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Integral submanifolds with closed conformal vector field in Sasakian manifolds

Gheorghe Pitiş

Abstract. A class of Legendrian submanifolds with closed conformal vector field in Sasakian space forms is studied. The existence of these submanifolds is analyzed and some topological and geometric properties are given. A characterization up to conformal transformation by the Maslov form is also given.

Contents

1. Introduction

An important topic in contact geometry is the study of integral submanifolds of the contact distribution in a contact manifold $\widetilde{M}$ of dimension $2n+1$. Many remarkable results were obtained, starting from Sasaki's theorems (1964) which assert that the maximum dimension of such a submanifold is n and that any r linearly independent vectors ($r \leq n$) vanishing the contact form of $\widetilde{M}$ and its differential determine an r-dimensional integral submanifold of $\widetilde{M}$. Hence a contact manifold has a great wealth of integral submanifolds and this is a serious argument for the difficulty of their study. Moreover, these submanifolds are anti-invariant and then, by usual methods, we are guided to study their transverse geometry, but not the properties of the submanifold itself.

In maximum dimension (i.e., for Legendrian submanifolds) we know many results about the geometry of integral submanifolds, especially in Sasakian space forms. We also mention Vaisman's construction [Vai1], [Vai2] of the Maslov class

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for Legendrian submanifolds in the cotangent unit sphere bundle of a Riemannian manifold endowed with the classical contact structure. In [Pit3] some characteristic classes are associated to an integral submanifold of dimension $< n$ in a Sasakian manifold, while in [Pit2] the stability and different kinds of minimality for such submanifolds of dimension $\leq n$ are studied.

The first part of this paper is devoted to the study of some tensor fields related to the mean curvature vector of an integral submanifold in a Sasakian manifold and we prove that under a general hypothesis its first Betti number cannot be equal to zero (Theorem 2.4). Also, we prove that an integral submanifold with nowhere zero parallel mean curvature vector in a Sasakian manifold is locally conformal to the cylinder of a leaf of some foliation associated to a special closed and conformal vector field on the submanifold (Theorem 2.5). We also study the eigenvalues and the eigenvectors of the Weingarten operator A_{FX} of an integral submanifold, for a special closed and conformal vector field X. These results are applied to the case of integral submanifolds in a Sasakian space form with F-sectional curvature equal to -3. The local decomposition of integral submanifolds with a nowhere special parallel vector field is also given (Theorem 3.5).

A class of Legendrian submanifolds having a special closed and conformal vector field are studied in the second part. Their importance follows from the fact that generally ordinary spheres cannot be embedded in a Sasakian space form as Legendrian submanifolds (Propositions 4.1 and 4.3) and so these submanifolds are the best spherical type Legendrian submanifolds in the sense that these have a topological and geometric behaviour similar to the spheres. The existence of such submanifolds is analyzed and some examples are presented. A characterization up to conformal transformation by the Maslov form is given (Theorem 5.1) and their orientability is also studied (Theorem 4.10).

2. Some results about integral submanifolds

Let $\widetilde{M}$ be a Sasakian manifold of dimension $2n+1$ with the contact structure defined by the tensor fields F, ξ, η, g and denote by D its contact distribution. If M is an integral submanifold of dimension m ($m \leq n$) of D then at each point $x \in M$ the normal space of M has the following decomposition

$$T_x^{\perp} M = FT_x M \oplus \tau_x(M) \oplus \langle \xi_x \rangle \tag{1}$$

and the $2(n-m)$-dimensional vector spaces $\tau_x(M)$ span the so-called *maximal invariant normal bundle* of M, studied in [Pit3].

We denote by $\widetilde{\nabla}, \nabla, \nabla^{\perp}$ the Levi-Civita connections on $\widetilde{M}, M$, and respectively the normal connection of M, induced by $\widetilde{\nabla}$. Also, we denote by h, A, Ric the second fundamental form, the Weingarten endomorphism and the Ricci curvature of M, respectively. From the well-known equality

$$\widetilde{\nabla}_X \xi = -FX, \tag{2}$$

that holds for any vector field X in a Sasakian manifold $\widetilde{M}$, and taking into account the Gauss formula it follows that $h(X,Y)$ is orthogonal to ξ for X and Y tangent to M and then, from (1) we obtain the decomposition

$$h(X,Y) = h^F(X,Y) + h^{\tau}(X,Y) \tag{3}$$

with $h^F(X,Y) \in FT_xM$ and $h^\tau(X,Y) \in \tau_x(M)$. Moreover, by using the Weingarten formula and the expression of the tensor field $\widetilde{\nabla}_X F$, we obtain by straightforward computation:

Proposition 2.1. *On any integral submanifold of a Sasakian manifold we have*

$$\nabla^\perp \circ F = F \circ \nabla + F \circ h^\tau + g \otimes \xi, \qquad F \circ h^F + A_F = 0. \tag{4}$$

