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On the structure of derivations on certain nonamenable nuclear Banach algebras

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Abstract. We develop the structure of general bounded derivations on the algebra of $\mathfrak{X}^*$-nuclear operators on an infinite-dimensional Banach space $\mathfrak{X}$ that admits a shrinking basis.

Contents

1. Introduction

Let $(\mathfrak{X}, \mathfrak{Y}, \langle \circ, \circ \rangle)$ be a *dual pair of Banach spaces*, i.e., a pair of Banach spaces $(\mathfrak{X}, \mathfrak{Y})$ with a nondegenerate, bounded, bilinear map $\langle \circ, \circ \rangle : \mathfrak{X} \times \mathfrak{Y} \to \mathbb{C}$. By $\mathcal{N}_{\mathfrak{Y}}(\mathfrak{X})$ we denote the class of so-called $\mathfrak{Y}$-*nuclear operators* $T \in \mathcal{B}(\mathfrak{X})$ which can be written as $Tx = \sum_{n=1}^{\infty} \langle x, y_n \rangle x_n$ if $x \in \mathfrak{X}$, with $\{x_n\}_{n=1}^{\infty} \subseteq \mathfrak{X}$, $\{y_n\}_{n=1}^{\infty} \subseteq \mathfrak{Y}$ and $\sum_{n=1}^{\infty} \|x_n\| \|y_n\| < \infty$. The infimum of these series taking over all such representations of T furnish a norm $\|T\|$ of T so that $(\mathcal{N}_{\mathfrak{Y}}(\mathfrak{X}), \|\circ\|)$ becomes a Banach space. By the universal property of the tensor product there is a unique linear function $\tau : \mathfrak{X} \otimes \mathfrak{Y} \to \mathcal{N}_{\mathfrak{Y}}(\mathfrak{X})$ so that $\tau(x \otimes y) = x \odot y$ if $x \in \mathfrak{X}, y \in \mathfrak{Y}$, with $(x \odot y)(z) = \langle z, y \rangle x$ if

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$x, z \in \mathfrak{X}$, $y \in \mathfrak{Y}$. Since $\mathfrak{X} \otimes \mathfrak{Y}$ is dense in the *projective tensor product* then τ admits a unique extension Λ to $\mathfrak{X} \widehat{\otimes} \mathfrak{Y}$. If

$$\mathcal{N}(\mathfrak{X}, \mathfrak{Y}) = \left\{ \mathfrak{q} \in \mathfrak{X} \widehat{\otimes} \mathfrak{Y} : \mathfrak{q} = \sum_{n=1}^{\infty} x_n \otimes y_n, \ \sum_{n=1}^{\infty} \|x_n\| \, \|y_n\| < \infty \right\}$$

then $\mathcal{N}(\mathfrak{X}, \mathfrak{Y})$ is a linear subspace of $\mathfrak{X} \widehat{\otimes} \mathfrak{Y}$. Given $\mathfrak{q} \in \mathcal{N}(\mathfrak{X}, \mathfrak{Y})$ it is readily seen that

$$\|\mathfrak{q}\|_\pi = \inf \left\{ \sum_{n=1}^{\infty} \|x_n\| \, \|y_n\| : \mathfrak{q} = \sum_{n=1}^{\infty} x_n \otimes y_n \right\}$$

and that $\mathcal{N}(\mathfrak{X}, \mathfrak{Y})$ is closed in $\mathfrak{X} \widehat{\otimes} \mathfrak{Y}$. As $\mathfrak{X} \otimes \mathfrak{Y} \subseteq \mathcal{N}(\mathfrak{X}, \mathfrak{Y})$ then $\mathcal{N}(\mathfrak{X}, \mathfrak{Y}) = \mathfrak{X} \widehat{\otimes} \mathfrak{Y}$ and Λ becomes surjective. Indeed, by the same reasoning it is easy to see that $c_0(\mathfrak{X}) \otimes l^1(\mathfrak{Y}) \approx \mathfrak{X} \widehat{\otimes} \mathfrak{Y}$, where $\approx$ denotes an isomorphism of Banach spaces. Now, by the open mapping theorem we deduce that $\mathcal{N}_{\mathfrak{Y}}(\mathfrak{X}) \approx \mathfrak{X} \widehat{\otimes} \mathfrak{Y} / \ker \Lambda$.

The projective tensor product has a Banach algebra structure given by the multiplication $(x_1 \otimes y_1)(x_2 \otimes y_2) = \langle x_2, y_1 \rangle (x_1 \otimes y_2)$ if $x_1, x_2 \in \mathfrak{X}$, $y_1, y_2 \in \mathfrak{Y}$. So, $\mathfrak{X} \widehat{\otimes} \mathfrak{Y}$ becomes *biprojective* and hence *biflat*. Thus $\mathfrak{X} \widehat{\otimes} \mathfrak{Y}$ is *amenable* if and only if it has a *bounded approximate identity* (cf. [8], Theorem 2.21). In the case of Banach pairings this is indeed the case, so that $\mathfrak{X} \widehat{\otimes} \mathfrak{Y}$ is amenable if and only if $\dim(\mathfrak{X}) < \infty$ (cf. [6]). The amenability and *superamenability* of $\mathfrak{X} \widehat{\otimes} \mathfrak{Y}$ impose finite dimensionality of $\mathfrak{X}$ and $\mathfrak{Y}$. Moreover,

