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On the grades of order ideals

Alexandre Tchernev

Abstract. Let R be a commutative Noetherian local ring, let M be a finitely generated R-module of finite projective dimension, and let $u \in M$ be a minimal generator of M. We investigate in a characteristic free setting the grade of the order ideal $O_M(u) = \{f(u) \mid f \in \mathrm{Hom}_R(M, R)\}$. The main result is that when M is a k-th syzygy module and $\mathrm{pd}_R M \le 1$ then $\mathrm{grade}_R O_M(u) \ge k$; in particular if M is an ideal of projective dimension at most 1 then every minimal generator of M is a regular element of R. As an application we show that the minimal generators of M are regular elements of R also in the case when M is a Gorenstein ideal of grade 3, in the case when M is a three generated ideal, and in the case when M is an almost complete intersection ideal of grade 3 and R is Cohen–Macaulay.

Contents

Introduction 151

1. Syzygies of projective dimension at most 1 152

2. Syzygies of Gorenstein ideals 153

3. Three-generated ideals of finite projective dimension 154

4. Almost complete intersections 154

References 155

Introduction

Throughout this paper R is a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$, and M is a finitely generated R-module of finite projective dimension. A homomorphism of local rings $\phi\colon R \longrightarrow S$ is a *deformation* if it is surjective and $\mathrm{Ker}(\phi) = (x_1, \dots, x_s)$ for some regular sequence $(x_1, \dots, x_s)$ on R.

Recall that for an element $u \in M$ its *order ideal* $O_M(u)$ is the ideal

$$O_M(u) = \{f(u) \mid f \in \mathrm{Hom}_R(M, R)\}.$$

We are interested in the grade of $O_M(u)$ when M is a k-th syzygy module, and u is a minimal generator of M. The best known results in this direction are due

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to Evans and Griffith [6], and Hochster and Huneke [12]. We summarize them as follows.

Theorem 1. ([6], [12]). *The following conditions are equivalent:*

(1) *For every deformation $\phi\colon R \longrightarrow S$, every $k \geq 0$, every k-th syzygy S-module N of finite projective dimension over S, and every minimal generator v of N one has* $\operatorname{grade}_S O_N(v) \geq k$.
(2) *For every deformation $\phi\colon R \longrightarrow S$, every first syzygy S-module N of finite projective dimension over S, and every minimal generator n of N one has* $\operatorname{grade}_S O_N(v) \geq 1$.
(3) *For every deformation $\phi\colon R \longrightarrow S$, every ideal I in S with $\operatorname{pd}_S S/I < \infty$ and every minimal generator y of I one has that y is regular in S.*
(4) *The syzygy theorem* [5, Theorem 3.15] *holds for every ring S such that there is a deformation $\phi\colon R \longrightarrow S$.*

Furthermore, if $R/\mathfrak{p}$ has a balanced big Cohen–Macaulay module for every prime $\mathfrak{p}$ of R, then the conditions hold. In particular, they hold if R contains a field (*by* [9] *and* [7]), *or if* $\dim R \leq 3$ (*by* [8], [11], *and* [7]).

The difficulty of proving that the four conditions of Theorem 1 hold without the assumption that R contains a field is further illuminated by the result of Simon and Strooker [15] who show that Condition (1) is true for all Gorenstein rings R if and only if Hochster's Canonical Element Conjecture [10] is true for all rings R.

Our goal in this short note is to examine in a characteristic free setting special cases of the conditions. In Section 1 we show that the conclusion of Condition (1) holds in characteristic free setting when we consider syzygies of projective dimension at most one. In particular, this yields a nice "supplement" to the Hilbert–Burch theorem.

In Section 2 we examine an alternative proof of the results from Section 1 that was pointed out by the referee and use a key observation in this alternative argument to show that the conclusion of Condition (1) holds when we consider k-th syzygies of perfect Gorenstein ideals of grade $k+2$. In particular, the conclusion of Condition (3) holds for Gorenstein ideals of grade 3, which yields a supplement to the Buchsbaum–Eisenbud structure theorem [4] for these ideals.

In Section 3 we show that the conclusion of Condition (3) holds for three-generated ideals of finite projective dimension. This result is interesting in view of Bruns' theorem [1] that for $k \geq 3$ every k-th syzygy of finite projective dimension is also the k-th syzygy of R/I for some three-generated ideal I of finite projective dimension.