Remark. If M is a Legendrian submanifold of $\widetilde{M}$ (i.e., M has maximal dimension equal to n), then the first equality in (4) becomes

$$\nabla^\perp \circ F = F \circ \nabla + g \otimes \xi.$$

From the second equality in (4) we obtain

$$g(h^F(X,Y), FZ) = g(h^F(X,Z), FY)$$

for any vectors X, Y, Z tangent to M and therefore we can define a symmetric 3-linear form C^F on M by

$$C^F(X,Y,Z) = g(h^F(X,Y), FZ).$$

Proposition 2.2. *If $\widetilde{M}(c)$ is a Sasakian space form of F-sectional curvature c then the 3-linear form ∇C^F is also symmetric.*

Proof. Simple computations give

$$(\nabla_X C^F)(Y,Z,U) - (\nabla_Y C^F)(X,Z,U) = g((\nabla_X h^F)(Y,Z) - (\nabla_Y h^F)(X,Z), FU) \tag{5}$$

for X, Y, Z and U tangent to M. But from the well-known expression of the curvature tensor $\widetilde{R}$ of the Sasakian space form $\widetilde{M}(c)$ (see for instance [Bla1, p. 113])

$$\begin{aligned}\widetilde{R}(X,Y)Z = {} & \frac{c+3}{4}\,[g(Y,Z)X - g(X,Z)Y] \\ & + \frac{c-1}{4}[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ & - g(Y,Z)\eta(X)\xi + g(Z,FY)FX - g(Z,FX)FY + 2g(X,FY)FZ].\end{aligned} \tag{6}$$

We deduce

$$\left[\widetilde{R}(X,Y)Z\right]^\perp = 0$$

and then from (5) and the Codazzi equation we obtain the symmetry of ∇C^F with respect to X and Y. $\square$

In [Pit3], the 1-form $\alpha_{\vec{n}}$, defined on the integral submanifold M by the equality

$$\alpha_{\vec{n}}(X) = g(F\vec{n}, X), \quad X \in \mathcal{X}(M) \tag{7}$$

for a given normal vector field $\vec{n}$ to M, is used for the computation of some Chern classes of the maximal invariant normal bundle of M. It is proved that if the mean curvature vector H of M is parallel then α_H is closed ([Pit3], Proposition 5(b)). On the other hand, if $\widetilde{M}$ is a Sasakian space form and M is Legendrian then α_H is always closed ([Pit3], Proposition 5(c)) and $H^F = H$. Hence we have:

Proposition 2.3. *Let M be an integral submanifold of the Sasakian manifold $\widetilde{M}$. If one of the following conditions is satisfied, then FH^F is a closed vector field on M:*

(a) *M has parallel mean curvature vector.*

(b) *$\widetilde{M}$ is a Sasakian space form and M is Legendrian.*

On the other hand, from Theorems 1 and 2, [Pit2], it follows that if M is a compact integral submanifold with parallel mean curvature vector in a Sasakian manifold then $H^\tau = 0$, $\delta\alpha_H = 0$ and using Proposition 2.3 we deduce the following known result:

Theorem 2.4. *Let M be a compact integral submanifold with nontrivial parallel mean curvature vector in a Sasakian manifold. Then the first Betti number of M does not vanish.*

If $X = FH^F$ is a closed vector field on the integral submanifold M then from $d\alpha_{H^F} = 0$ it follows that the distribution

$$\mathcal{I} : x \in M \mapsto \mathcal{I}(x) = \{Y \in T_xM : g(X, Y) = 0\}$$

is integrable and then it defines a foliation on M, also denoted by $\mathcal{I}$. A characterization of integral submanifolds by this foliation is given in the following:

Theorem 2.5. *Let M be a compact integral submanifold of a Sasakian manifold. If M has nowhere zero parallel mean curvature vector then M is locally conformal to the Riemannian product $I \times N$ of an open interval I of the real line with a leaf N of the foliation $\mathcal{I}$, endowed with the natural metric $dt^2 \times g_N$.*

Proof. We have $H = H^F$ and as H is parallel, we obtain $\widetilde{\nabla}_Y H = -A_H Y$. Now, applying F to this formula and taking into account the well-known equality

$$(\widetilde{\nabla}_X F)Y = g(X, Y)\xi - \eta(Y)X \tag{8}$$

we deduce

$$\widetilde{\nabla}_Y(FH) = -F\,A_H Y \tag{9}$$

for any $Y \in \mathcal{X}(M)$. But $FH = FH^F$ is tangent to M and then, by using the Gauss formula, from (9) we deduce that FH is also parallel. Therefore FH is conformal and from Proposition 2.3 and taking into account Lemma 1 of [RUr1] it follows that $(M, \|H\|^{-2}g)$ is locally isometric to $(I \times N, dt^2 \times g_N)$, where g_N is the restriction of the metric g to the leaf N. □

Finally, in the case of maximal integral submanifolds we have:

Proposition 2.6. *There are no compact Legendrian submanifolds with* $\mathrm{Ric} > 0$ *in a Sasakian space form $\widetilde{M}(c)$ when $c \le -3$.*