Theorem 1 (cf. [10], Theorem 4.3.5, p. 98)**.** *For a dual Banach pair*

$$(\mathfrak{X}, \mathfrak{Y}, \langle \circ, \circ \rangle),$$

the following assertions are equivalent:

(a) $\mathfrak{X} \widehat{\otimes} \mathfrak{Y}$ *is superamenable.*
(b) $\mathfrak{X} \widehat{\otimes} \mathfrak{Y}$ *is amenable.*
(c) $\mathfrak{X} \widehat{\otimes} \mathfrak{Y}$ *has a bounded approximate identity.*
(d) $\mathfrak{X} \widehat{\otimes} \mathfrak{Y}$ *has a bounded left approximate identity.*
(e) $\mathcal{N}_{\mathfrak{Y}}(\mathfrak{X})$ *has a bounded left approximate identity.*
(f) $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) < \infty$.

Throughout this article by $\mathfrak{X}$ we will denote an infinite dimensional Banach space having a *shrinking basis* (see Section 3). The sets of all bounded finite rank operators on $\mathfrak{X}$ and of all bounded derivations on a Banach algebra $\mathfrak{U}$ will be denoted as $\mathfrak{F}(\mathfrak{X})$ and $\mathcal{D}(\mathfrak{U})$ respectively. As usual, the closure in $\mathcal{B}(\mathfrak{X})$ of $\mathfrak{F}(\mathfrak{X})$, the set of approximable operators on $\mathfrak{X}$, will be denoted as $\mathcal{A}(\mathfrak{X})$. We will be concerned about structure theorems of derivations on $\mathcal{N}_{\mathfrak{X}^*}(\mathfrak{X})$. For previous related studies on which the underlying space $\mathfrak{X}$ consists of nuclear or Hilbert–Schmidt operators on a separable Hilbert space the reader can see [1] or [2]. In Section 2 we will assume that $\mathfrak{X}$ satisfies the *approximation property* in the sense of A. Grothendieck, i.e., there is a net $\{S_a\}_{a \in A}$ in $\mathcal{F}(\mathfrak{X})$ such that $\lim_{a \in A} S_a = \mathrm{Id}_{\mathfrak{X}}$ uniformly on compact subsets of $\mathfrak{X}$ (cf. [7], p. 165). Then there is an isometric isomorphism of Banach algebras between $\mathcal{N}_{\mathfrak{X}^*}(\mathfrak{X})$ and $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ (cf. [10], Theorem C.1.5). This will allow us to transfer our investigation to the frame of the projective Banach algebra $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ (see Remark 5). By Theorem 1 the determination of structure theorems of derivations on $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ has its own interest when $\mathfrak{X}$ is infinite-dimensional. To this end we will focus on the case when $\mathfrak{X}$ admits a shrinking basis. So, in Section 3 we will consider an infinite-dimensional Banach space $\mathfrak{X}$ endowed with a shrinking basis and we will show briefly how the basis may be used to construct a basis for $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$. Our main results rely on Theorem 7 in Section 4. In this theorem we will develop the precise structure of general bounded derivations on $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$. Later, we will introduce the notion of Hadamard derivations and we will prove in Theorem 10 that they constitute a Banach complementary subspace of $\mathcal{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$. Finally, in Proposition 12 and Theorem 13 we will investigate how bounded operators on $\mathfrak{X}$ induce bounded derivations on $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$.

2. An isomorphism theorem

Proposition 2. *Let $\tau : \mathfrak{X}\widehat{\otimes}\mathfrak{X}^* \to \mathcal{N}_{\mathfrak{X}^*}(\mathfrak{X})$ be the unique bounded operator so that $\tau(x\otimes x^*) = x\odot x^*$ for all basic tensors. If $\mathfrak{X}$ satisfies the approximation property then τ is a Banach algebra isometric isomorphism.*

Proof. Let $u \in \mathfrak{X}\widehat{\otimes}\mathfrak{X}^* - \{0\}$, say $u = \sum_{n=1}^{\infty} x_n \otimes x_n^*$ with $\sum_{n=1}^{\infty} \|x_n\| \|x_n^*\| < \infty$. Indeed, we can assume that $x_n \to 0$ and $\sum_{n=1}^{\infty} \|x_n^*\| < \infty$. Since $\{x_n\}_{n=1}^{\infty} \cup \{0\}$ is compact and $\mathfrak{X}$ has the approximation property, we may find $S \in \mathcal{F}(\mathfrak{X})$ so that

$$\|Sx_n - x_n\| < \frac{\|u\|_\pi}{2\sum_{n=1}^{\infty} \|x_n^*\|}$$

for all $n \in \mathbb{N}$. If $v = \sum_{n=1}^{\infty} Sx_n \otimes x_n^*$ then

$$\|u - v\|_\pi \leq \sum_{n=1}^{\infty} \|Sx_n - x_n\| \|x_n^*\| < \|u\|_\pi /2,$$

