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T-homotopy and refinement of observation. III. Invariance of the branching and merging homologies

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Abstract. This series explores a new notion of T-homotopy equivalence of flows. The new definition involves embeddings of finite bounded posets preserving the bottom and the top elements and the associated cofibrations of flows. In this third part, it is proved that the generalized T-homotopy equivalences preserve the branching and merging homology theories of a flow. These homology theories are of interest in computer science since they detect the nondeterministic branching and merging areas of execution paths in the time flow of a higher-dimensional automaton. The proof is based on Reedy model category techniques.

Contents

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1. Outline of the paper

The main feature of the algebraic topological model of *higher dimensional automata* (or HDA) introduced in [Gau03], the category of *flows*, is to provide a framework for modelling continuous deformations of HDA corresponding to subdivision or refinement of observation. The equivalence relation induced by these deformations, called *dihomotopy*, preserves geometric properties like the *initial* or *final states*, and therefore computer-scientific properties like the presence or not of *deadlocks* or of *unreachable states* in concurrent systems [Gou03]. More generally, dihomotopy is designed to preserve all computer-scientific properties invariant under refinement of observation. Figure 2 represents a very simple example of refinement of observation, where a 1-dimensional transition from an initial state to a final state is identified with the composition of two such transitions.

In the framework of flows, there are two kinds of dihomotopy equivalences [Gau00]: the *weak S-homotopy equivalences* (the spatial deformations of [Gau00]) and the *T-homotopy equivalences* (the temporal deformations of [Gau00]). The geometric explanations underlying the intuition of S-homotopy and T-homotopy are given in the first part of this series [Gau05c], but the reference [GG03] must be preferred.

It is very fortunate that the class of weak S-homotopy equivalences can be interpreted as the class of weak equivalences of a model structure [Gau03] in the sense of Hovey's book [Hov99]. This fact makes their study easier. Moreover, this model structure is necessary for the formulation of the only known definition of T-homotopy.

The purpose of this paper is to prove that the new notion of T-homotopy equivalence is well-behaved with respect to the *branching and merging homologies* of a flow. The latter homology theories are able to detect the nondeterministic higher dimensional branching and merging areas of execution paths in the time flow of a higher-dimensional automaton [Gau05b]. More precisely, one has:

Theorem (Corollary 11.3). *Let $f : X \longrightarrow Y$ be a generalized T-homotopy equivalence. Then for any $n \geqslant 0$, the morphisms of abelian groups $H_n^-(f) : H_n^-(X) \longrightarrow H_n^-(Y)$, $H_n^+(f) : H_n^+(X) \longrightarrow H_n^+(Y)$ are isomorphisms of groups where H_n^- (resp. H_n^+) is the n-th branching (resp. merging) homology group.*

The core of the paper starts with Section 3 which recalls the definition of a flow and the description of the weak S-homotopy model structure. The latter is a fundamental tool for the sequel. Section 4 recalls the new notion of T-homotopy equivalence.

Section 5 recalls the definition of the branching space and the homotopy branching space of a flow. The same section explains the principle of the proof of the following theorem:

Theorem (Theorem 9.8). *The homotopy branching space of a full directed ball at any state different from the final state is contractible (it is empty at the final state).*

We give the idea of the proof for a full directed ball which is not too simple, and not too complicated. The latter theorem is the technical core of the paper because a generalized T-homotopy equivalence consists in replacing in a flow a full directed ball by a more refined full directed ball (Figure 3), and in iterating this replacement process transfinitely.

Section 6 introduces a diagram of topological spaces $\mathcal{P}^-_\alpha(X)$ whose colimit calculates the branching space $\mathbb{P}^-_\alpha X$ for every loopless flow X (Theorem 6.3) and every $\alpha \in X^0$. Section 7 builds a Reedy structure on the base category of the diagram $\mathcal{P}^-_\alpha(X)$ for any loopless flow X whose poset $(X^0, \leqslant)$ is locally finite so that the colimit functor becomes a left Quillen functor (Theorem 7.5). Section 8 then shows that the diagram $\mathcal{P}^-_\alpha(X)$ is Reedy cofibrant as soon as X is a cell complex of the model category **Flow** (Theorem 8.4). Section 9 completes the proof that the homotopy branching and homotopy merging spaces of every full directed ball are contractible (Theorem 9.8). Section 10 recalls the definition of the branching and merging homology theories. Finally, Section 11 proves the invariance of the branching and merging homology theories with respect to T-homotopy.

Warning. This paper is the third part of a series of papers devoted to the study of T-homotopy. Several other papers explain the geometrical content of T-homotopy. The best reference is probably [GG03] (it does not belong to the series). The knowledge of the first and second parts is not required, except for the left properness of the weak S-homotopy model structure of **Flow** available in [Gau05d]. The latter fact is used twice in the proof of Theorem 11.2. The material collected in the appendices A, B and C will be reused in the fourth part [Gau06b]. The proofs of these appendices are independent from the technical core of this part.

2. Prerequisites and notations

The initial object (resp. the terminal object) of a category $\mathcal{C}$, if it exists, is denoted by $\varnothing$ (resp. $\mathbf{1}$).

Let $\mathcal{C}$ be a cocomplete category. If K is a set of morphisms of $\mathcal{C}$, then the class of morphisms of $\mathcal{C}$ that satisfy the RLP (*right lifting property*) with respect to any morphism of K is denoted by $\mathbf{inj}(K)$ and the class of morphisms of $\mathcal{C}$ that are transfinite compositions of pushouts of elements of K is denoted by $\mathbf{cell}(K)$. Denote by $\mathbf{cof}(K)$ the class of morphisms of $\mathcal{C}$ that satisfy the LLP (*left lifting property*) with respect to the morphisms of $\mathbf{inj}(K)$. It is a purely categorical fact that $\mathbf{cell}(K) \subset \mathbf{cof}(K)$. Moreover, every morphism of $\mathbf{cof}(K)$ is a retract of a morphism of $\mathbf{cell}(K)$ as soon as the domains of K are small relative to $\mathbf{cell}(K)$ ([Hov99] Corollary 2.1.15). An element of $\mathbf{cell}(K)$ is called a *relative K-cell complex.* If X is an object of $\mathcal{C}$, and if the canonical morphism $\varnothing \longrightarrow X$ is a relative K-cell complex, then the object X is called a *K-cell complex.*

Let $\mathcal{C}$ be a cocomplete category with a distinguished set of morphisms I. Then let $\mathbf{cell}(\mathcal{C}, I)$ be the full subcategory of $\mathcal{C}$ consisting of the object X of $\mathcal{C}$ such that the canonical morphism $\varnothing \longrightarrow X$ is an object of $\mathbf{cell}(I)$. In other words, $\mathbf{cell}(\mathcal{C}, I) = (\varnothing{\downarrow}\mathcal{C}) \cap \mathbf{cell}(I)$.

It is obviously impossible to read this paper without a strong familiarity with *model categories.* Possible references for model categories are [Hov99], [Hir03] and [DS95]. The original reference is [Qui67] but Quillen's axiomatization is not used in this paper. The axiomatization from Hovey's book is preferred. If $\mathcal{M}$ is a *cofibrantly generated* model category with set of generating cofibrations I, let $\mathbf{cell}(\mathcal{M}) := \mathbf{cell}(\mathcal{M}, I)$: this is the full subcategory of *cell complexes* of the model category $\mathcal{M}$. A cofibrantly generated model structure $\mathcal{M}$ comes with a *cofibrant replacement functor* $Q : \mathcal{M} \longrightarrow \mathbf{cell}(\mathcal{M})$. For any morphism f of $\mathcal{M}$, the morphism $Q(f)$ is a cofibration, and even an inclusion of subcomplexes ([Hir03] Definition 10.6.7) because the cofibrant replacement functor Q is obtained by the small object argument.

A *partially ordered set* $(P, \leqslant)$ (or *poset*) is a set equipped with a reflexive antisymmetric and transitive binary relation $\leqslant$. A poset is *locally finite* if for any $(x, y) \in P \times P$, the set $[x, y] = \{z \in P, x \leqslant z \leqslant y\}$ is finite. A poset $(P, \leqslant)$ is *bounded* if there exist $\widehat{0} \in P$ and $\widehat{1} \in P$ such that $P = [\widehat{0}, \widehat{1}]$ and such that $\widehat{0} \neq \widehat{1}$. Let $\widehat{0} = \min P$ (the bottom element) and $\widehat{1} = \max P$ (the top element). In a poset P, the interval $]\alpha, -]$ (the sub-poset of elements of P strictly bigger than α) can also be denoted by $P_{>\alpha}$.

A poset P, and in particular an ordinal, can be viewed as a small category denoted in the same way: the objects are the elements of P and there exists a morphism from x to y if and only if $x \leqslant y$. If λ is an ordinal, a λ-*sequence* in a cocomplete category $\mathcal{C}$ is a colimit-preserving functor X from λ to $\mathcal{C}$. We denote by X_λ the colimit $\varinjlim X$ and the morphism $X_0 \longrightarrow X_\lambda$ is called the *transfinite composition* of the $X_\mu \longrightarrow X_{\mu+1}$.

Let $\mathcal{C}$ be a category. Let α be an object of $\mathcal{C}$. The *latching category* $\partial(\mathcal{C}\downarrow\alpha)$ at α is the full subcategory of $\mathcal{C}\downarrow\alpha$ containing all the objects except the identity map of α. The *matching category* $\partial(\alpha\downarrow\mathcal{C})$ at α is the full subcategory of $\alpha\downarrow\mathcal{C}$ containing all the objects except the identity map of α.

Let $\mathcal{B}$ be a small category. A *Reedy structure* on $\mathcal{B}$ consists of two subcategories $\mathcal{B}_-$ and $\mathcal{B}_+$, a map $d : \mathrm{Obj}(\mathcal{B}) \longrightarrow \lambda$ from the set of objets of $\mathcal{B}$ to some ordinal λ called the degree function, such that every nonidentity map in $\mathcal{B}_+$ raises the degree, every nonidentity map in $\mathcal{B}_-$ lowers the degree, and every map $f \in \mathcal{B}$ can be factored uniquely as $f = g \circ h$ with $h \in \mathcal{B}_-$ and $g \in \mathcal{B}_+$. A small category together with a Reedy structure is called a *Reedy category*.

If $\mathcal{C}$ is a small category and if $\mathcal{M}$ is a category, the notation $\mathcal{M}^{\mathcal{C}}$ is the category of functors from $\mathcal{C}$ to $\mathcal{M}$, i.e., the category of diagrams of objects of $\mathcal{M}$ over the small category $\mathcal{C}$.

Let $\mathcal{C}$ be a complete and cocomplete category. Let $\mathcal{B}$ be a Reedy category. Let i be an object of $\mathcal{B}$. The *latching space functor* is the composite $L_i : \mathcal{C}^{\mathcal{B}} \longrightarrow \mathcal{C}^{\partial(\mathcal{B}_+\downarrow i)} \longrightarrow \mathcal{C}$ where the latter functor is the colimit functor. The *matching space functor* is the composite $M_i : \mathcal{C}^{\mathcal{B}} \longrightarrow \mathcal{C}^{\partial(i\downarrow\mathcal{B}_-)} \longrightarrow \mathcal{C}$ where the latter functor is the limit functor.

A model category is *left proper* if the pushout of a weak equivalence along a cofibration is a weak equivalence. The model categories **Top** and **Flow** (see below) are both left proper.

In this paper, the notation $\hookrightarrow$ means *cofibration*, the notation $\twoheadrightarrow$ means *fibration*, the notation $\simeq$ means *weak equivalence*, and the notation $\cong$ means *isomorphism*.

A categorical adjunction $\mathbb{L} : \mathcal{M} \leftrightarrows \mathcal{N} : \mathbb{R}$ between two model categories is a *Quillen adjunction* if one of the following equivalent conditions is satisfied:

(1) $\mathbb{L}$ preserves cofibrations and trivial cofibrations.
(2) $\mathbb{R}$ preserves fibrations and trivial fibrations.

In that case, $\mathbb{L}$ (resp. $\mathbb{R}$) preserves weak equivalences between cofibrant (resp. fibrant) objects.

If P is a poset, let us denote by $\Delta(P)$ the *order complex* associated with P. Recall that the order complex is a simplicial complex having P as underlying set and having the subsets $\{x_0, x_1, \dots, x_n\}$ with $x_0 < x_1 < \dots < x_n$ as n-simplices [Qui78]. Such a simplex will be denoted by $(x_0, x_1, \dots, x_n)$. The order complex $\Delta(P)$ can be viewed as a poset ordered by the inclusion, and therefore as a small category. The corresponding category will be denoted in the same way. The opposite category $\Delta(P)^{\mathrm{op}}$ is freely generated by the morphisms $\partial_i : (x_0, \dots, x_n) \longrightarrow (x_0, \dots, \widehat{x_i}, \dots, x_n)$ for $0 \leqslant i \leqslant n$ and by the simplicial relations $\partial_i \partial_j = \partial_{j-1} \partial_i$ for any $i < j$, where the notation $\widehat{x_i}$ means that x_i is removed.

If $\mathcal{C}$ is a small category, then the *classifying space* of $\mathcal{C}$ is denoted by $B\mathcal{C}$ [Seg68] [Qui73].

The category **Top** of *compactly generated topological spaces* (i.e., of weak Hausdorff k-spaces) is complete, cocomplete and cartesian closed (more details for this kind of topological spaces in [Bro88, May99], the appendix of [Lew78] and also the preliminaries of [Gau03]). For the sequel, all topological spaces will be supposed to be compactly generated. A *compact space* is always Hausdorff.

3. Reminder about the category of flows

The category **Top** is equipped with the unique model structure having the *weak homotopy equivalences* as weak equivalences and having the *Serre fibrations*[1] as fibrations.

The time flow of a higher-dimensional automaton is encoded in an object called a *flow* [Gau03]. A flow X consists of a set X^0 called the 0-*skeleton* and whose elements correspond to the *states* (or *constant execution paths*) of the higher-dimensional automaton. For each pair of states $(\alpha, \beta) \in X^0 \times X^0$, there is a topological space $\mathbb{P}_{\alpha,\beta}X$ whose elements correspond to the (*nonconstant*) *execution paths* of the higher-dimensional automaton *beginning at* α and *ending at* β. For $x \in \mathbb{P}_{\alpha,\beta}X$, let $\alpha = s(x)$ and $\beta = t(x)$. For each triple $(\alpha, \beta, \gamma) \in X^0 \times X^0 \times X^0$, there exists a continuous map $* : \mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\beta,\gamma}X \longrightarrow \mathbb{P}_{\alpha,\gamma}X$ called the *composition law* which is supposed to be associative in an obvious sense. The topological space $\mathbb{P}X = \bigsqcup_{(\alpha,\beta) \in X^0 \times X^0} \mathbb{P}_{\alpha,\beta}X$ is called the *path space* of X. The category of flows is denoted by **Flow**. A point α of X^0 such that there are no nonconstant execution paths ending at α (resp. starting from α) is called an *initial state* (resp. a *final state*). A morphism of flows f from X to Y consists of a set map $f^0 : X^0 \longrightarrow Y^0$ and a continuous map $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ preserving the structure. A flow is therefore "almost" a small category enriched in **Top**.

