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Transparent rings and their extensions

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Abstract. Skew polynomial rings have invited attention of mathematicians and various properties of these rings have been discussed. The nature of ideals (in particular prime ideals, minimal prime ideals, associated prime ideals), primary decomposition and Krull dimension have been investigated in certain cases. In this article, we introduce a notion of primary decomposition of a noncommutative ring. We say that a Noetherian ring satisfying this type of primary decomposition is a *transparent ring*. We then show that if R is a commutative Noetherian $\mathbb{Q}$-algebra ($\mathbb{Q}$, the field of rational numbers) and σ is an automorphism of R, then there exists an integer $m \geq 1$ such that the Ore extension $R[x;\alpha,\delta]$ is a *transparent ring*, where $\sigma^m = \alpha$ and δ is an α-derivation of R such that $\alpha(\delta(a)) = \delta(\alpha(a))$, for all $a \in R$.

Contents

1. Introduction

Throughout this article R is an associative ring with identity and any R-module is unitary. Spec(R) denotes the set of prime ideals of R. Min.Spec(R) denotes the set of minimal prime ideals of R. The set of associated prime ideals of R (viewed as a right R-module over itself) is denoted by Ass(R_R). $C(0)$ denotes the set of regular elements of R. $C(I)$ denotes the set of elements of R regular modulo I, where I is an ideal of R. $N(R)$ denotes the

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prime radical of R. Let I and J be any two subsets of a ring R. Then $I \subset J$ means that I is strictly contained in J.

For any right R-module K, the right Krull dimension of K is denoted by $|K|_r$. Similarly if J is a left R-module then left Krull dimension of J is denoted by $|J|_l$. Recall that the right Krull dimension of a ring R is defined as the Krull dimension of R, viewed as a right module over itself. Left Krull dimension of a ring R is defined similarly. For further details on Krull dimension, the reader is referred to [10]. The field of rational numbers, the ring of integers and the set of positive integers are denoted by $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ respectively unless otherwise stated.

Let σ be an automorphism of R and δ a σ-derivation of R, i.e.,

$$\delta(ab) = \delta(a)\sigma(b) + a\delta(b) \quad \text{for all } a,\, b \in R.$$

Recall that the Ore-extension $R[x;\sigma,\delta] = \{f = \sum_{i=0}^{n} x^i a_i,\ a_i \in R\}$ subject to the relation ax $= x\sigma(a)+\delta(a)$ for all $a \in R$. We denote $R[x;\sigma,\delta]$ by $O(R)$. We take coefficients on the right as in McConnell and Robson [15]. In case σ is the identity map, we denote the differential operator ring $R[x;\delta]$ by $D(R)$. Alternatively, if δ is the zero map, we denote $R[x;\sigma]$ by $S(R)$. The skew Laurent polynomial ring $R[x;x^{-1},\sigma]$ is denoted by $L(R)$. For more details on Ore extensions (the skew polynomial rings), we refer the reader to Chapter 1 of McConnell and Robson [15]. The notion of the quotient ring of a ring, the contractions and extensions of ideals arising thereby appear in Chapter 9 of Goodearl and Warfield [8].

The classical study of any commutative Noetherian ring is done by studying its primary decomposition. Further there are other structural properties of rings, for example the existence of quotient rings or more particularly the existence of Artinian quotient rings etc. which can be nicely tied to primary decomposition of a Noetherian ring.

It is shown in Blair and Small [5] that if R is embeddable in a right Artinian ring and has characteristic zero, then the differential operator ring $R[x;\delta]$ embeds in a right Artinian ring, where δ is a derivation of R. It is also shown in Blair and Small [5] that if R is a commutative Noetherian ring and σ is an automorphism of R, then the skew-polynomial ring $R[x;\sigma]$ embeds in an Artinian ring. For more results on the existence of the Artinian quotient rings, the reader is referred to Robson [15].

In this paper the above mentioned properties have been studied with emphasis on primary decomposition of the Ore extension $O(R)$, where R is a commutative Noetherian $\mathbb{Q}$-algebra, σ and δ as usual.