and so $v \neq 0$. Let $S = \sum_{m=1}^{p} y_m \odot y_m^*$ with $y_m \in \mathfrak{X}$, $y_m^* \in \mathfrak{X}^*$. Since

$$v = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{p} \langle x_n, y_m^* \rangle y_m \right) \otimes x_n^* = \sum_{m=1}^{p} y_m \otimes \sum_{n=1}^{\infty} \langle x_n, y_m^* \rangle x_n^* \neq 0$$

there is $y^* \in \mathfrak{X}^*$ so that $\sum_{n=1}^{\infty} \langle x_n, y^* \rangle x_n^* \neq 0$ in $\mathfrak{X}^*$. Consequently, there exists $y^{**} \in \mathfrak{X}^{**}$ so that

$$0 \neq \left\langle \sum_{n=1}^{\infty} \langle x_n, y^* \rangle x_n^*, y^{**} \right\rangle = \sum_{n=1}^{\infty} \langle x_n, y^* \rangle \langle x_n^*, y^{**} \rangle = (y^* \otimes y^{**})(u),$$

i.e., $u \neq 0$ and the canonical map $\iota : \mathfrak{X}\widehat{\otimes}\mathfrak{X}^* \hookrightarrow \mathfrak{X} \overset{\vee}{\otimes} \mathfrak{X}^*$ is injective. But, if $w = \sum_{j=1}^{q} z_j \otimes z_j^*$ in $\mathfrak{X} \otimes \mathfrak{X}^*$ then

$$\begin{aligned}
\|\tau(w)\| &= \|\tau(w)^{**}\| \\
&= \sup_{\|x^{**}\|_{\mathfrak{X}^{**}}=1} \|x^{**} \circ \tau(w)^*\| \\
&= \sup_{\|x^{**}\|_{\mathfrak{X}^{**}}=1} \sup_{\|x^*\|_{\mathfrak{X}^*=1}} |\langle x^*, x^{**} \circ \tau(w)^* \rangle| \\
&= \sup_{\|x^{**}\|_{\mathfrak{X}^{**}}=1} \sup_{\|x^*\|_{\mathfrak{X}^*=1}} |\langle x^* \circ \tau(w), x^{**} \rangle| \\
&= \sup_{\|x^{**}\|_{\mathfrak{X}^{**}}=1} \sup_{\|x^*\|_{\mathfrak{X}^*=1}} \left| \sum_{j=1}^{q} \langle z_j, x^* \rangle \langle z_j^*, x^{**} \rangle \right| \\
&= \sup_{\|x^{**}\|_{\mathfrak{X}^{**}}=1} \sup_{\|x^*\|_{\mathfrak{X}^*=1}} |(x^* \otimes x^{**})(w)| = \|w\|_\epsilon,
\end{aligned}$$

i.e., $\tau\mid_{\mathfrak{X}\otimes\mathfrak{X}^*}: \mathfrak{X} \otimes \mathfrak{X}^* \hookrightarrow \mathcal{F}(\mathfrak{X})$ is an isometry. Therefore τ extends to an isometric isomorphism $\widetilde{\tau}$ between $\mathfrak{X} \overset{\vee}{\otimes} \mathfrak{X}^*$ and $\mathcal{A}(\mathfrak{X})$. Certainly, $\tau = \widetilde{\tau} \circ \iota$ becomes isometric. Indeed, we already know that τ is onto and by definition of the nuclear norm it is an isometric isomorphism. Finally, since

$$\begin{aligned}
\tau((x_1 \otimes x_1^*)(x_2 \otimes x_2^*)) &= \langle x_2, x_1^* \rangle \tau(x_1 \otimes x_2^*) \\
&= \langle x_2, x_1^* \rangle x_1 \odot x_2^* \\
&= (x_1 \odot x_1^*) \circ (x_2 \odot x_2^*) \\
&= \tau(x_1 \otimes x_1^*) \circ \tau(x_2 \otimes x_2^*)
\end{aligned}$$

if $x_1, x_2 \in \mathfrak{X}$, $x_1^*, x_2^* \in \mathfrak{X}^*$ the assertion follows. □

3. Shrinking basis and tensor products

If $\{x_n\}_{n=1}^{\infty}$ is a *basis* of $\mathfrak{X}$ there is $\{x_n^*\}_{n=1}^{\infty} \subseteq \mathfrak{X}^*$ so that $\langle x_n, x_m^* \rangle = \delta_{n,m}$ with $n, m \in \mathbb{N}$, i.e., $\{x_n^*\}_{n=1}^{\infty}$ is *the associated sequence of coefficient functionals* (*a.s.c.f.*) *of* $\{x_n\}_{n=1}^{\infty}$ (see [12], Theorem 3.1, p. 20). Certainly, $\{x_n^*\}_{n=1}^{\infty}$ need not be a basis of $\mathfrak{X}^*$ since $\mathfrak{X}^*$ may be nonseparable and so it may have no basis at all. However, $\{x_n^*\}_{n=1}^{\infty}$ is a basis if $\mathfrak{X}$ is a reflexive Banach space (cf. [9]). In the sequel, we will assume that $\{x_n\}_{n=1}^{\infty}$ is a *shrinking basis*. This means, by definition, that the a.s.c.f., $\{x_n^*\}_{n=1}^{\infty}$, is a basis for $\mathfrak{X}^*$ (cf. [3], [5]).