Proof. By the Bochner theorem the first de Rham cohomology group of the submanifold M is zero. But FH is closed (Proposition 2.3), hence it is the gradient of a function $f \in \mathcal{F}(M)$. On the other hand M is compact and then f has critical points which are zeroes for H. Now, if $\{e_1, \ldots, e_n\}$ is a local orthonormal frame in M then by using the Gauss equation we have

$$n\,g(h(U,U), H) = \mathrm{Ric}(U) + \sum_{i=1}^{n} \|h(U, e_i)\|^2 - \frac{c+3}{4}\,(n-1)\|U\|^2 \; > 0$$

for any nonzero tangent vector U at the point $x \in M$. We deduce that $\|H\| > 0$ for all $x \in M$ and our assertion is proved. □

3. Integral submanifolds with special closed conformal vector field

Let X be a closed and conformal nontrivial vector field defined on the integral submanifold M of the Sasakian manifold $\widetilde{M}$. This means that X satisfies the following conditions:

(i) X is a closed vector field.

(ii) $g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = \dfrac{2}{\dim M} \operatorname{div} X$ for any $Y, Z \in \mathcal{X}(M)$.

In this paper, a closed conformal vector field X with the property that

$$h(X, X) = f\, FX$$

for some real-valued function $f \in \mathcal{F}(M)$, is called *special closed conformal vector field.*

It is well-known (see for instance [RUr1], Lemma 1) that, for the closed conformal vector field X, the foliation $\mathcal{I}$ is umbilical.

In the following we shall study the eigenvalues and the eigenvectors of the Weingarten operator A_{FX} associated to this kind of vector field.

Proposition 3.1. *Let M be an integral submanifold of dimension $m \geq 2$ in the Sasakian manifold $\widetilde{M}$. If X is a special closed and conformal vector field on M then:*

(a) *X is an eigenvector of A_{FX}, corresponding to the eigenvalue f.*

(b) *Y is an eigenvector of A_{FX}, corresponding to the eigenvalue λ if and only if $h^F(X, Y) = \lambda\, FX$.*

(c) *Y is an eigenvector of A_{FX} if and only if*

$$\nabla^{\perp}_Y (FX) = \frac{\operatorname{div} X}{m}\, FY + F\, h^{\nu}(X, Y) + g(X, Y)\xi.$$

(d) *Any eigenvectors $Y, Z \in \mathcal{I}$, corresponding to different eigenvalues of A_{FX}, are orthogonal and $C^F(X, Y, Z) = 0$.*

Proof. (a) and (b) follow easily from the second equality in (4), taking into account the symmetry of h^F.

(c) follows from the Weingarten formula by using (b) and from the characterization of the closed and conformal vector field X (conditions (i) and (ii) at the beginning of this section) by the well-known equality

$$\nabla_Y X = \frac{\operatorname{div} X}{m}\, Y, \qquad Y \in \mathcal{X}(M). \tag{10}$$

If Y, Z are eigenvectors corresponding to the eigenvalues λ_1, λ_2 respectively, then by using the Gauss formula we have

$$g\,(A_{FX} Y, Z) = g\,(h(Y, Z), FX) = C^F(Y, Z, X) = \lambda_1\, g(Y, Z),$$
$$g\,(A_{FX} Z, Y) = g\,(h(Z, Y), FX) = C^F(Z, Y, X) = \lambda_2\, g(Z, Y),$$

and because C^F is symmetric, we deduce (d). □

Proposition 3.2. *Under the assumptions of Proposition* 3.1*, if* $\widetilde{M} = \widetilde{M}(c)$ *is a Sasakian space form of constant* F*-sectional curvature* c *then we have:*

(a) A_{FX} *has at most three eigenvalues* f, λ_1, λ_2 *at any point of* M*, where* λ_1, λ_2 *are the roots of the equation*

$$\lambda^2 - f\,\lambda + \frac{\mathrm{Ric}(X)}{m-1} - \frac{c+3}{4}\,\|X\|^2 = 0. \tag{11}$$

(b) *The eigenvalues* λ_1, λ_2 *of* A_{FX} *are constant on the connected leaves of the foliation* $\mathcal{I}$*.*

Proof. (a) Let $Y \in \mathcal{I}_x$ be an eigenvector corresponding to the eigenvalue λ of A_{FX} at $x \in M$. From the Gauss equation and by using Proposition 3.1(b), we obtain

$$g(\widetilde{R}(X,Y)X,Y) = g(R(X,Y)X,Y) - \lambda^2\|Y\|^2 + f\,\lambda\,\|Y\|^2, \tag{12}$$

where R denotes the curvature tensor of the submanifold M. On the other hand we have ([RUr1], Lemma 1)

$$\|X\|^2 R(U,V)X = \frac{\mathrm{Ric}\,(X)}{m-1}[g(V,X)U - g(U,X)V] \tag{13}$$

for any U, $V \in T_xM$. Then

$$R(X,Y)X = -\frac{\mathrm{Ric}(X)}{m-1}\,Y \tag{14}$$

and from (6) we deduce

$$\widetilde{R}(X,Y)X = -\frac{c+3}{4}\,\|X\|^2 Y. \tag{15}$$

Now, from (12), (14), (15) we obtain (11).