Finally, in Section 4 we show that the conclusion of Condition (3) holds for almost complete intersection ideals of grade 3 and of finite projective dimension over Cohen–Macaulay rings.

I would like to thank the referee for pointing out the existence of an alternative proof to Theorem 2; that remark inspired the content of Section 2.

1. Syzygies of projective dimension at most 1

Our main result is the following theorem.

Theorem 2. *Let M be a k-th syzygy R-module with $\operatorname{pd}_R M \leq 1$. Let u be a minimal generator of M. Then* $\operatorname{grade}_R O_M(u) \geq k$.

Proof. By a standard reduction argument, see, e.g., [6, Lemma 2.5], it suffices to consider the case when $k = 1$ and $\operatorname{pd}_R M = 1$. Let

$$0 \longrightarrow F \xrightarrow{\phi} G \xrightarrow{\pi} M \longrightarrow 0$$

be a minimal free resolution of M. We can choose a basis $g_1, \ldots, g_n$ of G such that $\pi(g_n) = u$. Let $I = O_M(u)$ and let $J = (0 : I)$. Since M is a first syzygy, we may assume without loss of generality that $M \subset \mathfrak{m}H$ for some free R-module H, and therefore I is generated by the entries in the last column of a matrix for π, considered as a minimal map of free R-modules $\pi : G \longrightarrow H$. Then we have $Jg_n \subset \phi(F) \cap JG$. However, $L = \phi(F)$ is a free R-module. Thus $N = Jg_n$ is a submodule of the free R-module L such that $IN = 0$, therefore $N \subset JL$. It follows that Jg_n is inside $J\mathfrak{R}$, where $\mathfrak{R}$ is the ideal generated by the elements in the last row of a matrix for ϕ. Since the map ϕ is minimal it follows that $J \subseteq \mathfrak{m}J$, therefore $J = 0$. It follows that $\operatorname{grade}_R I \geq 1$. □

As an imediate consequence we obtain the following nice “supplement” to the Hilbert–Burch theorem:

Corollary 3. *Let I be an ideal in a Noetherian local ring R such that $\operatorname{pd}_R R/I \leq 2$. Then every minimal generator of I is a regular element of R.* □

2. Syzygies of Gorenstein ideals

The main result here is:

Theorem 4. *Let I be a perfect Gorenstein ideal of grade $k + 2$, and let M be the k-th syzygy of R/I. Then $\operatorname{grade}_R O_M(u) \geq k$ for every minimal generator u of M.*

In order to prove this theorem we examine a slightly more complicated alternative proof of Theorem 2 that was pointed out by the referee, and that is based on a refined version (see, e.g., [5, Lemma 2.9]) of the Eagon–Northcott bound on the height of determinantal ideals. A main point in that proof is the following observation:

Observation 5. Let u be a minimal generator of an R-module M, let

$$F \xrightarrow{\phi} G \xrightarrow{\pi} M \longrightarrow 0$$

be a minimal free presentation of M with $r = \operatorname{rank} \phi$, and choose a basis $g_1, \ldots, g_n$ of G such that $\pi(g_n) = u$. Choose also a basis for F and let X be the $n \times m$ matrix for ϕ in these bases. Let X' be the $(n-1) \times m$ matrix obtained by removing the last row of X. As shown in [5, Proof of Theorem 2.8], the ideal of maximal minors $I' = I_r(X')$ is contained inside the order ideal $O_M(u)$. Thus we obtain the key inequalities

$$\begin{aligned} \operatorname{grade}_R O_M(u) \geq \operatorname{grade}_R I' &\geq \operatorname{grade}_R I_r(\phi) - \operatorname{height}_{R/I'}\bigl(I_r(\phi)/I'\bigr) \\ &\geq \operatorname{grade}_R I_r(\phi) - (m - r + 1) \end{aligned} \tag{6}$$

where the last inequality follows by [5, Lemma 2.9].

Proof of Theorem 4. It is immediate from the assumptions of the theorem and from the acyclicity criterion of Buchsbaum and Eisenbud [2] that M has a minimal free resolution of the form

$$0 \longrightarrow R \xrightarrow{\psi} F \xrightarrow{\phi} G \longrightarrow 0$$

such that $\operatorname{grade} I_r(\phi) \geq k+2$. The desired conclusion is now immediate from the inequalities (6). □

The following special case of Theorem 4 is worth explicit mention, as it supplements the Buchsbaum–Eisenbud structure theorem [4].

