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The Gauss–Bonnet theorem for Cayley–Klein geometries of dimension two

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Abstract. We extend the classical Gauss–Bonnet theorem for the Euclidean, elliptic, hyperbolic, and Lorentzian planes to the other three Cayley–Klein geometries of dimension two, all three of which are absolute-time spacetimes, providing one proof for all nine geometries. Suppose that M is a polygon in any one of the nine geometries. Let Γ, the boundary of M, have length element ds, discontinuities θ_i, and signed geodesic curvature κ_g, where M and Γ are oriented according to Stokes' theorem. Let K denote the constant Gaussian curvature of the geometry with area form dA. Then

$$\int_\Gamma \kappa_g \, ds + \sum_i \theta_i + \int\int_M K \, dA = 2\pi$$

for the nonspacetimes and

$$\int_\Gamma \kappa_g \, ds + \sum_i \theta_i + \int\int_M K \, dA = 0$$

for the spacetimes, where we assume that Γ is timelike.

Contents

1. Introduction to the Cayley–Klein geometries

Our goal in this first section is to give a brief introduction to those Cayley–Klein geometries that are two-dimensional, and the interested reader who is unfamiliar with this set of geometries is encouraged to read the three appendices in Yaglom's book [10], or to read the recent articles [4], [5], and [6] by Herranz, Ortega and

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Santander from which the material in this section was taken. We begin by placing a constant (and possibly degenerate) metric g on $\mathbb{R}^3$ by setting

$$ds^2 = dz^2 + \kappa_1 dt^2 + \kappa_1\kappa_2 dx^2,$$

where κ_1 and κ_2 are real constants (though the reader may assume that $\kappa_1, \kappa_2 \in \{-1, 0, 1\}$ for simplicity).

The nine geometries:

Measure of angles	Measure of lengths: Elliptic $\kappa_1 > 0$	Parabolic $\kappa_1 = 0$	Hyperbolic $\kappa_1 < 0$
Elliptic $\kappa_2 > 0$	elliptic geometries	Euclidean geometries	hyperbolic geometries
Parabolic $\kappa_2 = 0$	oscillating Newton–Hooke spacetimes	Galilean spacetime	expanding Newton–Hooke spacetimes
Hyperbolic $\kappa_2 < 0$	anti-de Sitter spacetimes	Minkowski spacetimes	de Sitter spacetimes

The motion group consisting of all real-linear isometries of g that preserve the orientation of $\mathbb{R}^3$ will be denoted by $\mathrm{SO}_{\kappa_1,\kappa_2}(3)$, and its lie algebra will be denoted by $so_{\kappa_1,\kappa_2}(3)$. A matrix representation of $so_{\kappa_1,\kappa_2}(3)$ is given by the matrices

$$P_1 = \begin{pmatrix} 0 & -\kappa_1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & -\kappa_1\kappa_2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad J_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\kappa_2 \\ 0 & 1 & 0 \end{pmatrix},$$

where the structure constants are given by the commutators

$$[J_{12}, P_1] = P_2, \quad [J_{12}, P_2] = -\kappa_2 P_1, \quad \text{and} \quad [P_1, P_2] = \kappa_1 J_{12},$$

and the expression $\kappa_2 P_1^2 + P_2^2 + \kappa_1 J_{12}^2$ is a Casimir invariant.

The one-parameter subgroups $H_{(2)}$, $H_{(02)}$, and $H_{(1)}$ of $\mathrm{SO}_{\kappa_1,\kappa_2}(3)$ consist of, by definition, matrices of the form

$$e^{\alpha P_1} = \begin{pmatrix} C_{\kappa_1}(\alpha) & -\kappa_1 S_{\kappa_1}(\alpha) & 0 \\ S_{\kappa_1}(\alpha) & C_{\kappa_1}(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$e^{\beta P_2} = \begin{pmatrix} C_{\kappa_1\kappa_2}(\beta) & 0 & -\kappa_1\kappa_2 S_{\kappa_1\kappa_2}(\beta) \\ 0 & 1 & 0 \\ S_{\kappa_1\kappa_2}(\beta) & 0 & C_{\kappa_1\kappa_2}(\beta) \end{pmatrix},$$

and

$$e^{\theta J_{12}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{\kappa_2}(\theta) & -\kappa_2 S_{\kappa_2}(\theta) \\ 0 & S_{\kappa_2}(\theta) & C_{\kappa_2}(\theta) \end{pmatrix},$$

respectively, where the generalized cosine $C_\kappa(x)$ and sine $S_\kappa(x)$ functions are defined by

$$C_\kappa(\theta) = \begin{cases} \cos\left(\sqrt{\kappa}\,\theta\right), & \text{if } \kappa > 0 \\ 1, & \text{if } \kappa = 0 \\ \cosh\left(\sqrt{-\kappa}\,\theta\right), & \text{if } \kappa < 0 \end{cases}$$

and

$$S_\kappa(\theta) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin\left(\sqrt{\kappa}\,\theta\right), & \text{if } \kappa > 0 \\ \theta, & \text{if } \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh\left(\sqrt{-\kappa}\,\theta\right), & \text{if } \kappa < 0. \end{cases}$$