An important example is the flow $\mathrm{Glob}(Z)$ defined by

$$\begin{aligned} &\mathrm{Glob}(Z)^0 = \{\widehat{0}, \widehat{1}\} \\ &\mathbb{P}\mathrm{Glob}(Z) = Z \\ &s = \widehat{0} \\ &t = \widehat{1} \end{aligned}$$

and a trivial composition law (cf. Figure 1).

[1] That is a continuous map having the RLP with respect to the inclusion $\mathbf{D}^n \times 0 \subset \mathbf{D}^n \times [0,1]$ for any $n \geqslant 0$ where $\mathbf{D}^n$ is the n-dimensional disk.

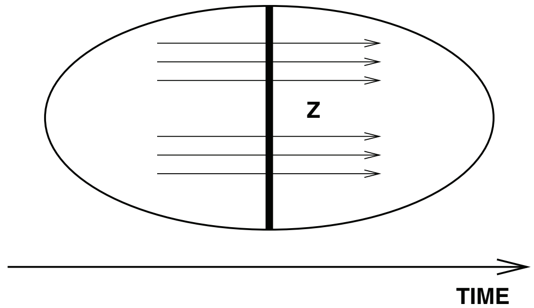

FIGURE 1. Symbolic representation of $\mathrm{Glob}(Z)$ for some topological space Z

$$\widehat{0} \xrightarrow{U} \widehat{1}$$

$$\widehat{0} \xrightarrow{U'} A \xrightarrow{U''} \widehat{1}$$

FIGURE 2. The simplest example of refinement of observation

The category **Flow** is equipped with the unique model structure such that [Gau03]:

- The weak equivalences are the *weak S-homotopy equivalences*, i.e., the morphisms of flows $f : X \longrightarrow Y$ such that $f^0 : X^0 \longrightarrow Y^0$ is a bijection and such that $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a weak homotopy equivalence.
- The fibrations are the morphisms of flows $f : X \longrightarrow Y$ such that $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a Serre fibration.

This model structure is cofibrantly generated. The set of generating cofibrations is the set $I^{\mathrm{gl}}_{+} = I^{\mathrm{gl}} \cup \{R : \{0,1\} \longrightarrow \{0\}, C : \varnothing \longrightarrow \{0\}\}$ with

$$I^{\mathrm{gl}} = \{\mathrm{Glob}(\mathbf{S}^{n-1}) \subset \mathrm{Glob}(\mathbf{D}^n), n \geqslant 0\}$$

where $\mathbf{D}^n$ is the n-dimensional disk and $\mathbf{S}^{n-1}$ the $(n-1)$-dimensional sphere. The set of generating trivial cofibrations is

$$J^{\mathrm{gl}} = \{\mathrm{Glob}(\mathbf{D}^n \times \{0\}) \subset \mathrm{Glob}(\mathbf{D}^n \times [0,1]), n \geqslant 0\}.$$

If X is an object of **cell**(**Flow**), then a presentation of the morphism $\varnothing \longrightarrow X$ as a transfinite composition of pushouts of morphisms of I^{gl}_{+} is called a *globular decomposition* of X.

4. Generalized T-homotopy equivalence

We recall here the definition of a T-homotopy equivalence already given in [Gau05c] and [Gau05d].

Definition 4.1. A flow X is loopless if for any $\alpha \in X^0$, the space $\mathbb{P}_{\alpha,\alpha}X$ is empty.

Recall that a flow is a small category without identity morphisms enriched over a category of topological spaces. So the preceding definition is meaningful.

Lemma 4.2. *If a flow X is loopless, then the transitive closure of the set*

$$\{(\alpha,\beta)\in X^0\times X^0 \text{ such that } \mathbb{P}_{\alpha,\beta}X\neq\varnothing\}$$

induces a partial ordering on X^0.

Proof. If (α,β) and (β,α) with $\alpha\neq\beta$ belong to the transitive closure, then there exists a finite sequence $(x_1,\dots,x_\ell)$ of elements of X^0 with $x_1=\alpha$, $x_\ell=\alpha$, $\ell>1$ and for any m, $\mathbb{P}_{x_m,x_{m+1}}X$ is nonempty. Consequently, the space $\mathbb{P}_{\alpha,\alpha}X$ is nonempty because of the existence of the composition law of X: contradiction. $\square$

Definition 4.3. [2] A full directed ball is a flow $\overrightarrow{D}$ such that:

- $\overrightarrow{D}$ is loopless (so by Lemma 4.2, the set $\overrightarrow{D}^0$ is equipped with a partial ordering $\leqslant$).
- $(\overrightarrow{D}^0,\leqslant)$ is finite bounded.
- for all $(\alpha,\beta)\in\overrightarrow{D}^0\times\overrightarrow{D}^0$, the topological space $\mathbb{P}_{\alpha,\beta}\overrightarrow{D}$ is weakly contractible if $\alpha<\beta$, and empty otherwise by definition of $\leqslant$.

Let $\overrightarrow{D}$ be a full directed ball. Then by Lemma 4.2, the set $\overrightarrow{D}^0$ can be viewed as a finite bounded poset. Conversely, if P is a finite bounded poset, let us consider the *flow $F(P)$ associated with P*: it is of course defined as the unique flow $F(P)$ such that $F(P)^0=P$ and $\mathbb{P}_{\alpha,\beta}F(P)=\{u_{\alpha,\beta}\}$ if $\alpha<\beta$ and $\mathbb{P}_{\alpha,\beta}F(P)=\varnothing$ otherwise. Then $F(P)$ is a full directed ball and for any full directed ball $\overrightarrow{D}$, the two flows $\overrightarrow{D}$ and $F(\overrightarrow{D}^0)$ are weakly S-homotopy equivalent.

Let $\overrightarrow{E}$ be another full directed ball. Let $f:\overrightarrow{D}\longrightarrow\overrightarrow{E}$ be a morphism of flows preserving the initial and final states. Then f induces a morphism of posets from $\overrightarrow{D}^0$ to $\overrightarrow{E}^0$ such that $f(\min\overrightarrow{D}^0)=\min\overrightarrow{E}^0$ and $f(\max\overrightarrow{D}^0)=\max\overrightarrow{E}^0$. Hence the following definition:

Definition 4.4. Let $\mathcal{T}$ be the class of morphisms of posets $f:P_1\longrightarrow P_2$ such that:

(1) The posets P_1 and P_2 are finite and bounded.
(2) The morphism of posets $f:P_1\longrightarrow P_2$ is one-to-one; in particular, if x and y are two elements of P_1 with $x<y$, then $f(x)<f(y)$.
(3) One has $f(\min P_1)=\min P_2$ and $f(\max P_1)=\max P_2$.

Then a generalized T-homotopy equivalence is a morphism of

$$\mathbf{cof}(\{Q(F(f)),f\in\mathcal{T}\})$$

where Q is the cofibrant replacement functor of the model category **Flow**.

One can choose a *set* of representatives for each isomorphism class of finite bounded posets. One obtains a *set* of morphisms $\overline{\mathcal{T}}\subset\mathcal{T}$ such that there is the equality of classes

$$\mathbf{cof}(\{Q(F(f)),f\in\overline{\mathcal{T}}\})=\mathbf{cof}(\{Q(F(f)),f\in\mathcal{T}\}).$$

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[2] The statement of the definition is slightly different, but equivalent to the statement given in other parts of this series.

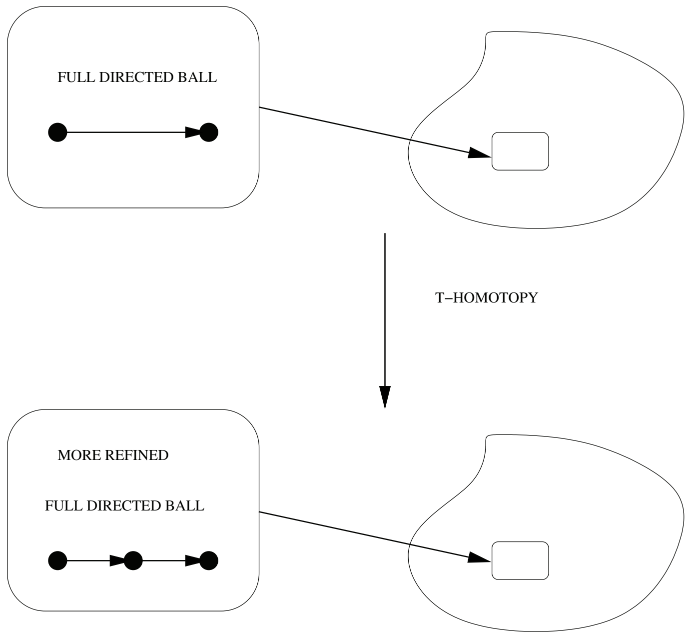

FIGURE 3. Replacement of a full directed ball by a more refined one

By [Gau03] Proposition 11.5, the set of morphisms $\{Q(F(f)), f \in \overline{\mathcal{T}}\}$ permits the small object argument. Thus, the class of morphisms $\mathbf{cof}(\{Q(F(f)), f \in \mathcal{T}\})$ contains exactly the retracts of the morphisms of $\mathbf{cell}(\{Q(F(f)), f \in \mathcal{T}\})$ by [Hov99] Corollary 2.1.15.

The inclusion of posets $\{\widehat{0} < \widehat{1}\} \subset \{\widehat{0} < A < \widehat{1}\}$ corresponds to the case of Figure 2.

A T-homotopy consists in locally replacing in a flow a full directed ball by a more refined one (cf. Figure 3), and in iterating the process transfinitely.

5. Principle of the proof of the main theorem

In this section, we collect the main ideas used in the proof of Theorem 9.3. These ideas are illustrated by the case of the flow $F(P)$ associated with the poset P of Figure 4. More precisely, we will explained the reason for the contractibility of the homotopy branching space $\mathrm{ho}\mathbb{P}^-_{\widehat{0}}\, F(P)$ of the flow $F(P)$ at the initial state $\widehat{0}$.

First of all, we recall the definition of the branching space functor. Roughly speaking, the branching space of a flow is the space of germs of nonconstant execution paths beginning in the same way.

Proposition 5.1 ([Gau05b] Proposition 3.1)**.** *Let X be a flow. There exists a topological space $\mathbb{P}^-X$ unique up to homeomorphism and a continuous map $h^- : \mathbb{P}X \longrightarrow \mathbb{P}^-X$ satisfying the following universal property:*

(1) *For any x and y in $\mathbb{P}X$ such that $t(x) = s(y)$, the equality $h^-(x) = h^-(x*y)$ holds.*

$$\begin{tikzcd}
& A \arrow[r, "\leqslant"] & B \arrow[dr, "\leqslant"] & \\
\widehat{0} \arrow[ur, "\leqslant"] \arrow[dr, "\leqslant"] & & & \widehat{1} \\
& C \arrow[urr, "\leqslant"] & &
\end{tikzcd}$$

FIGURE 4. Example of finite bounded poset

(2) *Let $\phi : \mathbb{P}X \longrightarrow Y$ be a continuous map such that for any x and y of $\mathbb{P}X$ such that $t(x) = s(y)$, the equality $\phi(x) = \phi(x * y)$ holds. Then there exists a unique continuous map $\overline{\phi} : \mathbb{P}^-X \longrightarrow Y$ such that $\phi = \overline{\phi} \circ h^-$.*

Moreover, one has the homeomorphism

$$\mathbb{P}^-X \cong \bigsqcup_{\alpha \in X^0} \mathbb{P}^-_\alpha X$$

where $\mathbb{P}^-_\alpha X := h^- \left(\bigsqcup_{\beta \in X^0} \mathbb{P}^-_{\alpha,\beta} X\right)$. The mapping $X \mapsto \mathbb{P}^-X$ yields a functor $\mathbb{P}^-$ from **Flow** *to* **Top**.

Definition 5.2. Let X be a flow. The topological space $\mathbb{P}^-X$ is called the branching space of the flow X. The functor $\mathbb{P}^-$ is called the branching space functor.

Theorem 5.3 ([Gau05b] Theorem 5.5)**.** *The branching space functor*

$$\mathbb{P}^- : \mathbf{Flow} \longrightarrow \mathbf{Top}$$

is a left Quillen functor.

Definition 5.4. The homotopy branching space $\mathrm{ho}\mathbb{P}^- X$ of a flow X is by definition the topological space $\mathbb{P}^-Q(X)$. For $\alpha \in X^0$, let $\mathrm{ho}\mathbb{P}^-_\alpha X = \mathbb{P}^-_\alpha Q(X)$.

The first idea would be to replace the calculation of $\mathbb{P}^-_{\widehat{0}} Q(F(P))$ by the calculation of $\mathbb{P}^-_{\widehat{0}} F(P)$ because there exists a natural weak S-homotopy equivalence $Q(F(P)) \longrightarrow F(P)$. However, the flow $F(P)$ is not cofibrant because its composition law contains relations, for instance $u_{\widehat{0},A} * u_{A,\widehat{1}} = u_{\widehat{0},C} * u_{C,\widehat{1}}$. In any cofibrant replacement of $F(P)$, a relation like $u_{\widehat{0},A} * u_{A,\widehat{1}} = u_{\widehat{0},C} * u_{C,\widehat{1}}$ is always replaced by a S-homotopy between $u_{\widehat{0},A} * u_{A,\widehat{1}}$ and $u_{\widehat{0},C} * u_{C,\widehat{1}}$. Moreover, it is known from [Gau05b] Theorem 4.1 that the branching space functor does not necessarily send a weak S-homotopy equivalence of flows to a weak homotopy equivalence of topological spaces. So this first idea fails, or at least it cannot work directly.

Let $X = Q(F(P))$ be the cofibrant replacement of $F(P)$. Another idea that we did not manage to work out can be presented as follows. Every nonconstant execution path γ of $\mathbb{P}X$ such that $s(\gamma) = \widehat{0}$ is in the same equivalence class as an execution path of $\mathbb{P}_{\widehat{0},\widehat{1}}X$ since the state $\widehat{1}$ is the only final state of X. Therefore, the topological space $\mathrm{ho}\mathbb{P}^-_{\widehat{0}} F(P) = \mathbb{P}^-_{\widehat{0}} X$ is a quotient of the contractible cofibrant space $\mathbb{P}_{\widehat{0},\widehat{1}}X$. However, the quotient of a contractible space is not necessarily contractible. For example, identifying in the 1-dimensional disk $\mathbf{D}^1$ the points -1 and $+1$ gives the 1-dimensional sphere $\mathbf{S}^1$.