The argument for (b) is essentially the same as in [RUr1, proof of Lemma 2]. □

For a simple case of special closed conformal vector field, we have the following characterization of the function f:

Proposition 3.3. *Let* X *be a nowhere zero parallel vector field on the connected integral submanifold* $M, \dim M \geq 2$*, in a Sasakian space form. If there exists a real-valued function* f *such that* $h(X,X) = f\,FX$*, then* f *is constant on* M*.*

Proof. For any Y, $Z \in \mathcal{X}(M)$ we have

$$g\,(\nabla_Y(h(X,X)), FZ) = g\,(\nabla_Y(f\,FX), FZ)$$

and because X is parallel, we deduce

$$\left(\nabla_Y C^F\right)(X,X,Z) = (Yf)\,g(X,Z). \tag{16}$$

But $\nabla_Y C^F$ is symmetric (Proposition 2.2) and then from (16) we deduce

$$(Yf)\,g(X,Z) = (Xf)\,g(Y,Z)$$

for any $Z \in \mathcal{X}(M)$, hence $(Xf)Y - (Yf)X = 0$ for all $Y \in \mathcal{X}(M)$ and taking into account X has no zeros and $\dim M \geq 2$, it follows $f = $ constant. □

The above results will be applied to integral submanifolds having a special closed and conformal vector field, namely we shall obtain a characterization of connected integral submanifolds with nowhere zero parallel vector field X with the property $h(X,X) = f\,FX$ in a Sasakian space form $\widetilde{M}(-3)$. By Proposition 3.3 we have the following two possibilities for the function f:

Case 1: $f \equiv 0$. From (12), (14), (15) it follows $\mathrm{Ric}\,(X) = 0$ and taking into account (11) we deduce $A_{FX} = 0$, hence A_{FX} has only 0 as eigenvalue. But it is known ([RUr1], formula (3)) that the second fundamental form h_0 of a leaf N of the distribution $\mathcal{I}$, corresponding to the vector field X in a m-dimensional submanifold M of the Riemannian manifold $\widetilde{M}$, is given by

$$h_0(V,W) = -\frac{\operatorname{div} X}{m\|X\|^2}\, g(V,W)\, X \tag{17}$$

for any V, $W \in \mathcal{X}(N)$.

Now, taking into account formula (17), we deduce that M is locally isometric to the Riemannian product $\Gamma \times N$, where Γ is a geodesic of $\widetilde{M}$ (integral curve of X).

Case 2: $f = \text{constant} \neq 0$. We have $\mathrm{Ric}\,(X) = 0$ in this case too and then from (11) it follows that A_{FX} has two different eigenvalues f and 0. We denote by m_1, $m - m_1$ their multiplicities, respectively and by $\{e_1, \dots, e_{m_1}, e_{m_1+1}, \dots, e_m\}$ a local orthonormal basis with the property that $e_1, \dots, e_{m_1}$ are eigenvectors of f and $e_{m_1+1}, \dots, e_m$ are eigenvectors of 0. We remark that m_1 is constant because M is connected and then we can define two distributions $\mathcal{D}_1$, $\mathcal{D}_2$ on M, where $\mathcal{D}_1(x) =$ eigenspace of f and $\mathcal{D}_2(x) =$ eigenspace of 0 at $x \in M$. These distributions have the following properties:

Proposition 3.4. *$\mathcal{D}_1$ and $\mathcal{D}_2$ are orthogonal, parallel and define two totally geodesic foliations on M.*

Proof. For any $V \in \mathcal{X}(M)$ we have

$$(\nabla h)(V, X, e_i) + h(X, \nabla_V e_i) = 0, \quad i = m_1 + 1, \dots, m$$

and taking into account the symmetry of $(\nabla h)(V, X, e_i)$ in X, e_i and because X is parallel, we deduce $h(X, \nabla_V e_i) = 0$. But by Proposition 3.1(b), and because $\nabla_V e_i$ is orthogonal to X, this happens if and only if $\nabla_V e_i \in \mathcal{D}_2$, hence $\mathcal{D}_2$ is parallel and then, by the Gauss formula, it is totally geodesic. On the other hand, for any $j = 1, \dots, m_1$ we have

$$g(\nabla_V e_j, e_i) = -g(e_j, \nabla_V e_i) = 0, \quad i = m_1 + 1, \dots, m$$

and then $\mathcal{D}_1$ is also parallel and totally geodesic.

Now, taking into account ∇ is torsionfree, it follows that $\mathcal{D}_1$ and $\mathcal{D}_2$ are integrable and then these define two foliations on M. By Proposition 3.1(d), $\mathcal{D}_1$ and $\mathcal{D}_2$ are also orthogonal. $\square$

From the above argument we deduce:

Theorem 3.5. *Let M be a connected m-dimensional integral submanifold in a Sasakian space form $\widetilde{M}(-3)$. If M has a nowhere zero parallel vector field X with the property $h(X,X) = fFX$ then M is locally isometric to one of the following manifolds:*

(a) *the Riemannian product of a geodesic of* $\widetilde{M}(-3)$ *with an* $(m-1)$*-dimensional integral submanifold of* $\widetilde{M}(-3)$*, totally geodesic in* M*,*
(b) *the Riemannian product* $M_1 \times M_2$ *of two leaves* M_1*,* M_2 *of the foliations* $\mathcal{D}_1$ *and* $\mathcal{D}_2$*, respectively.*

4. Spherical type Legendrian submanifolds

In this section we study the existence of imbeddings of the sphere S^n as a Legendrian submanifold in a $(2n+1)$-dimensional Sasakian space form. Because the answer to this problem is generally negative, we consider a kind of Legendrian submanifold having some topological and geometric properties similar to those of the sphere.