We record here a few trigonometric formulas that will prove useful when we prove the Gauss–Bonnet theorem:

$$\begin{aligned} \frac{d}{d\theta} C_\kappa(\theta) &= -\kappa S_\kappa(\theta) \\ \frac{d}{d\theta} S_\kappa(\theta) &= C_\kappa(\theta) \\ {C_\kappa}^2(\theta) + \kappa {S_\kappa}^2(\theta) &= 1. \end{aligned}$$

We can then model each Cayley–Klein geometry by the space

$$S^2_{[\kappa_1],\kappa_2} \equiv \mathrm{SO}_{\kappa_1,\kappa_2}(3)/H_{(1)},$$

where the motion $\exp(\theta J_{12})$ is a rotation (or boost for a spacetime) of $S^2_{[\kappa_1],\kappa_2}$, and where $\exp(\alpha P_1)$ and $\exp(\beta P_2)$ are translations of $S^2_{[\kappa_1],\kappa_2}$ (time and space translations respectively for a spacetime). These rotations naturally give an orientation to $S^2_{[\kappa_1],\kappa_2}$ as well as an origin point $\mathcal{O}$. The parameters κ_1 and κ_2 are, for the spacetimes, related to the universe time radius τ and speed of light c by

$$\kappa_1 = \pm\frac{1}{\tau^2} \qquad \text{and} \qquad \kappa_2 = -\frac{1}{c^2}.$$

The lie algebra $so_{\kappa_1,\kappa_2}(3)$ is acted upon by the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ group of involutions generated by

$$\Pi_{(1)} : (P_1, P_2, J_{12}) \to (-P_1, -P_2, J_{12})$$

and

$$\Pi_{(2)} : (P_1, P_2, J_{12}) \to (P_1, -P_2, -J_{12}).$$

The three subgroups $H_{(1)}$, $H_{(2)}$, and $H_{(02)}$ are generated by the three lie subalgebras determined, respectively, by $\Pi_{(1)}$, $\Pi_{(2)}$, and $\Pi_{(02)} \equiv \Pi_{(1)} \cdot \Pi_{(2)}$: each subalgebra consists of those elements that are invariant under the involution. We can now naturally introduce the Cayley–Klein geometry $S^2_{\kappa_1,[\kappa_2]} \equiv \mathrm{SO}_{\kappa_1,\kappa_2}(3)/H_{(2)}$, the space of *first-kind lines* of $S^2_{[\kappa_1],\kappa_2}$. Here $\exp(\alpha P_1)$ are rotations about an "origin" line l_1, a line which is moved in two distinct directions by the motions $\exp(\theta J_{12})$ and $\exp(\beta P_2)$.

We may similarly introduce the Cayley–Klein geometry $\mathrm{SO}_{\kappa_1,\kappa_2}(3)/H_{(02)}$, the space of *second-kind lines* of $S^2_{[\kappa_1],\kappa_2}$. Here $\exp(\beta P_2)$ are rotations about an origin line l_2, a line which is moved in two distinct directions by the motions $\exp(\theta J_{12})$ and $\exp(\alpha P_1)$. The lines l_1 and l_2 in $S^2_{[\kappa_1],\kappa_2}$ intersect orthogonally at $\mathcal{O}$.

The set of Cayley–Klein geometries has a duality property that is defined by the lie algebra isomorphism of $so_{\kappa_1,\kappa_2}(3)$ that is determined by the involutions

$$(P_1, P_2, J_{12}) \to (-J_{12}, -P_2, -P_1) \qquad \text{and} \qquad (\kappa_1, \kappa_2) \to (\kappa_2, \kappa_1)\,.$$

This isomorphism preserves the structure constants and thereby exchanges the set of geometries $S^2_{[\kappa_1],\kappa_2}$ with that of $S^2_{\kappa_1,[\kappa_2]}$ under the correspondence $S^2_{[\kappa_1],\kappa_2} \leftrightarrow S^2_{\kappa_1,[\kappa_2]}$, while preserving the set of second-kind lines. The space $S^2_{\kappa_1,[\kappa_2]}$ is the space of nonoriented lines in $S^2_{[\kappa_1],\kappa_2}$ if $\kappa_2 > 0$, and is the space of nonoriented timelike lines in $S^2_{[\kappa_1],\kappa_2}$ if $\kappa_2 \le 0$. When $\kappa_2 \le 0$, the space of second-kind lines is the space of nonoriented spacelike lines in $S^2_{[\kappa_1],\kappa_2}$, and when $\kappa_2 > 0$ the space of second-kind lines is the same as the space of first-kind lines, as the generators P_1 and P_2 are conjugates of one another.

As a set of geometries, elliptical and also de Sitter geometries are each self-dual under this transformation, and the Galilean plane is self-dual. The set of euclidean geometries is dual to that of oscillating Newton–Hooke spacetimes, the set of hyperbolic geometries is dual to that of anti-de Sitter spacetimes, and the set of expanding Newton–Hooke spacetimes is dual to that of Minkowski spacetimes.