The principle of the proof given in this paper consists in finding a diagram of topological spaces $\mathcal{P}^-_{\widehat{0}}(X)$ satisfying the following properties:

(1) There is an isomorphism of topological spaces
$$\mathbb{P}^-_{\widehat{0}}X \cong \varinjlim \mathcal{P}^-_{\widehat{0}}(X).$$

(2) There is a weak homotopy equivalence of topological spaces
$$\varinjlim \mathcal{P}^-_{\widehat{0}}(X) \simeq \underrightarrow{\operatorname{holim}}\, \mathcal{P}^-_{\widehat{0}}(X)$$
because the diagram of topological spaces $\mathcal{P}^-_{\widehat{0}}(X)$ is cofibrant for an appropriate model structure and because for this model structure, the colimit functor is a left Quillen functor.

(3) Each vertex of the diagram of topological spaces $\mathcal{P}^-_{\widehat{0}}(X)$ is contractible. Hence, its homotopy colimit is weakly homotopy equivalent to the classifying space of the underlying category of $\mathcal{P}^-_{\widehat{0}}(X)$.

(4) The underlying category of the diagram $\mathcal{P}^-_{\widehat{0}}(X)$ is contractible.

To prove the second assertion, we will build a Reedy structure on the underlying category of the diagram $\mathcal{P}^-_{\widehat{0}}(X)$. The main ingredient (but not the only one) of this construction will be that for every triple $(\alpha, \beta, \gamma) \in X^0 \times X^0 \times X^0$, the continuous map $\mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\beta,\gamma}X \longrightarrow \mathbb{P}_{\alpha,\gamma}X$ induced by the composition law of X is a cofibration of topological spaces since X is cofibrant.

The underlying category of the diagram of topological spaces $\mathcal{P}^-_{\widehat{0}}(X)$ will be the opposite category $\Delta(P\backslash\{\widehat{0}\})^{\mathrm{op}}$ of the order complex of the poset $P\backslash\{\widehat{0}\}$. The latter looks as follows (it is the opposite category of the category generated by the inclusions, therefore all diagrams are commutative):

$$\begin{tikzcd}
 & & (A,B,\widehat{1}) \arrow[dl] \arrow[d] \arrow[dr, dashed] & \\
(C,\widehat{1}) \arrow[d, dashed] \arrow[dr] & (A,\widehat{1}) \arrow[d] \arrow[dr, dashed] & (B,\widehat{1}) \arrow[dl] \arrow[dr, dashed] & (A,B) \arrow[dl, dashed] \arrow[d] \\
(C) & (\widehat{1}) & (A) & (B)
\end{tikzcd}$$

The diagram $\mathcal{P}^-_{\widehat{0}}(X)$ is then defined as follows:

- $\mathcal{P}^-_{\widehat{0}}(X)(A,B,\widehat{1}) = \mathbb{P}_{\widehat{0},A}X \times \mathbb{P}_{A,B}X \times \mathbb{P}_{B,\widehat{1}}X$.
- $\mathcal{P}^-_{\widehat{0}}(X)(A) = \mathbb{P}_{\widehat{0},A}X$.
- $\mathcal{P}^-_{\widehat{0}}(X)(B) = \mathbb{P}_{\widehat{0},B}X$.
- $\mathcal{P}^-_{\widehat{0}}(X)(C) = \mathbb{P}_{\widehat{0},C}X$.
- $\mathcal{P}^-_{\widehat{0}}(X)(\widehat{1}) = \mathbb{P}_{\widehat{0},\widehat{1}}X$.
- $\mathcal{P}^-_{\widehat{0}}(X)(A,B) = \mathbb{P}_{\widehat{0},A}X \times \mathbb{P}_{A,B}X$.
- $\mathcal{P}^-_{\widehat{0}}(X)(B,\widehat{1}) = \mathbb{P}_{\widehat{0},B}X \times \mathbb{P}_{B,\widehat{1}}X$.

- $\mathcal{P}^-_{\widehat{0}}(X)(A,\widehat{1}) = \mathbb{P}_{\widehat{0},A}X \times \mathbb{P}_{A,\widehat{1}}X$.
- $\mathcal{P}^-_{\widehat{0}}(X)(C,\widehat{1}) = \mathbb{P}_{\widehat{0},C}X \times \mathbb{P}_{C,\widehat{1}}X$.
- The morphisms $\dashrightarrow$ are induced by the projection.
- The morphisms $\longrightarrow$ are induced by the composition law.

Note that the restriction $\mathbb{P}^-_{\widehat{0}}(X) \restriction_{p^-_{\widehat{0}}(X)}$ of the diagram of topological spaces $\mathcal{P}^-_{\widehat{0}}(X)$ to the small category $p^-_{\widehat{0}}(X) \subset \Delta(P\backslash\{\widehat{0}\})^{\mathrm{op}}$

$$\begin{tikzcd}
(C,\widehat{1}) \arrow[d, dashed] \arrow[dr] & (A,\widehat{1}) \arrow[d] \arrow[dr, dashed] & (B,\widehat{1}) \arrow[dl] \arrow[dr, dashed] & (A,B) \arrow[dl, dashed] \arrow[d] \\
(C) & (\widehat{1}) & (A) & (B)
\end{tikzcd}$$

has the same colimit, that is $\mathbb{P}^-_{\widehat{0}}(X)$, since the category $p^-_{\widehat{0}}(X)$ is a final subcategory of $\Delta(P\backslash\{\widehat{0}\})^{\mathrm{op}}$. However, the latter restriction cannot be Reedy cofibrant because of the associativity of the composition law. Indeed, the continuous map

$$\mathbb{P}_{\widehat{0},A}X \times \mathbb{P}_{A,\widehat{1}}X \sqcup \mathbb{P}_{\widehat{0},B}X \times \mathbb{P}_{B,\widehat{1}}X \longrightarrow \mathbb{P}^-_{\widehat{0}}(X) \restriction_{p^-_{\widehat{0}}(X)} (\widehat{1}) = \mathbb{P}_{\widehat{0},\widehat{1}}X$$

induced by the composition law of X is not even a monomorphism: if $(u,v,w) \in \mathbb{P}_{\widehat{0},A}X \times \mathbb{P}_{A,B}X \times \mathbb{P}_{B,\widehat{1}}X$, then $u * v * w = (u * v) * w \in \mathbb{P}_{\widehat{0},B}X \times \mathbb{P}_{B,\widehat{1}}X$ and $u*v*w = u*(v*w) \in \mathbb{P}_{\widehat{0},A}X \times \mathbb{P}_{A,\widehat{1}}X$. So the second assertion of the main argument cannot be true. Moreover, the classifying space of $p^-_{\widehat{0}}(X)$ is not contractible: it is homotopy equivalent to the circle $\mathbf{S}^1$. So the fourth assertion of the main argument cannot be applied either.

On the other hand, the continuous map

$$(\mathbb{P}_{\widehat{0},A}X \times \mathbb{P}_{A,\widehat{1}}X) \sqcup_{(\mathbb{P}_{\widehat{0},A}X \times \mathbb{P}_{A,B}X \times \mathbb{P}_{B,\widehat{1}}X)} (\mathbb{P}_{\widehat{0},B}X \times \mathbb{P}_{B,\widehat{1}}X) \longrightarrow \mathbb{P}^-_{\widehat{0}}(X)(\widehat{1}) = \mathbb{P}_{\widehat{0},\widehat{1}}X$$

is a cofibration of topological spaces and the classifying space of the order complex of the poset $P\backslash\{\widehat{0}\}$ is contractible since the poset $P\backslash\{\widehat{0}\} =]\widehat{0},\widehat{1}]$ has a unique top element $\widehat{1}$ [Qui78].

6. Calculating the branching space of a loopless flow

Theorem 6.1. *Let X be a loopless flow. Let $\alpha \in X^0$. There exists one and only one functor*

$$\mathcal{P}^-_{\alpha}(X) : \Delta(X^0_{>\alpha})^{\mathrm{op}} \longrightarrow \mathbf{Top}$$

satisfying the following conditions:

1. $\mathcal{P}^-_{\alpha}(X)_{(\alpha_0,\dots,\alpha_p)} := \mathbb{P}_{\alpha,\alpha_0}X \times \mathbb{P}_{\alpha_0,\alpha_1}X \times \dots \times \mathbb{P}_{\alpha_{p-1},\alpha_p}X$.
2. *The morphism $\partial_i : \mathcal{P}^-_{\alpha}(X)_{(\alpha_0,\dots,\alpha_p)} \longrightarrow \mathcal{P}^-_{\alpha}(X)_{(\alpha_0,\dots,\widehat{\alpha_i},\dots,\alpha_p)}$ for $0 < i < p$ is induced by the composition law of X, more precisely by the morphism*
$$\mathbb{P}_{\alpha_{i-1},\alpha_i}X \times \mathbb{P}_{\alpha_i,\alpha_{i+1}}X \longrightarrow \mathbb{P}_{\alpha_{i-1},\alpha_{i+1}}X.$$
3. *The morphism $\partial_0 : \mathcal{P}^-_{\alpha}(X)_{(\alpha_0,\dots,\alpha_p)} \longrightarrow \mathcal{P}^-_{\alpha}(X)_{(\widehat{\alpha_0},\dots,\alpha_i,\dots,\alpha_p)}$ is induced by the composition law of X, more precisely by the morphism*
$$\mathbb{P}_{\alpha,\alpha_0}X \times \mathbb{P}_{\alpha_0,\alpha_1}X \longrightarrow \mathbb{P}_{\alpha,\alpha_1}X.$$

(4) *The morphism* $\partial_p : \mathcal{P}^-_\alpha(X)_{(\alpha_0,\dots,\alpha_p)} \longrightarrow \mathcal{P}^-_\alpha(X)_{(\alpha_0,\dots,\alpha_{p-1},\widehat{\alpha_p})}$ *is the projection map obtained by removing the component* $\mathbb{P}_{\alpha_{p-1},\alpha_p}X$.

Proof. The uniqueness on objects is exactly the first assertion. The uniqueness on morphisms comes from the fact that every morphism of $\Delta(X^0_{>\alpha})^{\mathrm{op}}$ is a composite of ∂_i. We have to prove existence.

The diagram of topological spaces

$$\begin{tikzcd}
\mathcal{P}^-_\alpha(X)_{(\alpha_0,\dots,\alpha_p)} \arrow[r, "\partial_i"] \arrow[d, "\partial_j"'] & \mathcal{P}^-_\alpha(X)_{(\alpha_0,\dots,\widehat{\alpha_i},\dots,\alpha_p)} \arrow[d, "\partial_{j-1}"] \\
\mathcal{P}^-_\alpha(X)_{(\alpha_0,\dots,\widehat{\alpha_j},\dots,\alpha_p)} \arrow[r, "\partial_i"] & \mathcal{P}^-_\alpha(X)_{(\alpha_0,\dots,\widehat{\alpha_i},\dots,\widehat{\alpha_j},\dots,\alpha_p)}
\end{tikzcd}$$

is commutative for any $0 < i < j < p$ and any $p \geqslant 2$. Indeed, if $i < j-1$, then one has

$$\boxed{\partial_i\partial_j(\gamma_0,\dots,\gamma_p) = \partial_{j-1}\partial_i(\gamma_0,\dots,\gamma_p) = (\gamma_0,\dots,\gamma_i\gamma_{i+1},\dots,\gamma_j\gamma_{j+1},\dots,\gamma_p)}$$

and if $i = j-1$, then one has

$$\boxed{\partial_i\partial_j(\gamma_0,\dots,\gamma_p) = \partial_{j-1}\partial_i(\gamma_0,\dots,\gamma_p) = (\gamma_0,\dots,\gamma_{j-1}\gamma_j\gamma_{j+1},\dots,\gamma_p)}$$

because of the associativity of the composition law of X. (This is the only place in this proof where this axiom is required.)

The diagram of topological spaces

$$\begin{tikzcd}
\mathcal{P}^-_\alpha(X)_{(\alpha_0,\dots,\alpha_p)} \arrow[r, "\partial_i"] \arrow[d, "\partial_p"'] & \mathcal{P}^-_\alpha(X)_{(\alpha_0,\dots,\widehat{\alpha_i},\dots,\alpha_p)} \arrow[d, "\partial_{p-1}"] \\
\mathcal{P}^-_\alpha(X)_{(\alpha_0,\dots,\alpha_{p-1})} \arrow[r, "\partial_i"] & \mathcal{P}^-_\alpha(X)_{(\alpha_0,\dots,\widehat{\alpha_i},\dots,\alpha_{p-1})}
\end{tikzcd}$$

is commutative for any $0 < i < p-1$ and any $p > 2$. Indeed, one has

$$\boxed{\partial_i\partial_p(\gamma_0,\dots,\gamma_p) = \partial_{p-1}\partial_i(\gamma_0,\dots,\gamma_p) = (\gamma_0,\dots,\gamma_i\gamma_{i+1},\dots,\gamma_{p-1})}.$$

Finally, the diagram of topological spaces

$$\begin{tikzcd}
\mathcal{P}^-_\alpha(X)_{(\alpha_0,\dots,\alpha_p)} \arrow[r, "\partial_{p-1}"] \arrow[d, "\partial_p"'] & \mathcal{P}^-_\alpha(X)_{(\alpha_0,\dots,\widehat{\alpha_{p-1}},\alpha_p)} \arrow[d, "\partial_{p-1}"] \\
\mathcal{P}^-_\alpha(X)_{(\alpha_0,\dots,\alpha_{p-1})} \arrow[r, "\partial_{p-1}"] & \mathcal{P}^-_\alpha(X)_{(\alpha_0,\dots,\alpha_{p-2})}
\end{tikzcd}$$

is commutative for any $p \geqslant 2$. Indeed, one has

$$\boxed{\partial_{p-1}\partial_p(\gamma_0,\dots,\gamma_p) = (\gamma_0,\dots,\gamma_{p-2})}$$

and

$$\boxed{\partial_{p-1}\partial_{p-1}(\gamma_0,\dots,\gamma_p) = \partial_{p-1}(\gamma_0,\dots,\gamma_{p-2},\gamma_{p-1}\gamma_p) = (\gamma_0,\dots,\gamma_{p-2})}.$$

In other words, the ∂_i maps satisfy the simplicial identities. Hence the result. $\square$

The following theorem is used in the proofs of Theorem 6.3 and Theorem 8.3.