Proposition 3 (cf. [11], [12])**.** *Let $\mathfrak{X}$ be a Banach space, let $\{x_n\}_{n=1}^{\infty}$ be a shrinking basis of $\mathfrak{X}$ and let $\{x_n^*\}_{n=1}^{\infty}$ be the a.s.c.f. Then the system of all basic tensor products $x_n \otimes x_m^*$ is a basis of $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$, when arranged into a single sequence as follows: if $m \in \mathbb{N}$ let $n \in \mathbb{N}$ so that $(n-1)^2 < m \leq n^2$ and then write*

$$z_m = x_{\sigma_1(m)} \otimes x_{\sigma_2(m)},$$

with

$$\sigma(m) = \begin{cases} \big(m-(n-1)^2, n\big) & \textit{if } (n-1)^2+1 \le m \le (n-1)^2+n, \\ \big(n, n^2-m+1\big) & \textit{if } (n-1)^2+n \le m \le n^2. \end{cases}$$

Remark 4. In particular, $\sigma : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ becomes a bijective function. Since $\mathfrak{X}^*\widehat{\otimes}\mathfrak{X} \hookrightarrow (\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)^*$ we will also write $z_m^* = x^*_{\sigma_1(m)} \otimes x_{\sigma_2(m)}$, $m \in \mathbb{N}$. Thus $\{z_m^*\}_{n=1}^\infty$ becomes the a.s.c.f. of $\{z_m\}_{n=1}^\infty$.

Remark 5. A Banach space $\mathfrak{X}$ with a shrinking basis $\{x_n\}_{n=1}^\infty$ satisfies the approximation property. For, it suffices to observe that if $\{x_n^*\}_{n=1}^\infty$ is the corresponding a.s.c.f. the sequence $\{\sum_{n=1}^m x_n \odot x_n^*\}_{m=1}^\infty$ in $\mathcal{F}(\mathfrak{X})$ converges uniformly on compact subsets of $\mathfrak{X}$ to $\mathrm{Id}_{\mathfrak{X}}$ (cf. [10], p. 255).

The proof of the following is straightforward.

Proposition 6. *With the above notation, for $n, m \in \mathbb{N}$ the following assertions hold:*

$$\sigma_1^{-1}(\{n\}) = \left\{n^2-k+1\right\}_{1\le k\le n} \cup \left\{k^2+n\right\}_{k\ge n},$$
$$\sigma_2^{-1}(\{m\}) = \left\{(m-1)^2+k\right\}_{1\le k\le m} \cup \left\{k^2-m+1\right\}_{k>m}.$$

In particular

$$\sigma^{-1}(n,m) = \begin{cases} n^2-m+1 & \textit{if } 1 \le m \le n, \\ (m-1)^2+n & \textit{if } m > n. \end{cases} \tag{1}$$

4. Some structure theorems

Theorem 7. *Let $\mathfrak{X}$ be an infinite-dimensional Banach space with a shrinking basis $\{x_n\}_{n=1}^\infty$. Given $\delta \in \mathcal{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ there are unique sequences $\{\mathfrak{h}_n\}_{n\in\mathbb{N}}$ and $\{\mathfrak{y}_u^v\}_{u,v\in\mathbb{N}}$ so that if $u, v \in \mathbb{N}$ then*

$$\delta\left(z_{\sigma^{-1}(u,v)}\right) = \left(\mathfrak{h}_u - \mathfrak{h}_v\right) z_{\sigma^{-1}(u,v)} + \sum_{n=1}^\infty \left(\mathfrak{y}_u^n \cdot z_{\sigma^{-1}(n,v)} - \mathfrak{y}_n^v \cdot z_{\sigma^{-1}(u,n)}\right). \tag{2}$$

In the sequel we will say that they are the $\mathfrak{h}$ and $\mathfrak{y}$ sequences of δ.

Proof. Let $\left\{\mathfrak{h}_{u,v}^n\right\}_{u,v,n\in\mathbb{N}} \subseteq \mathbb{C}$ so that $\delta\left(z_{\sigma^{-1}(u,v)}\right) = \sum_{n=1}^\infty \mathfrak{h}_{u,v}^n z_n$ if $u, v \in \mathbb{N}$. If $u, v, t \in \mathbb{N}$ then $x_u \otimes x_v^* = (x_u \otimes x_t^*)(x_t \otimes x_v^*)$ and so