In order to understand the nature of the metric on $S^2_{[\kappa_1],\kappa_2}$ a bit better, let us begin by defining the projective quadric Σ as the set of points

$$\{(z,t,x) \in \mathbb{R}^3 \mid z^2 + \kappa_1 t^2 + \kappa_1\kappa_2 x^2 = 1\}$$

that have been identified by the equivalence relation $(z,t,x) \sim (-z,-t,-x)$. The group $\mathrm{SO}_{\kappa_1,\kappa_2}(3)$ does not act transitively on $\mathbb{R}^3$, for $\mathrm{SO}_{\kappa_1,\kappa_2}(3)$ acts on Σ. Since the subgroup $H_{(1)}$ is the isotropy subgroup of the equivalence class $\mathcal{O} = [(1,0,0)] \in \Sigma$, $\mathrm{SO}_{\kappa_1,\kappa_2}(3)$ does act transitively on Σ, and so we identify $S^2_{[\kappa_1],\kappa_2}$ with Σ. The metric g on $\mathbb{R}^3$ induces a metric on Σ that has κ_1 as a factor. We define the *main metric* g_1 on Σ by setting

$$\left(ds^2\right)_1 = \frac{1}{\kappa_1} ds^2,$$

and the surface Σ, along with its main metric, is the Cayley–Klein geometry $S^2_{[\kappa_1],\kappa_2}$. Note that in general g_1 can be indefinite as well as nondegenerate. The surface Σ has constant curvature κ_1 (in fact, we will derive this result later) and g_1 has signature $\mathrm{diag}(+,\kappa_2)$. When $\kappa_1 = 0$ we may identify Σ with the plane $z = 1$ in $\mathbb{R}^3$.

For the absolute-time spacetimes where $\kappa_2 = 0$ and $c = \infty$, we foliate $S^2_{[\kappa_1],\kappa_2}$ so that each leaf consists of all points that are simultaneous with one another, and then $\mathrm{SO}_{\kappa_1,\kappa_2}(3)$ acts transitively on each leaf. We then define the *subsidiary metric* g_2 along each leaf of the foliation by setting

$$\left(ds^2\right)_2 = \frac{1}{\kappa_2}\left(ds^2\right)_1.$$

These leaves are given by the collection of straight lines $t = S_{\kappa_1}(\phi)$, $z = C_{\kappa_1}(\phi)$, where ϕ is a constant. Of course when $\kappa_2 \neq 0$, the subsidiary metric can be defined on all of Σ. The group $\mathrm{SO}_{\kappa_1,\kappa_2}(3)$ acts on $S^2_{[\kappa_1],\kappa_2}$ by isometries of g_1, by isometries of g_2 when $\kappa_2 \neq 0$ and, when $\kappa_2 = 0$, on the leaves of the foliation by isometries of g_2.

There exists a unique connection for $S^2_{[\kappa_1],\kappa_2}$ that is invariant under $\mathrm{SO}_{\kappa_1,\kappa_2}(3)$ and that is also compatible with both main and subsidiary metrics. We will derive the Maurer–Cartan structure equations for this connection below.

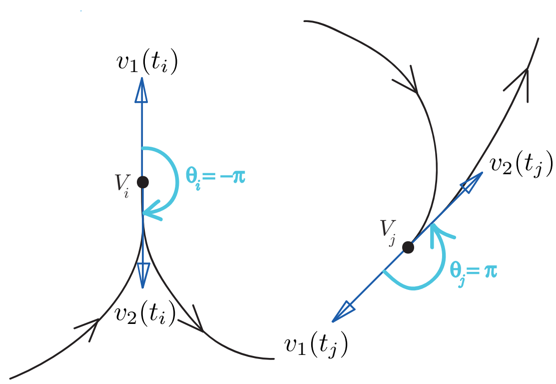
2. Discontinuities at vertices of polygons

Our aim in this paper is to state and to prove a meaningful Gauss–Bonnet formula for any polygonal region M of a Cayley–Klein geometry.

Definition 1. A region M of a two-dimensional geometry is said to be a *polygon* if ∂M is connected and if there is a parametrization $\gamma : [t_0, t_n] \to \partial M$ that is one-to-one and onto save that $\gamma(t_0) = \gamma(t_n)$. Furthermore we require that γ is a smooth imbedding on each interval $[t_{i-1}, t_i]$ of some partition $t_0 < t_1 < \cdots < t_{n-1} < t_n$ of $[t_0, t_n]$. We will call the points $\gamma(t_i)$, $i = 0, \ldots, n$, the *vertices* of $\Gamma \equiv \partial M$.

Our proof will emulate the standard book proof that one sees for Riemannian surfaces. Our goal in this section then is to formulate a precise definition for discontinuity at any vertex of Γ. Following Spivak [9] we give a few definitions in the next paragraph.

At each vertex $V_i \equiv \gamma(t_i)$ of Γ, let $v_1(t_i)$ denote the vector $\gamma'(t_i^-)$, the left-hand derivative of γ at t_i. Similarly, let $v_2(t_i)$ denote the right-hand derivative of γ at t_i. When $\kappa_2 > 0$, we then define the *discontinuity* θ_i as the signed angle of rotation needed to rotate $v_1(t_i)$ to $v_2(t_i)$, assuming that $v_1(t_i) \neq v_2(t_i)$. For the case where $v_i(t_i) = v_2(t_i)$, let $w_1^\epsilon(t_i)$ be the tangent vector of the geodesic from $\gamma(t_i - \epsilon)$ to $\gamma(t_i)$ at $\gamma(t_i)$, and let $w_2^\epsilon(t_i)$ be the tangent vector of the geodesic from $\gamma(t_i)$ to $\gamma(t_1 + \epsilon)$ at $\gamma(t_i)$. For $\epsilon > 0$ sufficiently small, $w_1^\epsilon(t_i)$ must be distinct from $w_2^\epsilon(t_i)$. So we can meaningfully define θ_i^ϵ as the signed angle of rotation needed to rotate $w_1^\epsilon(t_i)$ to $w_2^\epsilon(t_i)$, and we then define the discontinuity θ_i to equal $\lim_{\epsilon\to 0^+} \theta_i^\epsilon$.