Theorem 6.2 ([ML98] Theorem 1, p. 213). *Let $L : J' \longrightarrow J$ be a final functor between small categories, i.e., such that for any $k \in J$, the comma category $(k{\downarrow}L)$ is nonempty and connected. Let $F : J \longrightarrow \mathcal{C}$ be a functor from J to a cocomplete category $\mathcal{C}$. Then L induces a canonical morphism $\varinjlim(F \circ L) \longrightarrow \varinjlim F$ which is an isomorphism.*

Theorem 6.3. *Let X be a loopless flow. Then there exists an isomorphism of topological spaces $\mathbb{P}^-_\alpha X \cong \varinjlim \mathcal{P}^-_\alpha(X)$ for any $\alpha \in X^0$.*

Proof. Let $p^-_\alpha(X)$ be the full subcategory of $\Delta(X^0_{>\alpha})^{\mathrm{op}}$ generated by the arrows $\partial_0 : (\alpha_0, \alpha_1) \longrightarrow (\alpha_1)$ and $\partial_1 : (\alpha_0, \alpha_1) \longrightarrow (\alpha_0)$.

Let $k = (k_0, \dots, k_q)$ be an object of $\Delta(X^0_{>\alpha})^{\mathrm{op}}$. Then $k \to (k_0)$ is an object of the comma category $(k{\downarrow}p^-_\alpha(X))$. So the latter category is not empty. Let $k \to (x_0)$ and $k \to (y_0)$ be distinct elements of $(k{\downarrow}p^-_\alpha(X))$. The pair $\{x_0, y_0\}$ is therefore a subset of $\{k_0, \dots, k_q\}$. So either $x_0 < y_0$ or $y_0 < x_0$. Without loss of generality, one can suppose that $x_0 < y_0$. Then one has the commutative diagram

$$\begin{tikzcd} k \arrow[r, equal] \arrow[d] & k \arrow[r, equal] \arrow[d] & k \arrow[d] \\ (x_0) & (x_0, y_0) \arrow[l, "\partial_1"'] \arrow[r, "\partial_0"] & (y_0). \end{tikzcd}$$

Therefore, the objects $k \to (x_0)$ and $k \to (y_0)$ are in the same connected component of $(k{\downarrow}p^-_\alpha(X))$. Let $k \to (x_0)$ and $k \to (y_0, y_1)$ be distinct elements of $(k{\downarrow}p^-_\alpha(X))$. Then $k \to (x_0)$ is in the same connected component as $k \to (y_0)$ by the previous calculation. Moreover, one has the commutative diagram

$$\begin{tikzcd} k \arrow[r, equal] \arrow[d] & k \arrow[d] \\ (y_0) & (y_0, y_1). \arrow[l, "\partial_1"'] \end{tikzcd}$$

Thus, the objects $k \to (x_0)$ and $k \to (y_0, y_1)$ are in the same connected component of $(k{\downarrow}p^-_\alpha(X))$. So the comma category $(k{\downarrow}p^-_\alpha(X))$ is connected and nonempty. Thus for any functor $F : \Delta(X^0_{>\alpha})^{\mathrm{op}} \longrightarrow \mathbf{Top}$, the inclusion functor $i : p^-_\alpha(X) \longrightarrow \Delta(X^0_{>\alpha})^{\mathrm{op}}$ induces an isomorphism of topological spaces $\varinjlim(F \circ i) \longrightarrow \varinjlim F$ by Theorem 6.2.

Let $\widehat{p^-_\alpha}(X)$ be the full subcategory of $p^-_\alpha(X)$ consisting of the objects (α_0). The category $\widehat{p^-_\alpha}(X)$ is discrete because it does not contain any nonidentity morphism. Let $j : \widehat{p^-_\alpha}(X) \longrightarrow p^-_\alpha(X)$ be the canonical inclusion functor. It induces a canonical continuous map $\varinjlim(F \circ j) \longrightarrow \varinjlim(F \circ i)$ for any functor $F : \Delta(X^0_{>\alpha})^{\mathrm{op}} \longrightarrow \mathbf{Top}$.

For $F = \mathcal{P}^-_\alpha(X)$, one obtains the diagram of topological spaces

$$\varinjlim(\mathcal{P}^-_\alpha(X) \circ j) \longrightarrow \varinjlim(\mathcal{P}^-_\alpha(X) \circ i) \cong \varinjlim \mathcal{P}^-_\alpha(X).$$

It is clear that $\varinjlim(\mathcal{P}^-_\alpha(X) \circ j) \cong \bigsqcup_{\alpha_0} \mathbb{P}_{\alpha,\alpha_0} X$. Let $g : \varinjlim(\mathcal{P}^-_\alpha(X) \circ j) \longrightarrow Z$ be a continuous map such that $g(x * y) = g(x)$ for any x and any y such that $t(x) = s(y)$.

So there exists a commutative diagram

$$\begin{tikzcd}
\mathbb{P}_{s(x),t(x)}X \times \mathbb{P}_{t(x),t(y)}X \arrow[r, "\partial_0"] \arrow[d, "\partial_1"'] \arrow[dr] & \mathbb{P}_{s(x),t(y)}X \arrow[d, "g"] \\
\mathbb{P}_{s(x),t(x)}X \arrow[r, "g"] & Z
\end{tikzcd}$$

for any x and y as above. Therefore, the topological space $\varinjlim(\mathcal{P}^-_\alpha(X)\circ i)$ satisfies the same universal property as the topological space $\mathbb{P}^-_\alpha X$ (cf. Proposition 5.1). □

7. Reedy structure and homotopy colimit

Lemma 7.1. *Let X be a loopless flow such that $(X^0,\leqslant)$ is locally finite. If (α,β) is a 1-simplex of $\Delta(X^0)$ and if $(\alpha_0,\dots,\alpha_p)$ is a p-simplex of $\Delta(X^0)$ with $\alpha_0=\alpha$ and $\alpha_p=\beta$, then p is at most the cardinal* $\mathrm{card}(]\alpha,\beta])$ *of* $]\alpha,\beta]$.

Proof. If $(\alpha_0,\dots,\alpha_p)$ is a p-simplex of $\Delta(X^0)$, then one has $\alpha_0<\dots<\alpha_p$ by definition of the order complex. So one has the inclusion $\{\alpha_1,\dots,\alpha_p\}\subset]\alpha,\beta]$, and therefore $p\leqslant \mathrm{card}(]\alpha,\beta])$. □

The following choice of notation is therefore meaningful.

Notation 7.2. Let X be a loopless flow such that $(X^0,\leqslant)$ is locally finite. Let (α,β) be a 1-simplex of $\Delta(X^0)$. We denote by $\ell(\alpha,\beta)$ the maximum of the set of integers

$$\big\{p\geqslant 1, \exists(\alpha_0,\dots,\alpha_p)\ p\text{-simplex of }\Delta(X^0)\text{ s.t. }(\alpha_0,\alpha_p)=(\alpha,\beta)\big\}$$

One always has $1\leqslant \ell(\alpha,\beta)\leqslant \mathrm{card}(]\alpha,\beta])$.

Lemma 7.3. *Let X be a loopless flow such that $(X^0,\leqslant)$ is locally finite. Let (α,β,γ) be a 2-simplex of $\Delta(X^0)$. Then one has*

$$\ell(\alpha,\beta)+\ell(\beta,\gamma)\leqslant \ell(\alpha,\gamma).$$

Proof. Let $\alpha=\alpha_0<\dots<\alpha_{\ell(\alpha,\beta)}=\beta$. Let $\beta=\beta_0<\dots<\beta_{\ell(\beta,\gamma)}=\gamma$. Then

$$(\alpha_0,\dots,\alpha_{\ell(\alpha,\beta)},\beta_1,\dots,\beta_{\ell(\beta,\gamma)})$$

is a simplex of $\Delta(X^0)$ with $\alpha=\alpha_0$ and $\beta_{\ell(\beta,\gamma)}=\gamma$. So $\ell(\alpha,\beta)+\ell(\beta,\gamma)\leqslant\ell(\alpha,\gamma)$. □

Proposition 7.4. *Let X be a loopless flow such that $(X^0,\leqslant)$ is locally finite. Let $\alpha\in X^0$. Let $\Delta(X^0_{>\alpha})^{\mathrm{op}}_+$ be the subcategory of $\Delta(X^0_{>\alpha})^{\mathrm{op}}$ generated by the*

$$\partial_i:(\alpha_0,\dots,\alpha_p)\longrightarrow(\alpha_0,\dots,\widehat{\alpha_i},\dots,\alpha_p)$$

for any $p\geqslant 1$ and $0\leqslant i<p$. Let $\Delta(X^0_{>\alpha})^{\mathrm{op}}_-$ be the subcategory of $\Delta(X^0_{>\alpha})^{\mathrm{op}}$ generated by the

$$\partial_p:(\alpha_0,\dots,\alpha_p)\longrightarrow(\alpha_0,\dots,\alpha_{p-1})$$

for any $p\geqslant 1$. If $(\alpha_0,\dots,\alpha_p)$ is an object of $\Delta(X^0_{>\alpha})^{\mathrm{op}}$, let:

$$d(\alpha_0,\dots,\alpha_p)=\ell(\alpha,\alpha_0)^2+\ell(\alpha_0,\alpha_1)^2+\dots+\ell(\alpha_{p-1},\alpha_p)^2.$$

Then the triple $(\Delta(X^0_{>\alpha})^{\mathrm{op}},\Delta(X^0_{>\alpha})^{\mathrm{op}}_+,\Delta(X^0_{>\alpha})^{\mathrm{op}}_-)$ together with the degree function d is a Reedy category.

Note that the subcategory $\Delta(X^0_{>\alpha})^{\mathrm{op}}_+$ is precisely generated by the morphisms sent by the functor $\mathcal{P}^-_\alpha(X)$ (Theorem 6.1) to continuous maps induced by the composition law of the flow X. And note that the subcategory $\Delta(X^0_{>\alpha})^{\mathrm{op}}_-$ is precisely generated by the morphisms sent by the functor $\mathcal{P}^-_\alpha(X)$ (Theorem 6.1) to continuous maps induced by the projection obtained by removing the last component on the right.

Proof. Let $\partial_i : (\alpha_0, \dots, \alpha_p) \longrightarrow (\alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_p)$ be a morphism of $\Delta(X^0_{>\alpha})^{\mathrm{op}}_+$ with $p \geqslant 1$ and $0 \leqslant i < p$. Then (with the convention $\alpha_{-1} = \alpha$)

$$d(\alpha_0, \dots, \alpha_p) = \ell(\alpha, \alpha_0)^2 + \cdots + \ell(\alpha_{p-1}, \alpha_p)^2$$
$$d(\alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_p) = \ell(\alpha, \alpha_0)^2 + \cdots + \ell(\alpha_{i-1}, \alpha_{i+1})^2 + \cdots + \ell(\alpha_{p-1}, \alpha_p)^2.$$

So one obtains

$$d(\alpha_0, \dots, \alpha_p) - d(\alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_p) = \ell(\alpha_{i-1}, \alpha_i)^2 + \ell(\alpha_i, \alpha_{i+1})^2 - \ell(\alpha_{i-1}, \alpha_{i+1})^2.$$

By Lemma 7.3, one has

$$(\ell(\alpha_{i-1}, \alpha_i) + \ell(\alpha_i, \alpha_{i+1}))^2 \leqslant \ell(\alpha_{i-1}, \alpha_{i+1})^2$$

and one has

$$\ell(\alpha_{i-1}, \alpha_i)^2 + \ell(\alpha_i, \alpha_{i+1})^2 < (\ell(\alpha_{i-1}, \alpha_i) + \ell(\alpha_i, \alpha_{i+1}))^2$$

since $2\ell(\alpha_{i-1}, \alpha_i)\ell(\alpha_i, \alpha_{i+1}) \geqslant 2$. So every morphism of $\Delta(X^0_{>\alpha})^{\mathrm{op}}_+$ raises degree.

Let $\partial_p : (\alpha_0, \dots, \alpha_p) \longrightarrow (\alpha_0, \dots, \alpha_{p-1})$ with $p \geqslant 1$ be a morphism of $\Delta(X^0_{>\alpha})^{\mathrm{op}}_-$. Then one has

$$d(\alpha_0, \dots, \alpha_p) = \ell(\alpha, \alpha_0)^2 + \cdots + \ell(\alpha_{p-2}, \alpha_{p-1})^2 + \ell(\alpha_{p-1}, \alpha_p)^2$$
$$d(\alpha_0, \dots, \alpha_{p-1}) = \ell(\alpha, \alpha_0)^2 + \cdots + \ell(\alpha_{p-2}, \alpha_{p-1})^2.$$

So $d(\alpha_0, \dots, \alpha_p) - d(\alpha_0, \dots, \alpha_{p-1}) = \ell(\alpha_{p-1}, \alpha_p)^2 > 0$. Thus, every morphism of $\Delta(X^0_{>\alpha})^{\mathrm{op}}_-$ lowers degree.

Let $f : (\alpha_0, \dots, \alpha_p) \longrightarrow (\beta_0, \dots, \beta_q)$ be a morphism of $\Delta(X^0_{>\alpha})^{\mathrm{op}}$. Then one has $p \geqslant q$ and $(\beta_0, \dots, \beta_q) = (\alpha_{\sigma(0)}, \dots, \alpha_{\sigma(q)})$ where $\sigma : \{0, \dots, q\} \longrightarrow \{0, \dots, p\}$ is a strictly increasing set map. Then f can be written as a composite

$$(\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_p) \longrightarrow (\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{\sigma(q)}) \xrightarrow{\partial_{j_1}\partial_{j_2}\dots\partial_{j_r}} (\beta_1, \dots, \beta_q)$$

where $0 \leqslant j_1 < j_2 < \cdots < j_r < \sigma(q)$ and $\{j_1, j_2, \dots, j_r\} \cup \{\sigma(1), \sigma(2), \dots, \sigma(q)\} = \{0, 1, 2, \dots, \sigma(q)\}$ and this is the unique way of decomposing f as a morphism of $\Delta(X^0_{>\alpha})^{\mathrm{op}}_-$ followed by a morphism of $\Delta(X^0_{>\alpha})^{\mathrm{op}}_+$. $\square$

Theorem 7.5. *Let X be a loopless flow such that $(X^0, \leqslant)$ is locally finite. Let $\alpha \in X^0$. Then the colimit functor*

$$\varinjlim : \mathbf{Top}^{\Delta(X^0_{>\alpha})^{\mathrm{op}}} \longrightarrow \mathbf{Top}$$

is a left Quillen functor if the category of diagrams $\mathbf{Top}^{\Delta(X^0_{>\alpha})^{\mathrm{op}}}$ is equipped with the Reedy model structure.