Proposition 4.1. *Let* $\widetilde{M}(c)$ *be a* $(2n+1)$*-dimensional Sasakian space form. If* n *is even and* $c < 1$ *then the sphere* S^n *cannot be embedded in* $\widetilde{M}(c)$ *as a Legendrian submanifold.*

Proof. If S^n is a Legendrian submanifold of $\widetilde{M}(c)$ then from the Gauss equation it follows that

$$g\left(\widetilde{R}(X,Y)Y,X\right) = 1 - g\left(h(X,X),h(Y,Y)\right) + \|h(X,Y)\|^2 \tag{18}$$

for any orthogonal unit vectors X, Y tangent to S^n. Via (6), we deduce

$$\frac{c+3}{4} + g\left(h(X,X),h(Y,Y)\right) = 1 + \|h(X,Y)\|^2. \tag{19}$$

If $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of T_xS^n, $x \in S^n$, (19) now gives

$$n^2 g(H,H) = \sum_{i=1}^{n} \|h\left(e_i,e_i\right)\|^2 + 2\sum_{1\le i<j\le n} \|h\left(e_i,e_j\right)\|^2 + \frac{1-c}{2}\left(n^2-n\right). \tag{20}$$

But for n even, every vector field of S^n has at least one zero and then the last equality is impossible on S^n, for any $c < 1$. □

$\mathbf{R}^{2n+1}$ admits a standard contact structure (see for instance [Bla1, p. 114] and with respect to this structure it is a Sasakian space form of constant F-sectional curvature $c = -3$. Then from Proposition 4.1 we deduce:

Corollary 4.2. *For* n *even the sphere* S^n *cannot be embedded as a Legendrian submanifold of* $\mathbf{R}^{2n+1}(-3)$*.*

Concerning the problem on nonimbeddability of spheres in an arbitrary Sasakian space form, we have the following result:

Proposition 4.3. *Let* $\widetilde{M}(c)$ *be a* $(2n+1)$*-dimensional Sasakian space form with* n *odd. The sphere* S^n *cannot be embedded in* $\widetilde{M}(c)$ *as:*

(a) *a totally geodesic Legendrian submanifold for* $c \neq 1$*,*
(b) *a minimal Legendrian submanifold for* $c \le 1$*,*
(c) *a Legendrian submanifold with parallel mean curvature vector for any* c *and* $n > 1$*,*
(d) *a totally umbilical Legendrian submanifold for* $c \ge 1$*,*
(e) *a parallel Legendrian submanifold for any* $c \neq 1$*.*

Proof. (a) and (b) follow from (19) and (20), respectively. (c) is a consequence of Theorem 2.4 because the first Betti number of S^n is zero. (d) follows from (19).

(e) By [Pit1, Corollary 1], S^n must be totally geodesic in $\widetilde{M}(c)$ and then we use (a). □

Now, we shall introduce and study a class of Legendrian submanifolds whose topological behaviour is similar to the sphere. From the point of view of the their geometry, these submanifolds have an expression of the fundamental form which is the simplest one, except the case of totally geodesic submanifolds. Moreover, these submanifolds have a remarkable special conformal vector field.

Definition. Let $\widetilde{M}(c)$ be a Sasakian space form. A $*$-*Legendrian submanifold* of $\widetilde{M}(c)$ is a Legendrian submanifold M whose second fundamental form is given by

$$h(Y,Z) = \alpha\,[g(Y,Z)H + g(FY,H)FZ + g(FZ,H)FY] \tag{$*$}$$

for any $X, Y \in \mathcal{X}(M)$ and for some function $\alpha \in \mathcal{F}(M)$.

Obviously, any totally geodesic integral submanifold satisfies the condition $(*)$. On the other hand, direct computation shows that $\alpha = n/(n+2)$. We also remark that any totally umbilical or minimal $*$-Legendrian submanifold is totally geodesic.

In low-dimensional Sasakian space forms we have the following result concerning the existence of $*$-Legendrian submanifolds.