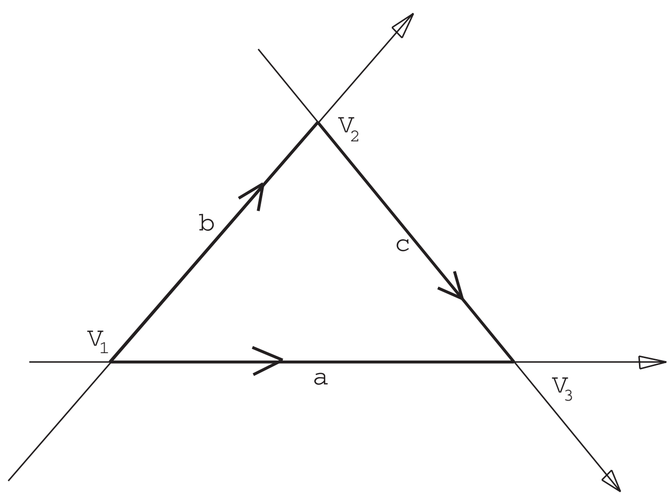


When $\kappa_2 \leq 0$, we cannot simply define θ_i as the angle of rotation needed to rotate $v_1(t_i)$ to $v_2(t_i)$, as each vector could be timelike, lightlike, or spacelike. The only obvious possibility from a physical point of view is when both vectors are future-directed and timelike, in which case θ_i should be taken as the relative rapidity between the two timelike vectors at that vertex. Birman and Nomizu's definition for θ_i in [1] is equivalent to reversing the orientation of all timelike vectors so that they are future-directed, and then defining θ_i as the rapidity. Herranz, Ortega, and Santander [5] give a Gauss–Bonnet theorem for geodesic triangles in $S^2_{[\kappa_1],\kappa_2}$ that, for spacetimes, also gives a definition for θ_i in agreement with Birman and Nomizu. Helzer [3], Dzan [2], and Law [7] also define the discontinuity between spacelike and timelike vectors.

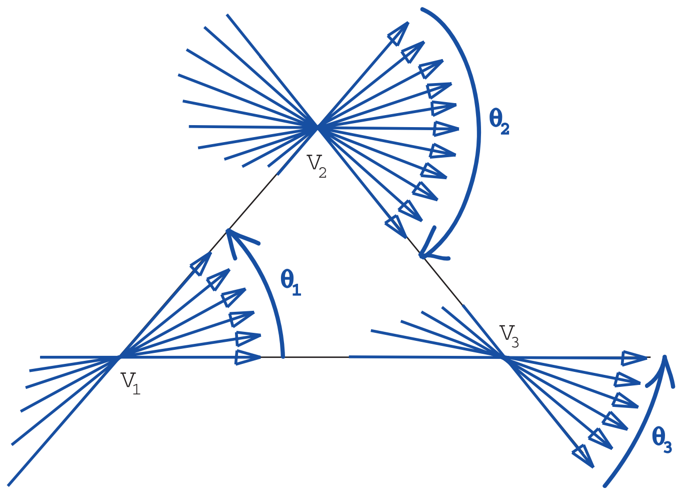
3. The Gauss–Bonnet theorem for triangles

In their paper on the trigonometry of spacetimes [5], Herranz, Ortega, and Santander derive a nice Gauss–Bonnet formula for geodesic triangles in real Cayley–Klein geometries. (Ortega and Santander derive a similar formula for complex Cayley–Klein geometries in [8].) More exactly they derive a Gauss–Bonnet formula for *triangular point loops*, as we cannot unambiguously define a triangle in a Cayley–Klein geometry as simply three distinct noncollinear points V_1, V_2, and V_3. For example, on a projective plane two distinct points determine a geodesic line, but there are two geodesic segments joining each point to the other.

A triangular point loop is defined as two different oriented and co-oriented paths (which are timelike and future-directed if $\kappa_2 \leq 0$) for a point going from vertex V_1 to vertex V_3, where one path is a geodesic segment V_1V_3 of positive length a



and the other path is composed of two geodesic segments V_1V_2 and V_2V_3 of positive lengths b and c respectively. The dual of the triangular point loop (see [5] for details concerning this duality) is called a *triangular line loop*, and consists of a closed loop of oriented lines obtained by rotating line $\overrightarrow{V_1V_3}$ to line $\overrightarrow{V_1V_2}$ at the vertex V_1, rotating line $\overrightarrow{V_1V_2}$ to line $\overrightarrow{V_2V_3}$ at the vertex V_2, and finally rotating line $\overrightarrow{V_2V_3}$ back to line $\overrightarrow{V_1V_3}$ at the vertex V_3. In the event that $\kappa_2 \leq 0$, these rotations will be through future-directed timelike lines. In all cases these rotations are through signed angles θ_1, θ_2, and θ_3 respectively, where the sign is determined by the orientation and co-orientation of the loop. Note that for Cayley–Klein spacetimes, a triangular point



loop gives the event lines for the twins of the Twin Paradox, where a is the proper time for one twin and $b+c$ is the proper time for the other twin, and where θ_i are the relative rapidities between the worldlines at each vertex.