$$\begin{aligned}(3)\qquad \delta\left(x_u\otimes x_v^*\right) &= \sum_{n=1}^{\infty}\mathfrak{h}_{u,v}^{n}\left(x_{\sigma_1(n)}\otimes x_{\sigma_2(n)}^*\right)\\ &= \delta\left(x_u\otimes x_t^*\right)\left(x_t\otimes x_v^*\right)+\left(x_u\otimes x_t^*\right)\delta\left(x_t\otimes x_v^*\right)\\ &= \sum_{n\in\sigma_1^{-1}(\{t\})}\mathfrak{h}_{t,v}^{n}\left(x_u\otimes x_{\sigma_2(n)}^*\right)\\ &\quad+\sum_{n\in\sigma_2^{-1}(\{t\})}\mathfrak{h}_{u,t}^{n}\left(x_{\sigma_1(n)}\otimes x_v^*\right)\\ &= \sum_{n=1}^{t}\left\{\mathfrak{h}_{u,t}^{(t-1)^2+n}z_{\sigma^{-1}(n,v)}+\mathfrak{h}_{t,v}^{t^2-n+1}z_{\sigma^{-1}(u,n)}\right\}\\ &\quad+\sum_{n=t}^{\infty}\left\{\mathfrak{h}_{u,t}^{n^2-t+1}z_{\sigma^{-1}(n,v)}+\mathfrak{h}_{t,v}^{n^2+t}z_{\sigma^{-1}(u,n+1)}\right\}.\end{aligned}$$

Therefore by (3) we get $\mathfrak{h}_{u,v}^{n}=0$ if $n\notin\sigma_1^{-1}(\{u\})\cup\sigma_2^{-1}(\{v\})$ for all $u,v,n\in\mathbb{N}$. Now, by evaluating $\mathfrak{h}_{u,v}^{\sigma^{-1}(u,v)}$ from (3) the following assertions hold:

$$\begin{aligned}(4)\qquad t\le u<v &\Rightarrow \mathfrak{h}_{u,v}^{(v-1)^2+u}=\mathfrak{h}_{t,v}^{(v-1)^2+t}+\mathfrak{h}_{u,t}^{u^2-t+1},\\ u<t<v &\Rightarrow \mathfrak{h}_{u,v}^{(v-1)^2+u}=\mathfrak{h}_{t,v}^{(v-1)^2+t}+\mathfrak{h}_{u,t}^{(t-1)^2+u},\\ u<v\le t &\Rightarrow \mathfrak{h}_{u,v}^{(v-1)^2+u}=\mathfrak{h}_{t,v}^{t^2-v+1}+\mathfrak{h}_{u,t}^{(t-1)^2+u}.\end{aligned}$$

In particular, $\mathfrak{h}_{n,n}^{\sigma^{-1}(n,n)}=0$ if $n\in\mathbb{N}$. Likewise,

$$\begin{aligned}(5)\qquad t<u=v &\Rightarrow \mathfrak{h}_{t,u}^{(u-1)^2+t}+\mathfrak{h}_{u,t}^{u^2-t+1}=0,\\ t=u=v &\Rightarrow \mathfrak{h}_{t,t}^{t^2-t+1}=0,\\ u=v<t &\Rightarrow \mathfrak{h}_{t,u}^{t^2-u+1}+\mathfrak{h}_{u,t}^{(t-1)^2+u}=0.\end{aligned}$$

Finally,

$$\begin{aligned}(6)\qquad t\le v<u &\Rightarrow \mathfrak{h}_{u,v}^{u^2-v+1}=\mathfrak{h}_{t,v}^{(v-1)^2+t}+\mathfrak{h}_{u,t}^{u^2-t+1},\\ v<t\le u &\Rightarrow \mathfrak{h}_{u,v}^{u^2-v+1}=\mathfrak{h}_{t,v}^{t^2-v+1}+\mathfrak{h}_{u,t}^{u^2-t+1},\\ v<u<t &\Rightarrow \mathfrak{h}_{u,v}^{u^2-v+1}=\mathfrak{h}_{t,v}^{t^2-v+1}+\mathfrak{h}_{u,t}^{(t-1)^2+u}.\end{aligned}$$

Consequently, from (4), (5) and (6) we deduce that

$$(7)\qquad \mathfrak{h}_{u,v}^{\sigma^{-1}(u,v)}=\mathfrak{h}_{u,t}^{\sigma^{-1}(u,t)}+\mathfrak{h}_{t,v}^{\sigma^{-1}(t,v)}\quad\text{if}\quad u,v\in\mathbb{N}.$$

By (7) we can write

$$(8)\qquad \mathfrak{h}_{u,v}^{\sigma^{-1}(u,v)}=\mathfrak{h}_{u,1}^{\sigma^{-1}(u,1)}+\mathfrak{h}_{1,v}^{\sigma^{-1}(1,v)}=\mathfrak{h}_{u,1}^{\sigma^{-1}(u,1)}-\mathfrak{h}_{v,1}^{\sigma^{-1}(v,1)}.$$

Let us write

$$\mathfrak{h}_n = \mathfrak{h}_{n,1}^{\sigma^{-1}(n,1)}, n \in \mathbb{N}. \tag{9}$$