A noncommutative analogue of associated prime ideals of a Noetherian ring has also been discussed. We would like to note that considerable work has been done in the investigation of prime ideals (in particular minimal prime ideals and associated prime ideals) of skew polynomial rings (K. R.

Goodearl and E. S. Letzter [9], C. Faith [6], S. Annin [1], Leroy and Matczuk [12], Nordstrom [14] and Bhat [4]).

In Section 4 of [9] Goodearl and Letzter have proved that if R is a Noetherian ring, then for each prime ideal P of $O(R)$, the prime ideals of R minimal over $P \cap R$ are contained within a single σ-orbit of Spec(R).

The author has proved in Theorem 2.4 of [4] that if σ is an automorphism of a Noetherian ring R and $K(R)$ is any of $S(R)$ or $L(R)$, then $P \in \mathrm{Ass}(K(R)_{K(R)})$ if and only if there exists $U \in \mathrm{Ass}(R_R)$ such that $K(P \cap R) = P$ and $(P \cap R) = \cap_{i=0}^{m} \sigma^i(U)$, where $m \geq 1$ is an integer such that $\sigma^m(V) = V$ for all $V \in \mathrm{Ass}(R_R)$. (The same result has been proved for minimal prime ideal case).

Carl Faith has proved in [6] that if R is a commutative ring, then the associated prime ideals of the usual polynomial ring $R[x]$ (viewed as a module over itself) are precisely the ideals of the form $P[x]$, where P is an associated prime ideal of R.

S. Annin has proved in Theorem 2.2 of [1] that if R is a ring and M a right R-module. If σ is an endomorphism of R and $S = R[x;\sigma]$ and M_R is σ-compatible, then $\mathrm{Ass}(M[x]_S) = \{P[x]$ such that $P \in \mathrm{Ass}(M_R)\}$.

In [12] Leroy and Matczuk have investigated the relationship between the associated prime ideals of an R-module M_R and that of the induced S-module M_S, where $S = R[x;\sigma]$ (σ an automorphism of a ring R). They have proved the following:

Theorem 1.1 (Theorem 5.7 of [12]). *Suppose M_R contains enough prime submodules and let $Q \in \mathrm{Ass}(M_S)$. If for every $P \in \mathrm{Ass}(M_R)$, $\sigma(P) = P$, then $Q = PS$ for some $P \in \mathrm{Ass}(M_R)$.*

In Theorem 1.2 of [14] Nordstrom has proved that if R is a ring with identity and σ is a surjective endomorphism of R, then for any right R-module M, $\mathrm{Ass}(M[x;\sigma]) = \{I[x;\sigma], I \in \sigma - \mathrm{Ass}(M)\}$. In Corollary 1.5 of [14] it has been proved that if R is Noetherian and σ an automorphism of R, then $\mathrm{Ass}(M[x,\sigma]_S) = \{P_\sigma[x;\sigma], P \in \mathrm{Ass}(M)\}$, where $P_\sigma = \cap_{i \in \mathbb{N}} \sigma^{-i}(P)$ and $S = R[x;\sigma]$.

The above discussion leads to a stronger type of primary decomposition of a Noetherian ring. We call such a ring a *transparent ring*.

Definition 1.2. A Noetherian ring R is said to be a *transparent ring* if there exist irreducible ideals I_j, $1 \leq j \leq n$ such that $\cap_{j=1}^{n} I_j = 0$ and each R/I_j has a right Artinian quotient ring.

It can be easily seen that an integral domain is a transparent ring, a commutative Noetherian ring is a transparent ring and so is a Noetherian ring having an Artinian quotient ring. A fully bounded Noetherian ring is also a transparent ring.

This type of decomposition was actually introduced by the author in [2]. Such a ring was called a decomposable ring, but in order to distinguish between one more definition of a decomposable ring given below and pointed out by the referee of one of the papers of the author, we now call such a ring a transparent ring.

Definition 1.3 (e.g., [11])**.** Let R be a ring. An R-module M is said to be decomposable if $M \simeq M_1 \oplus M_2$ of nonzero R-modules M_1 and M_2. A ring R is called a *decomposable ring* if it is a direct sum of two rings.