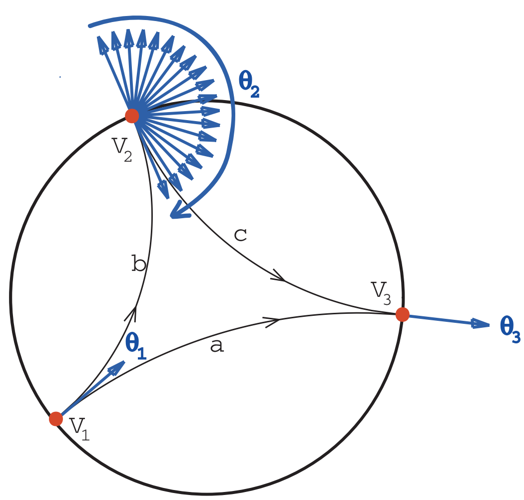
Herranz, Ortega, and Santander have shown that if we define the *angular excess* $\triangle$ (viewed as the *oriented* total angle turned by the line loop) and the *lateral excess* δ (viewed as *oriented* total length of the point loop) by the formulas

$$\triangle \equiv \theta_1 + \theta_2 + \theta_3 \qquad \text{and} \qquad \delta \equiv -a + b + c,$$

then the formulas

$$\kappa_1 \mathcal{S} \equiv \triangle \qquad \text{and} \qquad \kappa_2 \int \equiv \delta,$$

define the *area* $\mathcal{S}$ and *co-area* $\int$ of the triangular point loop. For example a nearly



ideal triangular point loop in the hyperbolic plane where $\kappa_1 = -1$ and $\kappa_2 = 1$ has an angular excess nearly equal to $-\pi$ so that the area of this loop is nearly equal to π.

In this paper we will insist that Γ be timelike, in agreement with Birman, Nomizu, Herranz, Ortega and Santander. The reason for our insistence is that for any polygonal curve Γ, we would like the polygonal curve of nonoriented lines tangent to Γ to lie in only one of the Cayley–Klein geometries. Thus, it is reasonable, when $\kappa_2 \leq 0$, to consider only timelike polygonal curves Γ and to define the discontinuities of Γ as follows. Suppose then that $v_1(t_i)$ and $v_2(t_i)$ are both timelike. We change the orientation of $v_1(t_i)$ or $v_2(t_i)$ if needed so that both are future-directed, and we then define θ_i to be the signed angle of rotation needed to rotate $v_1(t_i)$ to $v_2(t_i)$. Thus θ_i, for all cases, depends only on the orientation and co-orientation of Γ, not on the parametrization of Γ.

4. The Cartan connection for the Cayley–Klein geometries

Since the isometry group of $S^2_{[\kappa_1],\kappa_2}$ acts transitively on the frame bundle consisting of oriented, orthonormal frames, we can identify this frame bundle with $\mathrm{SO}_{\kappa_1,\kappa_2}(3)$. That is, this frame bundle is a homogeneous space for $\mathrm{SO}_{\kappa_1,\kappa_2}(3)$. As the space $\mathbb{R}^3$ with metric g is flat, the exterior derivative is compatible with g as a connection for $\mathbb{R}^3$. We then get a connection for $S^2_{[\kappa_1],\kappa_2}$ by taking the component (via orthogonal projection as determined by g) of the exterior derivative that is tangential to $S^2_{[\kappa_1],\kappa_2}$. This connection on $S^2_{[\kappa_1],\kappa_2}$ is compatible with g_1 and also with g_2 if $\kappa_2 \neq 0$, and also induces a connection on the leaves of the foliation of $S^2_{[\kappa_1],\kappa_2}$ when $\kappa_2 = 0$ that is compatible with g_2.

The tangent plane to Σ at a point E_3 is perpendicular to the vector E_3 as

$$\langle \operatorname{grad}\left(z^2 + \kappa_1 t^2 + \kappa_1\kappa_2 x^2\right), X\rangle = 0,$$

where X is a vector tangent to Σ at E_3 and where $\langle , \rangle$ denotes the standard inner product. Using the metric g_1 (and g_2 if necessary) we can represent an oriented frame on Σ by the matrix $\mathcal{F} = (E_1, E_2, E_3)$ where E_3 is a point on Σ, E_1 and E_2 are orthogonal unit vectors spanning the tangent plane to Σ at E_3, and E_1 is future-directed if $\kappa_2 \leq 0$. Regardless of the value of κ_2, we choose E_1 and E_2 in agreement with the orientation for $S^2_{[\kappa_1],\kappa_2}$. Let Θ_i denote the basis that is dual to the E_i, for $i = 1, 2, 3$.