Proposition 4.4. *Any curve orthogonal to the structure vector field of a 3-dimensional Sasakian space form is a $*$-Legendrian submanifold.*

Proposition 4.5. *Let M be a $*$-Legendrian submanifold of dimension ≥ 2 in a Sasakian space form. Then either:*

(a) *M is totally geodesic, or*
(b) *FH is a nonparallel special closed and conformal vector field on M.*

Proof. From (6) we have $\left[\widetilde{R}(Z,V)V\right]^{\perp} = 0$ and then, assuming that Z and V are orthogonal and $\|V\| = 1$, from the Codazzi equation we deduce

$$\begin{aligned}\nabla^{\perp}_Z H + 2g(FV, \nabla^{\perp}_Z H)FV = g(FV, \nabla^{\perp}_V H)FZ + g(FZ, \nabla^{\perp}_V H)FV \\ + g(FZ,H)\xi.\end{aligned} \tag{21}$$

But by Proposition 2.3, FH is closed and then

$$g(FV, \nabla^{\perp}_Z H) = g(FZ, \nabla^{\perp}_V H). \tag{22}$$

On the other hand, by using (8) and the Gauss formula, we have

$$g(FV, \nabla^{\perp}_Z H) = 0. \tag{23}$$

Now, taking into account (8), (21), (22), (23) and the Weingarten formula, a straightforward computation says that

$$\nabla_Z(FH) = -g(FV, \nabla^{\perp}_V H)Z$$

hence FH is a conformal vector field. Moreover, from $(*)$ it follows

$$h(FH,FH) = \frac{3n}{n+2}\,\|H\|^2 H, \quad h(V,W) = \frac{n}{n+2}\,g(V,W)H \tag{24}$$

for any V, W orthogonal to FH. Now, comparing (17) with the second equality in (24), from (10) it follows that FH is nonparallel if M is nontotally geodesic. □

From $(*)$ it follows that on a $*$-Legendrian submanifold the Ricci curvature is given by

$$\mathrm{Ric}(V) = (n-1)\frac{c+3}{4}\|V\|^2 + \left(\frac{n}{n+2}\right)^2 \left[n\|V\|^2\|H\|^2 + 2(n-2)\, g^2(FV, H)\right]$$

for any $V \in \mathcal{X}(M)$. From this equality we deduce some sharp relationships between the Ricci curvature, sectional curvature or scalar curvature and the squared mean curvature of a $*$-Legendrian submanifold.

Proposition 4.6. *Let M be a $*$-Legendrian submanifold of the Sasakian space form $\widetilde{M}(c)$. Then:*

(a) $(n-1)\frac{c+3}{4} + \frac{n^3}{(n+2)^2}\|H\|^2 \leq \mathrm{Ric} \leq (n-1)\frac{c+3}{4} + \left(\frac{n}{n+2}\right)^2 (3n-4)\|H\|^2$.

(b) *sectional curvature* $\geq \frac{c+3}{4} + \left(\frac{n}{n+2}\right)^2 \|H\|^2$.

(c) *scalar curvature* $\geq n(n-1)\frac{c+3}{4} + \left[n^2\left(\frac{n}{n+2}\right)^2 + 2(n-2)\right]\|H\|^2$.

Remark. Similar inequalities concerning Ricci curvature (second part of (a) from the above Proposition) were obtained by B.-Y. Chen, [Che1], [Che2], in the case of submanifolds of real space forms and for isotropic and Lagrangian submanifolds in complex space forms. For integral submanifolds of arbitrary dimensions in Sasakian space forms such an inequality was proved by K. Matsumoto and I. Mihai, [KMi1], but our result obtained in Proposition 4.6(a), for $*$-Legendrian submanifolds is a stronger version.

Proposition 4.7. *Let $\widetilde{M}(c)$ be a $(2n+1)$-dimensional Sasakian space form.*

(a) *There are no compact $*$-Legendrian submanifolds with* $\mathrm{Ric} > 0$ *if* $c \leq -3$ *or with* $\mathrm{Ric} \leq 0$ *if* $c < -3$.

(b) *For $c > -3$ the mean curvature vector field of a compact $*$-Legendrian submanifold of $\widetilde{M}(c)$ has zeros.*

Proof. (a) is obvious by Proposition 2.6.

(b) From Proposition 4.6(a), it follows that $\mathrm{Ric} > 0$ and then $H^1(M; \mathbf{R}) = 0$, hence the closed vector field FH is the gradient of a function $f : M \to \mathbf{R}$. Now, as M is compact f has at least two critical points and at these points H vanish. □

When the mean curvature vector field of a $*$-Legendrian (or Legendrian) submanifold is nontrivial (the submanifold is nonminimal), it is called a *proper $*$-Legendrian (or Legendrian) submanifold.*

Theorem 4.8. *Let $\widetilde{M}(c)$ be a $(2n+1)$-dimensional Sasakian space form, $n \geq 2$. For n even and $c > -3$, any compact and connected proper $*$-Legendrian submanifold is conformal to the ordinary sphere S^n.*

Proof. By Proposition 4.7(b), FH is a nontrivial closed conformal vector field with zeros. Then M is diffeomorphic to a sphere ([STs1], Theorem 3). It follows $H^1(M; \mathbf{R}) = 0$ and as in the proof of Proposition 4.7, we have $FH = \mathrm{grad} f$. Then, taking into account (10), we deduce

$$\mathrm{Hess} f(Y, Z) = \frac{\mathrm{div} FH}{n} g(X, Z)$$

for any vectors Y, Z tangent to M. Then by Theorem 1, [Tas1], the function $\frac{\mathrm{div} FH}{n}$ has at most two critical points and because M is compact, it has exactly two. Then the same theorem asserts that M is conformal to an n-dimensional sphere. $\square$

