^+_{q(1)}U$ are not singletons. So $q(0)$ and $q(1)$ belong to $g(X^0) = f(X^0)$. Then $\mathbb{P}(g^{-1} \circ q)(T) = (g^{-1} \circ \mathbb{P}q)(T)$ is a compact of $\mathbb{P}X$ (since g is an homeomorphism !). By Proposition III.5.1, there exists a continuous map $\omega : T \longrightarrow \mathbf{TOP}([0,1],[0,1])$ such that $\omega(0) = 0$, $\omega(1) = 1$ and such that $\omega(z)$ is nondecreasing for any $z \in T$ and such that

$$i_U(\mathbb{P}q(z)) = f \circ i_X(g^{-1} \circ \mathbb{P}q(z)) \circ \omega(z)$$

for any $z \in T$. The map $z \mapsto i_X(g^{-1} \circ \mathbb{P}q(z)) \circ \omega(z)$ is mapped by the set map

$$\mathbf{Top}(T, \mathbb{P}^{\mathrm{top}}X) \longrightarrow \mathbf{Top}(T \times [0,1], X)$$

to a function $\omega' \in \mathbf{Top}(T \times [0,1], X)$. Therefore

$$f^{-1} \circ q(\mathrm{Glob}^{\mathrm{top}}(T)) = \omega'(T \times [0,1])$$

is again a compact of X.

To conclude, let F be a closed subset of X. Then

$$f(F) \cap q(\mathrm{Glob}^{\mathrm{top}}(T)) = f(F \cap (f^{-1} \circ q)(\mathrm{Glob}^{\mathrm{top}}(T))).$$

Since $f^{-1} \circ q(\mathrm{Glob}^{\mathrm{top}}(T))$ is always compact, the set $f(F) \cap q(\mathrm{Glob}^{\mathrm{top}}(T))$ is compact as well. Since U is equipped with the weak topology induced by its globular decomposition, the set $f(F)$ is a closed subspace of U. So f^{-1} is continuous. $\quad\square$

4. Conclusion

We have defined in this part a class of morphisms of flows, the T-homotopy equivalences, such that there exists a T-homotopy equivalence between two globular complexes if and only if there exists a T-homotopy equivalence between the corresponding flows. So not only the category of flows allows the study of S-homotopy of globular complexes, but also the study of T-homotopy of globular complexes.

Part VII. Application: the underlying homotopy type of a flow

1. Introduction

The main theorem of this paper (Theorem V.4.2) establishes the equivalence of two approaches of dihomotopy. The first one uses the category of globular complexes in which the concurrent processes are modelled by topological spaces equipped with an additional structure, the globular decomposition, encoding the time flow, and in which the execution paths are "locally strictly increasing" continuous maps. The second one uses the category of flows in which the concurrent processes are modelled by categorical-like objects and in which it is possible to define a model structure relevant for the study of dihomotopy.

Another interest of this equivalence is that it makes the construction of the *underlying homotopy type* of a flow possible. Indeed, loosely speaking, a dihomotopy type is an homotopy type equipped with an additional structure encoding the time flow. So there must exist a forgetful functor $|-| : \textbf{Flow} \longrightarrow \textbf{Ho}(\textbf{Top})$ from the category of flows to the category of homotopy types which is also a dihomotopy invariant, i.e., sending weak S-homotopy and T-homotopy equivalences to isomorphisms.

2. Construction of the underlying homotopy type functor

Definition VII.2.1. (cf. Part II Section 2) Let (X, X^0) be a multipointed topological space. Then the mapping

$$(X, X^0) \mapsto X$$

induces a functor $|-| : \textbf{Top}^{\mathrm{m}} \longrightarrow \textbf{Top}$ called the *underlying topological space* of (X, X^0).

Proposition VII.2.2. *Passing to the underlying topological space induces a functor $|-| : \textbf{glCW} \longrightarrow \textbf{Top}$ from the category of globular CW-complexes to the category of topological spaces. Moreover, for any S-homotopy equivalence $f : X \longrightarrow U$ of globular CW-complexes, the continuous map $|f| : |X| \longrightarrow |U|$ is a homotopy equivalence of topological spaces.*

Proof. It suffices to prove that if f and g are two morphisms of globular complexes which are S-homotopy equivalent, then $|f|$ and $|g|$ are two homotopy equivalent continuous maps. Let H be a S-homotopy between f and g. By Proposition II.4.6, H induces a continuous map $h \in \textbf{Top}([0, 1], \textbf{glTOP}(X, Y))$, so a continuous map $h \in \textbf{Top}([0, 1], \textbf{TOP}(|X|, |Y|))$. Hence an homotopy between the continuous maps $|f|$ and $|g|$. □

Corollary VII.2.3. *The functor $|-| : \textbf{glCW} \longrightarrow \textbf{Top}$ induces a unique functor $|-| : \textbf{glCW}[\mathcal{SH}^{-1}] \longrightarrow \textbf{Ho}(\textbf{Top})$ making the following diagram commutative:*

$$\begin{array}{ccc} \textbf{glCW} & \longrightarrow & \textbf{Ho}(\textbf{Top}) \\ \downarrow & \nearrow & \\ \textbf{glCW}[\mathcal{SH}^{-1}]. & & \end{array}$$

Definition VII.2.4. The composite functor

$$|-| : \mathbf{Flow} \longrightarrow \mathbf{Flow}[\mathcal{S}^{-1}] \simeq \mathbf{glCW}[\mathcal{SH}^{-1}] \xrightarrow{|-|} \mathbf{Ho}(\mathbf{Top})$$

is called the *underlying homotopy type functor*. If X is a flow, then $|X|$ is called the *underlying homotopy type* of X.

Proposition VII.2.5. *If $f : X \longrightarrow Y$ is a weak S-homotopy equivalence of flows, then $|f|$ is an isomorphism of $\mathbf{Ho}(\mathbf{Top})$. If $g : \mathrm{cat}(X) \longrightarrow \mathrm{cat}(Y)$ is a T-homotopy equivalence of flows, then $|g|$ is an isomorphism of $\mathbf{Ho}(\mathbf{Top})$ as well.*

Proof. Obvious with Theorem VI.3.5. $\qquad\qquad\square$

Figure 2 represents the simplest example of T-homotopy equivalence. The underlying homotopy types of its source and its target are both equal to the homotopy type of the point.

Notice that the functor from **Flow** to **Ho**(**Top**) defined by associating to a flow X the homotopy type of the disjoint sum $\mathbb{P}X \sqcup X^0$ is not a dihomotopy invariant. Therefore the functor $X \mapsto$ "homotopy type of $\mathbb{P}X \sqcup X^0$" has no relation with the underlying homotopy type functor. In the case of Figure 2, the discrete space $\{u, s(u), t(u)\}$ becomes the discrete space $\{v, w, v * w, s(v), t(w), t(v) = s(w)\}$.

Question VII.2.6. *How can one define the underlying homotopy type of a flow without using the category of globular complexes?*

3. Conclusion

The underlying homotopy type functor is a new dihomotopy invariant which can be useful for the study of flows up to dihomotopy.

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