If we take the exterior derivative of $\mathcal{F} \in \mathrm{SO}_{\kappa_1,\kappa_2}(3)$, we may write

$$d\mathcal{F} = (dE_1, dE_2, dE_3) = (E_1, E_2, E_3)\,\Omega,$$

where

$$\Omega = \begin{pmatrix} \omega_{11} & \omega_{21} & \omega_{31} \\ \omega_{12} & \omega_{22} & \omega_{32} \\ \omega_{13} & \omega_{23} & \omega_{33} \end{pmatrix}$$

and where $dE_i = \sum_{i=1}^{3} \omega_{ij} E_j$. If $g(E_i, E_j) = g_{ij}$, which is a constant for each $i, j \in \{1, 2, 3\}$, then $dg(E_i, E_j) = g(dE_i, E_j) + g(E_i, dE_j) = 0$. Thus

$$\sum_{k=1}^{3} \left[g(\omega_{ik} E_k, E_j) + g(E_i, \omega_{jk} E_k)\right] = \omega_{ij} K_j + \omega_{ji} K_i = 0,$$

where

$$K_i = \begin{cases} \kappa_1, & \text{if } i = 1 \\ \kappa_1\kappa_2, & \text{if } i = 2 \\ 1, & \text{if } i = 3. \end{cases}$$

We also write $\Omega^T g + g\Omega = 0$, where

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_1\kappa_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We then have the following table of formulae:

	$i=1$	$i=2$	$i=3$
$j=1$	$\kappa_1\omega_{11}=0$	$\kappa_1\kappa_2\omega_{12}+\kappa_1\omega_{21}=0$	$\omega_{13}+\kappa_1\omega_{31}=0$
$j=2$	$\kappa_1\kappa_2\omega_{12}+\kappa_1\omega_{21}=0$	$\kappa_1\kappa_2\omega_{22}=0$	$\omega_{23}+\kappa_1\kappa_2\omega_{32}=0$
$j=3$	$\omega_{13}+\kappa_1\omega_{31}=0$	$\omega_{23}+\kappa_1\kappa_2\omega_{32}=0$	$\omega_{33}=0.$

As the connection on $S^2_{[\kappa_1],\kappa_2}$ is compatible with both main and subsidiary metrics, we may simplify Ω to

$$\Omega = \begin{pmatrix} 0 & -\kappa_2\omega_{12} & \Theta_1 \\ \omega_{12} & 0 & \Theta_2 \\ -\kappa_1\Theta_1 & -\kappa_1\kappa_2\Theta_2 & 0 \end{pmatrix},$$

where $\Theta_1 = \omega_{31}$ and $\Theta_2 = \omega_{32}$, for E_3 is a point on Σ and so $dE_3 = \Theta_1 E_1 + \Theta_2 E_2$. Recalling that

$$d\mathcal{F} = (E_1, E_2, E_3)\,\Omega,$$

it follows that

$$\begin{aligned} d^2\mathcal{F} &= 0 \\ &= (dE_1, dE_2, dE_3) \wedge \Omega + (E_1, E_2, E_3)\, d\Omega \\ &= (E_1, E_2, E_3)\,\Omega \wedge \Omega + (E_1, E_2, E_3)\, d\Omega \end{aligned}$$

so that

$$d\Omega + \Omega \wedge \Omega = 0.$$

Expanding this last formula gives us

$$
\begin{pmatrix} 0 & -\kappa_2 d\omega_{12} & d\Theta_1 \\ d\omega_{12} & 0 & d\Theta_2 \\ -\kappa_1 d\Theta_1 & -\kappa_1\kappa_2 d\Theta_2 & 0 \end{pmatrix}
+ \begin{pmatrix} 0 & -\kappa_2 \omega_{12} & \Theta_1 \\ \omega_{12} & 0 & \Theta_2 \\ -\kappa_1 \Theta_1 & -\kappa_1\kappa_2 \Theta_2 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & -\kappa_2 \omega_{12} & \Theta_1 \\ \omega_{12} & 0 & \Theta_2 \\ -\kappa_1 \Theta_1 & -\kappa_1\kappa_2 \Theta_2 & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & -\kappa_2 d\omega_{12} & d\Theta_1 \\ d\omega_{12} & 0 & d\Theta_2 \\ -\kappa_1 d\Theta_1 & -\kappa_1\kappa_2 d\Theta_2 & 0 \end{pmatrix}
+ \begin{pmatrix} 0 & -\kappa_1\kappa_2 \Theta_1 \wedge \Theta_2 & -\kappa_2 \omega_{12} \wedge \Theta_2 \\ -\kappa_1 \Theta_2 \wedge \Theta_1 & 0 & \omega_{12} \wedge \Theta_1 \\ -\kappa_1\kappa_2 \Theta_2 \wedge \omega_{12} & \kappa_1\kappa_2 \Theta_1 \wedge \omega_{12} & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & -\kappa_2 d\omega_{12} - \kappa_1\kappa_2 \Theta_1 \wedge \Theta_2 & d\Theta_1 - \kappa_2 \omega_{12} \wedge \theta_2 \\ d\omega_{12} + \kappa_1 \Theta_1 \wedge \Theta_2 & 0 & d\theta_2 + \omega_{12} \wedge \Theta_1 \\ -\kappa_1 d\Theta_1 - \kappa_1\kappa_2 \Theta_2 \wedge \omega_{12} & -\kappa_1\kappa_2 d\Theta_2 + \kappa_1\kappa_2 \Theta_1 \wedge \omega_{12} & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$

so that the Maurer–Cartan equations are

$$
\begin{aligned}
d\omega_{12} + \kappa_1 \Theta_1 \wedge \Theta_2 &= 0 \\
d\Theta_1 - \kappa_2 \omega_{12} \wedge \Theta_2 &= 0 \\
d\Theta_2 + \omega_{12} \wedge \Theta_1 &= 0.
\end{aligned}
$$