Proposition 4.9. *Any compact $*$-Legendrian submanifold with* $\mathrm{Ric}(FH) \le 0$ *in a Sasakian space form has constant mean curvature.*

Proof. If X is a closed conformal vector field then ([RUr1], Lemma 1 and its proof; see also [Udr1] for a more general form)

$$\mathrm{Hess}\|X\|^2(Y,Z) = 2\left(\frac{\mathrm{div}X}{n}\right)^2 g(Y,Z) + \frac{2}{n} g\left(\mathrm{grad}(\mathrm{div}X), Y\right) g(X,Z) \tag{25}$$

$$\|X\|^2 \mathrm{grad}(\mathrm{div}X) = -\frac{n}{n-1}\mathrm{Ric}(X)X \tag{26}$$

for any $Y, Z \in \mathcal{X}(M)$. But it is well-known that a closed and conformal vector field X satisfies

$$\mathrm{grad}\|X\|^2 = \frac{2\mathrm{div}X}{n} X \tag{27}$$

hence the critical points of the function $\|X\|^2 : M \to \mathbf{R}$ are either zeros of div X or zeros of X ([Udr1], Lemma 1; see also [RUr1], Lemma 1 for a summary of the most important properties of manifolds admitting closed and conformal vector fields). The function $\frac{1}{2}\|X\|^2$ is sometimes called the *energy* of the conformal vector field X (see for example [Udr1]).

Now, let x_0 be a critical point of $\|X\|^2$, where $X = FH$. By Proposition 4.5, it is a closed conformal vector field and if $X_{x_0} = 0$ then from (25) we deduce

$$\mathrm{Hess}_{x_0}\|X\|^2(Y_{x_0}, Y_{x_0}) = 2\left(\frac{\mathrm{div}_{x_0}X}{n}\right)^2 \|Y_{x_0}\|^2 > 0$$

for any $Y_{x_0} \in T_{x_0}M - 0$, hence x_0 is a minimum.

Assume $\mathrm{div}_{x_0}X = 0$. Then $X_{x_0} \neq 0$ and from (25), (26) we obtain

$$\mathrm{Hess}_{x_0}\|X\|^2(Y_{x_0}, Y_{x_0}) = -\frac{2}{n-1}\frac{1}{\|X_{x_0}\|^2}\mathrm{Ric}_{x_0}(X)g^2(X_{x_0}, Y_{x_0}) \ge 0$$

hence x_0 is a minimum, too. It follows that $\|FH\| = \|H\|$ is constant. $\square$

Theorem 4.10. *Any compact connected proper $*$-Legendrian submanifold M of the $(2n+1)$-dimensional Sasakian space form $\widetilde{M}(c)$ with $c > -3$ is orientable. Moreover, if n is even then M is simply connected.*

Proof. By Proposition 4.6(b), M has positive sectional curvature and then from the Synge theorem (see for instance [Car1, p. 206]) it follows that for n odd, M is orientable.

For n even M is orientable by Theorem 4.8 and then it is simply connected by the same Synge theorem. $\square$

5. Legendrian submanifolds with conformal Maslov form

By Proposition 2.3, if M is a Legendrian submanifold in a Sasakian space form then FH is a closed vector field on M (or Legendrian variation of M, [Pit2]), hence the 1-form α_H defines a 1-dimensional cohomology class $[\alpha_H] \in H^1(M, \mathbf{R})$. It is proved in [Vai1] (the main theorem) that if M is a Legendrian submanifold of the cotangent unit sphere bundle S^*M^* over a flat Riemannian manifold M^*, with the natural contact structure (see for instance [Bla1], Chap. VII), then the Maslov class $m^*(M)$ of M can be simply expressed by α_H, namely

$$m^*(M) = \frac{n}{\pi}\,[\alpha_H].$$

This result suggest to call *Legendrian submanifolds with conformal Maslov form* the Legendrian submanifolds with the property that the vector field FH is conformal. Their properties and some relations with $*$-Legendrian submanifolds are studied in this section.

Theorem 5.1. *A compact and complete Legendrian submanifold with nontrivial conformal Maslov form in a Sasakian space form of dimension $2n+1$, $n \geq 3$, is conformal to a $*$-Legendrian submanifold.*

Proof. On the Legendrian submanifold M we consider the vector field

$$Z = F\,h(X,X) \,-\, \frac{3n}{n+2}\,\|X\|^2 X, \quad X = FH.$$

By using (10) and the symmetry of C^F we have

$$g(\nabla_V(Fh(X,X)),W) = -(\nabla_V C^F)(X,X,W) + 2\,\frac{\operatorname{div} X}{n}\,g(h(V,W),H) \tag{28}$$

$$g(\nabla_V(\|X\|^2 X),W) = \frac{\operatorname{div} X}{n}\,\big[2g(V,X)g(W,X) + \|X\|^2 g(V,W)\big] \tag{29}$$

for any V, $W \in \mathcal{X}(M)$.