Now there arises a natural question: if R is a transparent ring. Is $O(R)$ a transparent ring? We have not been able to answer this question in general, however, in the commutative case we have the following:

Theorem 1.4. *If R is a commutative Noetherian $\mathbb{Q}$-algebra and σ is an automorphism of R, then there exists an integer $m \geq 1$ such that the skew-polynomial ring $R[x;\alpha,\delta]$ is a transparent ring, where $\sigma^m = \alpha$ and δ is an α-derivation of R.*

This is proved in Theorem 3.4. Before proving the main result, we prove that if R is a ring which is an order in an Artinian ring S, σ is an automorphism of R and δ a σ-derivation of R, then σ can be extended to an automorphism τ (say) of S and δ can be extended to a τ-derivation (say) ρ of S. This is proved in Lemma 3.2.

2. Preliminaries

We begin with the following lemma:

Lemma 2.1. *Let R be a Noetherian $\mathbb{Q}$-algebra, σ an automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$. Then $e^{t\delta}$ is an automorphism of $T = R[[t;\sigma]]$, the skew power series ring, where $e^{t\delta} = 1 + t\delta + \frac{t^2\delta^2}{2!} + \cdots$.*

Proof. The proof is on the same lines as in Seidenberg [16] and in noncommutative case, it is similar to the sketch of the proof provided in Blair and Small [5]. □

Henceforth we denote $R[[t,\sigma]]$ by T.

Lemma 2.2. *Let R be a Noetherian $\mathbb{Q}$-algebra, σ and δ as in Lemma* 2.1. *Let I be an ideal of R such that $\sigma(I) = I$. Then I is δ-invariant if and only if TI is $e^{t\delta}$-invariant.*

Proof. Let TI be $e^{t\delta}$-invariant. Let $a \in I$. Then $a \in TI$. So $e^{t\delta}(a) \in TI$; i.e., $a + t\delta(a) + (t^2/2!)\delta^2(a) + \cdots \in TI$, which implies that $\delta(a) \in I$.

Conversely suppose that $\delta(I) \subseteq I$ and let $f = \sum_{j=0}^{\infty} t^j a_j \in TI$. Then

$$\begin{aligned} e^{t\delta}(f) &= f + t\delta(f) + (t^2\delta^2/2!)(f) + \cdots \\ &= \sum_{j=0}^{\infty} t^j a_j + t\left(\sum_{j=0}^{\infty} t^j \delta(a_j) + \cdots\right). \end{aligned}$$

This lies in TI, as $\delta(a_i) \in I$. Therefore $e^{t\delta}(TI) \subseteq TI$. Replacing $e^{t\delta}$ by $e^{-t\delta}$, we get that $e^{t\delta}(TI) = TI$. □

Let σ be an automorphism of a ring R, and I be an ideal of R such that $\sigma(I) = I$. Then it is easy to see that $TI \subseteq IT$ and $IT \subseteq TI$. Hence $TI = IT$ is an ideal of T.

Lemma 2.3. *Let R be a semiprime Noetherian ring and σ an automorphism of R. Then $L(R) = R[[x, x^{-1}; \sigma]]$ is also a semiprime Noetherian ring.*

Proof. Let $f(x) = x^i a + x^{i+1} a_{i+1} + \cdots$ be a nonzero series in $L(R)$ with initial nonzero term of degree $i \in \mathbb{Z}$, and with initial coefficient a. Suppose $f(x)L(R)f(x) = 0$. Then $f(x)(x^{-i}R)f(x) = 0$, and therefore $aRa = 0$, a contradiction since R is semiprime. Therefore $f(x)L(R)f(x) \neq 0$ for all nonzero $f(x) \in L(R)$. Hence $L(R)$ is semiprime. □

Corollary 2.4. *Let R be a semiprime Noetherian ring and σ an automorphism of R. Then $T = R[[x; \sigma]]$ is also a semiprime Noetherian ring.*

Lemma 2.5. *Let R be a Noetherian ring and T as usual. Then:*

(1) *Let $U \in \text{Min.Spec}(R)$ be such that $\sigma(U) = U$. Then*

$$UT \in \text{Min.Spec}(T).$$

(2) *$P \in \text{Min.Spec}(T)$ implies $P \cap R \in \text{Min.Spec}(R)$ and $P = (P \cap R)T$.*

Proof. (1) Let $U \in \text{Min.Spec}(R)$. Then $UT \in \text{Spec}(T)$ by Corollary 2.4. Suppose $UT \notin \text{Min.Spec}(T)$. Let $U_1 \subset UT$ be a minimal prime ideal of T. Then $U_1 \cap R \subset UT \cap R = U$ which is not possible as $U_1 \cap R \in \text{Spec}(R)$ and $U \in \text{Min.Spec}(R)$. Therefore $UT \in \text{Min.Spec}(T)$.