We have thus proved the following theorem:

Theorem 1. *The Maurer–Cartan equations for the two-dimensional Cayley–Klein geometry* $S^2_{[\kappa_1],\kappa_2}$ *with real parameters* κ_1 *and* κ_2 *are given by*

$$
\begin{aligned}
d\Theta_1 &= \kappa_2 \omega_{12} \wedge \Theta_2 \\
d\Theta_2 &= -\omega_{12} \wedge \Theta_1 \\
d\omega_{12} &= -\kappa_1 \Theta_1 \wedge \Theta_2.
\end{aligned}
$$

We can see now that the Gaussian curvature is the constant κ_1, as claimed earlier.

5. Proof of the Gauss–Bonnet theorem

We wish to derive a Gauss–Bonnet formula for each of the nine Cayley–Klein geometries that are two-dimensional. We will show that the following theorem is valid.

Theorem 2. *Let* $M \subset S^2_{[\kappa_1],\kappa_2}$ *be a polygon. Let* Γ*, the boundary of* M*, have length element* ds*, discontinuities* θ_i*, and signed geodesic curvature* κ_g*, where* M *and* Γ

are oriented according to Stokes' theorem. Let K denote the constant Gaussian curvature of the geometry with area form dA. Then

$$\int_\Gamma \kappa_g\, ds + \sum_i \theta_i + \int\int_M K\, dA = 2\pi$$

for the nonspacetimes and

$$\int_\Gamma \kappa_g\, ds + \sum_i \theta_i + \int\int_M K\, dA = 0$$

for the spacetimes, where we assume that Γ is timelike.

Note that this theorem is in agreement with the usual Gauss–Bonnet formula for surfaces of constant curvature when $\kappa_2 > 0$. The Gauss–Bonnet theorem was proved by Helzer [3] and a decade later by Birman and Nomizu [1] for Lorentzian surfaces (which include de Sitter, anti-de Sitter, and Minkowski spacetimes), though Birman and Nomizu were unaware of Helzer's paper. A different Gauss–Bonnet formula for Lorentzian surfaces was proved by Dzan [2], and this formula was later generalized by Law [7] (both authors were also unaware of Helzer's contributions), where now the angle between timelike and spacelike vectors is defined, though these definitions do not agree with that of Helzer. For relative-time spacetimes (where $\kappa_2 < 0$) and where Γ is timelike, the results of Helzer, Birman, and Nomizu agree with ours, but a change in the orientation of M produces a formula that appears to be different. For example, Helzer shows that

$$\int_\Gamma \kappa_g\, ds + \sum_i \theta_i + \int\int_M K\, dA = 0$$

as he orients and co-orients Γ as we do, while Birman and Nomizu show that

$$\int_\Gamma \kappa_g\, ds + \sum_i \theta_i - \int\int_M K\, dA = 0,$$

as a change in co-orientation reverses the sign of the signed quantities κ_g and θ_i. Also note that for spacetimes the measure of an angle is unambiguous, but for nonspacetimes the measure is defined modulo 2π, explaining the need for separate Gauss–Bonnet formulae (but see the papers [2] by Dzan and [7] by Law).

Proof. In concert with the previous section on moving frames, choose a smooth section of the frame bundle over $S^2_{[\kappa_1],\kappa_2}$. It is possible to construct such a section since Σ can be covered by a single coordinate chart. The unit tangent T and unit normal N vectors along Γ are defined by

$$\begin{aligned} T(t) &= \pm\,[C_{\kappa_2}(\theta(t))\cdot E_1(t) + S_{\kappa_2}(\theta(t))\cdot E_2(t)] \\ N(t) &= \pm\,[-\kappa_2 S_{\kappa_2}(\theta(t))\cdot E_1(t) + C_{\kappa_2}(\theta(t))\cdot E_2(t)] \end{aligned}$$

according to the main metric g_1 and, if needed for N, the subsidiary metric g_2, where we use the $+$ sign if $T(t)$ is future-directed or if $\kappa_2 > 0$, and we use the $-$ sign if $T(t)$ is past-directed. Then we can define the signed geodesic curvature κ_g function along Γ by the formulae

$$\begin{aligned} \nabla_T T &= \pm\,\dot\theta\,(-\kappa_2 S_{\kappa_2} E_1 + C_{\kappa_2} E_2) \pm (C_{\kappa_2}\nabla_T E_1 + S_{\kappa_2}\nabla_T E_2) \\ &\equiv \kappa_g N. \end{aligned}$$