Now, let $n, u, v \in \mathbb{N}$, $n < u$. By (3) we have

$$\begin{aligned} \text{if } n \leq t \quad &\Rightarrow \quad \mathfrak{h}_{u,v}^{\sigma^{-1}(n,v)} = \mathfrak{h}_{u,t}^{(t-1)^2+n}, \\ \text{if } t < n \quad &\Rightarrow \quad \mathfrak{h}_{u,v}^{\sigma^{-1}(n,v)} = \mathfrak{h}_{u,t}^{n^2-t+1}, \end{aligned}$$

i.e., $\mathfrak{h}_{u,v}^{\sigma^{-1}(n,v)} = \mathfrak{h}_{u,t}^{\sigma^{-1}(n,t)}$. The same conclusion holds if $u < n$. Therefore we deduce the existence of doubly indexed sequences $\{\mathfrak{y}_u^p\}_{u,p\in\mathbb{N}}$, $\{\mathfrak{z}_v^q\}_{v,q\in\mathbb{N}}$ so that $\mathfrak{y}_u^p = \mathfrak{h}_{u,n}^{\sigma^{-1}(p,n)}$ and $\mathfrak{z}_v^q = \mathfrak{h}_{n,v}^{\sigma^{-1}(n,q)}$ if $u, v, p, q, n \in \mathbb{N}$. In particular, we already know that $\mathfrak{y}_n^n = \mathfrak{z}_n^n = 0$ if $n \in \mathbb{N}$. Thus

$$\delta\left(x_u \otimes x_v^*\right) = \mathfrak{h}_{u,v}^{\sigma^{-1}(u,v)} z_{\sigma^{-1}(u,v)} + \sum_{n=1}^{\infty}\left(\mathfrak{y}_u^n \cdot z_{\sigma^{-1}(n,v)} + \mathfrak{z}_v^n \cdot z_{\sigma^{-1}(u,n)}\right) \tag{10}$$

and

$$\delta_{u,v} \cdot \delta\left(x_u \otimes x_v^*\right) = \delta\left(x_u \otimes x_v^*\right)\left(x_u \otimes x_v^*\right) + \left(x_u \otimes x_v^*\right)\delta\left(x_u \otimes x_v^*\right).$$

Hence

$$\begin{aligned} &\delta_{u,v} \cdot \delta\left(x_u \otimes x_v^*\right) \\ &= \delta_{u,v}\left[2\mathfrak{h}_{u,v}^{\sigma^{-1}(u,v)} z_{\sigma^{-1}(u,v)} + \sum_{n=1}^{\infty}\left(\mathfrak{y}_u^n \cdot z_{\sigma^{-1}(n,v)} + \mathfrak{z}_v^n \cdot z_{\sigma^{-1}(u,n)}\right)\right] \\ &\quad + \left(\mathfrak{y}_u^v + \mathfrak{z}_v^u\right) \cdot \left(x_u \otimes x_v^*\right), \end{aligned}$$

i.e., $\mathfrak{y}_u^v + \mathfrak{z}_v^u = 0$ if $u \neq v$ in $\mathbb{N}$. Finally, (2) follows from (8), (9) and (10). $\square$

Remark 8. If $n, m \in \mathbb{N}$ then

$$\left\langle \delta\left(z_m\right), z_n^*\right\rangle = \begin{cases} \mathfrak{h}_{\sigma_1(m)} - \mathfrak{h}_{\sigma_2(m)} & \text{if } m = n, \\ \mathfrak{y}_{\sigma_1(m)}^{\sigma_1(n)} & \text{if } \sigma_1(n) \neq \sigma_1(m) \text{ and } \sigma_2(n) = \sigma_2(m), \\ -\mathfrak{y}_{\sigma_2(n)}^{\sigma_2(m)} & \text{if } \sigma_1(n) = \sigma_1(m) \text{ and } \sigma_2(n) \neq \sigma_2(m), \\ 0 & \text{if } \sigma_1(n) \neq \sigma_1(m) \text{ and } \sigma_2(n) \neq \sigma_2(m) \end{cases}$$

and

$$\delta(\mathfrak{q}) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\left\langle\delta\left(z_m\right), z_n^*\right\rangle\left\langle\mathfrak{q}, z_m^*\right\rangle\right) z_n \text{ if } \mathfrak{q} \in \mathfrak{X}\widehat{\otimes}\mathfrak{X}^*.$$

Definition 9. With the notation of Theorem 7 and if $\mathcal{X} = \{x_n\}_{n=1}^{\infty}$ is a fixed basis of a Banach space $\mathfrak{X}$, a derivation $\delta \in \mathcal{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ is called an $\mathcal{X}$-*Hadamard derivation* if its $\mathfrak{y}$-sequence is the null sequence. We will denote the set of all those derivations by $\mathcal{D}_{\mathcal{X}}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$.

Theorem 10. *Let $\mathfrak{X}$ be a Banach space with a shrinking basis $\mathcal{X} = \{x_n\}_{n=1}^{\infty}$ and let $\{z_n\}_{n=1}^{\infty}$ be the induced basis of $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ as in Proposition 3. Then*

(a) $\mathcal{D}_{\mathcal{X}}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ is a Banach subspace of $\mathcal{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$.
(b) $\mathcal{D}_{\mathcal{X}}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*) \hookrightarrow \mathrm{M}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*, \{z_n\}_{n=1}^{\infty})$, i.e., there is an isometric isomorphism from the space of *$\mathcal{X}$-Hadamard derivations into the multiplier Banach space of* $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ on the basis $\{z_n\}_{n=1}^{\infty}$.
(c) The Banach space $\mathcal{D}_{\mathcal{X}}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ is complementable.