Proof. The Reedy structure on $\Delta(X^0_{>\alpha})^{\mathrm{op}}$ provides a model structure on the category $\mathbf{Top}^{\Delta(X^0_{>\alpha})^{\mathrm{op}}}$ of diagrams of topological spaces over $\Delta(X^0_{>\alpha})^{\mathrm{op}}$ such that if $f : D \longrightarrow E$ is a morphism of diagrams, then:

(1) f is a weak equivalence if and only if for all objects $\underline{\alpha}$ of $\Delta(X^0_{>\alpha})^{\mathrm{op}}$, the morphism $D_{\underline{\alpha}} \longrightarrow E_{\underline{\alpha}}$ is a weak homotopy equivalence of **Top**, i.e., f is an objectwise weak homotopy equivalence.
(2) f is a cofibration if and only if for all objects $\underline{\alpha}$ of $\Delta(X^0_{>\alpha})^{\mathrm{op}}$, the morphism $D_{\underline{\alpha}} \sqcup_{L_{\underline{\alpha}}D} L_{\underline{\alpha}}E \longrightarrow E_{\underline{\alpha}}$ is a cofibration of **Top**.
(3) f is a fibration if and only if for all objects $\underline{\alpha}$ of $\Delta(X^0_{>\alpha})^{\mathrm{op}}$, the morphism $D_{\underline{\alpha}} \longrightarrow E_{\underline{\alpha}} \times_{M_{\underline{\alpha}}E} M_{\underline{\alpha}}D$ is a fibration of **Top**.

Consider the categorical adjunction $\varinjlim : \mathbf{Top}^{\Delta(X^0_{>\alpha})^{\mathrm{op}}} \leftrightarrows \mathbf{Top} : \mathrm{Diag}$ where Diag is the diagonal functor. Let $p : E \longrightarrow B$ be a continuous map. If $\underline{\alpha} = (\alpha_0)$, then the matching category of $\underline{\alpha}$ is the empty category. So

$$\mathrm{Diag}(B)_{\underline{\alpha}} \times_{M_{\underline{\alpha}}\mathrm{Diag}(B)} M_{\underline{\alpha}}\mathrm{Diag}(E) \cong \mathrm{Diag}(B)_{\underline{\alpha}} \cong B$$

since the spaces $M_{\underline{\alpha}}\mathrm{Diag}(B)$ and $M_{\underline{\alpha}}\mathrm{Diag}(E)$ are both equal to singletons. If $\underline{\alpha} = (\alpha_0, \dots, \alpha_p)$ with $p \geqslant 1$, then the matching category of $\underline{\alpha}$ looks like the following tower:

$$(\alpha_0, \dots, \alpha_{p-1}) \longrightarrow (\alpha_0, \dots, \alpha_{p-2}) \longrightarrow \cdots \longrightarrow (\alpha_0).$$

Therefore in that case, one has the isomorphisms

$$M_{\underline{\alpha}}\mathrm{Diag}(B) \cong \mathrm{Diag}(B)_{(\alpha_0,\dots,\alpha_{p-1})} \cong B$$

and

$$M_{\underline{\alpha}}\mathrm{Diag}(E) \cong \mathrm{Diag}(E)_{(\alpha_0,\dots,\alpha_{p-1})} \cong E.$$

Hence $\mathrm{Diag}(B)_{\underline{\alpha}} \times_{M_{\underline{\alpha}}\mathrm{Diag}(B)} M_{\underline{\alpha}}\mathrm{Diag}(E) \cong E$. Thus, the continuous map

$$E \cong \mathrm{Diag}(E)_{\underline{\alpha}} \longrightarrow \mathrm{Diag}(B)_{\underline{\alpha}} \times_{M_{\underline{\alpha}}\mathrm{Diag}(B)} M_{\underline{\alpha}}\mathrm{Diag}(E)$$

is either the identity of E, or p. So if p is a fibration (resp. a trivial fibration), then $\mathrm{Diag}(p) : \mathrm{Diag}(E) \longrightarrow \mathrm{Diag}(B)$ is a Reedy fibration (resp. a Reedy trivial fibration). One then deduces that Diag is a right Quillen functor and that the colimit functor is a left Quillen functor. □

8. Homotopy branching space of a full directed ball

Notation 8.1. Let X be a loopless flow. Let $\underline{\alpha} = (\alpha_0, \dots, \alpha_p)$ and $\underline{\beta} = (\beta_0, \dots, \beta_q)$ be two simplices of the order complex $\Delta(X^0)$ of the poset X^0. Then we define:

(1) $\underline{\beta} \supseteqq \underline{\alpha}$ if $\alpha_0 = \beta_0$, $\alpha_p = \beta_q$ and $\{\alpha_0, \dots, \alpha_p\} \subset \{\beta_0, \dots, \beta_q\}$.
(2) $\underline{\beta} \supsetneqq \underline{\alpha}$ if $\alpha_0 = \beta_0$, $\alpha_p = \beta_q$ and $\{\alpha_0, \dots, \alpha_p\} \subsetneqq \{\beta_0, \dots, \beta_q\}$.

Notation 8.2. Let X be a loopless flow. Let $\underline{\alpha} = (\alpha_0, \dots, \alpha_p)$ be a simplex of the order complex $\Delta(X^0)$ of the poset X^0. Let $\alpha < \alpha_0$. Then the notation $\alpha.\underline{\alpha}$ represents the simplex $(\alpha, \alpha_0, \dots, \alpha_p)$ of $\Delta(X^0)$.

Theorem 8.3. *Let X be a loopless flow such that $(X^0, \leqslant)$ is locally finite. Let $\alpha \in X^0$. Then:*

(1) *For any object $\underline{\alpha}$ of $\Delta(X^0_{>\alpha})^{\mathrm{op}}$, one has the isomorphism of topological spaces*

$$L_{\underline{\alpha}}\mathcal{P}^-_\alpha(X) \cong \varinjlim_{\alpha.\underline{\beta} \supsetneqq \alpha.\underline{\alpha}} \mathcal{P}^-_\alpha(X)_{\underline{\beta}}.$$

(2) *For any object* $\underline{\alpha} = (\alpha_0, \dots, \alpha_p)$*, the canonical continuous map*

$$i_{\alpha.\underline{\alpha}} : L_{\underline{\alpha}}\mathcal{P}^-_\alpha(X) \longrightarrow \mathcal{P}^-_\alpha(X)_{\underline{\alpha}}$$

is equal to the pushout product (*cf. Notation* B.2) *of the canonical continuous maps*

$$i_{(\alpha,\alpha_0)} \square i_{(\alpha_0,\alpha_1)} \square \dots \square i_{(\alpha_{p-1},\alpha_p)}.$$

Proof. Let $\underline{\alpha}$ be a fixed object of $\Delta(X^0_{>\alpha})^{\mathrm{op}}$. Since the subcategory $\Delta(X^0_{>\alpha})^{\mathrm{op}}_+$ contains only commutative diagrams, the latching category $\partial(\Delta(X^0_{>\alpha})^{\mathrm{op}}_+ {\downarrow} \underline{\alpha})$ is the full subcategory of $\Delta(X^0_{>\alpha})^{\mathrm{op}}_+$ consisting of the simplices $\underline{\beta}$ such that $\alpha.\underline{\beta} \supsetneqq \alpha.\underline{\alpha}$. Hence the first assertion.

Let $\boxed{\alpha_{-1} := \alpha}$. One has $\alpha.\underline{\beta} \supsetneqq \alpha.\underline{\alpha}$ if and only if the simplex $\alpha.\underline{\beta}$ can be written as an expression of the form

$$\alpha_{-1}.\underline{\delta_0}.\underline{\delta_1} \dots \underline{\delta_p}$$

with $\alpha_{i-1}.\underline{\delta_i} \supseteqq (\alpha_{i-1}, \alpha_i)$ for all $0 \leqslant i \leqslant p$ and such that at least for one i, one has $\alpha_{i-1}.\underline{\delta_i} \supsetneqq (\alpha_{i-1}, \alpha_i)$.

Let $\mathcal{E}$ be the set of subsets S of $\{0, \dots, p\}$ such that $S \neq \{0, \dots, p\}$. For such an S, let $I(S)$ be the full subcategory of $\Delta(X^0_{>\alpha})^{\mathrm{op}}_+$ consisting of the objects $\underline{\beta}$ such that:

- $\alpha.\underline{\beta} \supsetneqq \alpha.\underline{\alpha}$.
- For each $i \notin S$, one has $\alpha_{i-1}.\underline{\delta_i} \supsetneqq (\alpha_{i-1}, \alpha_i)$, and therefore $\underline{\delta_i} \neq (\alpha_i)$.
- For each $i \in S$, one has $\alpha_{i-1}.\underline{\delta_i} \supseteqq (\alpha_{i-1}, \alpha_i)$.

The full subcategory $\bigcup_{S \in \mathcal{E}} I(S)$ is exactly the subcategory of $\Delta(X^0_{>\alpha})^{\mathrm{op}}_+$ consisting of the objects $\underline{\beta}$ such that $\alpha.\underline{\beta} \supsetneqq \alpha.\underline{\alpha}$, that is to say the subcategory calculating $L_{\underline{\alpha}}\mathcal{P}^-_\alpha(X)$. In other words, one obtains the isomorphism

$$\varinjlim_{\bigcup_{S\in\mathcal{E}} I(S)} \mathcal{P}^-_\alpha(X) \cong L_{\underline{\alpha}}\mathcal{P}^-_\alpha(X). \tag{1}$$

The full subcategory $I(S)$ of $\Delta(X^0_{>\alpha})^{\mathrm{op}}_+$ has a final subcategory $\overline{I(S)}$ consisting of the objects $\underline{\beta}$ such that:

- $\alpha.\underline{\beta} \supsetneqq \alpha.\underline{\alpha}$.
- For each $i \notin S$, one has $\alpha_{i-1}.\underline{\delta_i} \supsetneqq (\alpha_{i-1}, \alpha_i)$, and therefore $\underline{\delta_i} \neq (\alpha_i)$.
- For each $i \in S$, one has $\alpha_{i-1}.\underline{\delta_i} = (\alpha_{i-1}, \alpha_i)$ and therefore $\underline{\delta_i} = (\alpha_i)$.

The subcategory $\overline{I(S)}$ is final in $I(S)$ because for any object $\underline{\beta}$ of $I(S)$, there exists a unique $\underline{\gamma}$ of $\overline{I(S)}$ and a unique arrow $\underline{\beta} \longrightarrow \underline{\gamma}$. Therefore, by Theorem 6.2, there is an isomorphism

$$\varinjlim_{I(S)} \mathcal{P}^-_\alpha(X) \cong \varinjlim_{\overline{I(S)}} \mathcal{P}^-_\alpha(X) \tag{2}$$

since the comma category $(\underline{\beta} {\downarrow} \overline{I(S)})$ is the one-object category.

For any object $\underline{\beta}$ of $\overline{I(S)}$, one gets

$$\begin{aligned}
&\mathcal{P}^-_\alpha(X)_{\underline{\beta}}\\
&= \prod_{i=0}^{i=p} \mathcal{P}^-_{\alpha_{i-1}}(X)_{\underline{\delta_i}} && \text{by definition of } \mathcal{P}^-\\
&\cong \left(\prod_{i\in S} \mathcal{P}^-_{\alpha_{i-1}}(X)_{(\alpha_i)}\right) \times \left(\prod_{i\notin S} \mathcal{P}^-_{\alpha_{i-1}}(X)_{\underline{\delta_i}}\right) && \text{by definition of } S.
\end{aligned}$$

Thus, since the category **Top** of compactly generated topological spaces is cartesian closed, one obtains

$$\begin{aligned}
&\varinjlim_{\overline{I(S)}} \mathcal{P}^-_\alpha(X)\\
&\cong \varinjlim_{\overline{I(S)}} \left(\left(\prod_{i\in S} \mathcal{P}^-_{\alpha_{i-1}}(X)_{(\alpha_i)}\right) \times \left(\prod_{i\notin S} \mathcal{P}^-_{\alpha_{i-1}}(X)_{\underline{\delta_i}}\right)\right)\\
&\cong \left(\prod_{i\in S} \mathcal{P}^-_{\alpha_{i-1}}(X)_{(\alpha_i)}\right) \times \varinjlim_{\substack{i\notin S\\ \alpha_{i-1}.\underline{\delta_i} \supsetneq (\alpha_{i-1},\alpha_i)}} \left(\prod_{i\notin S} \mathcal{P}^-_{\alpha_{i-1}}(X)_{\underline{\delta_i}}\right)\\
&\cong \left(\prod_{i\in S} \mathcal{P}^-_{\alpha_{i-1}}(X)_{(\alpha_i)}\right) \times \left(\prod_{i\notin S} \varinjlim_{\alpha_{i-1}.\underline{\delta_i} \supsetneq (\alpha_{i-1},\alpha_i)} \mathcal{P}^-_{\alpha_{i-1}}(X)_{\underline{\delta_i}}\right) && \text{by Lemma B.1.}
\end{aligned}$$

Therefore, one obtains the isomorphism of topological spaces

$$\varinjlim_{\overline{I(S)}} \mathcal{P}^-_\alpha(X) \cong \left(\prod_{i\in S} \mathcal{P}^-_{\alpha_{i-1}}(X)_{(\alpha_i)}\right) \times \left(\prod_{i\notin S} L_{(\alpha_i)} \mathcal{P}^-_{\alpha_{i-1}}(X)\right) \tag{3}$$

thanks to the first assertion of the theorem.