We can then write that

$$\begin{aligned}
g_2(\kappa_g N, N) &= \kappa_g \\
&= g_2(\nabla_T T, N) \\
&= \dot\theta \kappa_2 S_{\kappa_2}^2 + \dot\theta C_{\kappa_2}^2 + g_2\left(C_{\kappa_2}\nabla_T E_1 + S_{\kappa_2}\nabla_T E_2, -\kappa_2 S_{\kappa_2} E_1 + C_{\kappa_2} E_2\right) \\
&= \dot\theta + g_2\left(C_{\kappa_2}\nabla_T E_1 + S_{\kappa_2}\nabla_T E_2, -\kappa_2 S_{\kappa_2} E_1 + C_{\kappa_2} E_2\right) \\
&= \dot\theta + g_2\left(C_{\kappa_2}\omega_{12} E_2 - \kappa_2 S_{\kappa_2}\omega_{12} E_1, -\kappa_2 S_{\kappa_2} E_1 + C_{\kappa_2} E_2\right) \\
&= \dot\theta + \kappa_2 S_{\kappa_2}^2 \omega_{12} + C_{\kappa_2}^2 \omega_{12} \\
&= \dot\theta + \omega_{12} \\
&= \kappa_g .
\end{aligned}$$

Finally we apply Stokes' theorem to show that

$$\begin{aligned}
\int\int_M K\, dA &= \int\int \kappa_1 d\Theta_1 \wedge d\Theta_2 = \kappa_1 \mathrm{Area}(M) \\
&= -\int\int_M d\omega_{12} \\
&= -\int_\Gamma \omega_{12}\, dt \\
&= -\sum_{i=1}^n \left[\int_{t_{i-1}}^{t_i} \kappa_g(t)\, dt - \int_{t_{i-1}}^{t_i} \dot\theta(t)\, dt\right] \\
&= -\int_\Gamma \kappa_g\, dt + \sum_{i=1}^n \left(\theta_i(t_i) - \theta_i(t_{i-1})\right) \\
&= -\int_\Gamma \kappa_g\, dt - \sum_{i=1}^n \left(\theta_{i+1}(t_i) - \theta_i(t_i)\right) \\
&= \begin{cases} -\int_\Gamma \kappa_g\, dt - \sum_{i=1}^n \theta_i + 2\pi, & \text{if } \kappa_2 > 0 \\ -\int_\Gamma \kappa_g\, dt - \sum_{i=1}^n \theta_i, & \text{if } \kappa_2 \le 0 \end{cases}
\end{aligned}$$

since $\sum_{i=1}^n (\theta_{i+1}(t_i) - \theta_i(t_i)) = \sum_{i=1}^n \theta_i - 2\pi$ if $\kappa_2 > 0$ (see [9]), and where we identify $\theta_{n+1}(t)$ with $\theta_1(t)$. □

References

[1] Birman, Graciela S.; Nomizu, Katsumi. The Gauss–Bonnet theorem for 2-dimensional spacetimes. *Michigan Math. J.* **31** (1984), 77–81. MR0736471 (85g:53073), Zbl 0591.53053.

[2] Dzan, Jin Jee. Gauss–Bonnet formula for general Lorentzian surfaces. *Geometriae Dedicata* **15** (1984), no. 3, 215–231. MR0739926 (85k:53059), Zbl 0535.53051.

[3] Helzer, Garry. A relativistic version of the Gauss–Bonnet formula. *J. Differential Geom.* **9** (1974), 507–512. MR0348686 (50 #1183), Zbl 0318.53017.

[4] Herranz, Francisco J.; Ortega, Ramón; Santander, Mariano. Homogeneous phase spaces: the Cayley–Klein framework. *Memorias de la Real Academia de Ciencias Exactas, Físicas y Naturales de Madrid* **32** (1998) 59–84. MR1662738 (2000b:53071), Zbl 0958.51026.

[5] Herranz, Francisco J.; Ortega, Ramón; Santander, Mariano. Trigonometry of spacetimes: a new self-dual approach to a curvature/signature (in)dependent trigonometry. *J. Physics A* **33** (2000), 4525–4551, MR1768742 (2001k:53099), Zbl 0965.53049.

[6] Herranz, Francisco J.; Santander, Mariano. Conformal symmetries of spacetimes. *J. Physics A* **35** (2002), 6601–6618. MR1928851 (2004a:53091), Zbl 1039.53075.

[7] Law, Peter R. Neutral geometry and the Gauss–Bonnet theorem for two-dimensional pseudo-Riemannian manifolds. *Rocky Mountain J. Math.* **22** (1992), no. 4, 1365–1383. MR1201099 (94b:53103), Zbl 0772.53042.

[8] Ortega, Ramón; Santander, Mariano. Trigonometry of 'complex Hermitian' type homogeneous symmetric spaces. *J. Physics A* **33** (2000), 4525–4551. MR1768742 (2001k:53099), Zbl 0965.53049.

[9] Spivak, Michael. A comprehensive introduction to differential geometry, vol. III. *Publish or Perish, Inc., Boston, Massachusetts*, 1975, MR0532832 (82g:53003c), Zbl 0439.53003.

[10] Yaglom, I. M. A simple non-Euclidean geometry and its physical basis. An elementary account of Galilean geometry and the Galilean principle of relativity. Heidelberg Science Library. Translated from the Russian by Abe Shenitzer. With the editorial assistance of Basil Gordon. *Springer-Verlag, New York-Heidelberg*, 1979, MR0520230 (80c:51007), Zbl 0393.51013.

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