Corollary 7. *Let I be a perfect Gorenstein ideal of grade 3. Then every minimal generator of I is a regular element of R.* □

3. Three-generated ideals of finite projective dimension

As an application of Theorem 2 we consider ideals of finite projective dimension that are generated by three elements. This class of ideals is of interest because of Bruns' result [1] that when $k \geq 3$ every k-th syzygy of finite projective dimension is also the k-th syzygy of R/I for some ideal I from the class. We have the following result:

Theorem 8. *Let I be an ideal of finite projective dimension over a local ring R. If I is minimally generated by three elements then every minimal generator of I is a regular element of R.*

Proof. As a standard consequence of the structure theorems of Buchsbaum and Eisenbud, see [3, Corollary 5.2], we can always assume that $\operatorname{grade} I \geq 2$. If also $\operatorname{grade} I \geq 3$, then I is a complete intersection and the theorem is true. Thus we assume that $\operatorname{grade} I = 2$, and that y is a minimal generator of I such that the ideal $A = (0 : yR)$ is nonzero in R. We complete y to a minimal system of generators (y, x_1, x_2) of I such that (x_1, x_2) is a regular sequence. Let $J = ((x_1, x_2) : I)$, and let $B = (0 : A)$. If $\mathfrak{p}$ is a prime ideal of R such that $B + J \subset \mathfrak{p}$ then y is still a zerodivisor in $R_{\mathfrak{p}}$ and is still a minimal generator of $I_{\mathfrak{p}}$ which is an ideal in $R_{\mathfrak{p}}$ of grade at least 2 and of finite projective dimension; and therefore is minimally generated by three elements. Thus we may always assume that the maximal ideal of R is a minimal prime of $B + J$. Furthermore, if the projective dimension of R/I is two or less, the result follows from Theorem 2. Thus we may also assume that $d = \operatorname{pd}_R R/I \geq 3$. But then $\operatorname{pd}_R R/J = d - 1$, hence $\mathfrak{m}$ is not an associated prime ideal for R/J. Therefore, due to the results of Peskine and Szpiro [13], Hochster [9], and Roberts [14], there exists an associated prime $\mathfrak{p} \neq \mathfrak{m}$ of R/J which contains B. This however contradicts our assumptions on $\mathfrak{m}$ being the minimal prime of the ideal $B + J$. □

4. Almost complete intersections

Essentially the same proof as in Theorem 8 also yields:

Theorem 9. *Let R be a Cohen–Macaulay local ring, and let I be an almost complete intersection ideal of R of grade 3 (thus I is minimally generated by 4 elements) and of finite projective dimension. Then every minimal generator of I is a regular element of R.*

Proof. Let y be a minimal generator of I which is a zero divisor in R, and complete y to a minimal system of generators (y, x_1, x_2, x_3) of I such that (x_1, x_2, x_3) is a regular sequence in R. Let the ideals A, B, and J be defined as in the proof of Theorem 8, and the argument given there shows that we may assume that $\mathfrak{m}$ is the minimal prime of $B + J$. If $\operatorname{pd}_R R/I = 3$ it follows that I is a perfect ideal, hence so is J, therefore the ring R/J is Cohen–Macaulay and so the associated primes of R/J are the minimal primes of J. Thus (as in the proof of Theorem 8) there exists a minimal prime of J that contains B, hence $\mathfrak{m}$ must be that minimal prime of J, and therefore $\dim R = 3$. But then our result is true by Theorem 1. Thus we may assume that $d = \operatorname{pd}_R R/I \geq 4$. But then $\operatorname{pd}_R R/J = d - 1$, and we can reach contradiction with the minimality of $\mathfrak{m}$ over $B + J$ as in end of the proof of Theorem 8. □

Remark 10. It is clear from the proof above that the theorem will be true without the Cohen–Macaulay assumption on R if one can prove it for a perfect almost complete intersection of grade 3. Similarly, the theorem will be true for any almost complete intersection of finite projective dimension if one can prove it for the perfect ones.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY AT ALBANY, SUNY, ALBANY, NY 12222
tchernev@math.albany.edu http://math.albany.edu:8000/~tchernev/

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