If S and T are two elements of $\mathcal{E}$ such that $S \subset T$, then there exists a canonical morphism of diagrams $I(S) \longrightarrow I(T)$ inducing a canonical morphism of topological spaces

$$\varinjlim_{\underline{\beta}\in I(S)} \mathcal{P}^-(X)_{\underline{\beta}} \longrightarrow \varinjlim_{\underline{\beta}\in I(T)} \mathcal{P}^-(X)_{\underline{\beta}}.$$

Therefore, by Equation (2) and Equation (3), the double colimit

$$\varinjlim_{S\in\mathcal{E}} \left(\varinjlim_{I(S)} \mathcal{P}^-_\alpha(X)\right)$$

calculates the source of the morphism $i_{(\alpha,\alpha_0)} \square i_{(\alpha_0,\alpha_1)} \square \dots \square i_{(\alpha_{p-1},\alpha_p)}$ by Theorem B.3. It then suffices to prove the isomorphism

$$\varinjlim_{S\in\mathcal{E}} \left(\varinjlim_{I(S)} \mathcal{P}^-_\alpha(X)\right) \cong \varinjlim_{\alpha.\underline{\beta} \supsetneq \alpha.\underline{\alpha}} \mathcal{P}^-_\alpha(X)_{\underline{\beta}}$$

to complete the proof. For that purpose, it suffices to construct two canonical morphisms

$$\varinjlim_{S\in\mathcal{E}} \left(\varinjlim_{I(S)} \mathcal{P}^-_\alpha(X)\right) \longrightarrow \varinjlim_{\alpha.\underline{\beta} \supsetneq \alpha.\underline{\alpha}} \mathcal{P}^-_\alpha(X)_{\underline{\beta}}$$

and

$$\varinjlim_{\alpha.\underline{\beta} \supsetneq \alpha.\underline{\alpha}} \mathcal{P}^-_\alpha(X)_{\underline{\beta}} \longrightarrow \varinjlim_{S\in\mathcal{E}} \left(\varinjlim_{I(S)} \mathcal{P}^-_\alpha(X)\right).$$

The first morphism comes from the isomorphism of Equation (1). As for the second morphism, let us consider a diagram of flows of the form:

$$\begin{tikzcd}
\mathcal{P}^-_\alpha(X)_{\underline{\beta}} \arrow[r] \arrow[d] & \varinjlim_{S\in\mathcal{E}}\left(\varinjlim_{I(S)}\mathcal{P}^-_\alpha(X)\right) \\
\mathcal{P}^-_\alpha(X)_{\underline{\gamma}} \arrow[ur] &
\end{tikzcd}$$

One has to prove that it is commutative. Since one has $\bigcup_{S\in\mathcal{E}} I(S) = \Delta(X^0_{>\alpha})^{\mathrm{op}}_+$, there exists $S \in \mathcal{E}$ such that $\underline{\gamma}$ is an object of $I(S)$. So $\underline{\beta}$ is an object of $I(S)$ as well and there exists a commutative diagram

$$\begin{tikzcd}
\mathcal{P}^-_\alpha(X)_{\underline{\beta}} \arrow[r] \arrow[d] & \varinjlim_{I(S)}\mathcal{P}^-_\alpha(X) \\
\mathcal{P}^-_\alpha(X)_{\underline{\gamma}} \arrow[ur] &
\end{tikzcd}$$

since the subcategory $\Delta(X^0_{>\alpha})^{\mathrm{op}}_+$ is commutative. Hence the result. □

Theorem 8.4. *Let X be a loopless object of* **cell(Flow)** *such that $(X^0, \leqslant)$ is locally finite. Let $\alpha \in X^0$. Then the diagram of topological spaces $\mathcal{P}^-_\alpha(X)$ is Reedy cofibrant. In other words, for any object $\underline{\alpha}$ of $\Delta(X^0_{>\alpha})^{\mathrm{op}}$, the topological space $\mathcal{P}^-_\alpha(X)_{\underline{\alpha}}$ is cofibrant and the morphism $L_{\underline{\alpha}}\mathcal{P}^-_\alpha(X) \longrightarrow \mathcal{P}^-_\alpha(X)_{\underline{\alpha}}$ is a cofibration of topological spaces.*

Proof. Let X be an object of **cell(Flow)**. By Proposition A.3 and since the model category **Top** is monoidal, one deduces that for any object $\underline{\alpha}$ of $\Delta(X^0_{>\alpha})^{\mathrm{op}}$, the topological space $\mathcal{P}^-_\alpha(X)_{\underline{\alpha}}$ is cofibrant. The pushout product of two cofibrations of topological spaces is always a cofibration since the model category **Top** is monoidal. By Theorem 8.3, it then suffices to prove that for any loopless object X of **cell(Flow)** such that $(X^0, \leqslant)$ is locally finite, for any object (α_0) of $\Delta(X^0_{>\alpha})^{\mathrm{op}}$, the continuous map $L_{(\alpha_0)}\mathcal{P}^-_\alpha(X) \longrightarrow \mathcal{P}^-_\alpha(X)_{(\alpha_0)}$ is a cofibration of topological spaces. Let X be an object of **cell(Flow)**. Consider a pushout diagram of flows with $n \geqslant 0$ as follows:

$$\begin{tikzcd}
\mathrm{Glob}(\mathbf{S}^{n-1}) \arrow[r, "\phi"] \arrow[d] & X \arrow[d] \\
\mathrm{Glob}(\mathbf{D}^n) \arrow[r] & Y.
\end{tikzcd}$$

One then has to prove that if X satisfies this property, then Y satisfies this property as well. One has $X^0 = Y^0$ since the morphism $\mathrm{Glob}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n)$ restricts to the identity of $\{\widehat{0}, \widehat{1}\}$ on the 0-skeletons and since the 0-skeleton functor $X \mapsto X^0$

preserves colimits.[3] So one has the commutative diagram

$$\begin{tikzcd} L_{(\alpha_0)}\mathcal{P}^-_\alpha(X) \arrow[r] \arrow[d, hook] & L_{(\alpha_0)}\mathcal{P}^-_\alpha(Y) \arrow[d] \\ \mathcal{P}^-_\alpha(X)_{(\alpha_0)} \arrow[r] & \mathcal{P}^-_\alpha(Y)_{(\alpha_0)} \end{tikzcd}$$

where the symbol $\hookrightarrow$ means cofibration. There are two mutually exclusive cases:

(1) $(\phi(\widehat{0}), \phi(\widehat{1})) = (\alpha, \alpha_0)$. One then has the situation

$$\begin{tikzcd} L_{(\alpha_0)}\mathcal{P}^-_\alpha(X) \arrow[r, equal] \arrow[d, hook] & L_{(\alpha_0)}\mathcal{P}^-_\alpha(Y) \arrow[d] \\ \mathcal{P}^-_\alpha(X)_{(\alpha_0)} \arrow[r, hook] & \mathcal{P}^-_\alpha(Y)_{(\alpha_0)} \end{tikzcd}$$

where the bottom horizontal arrow is a cofibration since it is a pushout of the morphism of flows $\mathrm{Glob}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n)$. So the continuous map $L_{(\alpha_0)}\mathcal{P}^-_\alpha(Y) \longrightarrow \mathcal{P}^-_\alpha(Y)_{(\alpha_0)}$ is a cofibration.

(2) $(\phi(\widehat{0}), \phi(\widehat{1})) \neq (\alpha, \alpha_0)$. Then, one has the pushout diagram of flows

$$\begin{tikzcd} L_{(\alpha_0)}\mathcal{P}^-_\alpha(X) \arrow[r] \arrow[d, hook] & L_{(\alpha_0)}\mathcal{P}^-_\alpha(Y) \arrow[d] \\ \mathcal{P}^-_\alpha(X)_{(\alpha_0)} \arrow[r] & \mathcal{P}^-_\alpha(Y)_{(\alpha_0)}. \end{tikzcd}$$

So the continuous map $L_{(\alpha_0)}\mathcal{P}^-_\alpha(Y) \longrightarrow \mathcal{P}^-_\alpha(Y)_{(\alpha_0)}$ is again a cofibration. In this situation, it may happen that $L_{(\alpha_0)}\mathcal{P}^-_\alpha(X) = L_{(\alpha_0)}\mathcal{P}^-_\alpha(Y)$.

The proof is complete with Proposition C.1, and because the canonical morphism of flows $X^0 \longrightarrow X$ is a relative I^{gl}-cell complex, and at last because the property above is clearly satisfied for $X = X^0$. □

9. The end of the proof

The two following classical results about classifying spaces are going to be very useful.

Proposition 9.1 (e.g., [Hir03] Proposition 18.1.6)**.** *Let $\mathcal{C}$ be a small category. The homotopy colimit of the terminal object of $\mathbf{Top}^{\mathcal{C}}$ is homotopy equivalent to the classifying space of the opposite of the indexing category $\mathcal{C}$.*

Proposition 9.2 (e.g., [Hir03] Proposition 14.3.13)**.** *Let $\mathcal{C}$ be a small category having a terminal object. Then the classifying space of $\mathcal{C}$ is contractible.*

Theorem 9.3. *Let X be a loopless object of $\mathbf{cell}(\mathbf{Flow})$ such that $(X^0, \leqslant)$ is locally finite. Assume there is an element $\widehat{1}$ such that $\alpha \leqslant \widehat{1}$ for all $\alpha \in X^0$. For any 1-simplex (α, β) of $\Delta(X^0)$, let us suppose that $\mathbb{P}_{\alpha,\beta}X$ is weakly contractible. Then $\mathrm{ho}\mathbb{P}^-_\alpha X$ has the homotopy type of a point for any $\alpha \in X^0 \backslash \{\widehat{1}\}$.*

[3] One has the canonical bijection $\mathbf{Set}(X^0, Z) \cong \mathbf{Flow}(X, T(Z))$ where $T(Z)$ is the flow defined by $T(Z)^0 = Z$ and for any $(\alpha, \beta) \in Z \times Z$, $\mathbb{P}_{\alpha,\beta}T(Z) = \{0\}$.

Of course, with the hypothesis of the theorem, the topological space $\mathrm{ho}\mathbb{P}^-_{\widehat{1}}\, X$ is the empty space.

Proof. One has the sequence of weak homotopy equivalences (where Q is the cofibrant replacement functor, $\mathbf{1}$ is the terminal diagram, and $\alpha \in X^0\backslash\{\widehat{1}\}$):

$$\begin{aligned}
&\mathrm{ho}\mathbb{P}^-_\alpha\, X \\
&\simeq \mathbb{P}^-_\alpha Q(X) && \text{by definition of the homotopy branching space}\\
&\simeq \varinjlim_{\Delta(X^0_{>\alpha})^{op}} \mathcal{P}^-_\alpha(Q(X)) && \text{by Theorem 6.3}\\
&\simeq \underset{\Delta(X^0_{>\alpha})^{op}}{\underrightarrow{\mathrm{holim}}}\ \mathcal{P}^-_\alpha(Q(X)) && \text{by Theorem 7.5 and Theorem 8.4}\\
&\simeq \underset{\Delta(X^0_{>\alpha})^{op}}{\underrightarrow{\mathrm{holim}}}\ \mathbf{1} && \text{by homotopy invariance of the homotopy colimit}\\
&\simeq B\left(\Delta(X^0_{>\alpha})\right) && \text{by Proposition 9.1}\\
&\simeq B(X^0_{>\alpha}) && \text{since the barycentric subdivision is a homotopy invariant}\\
&\simeq B(]\alpha, \widehat{1}]) && \text{since } X^0 \text{ has exactly one maximal point}\\
&\simeq \{0\} && \text{by Proposition 9.2.}
\end{aligned}$$

□

Proposition 9.4 ([Gau05b] Proposition A.1)**.** *Let X be a flow. There exists a topological space $\mathbb{P}^+X$ unique up to homeomorphism and a continuous map $h^+ : \mathbb{P}X \longrightarrow \mathbb{P}^+X$ satisfying the following universal property:*

1. *For any x and y in $\mathbb{P}X$ such that $t(x) = s(y)$, the equality $h^+(y) = h^+(x*y)$ holds.*
2. *Let $\phi : \mathbb{P}X \longrightarrow Y$ be a continuous map such that for any x and y of $\mathbb{P}X$ such that $t(x) = s(y)$, the equality $\phi(y) = \phi(x * y)$ holds. Then there exists a unique continuous map $\overline{\phi} : \mathbb{P}^+X \longrightarrow Y$ such that $\phi = \overline{\phi} \circ h^+$.*

Moreover, one has the homeomorphism

$$\mathbb{P}^+X \cong \bigsqcup_{\alpha\in X^0} \mathbb{P}^+_\alpha X$$

where $\mathbb{P}^+_\alpha X := h^+\left(\bigsqcup_{\beta\in X^0} \mathbb{P}^+_{\alpha,\beta}X\right)$. The mapping $X \mapsto \mathbb{P}^+X$ yields a functor $\mathbb{P}^+$ from **Flow** *to* **Top**.

Roughly speaking, the merging space of a flow is the space of germs of nonconstant execution paths ending in the same way.

Definition 9.5. Let X be a flow. The topological space $\mathbb{P}^+X$ is called the merging space of the flow X. The functor $\mathbb{P}^+$ is called the merging space functor.

Theorem 9.6 ([Gau05b] Theorem A.4)**.** *The merging space functor $\mathbb{P}^+$:* **Flow** $\to$ **Top** *is a left Quillen functor.*

Definition 9.7. The homotopy merging space $\mathrm{ho}\mathbb{P}^+\, X$ of a flow X is by definition the topological space $\mathbb{P}^+Q(X)$. If $\alpha \in X^0$, let $\mathrm{ho}\mathbb{P}^+_\alpha\, X = \mathbb{P}^+_\alpha Q(X)$.

Theorem 9.8. *Let $\overrightarrow{D}$ be a full directed ball with initial state $\widehat{0}$ and final state $\widehat{1}$. Then one has the following homotopy equivalences:*

1. $\mathrm{ho}\mathbb{P}^-_\alpha\, \overrightarrow{D} \simeq \{0\}$ *for any $\alpha \in \overrightarrow{D}^0\backslash\{\widehat{1}\}$ and* $\mathrm{ho}\mathbb{P}^-_{\widehat{1}}\, \overrightarrow{D} = \varnothing$.

(2) $\mathrm{ho}\mathbb{P}^{+}_{\alpha}\,\overrightarrow{D} \simeq \{0\}$ *for any* $\alpha \in \overrightarrow{D}^0\backslash\{\widehat{0}\}$ *and* $\mathrm{ho}\mathbb{P}^{+}_{\widehat{0}}\,\overrightarrow{D} = \varnothing$.

Proof. The equalities $\mathrm{ho}\mathbb{P}^{-}_{\widehat{1}}\,\overrightarrow{D} = \varnothing$ and $\mathrm{ho}\mathbb{P}^{+}_{\widehat{0}}\,\overrightarrow{D} = \varnothing$ are obvious. The homotopy equivalence $\mathrm{ho}\mathbb{P}^{-}_{\alpha}\,\overrightarrow{D} \simeq \{0\}$ for any $\alpha \in \overrightarrow{D}^0\backslash\{\widehat{1}\}$ is the result of Theorem 9.3. Let us consider the opposite flow $\overrightarrow{D}^{\mathrm{op}}$ of $\overrightarrow{D}$ defined as follows:

(1) $(\overrightarrow{D}^{\mathrm{op}})^0 = \overrightarrow{D}^0$
(2) $\mathbb{P}_{\alpha,\beta}\overrightarrow{D}^{\mathrm{op}} = \mathbb{P}_{\beta,\alpha}\overrightarrow{D}$ with $t^{\mathrm{op}}(\gamma) = s(\gamma)$ and $s^{\mathrm{op}}(\gamma) = t(\gamma)$.