Proof. (a) Observe that

$$\mathcal{D}_{\mathcal{X}}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*) = \bigcap_{n=1}^{\infty} \left\{\delta \in \mathcal{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*) : \delta\left(z_{\sigma^{-1}(n,n)}\right) = 0\right\}.$$

(b) By Theorem 7 given $\delta \in \mathcal{D}_{\mathcal{X}}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ we can write

$$\delta(\mathfrak{q}) = \sum_{n=1}^{\infty} \left(\mathfrak{h}_{\sigma_1(n)} - \mathfrak{h}_{\sigma_2(n)}\right) \langle \mathfrak{q}, z_m^* \rangle \cdot z_n, \ \mathfrak{q} \in \mathfrak{X}\widehat{\otimes}\mathfrak{X}^*. \tag{11}$$

So, the sequence $\mathfrak{h}_\delta = \left\{\mathfrak{h}_{\sigma_1(n)} - \mathfrak{h}_{\sigma_2(n)}\right\}_{n=1}^{\infty}$ becomes a multiplier of $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$. Indeed, by (11) we have $\delta = \mathrm{M}_{\mathfrak{h}_\delta}$, where M is the usual isometric algebraic isomorphism (cf. [13]) of $\mathrm{M}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*, \{z_n\}_{n=1}^{\infty})$ into $\mathcal{B}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ given by

$$\mathrm{M}\left(\{c_n\}_{n=1}^{\infty}\right)(\mathfrak{q}) = \sum_{n=1}^{\infty} c_n \langle \mathfrak{q}, z_n^* \rangle \cdot z_n \text{ if } \mathfrak{q} \in \mathfrak{X}\widehat{\otimes}\mathfrak{X}^*.$$

(c) Let $\mathcal{D}_{\mathcal{X}}^{\perp}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ be the set of bounded derivations on $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ with null $\mathfrak{h}$-sequences. Since

$$\mathcal{D}_{\mathcal{X}}^{\perp}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*) = \bigcap_{n=1}^{\infty} \left\{\delta \in \mathcal{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*) : \left\langle \delta\left(z_{n^2}\right), z_{n^2}^* \right\rangle = 0\right\}$$

we deduce that $\mathcal{D}_{\mathcal{X}}^{\perp}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ is a Banach space and by Theorem 7 we have

$$\mathcal{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*) = \mathcal{D}_{\mathcal{X}}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*) + \mathcal{D}_{\mathcal{X}}^{\perp}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*).$$

Finally, it is immediate that $\mathcal{D}_{\mathcal{X}}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*) \cap \mathcal{D}_{\mathcal{X}}^{\perp}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*) = \{0\}$. □

Proposition 11. *Let $\mathfrak{X}$ be an infinite-dimensional Banach space with a shrinking basis $\{x_n\}_{n=1}^{\infty}$. Given $\delta \in \mathcal{D}_{\mathcal{X}}^{\perp}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ there are unique subsets $\{y_n\}_{n=1}^{\infty}$ and $\{y_n^*\}_{n=1}^{\infty}$ of $\mathfrak{X}$ and $\mathfrak{X}^*$ respectively so that*

$$\delta\left(z_{\sigma^{-1}(u,v)}\right) = y_u \otimes x_v^* - x_u \otimes y_v^* \textit{ if } u, v \in \mathbb{N}. \tag{12}$$

Proof. By Theorem 7 each series in (2) converges. Since

$$\begin{aligned}\overline{\lim}_{p,q\to\infty} \left\| \sum_{n=p}^{p+q} \mathfrak{y}_u^n \cdot x_n \right\| &= \frac{1}{\|x_v^*\|} \overline{\lim}_{p,q\to\infty} \left\| \left(\sum_{n=p}^{p+q} \mathfrak{y}_u^n \cdot x_n \right) \otimes x_v^* \right\|_\pi \\ &= \frac{1}{\|x_v^*\|} \overline{\lim}_{p,q\to\infty} \left\| \sum_{n=p}^{p+q} \mathfrak{y}_u^n \cdot (x_n \otimes x_v^*) \right\|_\pi = 0\end{aligned}$$

the series $\sum_{n=1}^{\infty} \mathfrak{y}_u^n \cdot x_n$ converges to an element $y_u \in \mathfrak{X}$ if $u \in \mathbb{N}$. Analogously, let $y_v^* = \sum_{n=1}^{\infty} \mathfrak{y}_n^v \cdot x_n^*$, $v \in \mathbb{N}$. Now (12) holds and our claim follows. □

The proof of the following is straightforward.