For $x \in M$ let $\mathcal{B}^0 = \{e^0_1, \ldots, e^0_n\}$ be an orthonormal basis of T_xM. In a normal coordinate neighbourhood of x we can consider a local orthonormal basis $\{e_1, \ldots, e_n\}$, obtained by parallel displacement of $\mathcal{B}^0$ along an integral curve of X. On the other hand, by using again (10), the symmetry of C^F and Proposition 2.2, we obtain

$$\sum_{i=1}^{n} (\nabla_{e_i} C^F)(X,X,e_i) = -\|H\|^2 \operatorname{div} X$$

and taking into account (28), (29), we deduce that $\operatorname{div} Z = 0$ and

$$d\,\alpha_{FZ}(V,W) = -g(\nabla_V Z,W) + g(\nabla_W Z,V) = 0$$

that is Z is closed. It follows that Z is harmonic. But M is compact and H is nontrivial with at least one zero (see the proof of Proposition 4.7), hence the only harmonic vector field is zero, and then we deduce that $X = FH$ satisfies the first equality in (24).

Denoting by A^0 the Weingarten operator of a leaf N of the foliation $\mathcal{I}$ in M and taking into account (17), we have

$$g(h(V,W),FU) = g(A_{FU}V,W) = g(A^0_{FU}V,W) = g(h^0(V,W),FU) = 0 \tag{30}$$

for any U, V, $W \in \mathcal{X}(M)$ orthogonal to FH.

From Proposition 3.1(b) and (d), we deduce

$$g(h(V,W),H) = -g(h(V,FH),FW) = -\lambda\,g(V,W) \tag{31}$$

if V, W are eigenvectors corresponding to the eigenvalue λ of A_{-H} and

$$g(h(V,W),H) = -C^F(V,W,FH) = -C^F(V,FH,W) = 0 \tag{32}$$

if V, W correspond to different eigenvalues (Proposition 3.2(a)).

If λ_1, λ_2 have the multiplicities p, $n-p-1$ respectively, then at a point $x \in M$ with $H_x \neq 0$, we can consider an orthonormal basis $\{e_1, \dots, e_n\}$ of T_xM so that $e_1, \dots, e_p$ are eigenvectors of λ_1, the vectors $e_{p+1}, \dots, e_{n-1}$ correspond to λ_2 and $e_n = FH/\|H\|$. From (31), (32) and (24) we deduce

$$p\,\lambda_1 + (n-p-1)\,\lambda_2 = -\frac{3n}{n+2}\|H\|^2$$

and, via (11), we obtain

$$\lambda_k = -\frac{n\{(3-k)(n-1)-3p\}}{(n+2)(n-2p-1)}\|H\|^2, \quad k=1,2. \tag{33}$$

From (30), (31) and using Proposition 3.1(b), we also deduce

$$h(V,W) = -\frac{\lambda}{\|H\|^2}g(V,W)H,\ h(V,FH) = \frac{\lambda}{\|H\|^2}g(FH,FH)FV \tag{34}$$

for any V, $W \in \mathcal{X}(M)$ corresponding to an eigenvalue λ of A_{-H}. But from (33) we have that $\lambda_1/\|H\|^2$ and $\lambda_2/\|H\|^2$ are constant on M except the points $x \in M$ with $H_x = 0$. Thus (34) and the first equality in (24) (satisfied by FH in this case) tell us that by a conformal change of the metric g of $\widetilde{M}$ we have

$$h(V,W) = \alpha\, g(V,W)H, \quad h(V,FH) = -\alpha\, g(FH,FH)FV \tag{35}$$

for some nonzero $\alpha \in \mathbf{R}$ and for any V, W orthogonal to FH.

A simple computation of H using (35) and the first equality in (24) shows that $\alpha = n/(n+2)$ and then $(*)$ is verified at any point of M where H is nontrivial. But H has only isolated zeros, hence by continuity $(*)$ is valid on the whole of M and therefore M is conformal to a $*$-Legendrian submanifold. □

Proposition 5.2. *Let M be a compact and complete proper Legendrian submanifold with conformal Maslov form in a Sasakian space form. If its first Betti number is zero then M is conformal to a $*$-Legendrian submanifold.*

Proof. Because $b_1 = 0$, the argument used in the proof of Proposition 2.6 shows that H has zeros. Now, we can apply Theorem 5.1. □

The proof of the following result is similar to the argument used in [RUr1], Corollary 5.

Proposition 5.3. *Let M be a compact and complete proper Legendrian submanifold with conformal Maslov form in the Sasakian space form $\widetilde{M}(c)$ with $c \geq -3$. The following assertions are equivalent:*

(a) $\mathrm{Ric}(FH) \geq 0$.

(b) *M has parallel mean curvature vector or M is conformal to a $*$-Legendrian submanifold of $\widetilde{M}(c)$.*

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Department of Differential Equations, Faculty of Mathematics and Informatics, University Transilvania of Braşov, Braşov, Romania
gh.pitis@info.unitbv.ro

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