The weak S-homotopy equivalence $Q(\overrightarrow{D}) \longrightarrow \overrightarrow{D}$ from the cofibrant replacement of $\overrightarrow{D}$ to $\overrightarrow{D}$ becomes a weak S-homotopy equivalence $Q(\overrightarrow{D})^{\mathrm{op}} \longrightarrow \overrightarrow{D}^{\mathrm{op}}$. Since one has the isomorphism $\mathrm{Glob}(Z)^{\mathrm{op}} \cong \mathrm{Glob}(Z)$ for any topological space Z (in particular, for $Z = \mathbf{S}^{n-1}$ and $Z = \mathbf{D}^n$ for all $n \geqslant 0$), then the transfinite composition $\varnothing \longrightarrow Q(\overrightarrow{D})$ of pushouts of morphisms of

$$\{\mathrm{Glob}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n), n \geqslant 0\} \cup \{R, C\}$$

allows us to view $\varnothing \longrightarrow Q(\overrightarrow{D})^{\mathrm{op}}$ as the transfinite composition of pushouts of the same set of morphisms. Therefore, the flow $Q(\overrightarrow{D})^{\mathrm{op}}$ is a cofibrant replacement functor of $\overrightarrow{D}^{\mathrm{op}}$. So one has the homotopy equivalences

$$\mathrm{ho}\mathbb{P}^{+}_{\alpha}\,\overrightarrow{D} \simeq \mathbb{P}^{+}_{\alpha}Q(\overrightarrow{D}) \simeq \mathbb{P}^{-}_{\alpha}Q(\overrightarrow{D})^{\mathrm{op}} \simeq \mathbb{P}^{-}_{\alpha}Q(\overrightarrow{D}^{\mathrm{op}}) \simeq \mathrm{ho}\mathbb{P}^{-}_{\alpha}\,\overrightarrow{D}^{\mathrm{op}}.$$

Thus, if α is not the final state of $\overrightarrow{D}^{\mathrm{op}}$, that is the initial state of $\overrightarrow{D}$, then we are reduced to verifying that $\overrightarrow{D}^{\mathrm{op}}$ is a full directed ball as well. The latter fact is clear. $\square$

10. The branching and merging homologies of a flow

We recall in this section the definition of the branching and merging homologies.

Definition 10.1 ([Gau05b])**.** Let X be a flow. Then the $(n+1)$-st branching homology group $H^{-}_{n+1}(X)$ is defined as the n-th homology group of the augmented simplicial set $\mathcal{N}^{-}_{*}(X)$ defined as follows:

(1) $\mathcal{N}^{-}_{n}(X) = \mathrm{Sing}_n(\mathrm{ho}\mathbb{P}^{-}\,X)$ for $n \geqslant 0$.
(2) $\mathcal{N}^{-}_{-1}(X) = X^0$.
(3) The augmentation map $\epsilon : \mathrm{Sing}_0(\mathrm{ho}\mathbb{P}^{-}\,X) \longrightarrow X^0$ is induced by the mapping $\gamma \mapsto s(\gamma)$ from $\mathrm{ho}\mathbb{P}^{-}\,X = \mathrm{Sing}_0(\mathrm{ho}\mathbb{P}^{-}\,X)$ to X^0.

Here, $\mathrm{Sing}(Z)$ denotes the singular simplicial nerve of a given topological space Z [GJ99]. In other words:

(1) For $n \geqslant 1$, $H^{-}_{n+1}(X) := H_n(\mathrm{ho}\mathbb{P}^{-}\,X)$.
(2) $H^{-}_{1}(X) := \ker(\epsilon)/\operatorname{im}\big(\partial : \mathcal{N}^{-}_{1}(X) \to \mathcal{N}^{-}_{0}(X)\big)$.
(3) $H^{-}_{0}(X) := \mathbb{Z}(X^0)/\operatorname{im}(\epsilon)$.

Here ∂ is the simplicial differential map, $\ker(f)$ is the kernel of f and $\operatorname{im}(f)$ is the image of f.

For any flow X, $H^{-}_{0}(X)$ is the free abelian group generated by the final states of X.

Definition 10.2 ([Gau05b])**.** Let X be a flow. Then the $(n+1)$-st merging homology group $H^{+}_{n+1}(X)$ is defined as the n-th homology group of the augmented simplicial set $\mathcal{N}^{+}_{*}(X)$ defined as follows:

(1) $\mathcal{N}_n^+(X) = \mathrm{Sing}_n(\mathrm{ho}\mathbb{P}^+ X)$ for $n \geqslant 0$.
(2) $\mathcal{N}_{-1}^+(X) = X^0$.
(3) The augmentation map $\epsilon : \mathrm{Sing}_0(\mathrm{ho}\mathbb{P}^+ X) \longrightarrow X^0$ is induced by the mapping $\gamma \mapsto t(\gamma)$ from $\mathrm{ho}\mathbb{P}^+ X = \mathrm{Sing}_0(\mathrm{ho}\mathbb{P}^+ X)$ to X^0.

Here, $\mathrm{Sing}(Z)$ denotes the singular simplicial nerve of a given topological space Z. In other words:

(1) For $n \geqslant 1$, $H_{n+1}^+(X) := H_n(\mathrm{ho}\mathbb{P}^+ X)$.
(2) $H_1^+(X) := \ker(\epsilon)/\operatorname{im}\left(\partial : \mathcal{N}_1^+(X) \to \mathcal{N}_0^+(X)\right)$.
(3) $H_0^+(X) := \mathbb{Z}(X^0)/\operatorname{im}(\epsilon)$.

Here, ∂ is the simplicial differential map, $\ker(f)$ is the kernel of f and $\operatorname{im}(f)$ is the image of f.

For any flow X, $H_0^+(X)$ is the free abelian group generated by the initial states of X.

11. Preservation of the branching and merging homologies

Definition 11.1 ([Gau05a]). Let X be a flow. Let A and B be two subsets of X^0. One says that A is surrounded by B (in X) if for any $\alpha \in A$, either $\alpha \in B$ or there exists execution paths γ_1 and γ_2 of $\mathbb{P}X$ such that $s(\gamma_1) \in B$, $t(\gamma_1) = s(\gamma_2) = \alpha$ and $t(\gamma_2) \in B$. We denote this situation by $A \lll B$.

Theorem 11.2. *Let $f : X \longrightarrow Y$ be a generalized T-homotopy equivalence. Then the morphism of flows f satisfies the following conditions (with $\epsilon = \pm$):*

(1) *$Y^0 \lll f(X^0)$.*
(2) *For $\alpha \in X^0$, f induces a weak homotopy equivalence $\mathrm{ho}\mathbb{P}^\epsilon_\alpha X \simeq \mathrm{ho}\mathbb{P}^\epsilon_{f(\alpha)} Y$.*
(3) *For $\alpha \in Y^0 \backslash f(X^0)$, the topological space $\mathrm{ho}\mathbb{P}^\epsilon_\alpha Y$ is contractible.*

Proof. First of all, let us suppose that f is a pushout of the form

$$\begin{tikzcd} Q(F(P_1)) \arrow[r] \arrow[d, hook, "Q(F(u))"'] & X \arrow[d, "f"] \\ Q(F(P_2)) \arrow[r] & Y \end{tikzcd}$$

where P_1 and P_2 are two finite bounded posets and where $u : P_1 \longrightarrow P_2$ belongs to $\mathcal{T}$ (Definition 4.4). Let us factor the morphism of flows $Q(F(P_1)) \longrightarrow X$ as a composite of a cofibration $Q(F(P_1)) \longrightarrow W$ followed by a trivial fibration $W \longrightarrow X$. Then one obtains the commutative diagram of flows

$$\begin{tikzcd} Q(F(P_1)) \arrow[r, hook] \arrow[d, hook, "Q(F(u))"'] & W \arrow[r, two heads, "\simeq"] \arrow[d, hook] & X \arrow[d, "f"] \\ Q(F(P_2)) \arrow[r, hook] & T \arrow[r, "\simeq"] & Y. \end{tikzcd}$$

The morphism $T \longrightarrow Y$ of the diagram above is a weak S-homotopy equivalence since the model category **Flow** is left proper by [Gau05d] Theorem 6.4. So the flows

W and X (resp. T and Y) have same homotopy branching and merging spaces and we are reduced to the following situation:

$$\begin{tikzcd} Q(F(P_1)) \arrow[r, hook] \arrow[d, hook, "Q(F(u))"'] & X \arrow[d, hook, "f"] \\ Q(F(P_2)) \arrow[r, hook] & Y \end{tikzcd}$$

where the square is both a pushout and a homotopy pushout diagram of flows. The 0-skeleton functor gives rise to the commutative diagram of set maps:

$$\begin{tikzcd} P_1 \arrow[r, "v"] \arrow[d, "u"'] & X^0 \arrow[d] \\ P_2 \arrow[r, "w"] & Y^0. \end{tikzcd}$$

Thus, one obtains the commutative diagram of topological spaces ($\epsilon \in \{-1, +1\}$)

$$\begin{tikzcd} \bigsqcup_{\beta \in v^{-1}(\alpha)} \mathbb{P}^{\epsilon}_{\beta} Q(F(P_1)) \arrow[r, hook] \arrow[d, hook, "\simeq"] & \mathbb{P}^{\epsilon}_{\alpha} X \arrow[d, hook, "\simeq"] \\ \bigsqcup_{\beta \in w^{-1}(f(\alpha))} \mathbb{P}^{\epsilon}_{\beta} Q(F(P_2)) \arrow[r, hook] & \mathbb{P}^{\epsilon}_{f(\alpha)} Y. \end{tikzcd}$$

The left vertical arrow is a weak homotopy equivalence for the following reasons:

(1) Theorem 9.8 says that each component of the domain and of the codomain is weakly contractible, or empty. And since $u(\widehat{0}) = \widehat{0}$ and $u(\widehat{1}) = \widehat{1}$, a component $\mathbb{P}^{\epsilon}_{\beta} Q(F(P_1))$ is empty (resp. weakly contractible) if and only if $\mathbb{P}^{\epsilon}_{u(\beta)} Q(F(P_2))$ is empty (resp. weakly contractible).

(2) The map u is one-to-one and therefore, the restriction
$$u : v^{-1}(\alpha) \longrightarrow w^{-1}(f(\alpha))$$
is bijective.

The left vertical arrow is also a cofibration. So the right vertical arrow $\mathbb{P}^{\epsilon}_{\alpha} X \longrightarrow \mathbb{P}^{\epsilon}_{f(\alpha)} Y$ is a trivial cofibration as well since the functors $\mathbb{P}^{\epsilon}$ with $\epsilon \in \{-1, +1\}$ are both left Quillen functors.

Let $\alpha \in Y^0 \backslash f(X^0)$, that is to say $\alpha \in P_2 \backslash u(P_1)$. Then one obtains the pushout diagram of topological spaces

$$\begin{tikzcd} \varnothing = \mathbb{P}^{\epsilon}_{\alpha} Q(\overrightarrow{I}) \arrow[r, hook] \arrow[d, hook] & \varnothing = \mathbb{P}^{\epsilon}_{\alpha} X \arrow[d, hook] \\ \mathbb{P}^{\epsilon}_{\alpha} Q(D) \arrow[r, hook] & \mathbb{P}^{\epsilon}_{\alpha} Y. \end{tikzcd}$$

So by Theorem 9.8 again, one deduces that $\mathbb{P}^{\epsilon}_{\alpha} Y$ is contractible as soon as $\alpha \in Y^0 \backslash f(X^0)$.

Now let us suppose that $f : X \longrightarrow Y$ is a transfinite composition of morphisms as above. Then there exists an ordinal λ and a λ-sequence $Z : \lambda \longrightarrow$ **Flow** with $Z_0 = X$, $Z_\lambda = Y$ and the morphism $Z_0 \longrightarrow Z_\lambda$ is equal to f. Since for any $u \in \mathcal{T}$, the morphism of flows $Q(F(u))$ is a cofibration, the morphism $Z_\mu \longrightarrow Z_{\mu+1}$ is a cofibration for any $\mu < \lambda$. Since the model category **Flow** is left proper by [Gau05d]

Theorem 6.4, there exists by [Hir03] Proposition 17.9.4 a λ-sequence $\widetilde{Z} : \lambda \longrightarrow \mathbf{Flow}$ and a morphism of λ-sequences $\widetilde{Z} \longrightarrow Z$ such that for any $\mu \leqslant \lambda$, the flow $\widetilde{Z}_\mu$ is cofibrant and the morphism $\widetilde{Z}_\mu \longrightarrow Z_\mu$ is a weak S-homotopy equivalence. So for any $\mu \leqslant \lambda$, one has $\mathbb{P}^\epsilon \widetilde{Z}_\mu \simeq \mathrm{ho}\mathbb{P}^\epsilon Z_\mu$ and for any $\mu < \lambda$, the continuous map $\mathbb{P}^\epsilon \widetilde{Z}_\mu \longrightarrow \mathbb{P}^\epsilon \widetilde{Z}_{\mu+1}$ is a cofibration. So for a given $\alpha \in Z_0^0 = X^0$, the continuous map $\mathrm{ho}\mathbb{P}^\epsilon_\alpha X \longrightarrow \mathrm{ho}\mathbb{P}^\epsilon_{f(\alpha)} Y$ is a transfinite composition of trivial cofibrations, and therefore a trivial cofibration as well.

The same argument proves that the continuous map $\mathrm{ho}\mathbb{P}^\epsilon_\alpha Z_\mu \longrightarrow \mathrm{ho}\mathbb{P}^\epsilon_{\alpha'} Y$ is a trivial cofibration for any $\mu \leqslant \lambda$ where $\alpha' \in Y^0$ is the image of $\alpha \in Z_\mu^0$ by the morphism $Z_\mu \longrightarrow Y$. Let $\alpha \in Y^0 \backslash f(X^0)$. Consider the set of ordinals

$$\big\{\mu \leqslant \lambda, \exists \beta_\mu \in Z_\mu \text{ mapped to } \alpha\big\}.$$

This nonempty set (it contains at least λ) has a smallest element μ_0. The ordinal μ_0 cannot be a limit ordinal. Otherwise, one would have $Z_{\mu_0} = \varinjlim_{\mu<\mu_0} Z_\mu$ and therefore there would exist a β_μ mapped to β_{μ_0} for some $\mu < \mu_0$: contradiction. So one can write $\mu_0 = \mu_1 + 1$. Then $\mathrm{ho}\mathbb{P}^\epsilon_{\beta_{\mu_0}} Z_{\mu_0}$ is contractible because of the first part of the proof applied to the morphism $Z_{\mu_1} \longrightarrow Z_{\mu_0}$. Therefore, $\mathrm{ho}\mathbb{P}^\epsilon_{\alpha'} Y$ is contractible as well.