Proposition 12. *Let $\mathfrak{X}$ be a Banach space, $T \in \mathcal{B}(\mathfrak{X})$. There exists a unique $\delta_T \in \mathcal{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ so that*

$$\delta_T (x \otimes x^*) = T(x) \otimes x^* - x \otimes T^*(x^*) \text{ if } x \in \mathfrak{X} \text{ and } x^* \in \mathfrak{X}^*.$$

Theorem 13. *Let $\mathfrak{X}$ be an infinite-dimensional Banach space with a shrinking basis $\{x_n\}_{n=1}^{\infty}$. Let $\delta \in \mathcal{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ whose associated $\mathfrak{h}$-sequence established in Theorem 7 is zero and let $\{y_n\}_{n=1}^{\infty}$, $\{y_n^*\}_{n=1}^{\infty}$ be the unique sequences determinated by δ in Proposition 11. Then, if*

$$\sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^{n} y_i \otimes x_i^* \right\|_{\pi} < \infty \tag{13}$$

there is a unique $T \in \mathcal{B}(\mathfrak{X})$ so that $\delta = \delta_T$.

Proof. Write $T(x_u) = y_u$, $u \in \mathbb{N}$. Then

$$\begin{aligned}
\sup_{\|x\|_{\mathfrak{X}} \le 1} \left\| \sum_{i=1}^{n} y_i \cdot \langle x, x_i^* \rangle \right\|_{\mathfrak{X}} &= \left\| \sum_{i=1}^{n} y_i \odot x_i^* \right\|_{\mathcal{B}(\mathfrak{X})} \\
&= \left\| \tau \left(\sum_{i=1}^{n} y_i \otimes x_i^* \right) \right\|_{\mathcal{B}(\mathfrak{X})} \\
&= \left\| \sum_{i=1}^{n} y_i \otimes x_i^* \right\|_{\pi} \\
&\le \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^{n} y_i \otimes x_i^* \right\|_{\pi} < \infty
\end{aligned}$$

and by linearity T becomes a bounded linear operator on $\mathfrak{X}$ (cf. [4]). So, if $x \in \mathfrak{X}$ and $v \in \mathbb{N}$ we have

$$\begin{aligned}
\langle x, T^*(x_v^*) \rangle &= \langle T(x), x_v^* \rangle \\
&= \sum_{u=1}^{\infty} \langle x, x_u^* \rangle \langle T(x_u), x_v^* \rangle \\
&= \sum_{u=1}^{\infty} \langle x, x_u^* \rangle \left\langle \sum_{n=1}^{\infty} \mathfrak{h}_u^n \cdot x_n, x_v^* \right\rangle \\
&= \sum_{u=1}^{\infty} \langle x, x_u^* \rangle \cdot \mathfrak{h}_u^v = \langle x, y_v^* \rangle,
\end{aligned}$$

i.e., $T^*(x_v^*) = y_v^*$. Thus by (12) $T_\delta(z_n) = \delta(z_n)$ if $n \in \mathbb{N}$, i.e., $T = T_\delta$. □

Example 14. Assume that $\{\mathfrak{y}_n^m\}_{n,m\in\mathbb{N}} \in l^1(\mathbb{N}\times\mathbb{N})$ and that $\{x_n\}_{n=1}^{\infty}$ is a bounded shrinking basis of $\mathfrak{X}$, i.e., assume that

$$0 < \inf_{n\in\mathbb{N}} \|x_n\| \triangleq \iota < \sigma \triangleq \sup_{n\in\mathbb{N}} \|x_n\| < \infty.$$

Then (13) holds. For, there exists $M > 0$ so that $1 \leq \|x_n\| \|x_n^*\| \leq M$ if $n \in \mathbb{N}$ (cf. [12], Theorem 3.1, p. 20). Thus,

$$\begin{aligned}\left\|\sum_{i=1}^{n} y_i \otimes x_i^*\right\|_\pi &\leq \frac{M}{\iota}\sum_{i=1}^{n}\|y_i\| \leq \frac{M}{\iota}\sum_{i=1}^{n}\sum_{j=1}^{\infty}\left|\mathfrak{y}_i^j\right| \|x_j\| \\ &\leq \frac{M\sigma}{\iota}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\mathfrak{y}_i^j\right| \ \forall n\in\mathbb{N}.\end{aligned}$$

Example 15. For $\mathfrak{q} \in \mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ we have

$$\begin{aligned}\delta_{x_1\odot x_2^*}(\mathfrak{q}) &= \langle \mathfrak{q}, z_4^*\rangle z_1 - \langle \mathfrak{q}, z_1^*\rangle z_2 \\ &\quad + \sum_{n=2}^{\infty}\left[\left\langle \mathfrak{q}, z_{(n-1)^2+2}^*\right\rangle z_{(n-1)^2+1} - \left\langle \mathfrak{q}, z_{n^2}^*\right\rangle z_{n^2-1}\right] \\ &= \sum_{n=1}^{\infty}\left[\left\langle \mathfrak{q}, z_{\sigma^{-1}(2,n)}\right\rangle x_1\otimes x_n^* - \left\langle \mathfrak{q}, z_{\sigma^{-1}(n,1)}\right\rangle x_n\otimes x_2^*\right].\end{aligned}$$

With the notation of Theorem 7, since $\mathfrak{y}_2^1 = 1$ then

$$\delta_{x_1\odot x_2^*} \in \mathcal{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*) - \mathcal{D}_{\mathcal{X}}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*).$$

Indeed, it is easy to see that $\mathfrak{h}_n = 0$ if $n \in \mathbb{N}$.

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