The condition $Y^0 \lll f(X^0)$ is always clearly satisfied.

This leaves the case where f is a retract of a generalized T-equivalence of the preceding kinds. The result follows from the fact that everything is functorial and that the retract of a weak homotopy equivalence (resp. a nonempty set) is a weak homotopy equivalence (resp. a nonempty set). ☐

Corollary 11.3. *Let $f : X \longrightarrow Y$ be a generalized T-homotopy equivalence. Then for any $n \geqslant 0$, one has the isomorphisms*

$$H_n^-(f) : H_n^-(X) \xrightarrow{\cong} H_n^-(Y)$$
$$H_n^+(f) : H_n^+(X) \xrightarrow{\cong} H_n^+(Y).$$

Proof. This is the same proof as for [Gau05b] Proposition 7.4 (the word contractible being replaced by singleton). ☐

12. Conclusion

This new definition of T-homotopy equivalence seems to be well-behaved because it preserves the branching and merging homology theories. For an application of this new approach of T-homotopy, see the proof of an analogue of Whitehead's theorem for the full dihomotopy relation in [Gau06a].

Appendix A. Elementary remarks about flows

Proposition A.1 ([Gau03] Proposition 15.1)**.** *If one has the pushout of flows*

$$\begin{tikzcd} \mathrm{Glob}(\partial Z) \arrow[r, "\phi"] \arrow[d] & A \arrow[d] \\ \mathrm{Glob}(Z) \arrow[r] & M \end{tikzcd}$$

then the continuous map $\mathbb{P}A \longrightarrow \mathbb{P}M$ is a transfinite composition of pushouts of continuous maps of the form $\mathrm{Id}\times\ldots\times\mathrm{Id}\times f\times\mathrm{Id}\times\ldots\times\mathrm{Id}$ *where* $f:\mathbb{P}_{\phi(\widehat{0}),\phi(\widehat{1})}A \longrightarrow T$ *is the canonical inclusion obtained with the pushout diagram of topological spaces*

$$\begin{tikzcd} \partial Z \arrow[r] \arrow[d] & \mathbb{P}_{\phi(\widehat{0}),\phi(\widehat{1})}A \arrow[d] \\ Z \arrow[r] & T. \end{tikzcd}$$

Proposition A.2. *Let Y be a flow such that $\mathbb{P}Y$ is a cofibrant topological space. Let $f:Y\longrightarrow Z$ be a pushout of a morphism of I^{gl}_+. Then the topological space $\mathbb{P}Z$ is cofibrant.*

Proof. By hypothesis, f is the pushout of a morphism of flows $g\in I^{\mathrm{gl}}_+$. So one has the pushout of flows

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ B \arrow[r, "\psi"'] & Z. \end{tikzcd}$$

If f is a pushout of $C:\varnothing\subset\{0\}$, then $\mathbb{P}Z=\mathbb{P}Y$. Therefore, the space $\mathbb{P}Z$ is cofibrant. If f is a pushout of $R:\{0,1\}\to\{0\}$ and if $\phi(0)=\phi(1)$, then $\mathbb{P}Z=\mathbb{P}Y$ again. Therefore, the space $\mathbb{P}Z$ is also cofibrant. If f is a pushout of $R:\{0,1\}\to\{0\}$ and if ϕ is one-to-one, then one has the homeomorphism

$$\begin{aligned}\mathbb{P}Z\cong\mathbb{P}Y\sqcup\bigsqcup_{r\geqslant 0}&\left(\mathbb{P}_{.,\phi(0)}Y\times\mathbb{P}_{\phi(1),\phi(0)}Y\times\mathbb{P}_{\phi(1),\phi(0)}Y\times\ldots(r\text{ times})\times\mathbb{P}_{\phi(1),.}Y\right)\\ \sqcup\bigsqcup_{r\geqslant 0}&\left(\mathbb{P}_{.,\phi(1)}Y\times\mathbb{P}_{\phi(0),\phi(1)}Y\times\mathbb{P}_{\phi(0),\phi(1)}Y\times\ldots(r\text{ times})\times\mathbb{P}_{\phi(0),.}Y\right).\end{aligned}$$

Therefore, the space $\mathbb{P}Z$ is again cofibrant since the model category **Top** is monoidal. This leaves the case where g is the inclusion $\mathrm{Glob}(\mathbf{S}^{n-1})\subset\mathrm{Glob}(\mathbf{D}^n)$ for some $n\geqslant 0$. Consider the pushout of topological spaces

$$\begin{tikzcd} \mathbf{S}^{n-1} \arrow[r, "\mathbb{P}\phi"] \arrow[d, "g"'] & \mathbb{P}_{\phi(\widehat{0}),\phi(\widehat{1})}Y \arrow[d, "f"] \\ \mathbf{D}^n \arrow[r, "\mathbb{P}\psi"'] & T. \end{tikzcd}$$

By Proposition A.1, the continuous map $\mathbb{P}Y\longrightarrow\mathbb{P}Z$ is a transfinite composition of pushouts of continuous maps of the form $\mathrm{Id}\times\mathrm{Id}\times\ldots\times f\times\ldots\times\mathrm{Id}\times\mathrm{Id}$ where f is a cofibration and the identity maps are the identity maps of cofibrant topological spaces. So it suffices to notice that if k is a cofibration and if X is a cofibrant topological space, then $\mathrm{Id}_X\times k$ is still a cofibration since the model category **Top** is monoidal. □

Proposition A.3. *Let X be a cofibrant flow. Then for any $(\alpha,\beta)\in X^0\times X^0$, the topological space $\mathbb{P}_{\alpha,\beta}X$ is cofibrant.*

Proof. A cofibrant flow X is a retract of a I^{gl}_{+}-cell complex Y and $\mathbb{P}X$ becomes a retract of $\mathbb{P}Y$. So it suffices to show that $\mathbb{P}Y$ is cofibrant. Proposition A.2 completes the proof. □

Appendix B. Calculating pushout products

Lemma B.1. *Let $D : I \longrightarrow \mathbf{Top}$ and $E : J \longrightarrow \mathbf{Top}$ be two diagrams in a complete cocomplete cartesian closed category. Let $D \times E : I \times J :\longrightarrow \mathbf{Top}$ be the diagram of topological spaces defined by $(D \times E)(x,y) := D(x) \times E(y)$ if (x,y) is either an object or an arrow of the small category $I \times J$. Then one has $\varinjlim(D \times E) \cong (\varinjlim D) \times (\varinjlim E)$.*

Proof. One has $\varinjlim(D \times E) \cong \varinjlim_i(\varinjlim_j D(i) \times E(j))$ by [ML98]. And one has $\varinjlim_j(D(i) \times E(j)) \cong D(i) \times (\varinjlim E)$ since the category is cartesian closed. So $\varinjlim(D \times E) \cong \varinjlim_i(D(i) \times (\varinjlim E)) \cong (\varinjlim D) \times (\varinjlim E)$. □

Notation B.2. If $f : U \longrightarrow V$ and $g : W \longrightarrow X$ are two morphisms of a complete cocomplete category, then let us denote by $f \square g : (U \times X) \sqcup_{(U \times W)} (V \times W) \longrightarrow V \times X$ the pushout product of f and g. The notation $f_0 \square \dots \square f_p$ is defined by induction on p by $f_0 \square \dots \square f_p := (f_0 \square \dots \square f_{p-1}) \square f_p$.

Theorem B.3 (Calculating a pushout product of several morphisms)**.** *Let*

$$f_i : A_i \longrightarrow B_i, \quad 0 \leqslant i \leqslant p$$

be $p+1$ morphisms of a complete cocomplete cartesian closed category $\mathcal{C}$. Let $S \subset \{0,\dots,p\}$. Let

$$C_p(S) := \left(\prod_{i \in S} B_i\right) \times \left(\prod_{i \notin S} A_i\right).$$

If S and T are two subsets of $\{0,\dots,p\}$ such that $S \subset T$, let $C_p(i_S^T) : C_p(S) \longrightarrow C_p(T)$ be the morphism

$$\left(\prod_{i \in S} \mathrm{Id}_{B_i}\right) \times \left(\prod_{i \in T \backslash S} f_i\right) \times \left(\prod_{i \notin T} \mathrm{Id}_{A_i}\right).$$

Then:

1. *The mappings $S \mapsto C_p(S)$ and $i_S^T \mapsto C_p(i_S^T)$ give rise to a functor from $\Delta(\{0,\dots,p\})$ (the order complex of the poset $\{0,\dots,p\}$) to $\mathcal{C}$.*
2. *There exists a canonical morphism*

$$\varinjlim_{S \subsetneqq \{0,\dots,p\}} C_p(S) \longrightarrow C_p(\{0,\dots,p\}).$$

and it is equal to the morphism $f_0 \square \dots \square f_p$.

Proof. The first assertion is clear. Moreover, for any subset S and T of $\{0,\dots,p\}$ such that $S \subset T$, the diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \{0,\dots,p\} \\ T \arrow[ur] & \end{tikzcd}$$

is commutative since there is at most one morphism between two objects of the category $\Delta(\{0,\dots,p\})$, hence the existence of the morphism

$$\varinjlim_{S\subsetneqq\{0,\dots,p\}} C_p(S) \longrightarrow C(\{0,\dots,p\}).$$

The second assertion is clear for $p=0$ and $p=1$. We are going to prove it by induction on p. By definition, the morphism $f_0\square\dots\square f_{p+1}$ is the canonical morphism from

$$\Big(\big(\varinjlim_{S\subsetneqq\{0,\dots,p\}} C_p(S)\big)\times B_{p+1}\Big) \sqcup_{\left(\left(\varinjlim_{S\subsetneqq\{0,\dots,p\}} C_p(S)\right)\times A_{p+1}\right)} \big(C_p(\{0,\dots,p\})\times A_{p+1}\big)$$

to $B_0\times\dots\times B_{p+1}$. By Lemma B.1, the source of the morphism $f_0\square\dots\square f_{p+1}$ is then equal to

$$\big(\varinjlim_{p+1\in S\subsetneqq\{0,\dots,p+1\}} C_{p+1}(S)\big) \sqcup_{\left(\varinjlim_{S\subsetneqq\{0,\dots,p\}} C_{p+1}(S)\right)} \big(C_{p+1}(\{0,\dots,p\})\big),$$

and the latter is equal to $\varinjlim_{S\subsetneqq\{0,\dots,p+1\}} C_{p+1}(S)$. $\square$

Appendix C. Mixed transfinite composition of pushouts and cofibrations

Proposition C.1. *Let $\mathcal{M}$ be a model category. Let λ be an ordinal. Let*

$$(f_\mu : A_\mu \longrightarrow B_\mu)_{\mu<\lambda}$$

be a λ-sequence of morphisms of $\mathcal{M}$. For $\mu<\lambda$, suppose that $A_\mu\to A_{\mu+1}$ is an isomorphism or the diagram of objects of $\mathcal{M}$

$$\begin{tikzcd} A_\mu \arrow[r] \arrow[d, "f_\mu"'] & A_{\mu+1} \arrow[d] \\ B_\mu \arrow[r] & B_{\mu+1} \end{tikzcd}$$

is a pushout, and for $\mu<\lambda$ suppose also that the map $B_\mu \longrightarrow B_{\mu+1}$ is a cofibration. Then: if $f_0 : A_0\longrightarrow B_0$ is a cofibration, then $f_\lambda : A_\lambda \longrightarrow B_\lambda$ is a cofibration as well, where of course $A_\lambda := \varinjlim A_\mu$ and $B_\lambda := \varinjlim B_\mu$.

Proof. It is clear that if $f_\mu : A_\mu \longrightarrow B_\mu$ is a cofibration, then $f_{\mu+1} : A_{\mu+1} \longrightarrow B_{\mu+1}$ is a cofibration as well. It then suffices to prove that if $\nu \leqslant \lambda$ is a limit ordinal such that $f_\mu : A_\mu \longrightarrow B_\mu$ is a cofibration for any $\mu<\nu$, then $f_\nu : A_\nu \longrightarrow B_\nu$ is a cofibration as well. Consider a commutative diagram

$$\begin{tikzcd} A_\nu \arrow[r] \arrow[d, "f_\nu"'] & C \arrow[d] \\ B_\nu \arrow[r] \arrow[ur, dashed, "k"] & D \end{tikzcd}$$

where $C\longrightarrow D$ is a trivial fibration of $\mathcal{M}$. Then one has to find $k : B_\nu \longrightarrow C$ making both triangles commutative. Recall that by hypothesis, $f_\nu = \varinjlim_{\mu<\nu} f_\mu$.

Since f_0 is a cofibration, there exists a map k_0 making both triangles of the diagram

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d, "f_0"'] & C \arrow[d] \\ B_0 \arrow[r] \arrow[ur, dashed, "k_0"] & D \end{tikzcd}$$

commutative. Let us suppose k_μ constructed. There are two cases. Either the diagram

$$\begin{tikzcd} A_\mu \arrow[r] \arrow[d, "f_\mu"'] & A_{\mu+1} \arrow[d] \\ B_\mu \arrow[r, hook] & B_{\mu+1} \end{tikzcd}$$

is a pushout, and one can construct a morphism $k_{\mu+1}$ making both triangles of the diagram

$$\begin{tikzcd} A_{\mu+1} \arrow[r] \arrow[d, "f_{\mu+1}"'] & C \arrow[d] \\ B_{\mu+1} \arrow[r] \arrow[ur, dashed, "k_{\mu+1}"] & D \end{tikzcd}$$

commutative and such that the composite $B_\mu \longrightarrow B_{\mu+1} \longrightarrow C$ is equal to k_μ by using the universal property satisfied by the pushout. Or the morphism $A_\mu \to A_{\mu+1}$ is an isomorphism. In that latter case, consider the commutative diagram

$$\begin{tikzcd} B_\mu \arrow[r, "k_\mu"] \arrow[d, hook] & C \arrow[d] \\ B_{\mu+1} \arrow[r] & D \end{tikzcd}$$

Since the morphism $B_\mu \longrightarrow B_{\mu+1}$ is a cofibration, there exists $k_{\mu+1} : B_{\mu+1} \longrightarrow C$ making the two triangles of the latter diagram commutative. So, once again, the composite $B_\mu \longrightarrow B_{\mu+1} \longrightarrow C$ is equal to k_μ.

The map $k := \varinjlim_{\mu<\nu} k_\mu$ is a solution. $\square$

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