(2) Let $P \in \text{Min.Spec}(T)$. Then $P \cap R \in \text{Spec}(R)$. Suppose $(P \cap R) \notin \text{Min.Spec}(R)$. Let $P_1 \subset P \cap R$ be a minimal prime ideal of R. Then $P_1 T \subset (P \cap R)T \subset P$ which is not possible as $P \in \text{Min.Spec}(T)$ and $P_1 T \in \text{Spec}(T)$. Therefore $P \cap R \in \text{Min.Spec}(R)$. □

We also know that if R is a Noetherian ring and $U \in \text{Min.Spec}(R)$, then $\sigma^j(U) \in \text{Min.Spec}(R)$ for all positive integers j. Also $\text{Min.Spec}(R)$ is finite by Theorem 2.4 of Goodearl and Warfield [8]. Therefore there exists a positive integer m such that $\sigma^m(U) = U$ for all $U \in \text{Min.Spec}(R)$. In Lemma 3.4 of [7], Gabriel proved that if R is a Noetherian $\mathbb{Q}$-algebra and δ is a derivation of R, then $\delta(P) \subseteq P$ for all $P \in \text{Min.Spec}(R)$. We generalize this results for a σ-derivation δ and prove the following:

Lemma 2.6. *Let R be a Noetherian $\mathbb{Q}$-algebra. Let σ be an automorphism of R and δ a σ-derivation of R. Then:*

(1) *$\sigma(N(R)) = N(R)$.*
(2) *If $P \in \text{Min.Spec}(R)$ is such that $\sigma(P) = P$, then $\delta(P) \subseteq P$.*

Proof. (1) The proof is obvious.

(2) Let T be as usual. Now by Lemma 2.1 $e^{t\delta}$ is an automorphism of T. Let $P \in \text{Min.Spec}(R)$. Then by Lemma 2.5 $PT \in \text{Min.Spec}(T)$. So there exists an integer $n \geq 1$ such that $(e^{t\delta})^n(PT) = PT$; i.e., $e^{nt\delta}(PT) = PT$. But R is a $\mathbb{Q}$-algebra, therefore $e^{t\delta}(PT) = PT$, and so Lemma 2.2 implies that $\delta(P) \subseteq P$. □

Lemma 2.7. *Let R be a right Noetherian ring. Then there exist irreducible ideals I_j, $1 \leq j \leq n$ of R such that $\cap_{j=1}^n I_j = 0$.*

Proof. The proof is obvious and we leave the details to the reader. □

Lemma 2.8. *Let R be a Noetherian ring having a right Artinian quotient ring. Then R is a transparent ring.*

Proof. Let $Q(R)$ be the right quotient ring of R . Now for any ideal J of $Q(R)$, the contraction J^c of J is an ideal of R and the extension of J^c is J; i.e., $J^{ce} = J$. For this see Proposition 9.19 of Goodearl and Warfield [8]. Let I_j, $1 \leq j \leq n$ be the irreducible ideals of $Q(R)$ such that $0 = \cap_{j=1}^n I_j$. Also each $Q(R)/I_j$ is an Artinian ring. Let $I_j^c = K_j$. Then it is not difficult to see that R/K_j has Artinian quotient ring $Q(R)/I_j$. Moreover $\cap_{j=1}^n K_j = 0$. Hence R is a transparent ring. □

Definition 2.9. Let P be a prime ideal of a commutative ring R. Then the symbolic power of P for a positive integer n is denoted by $P^{(n)}$ and is defined as $P^{(n)} = \{a \in R$ such that there exists some $d \in R$, $d \notin P$ such that $da \in P^n\}$. Also if I is an ideal of R, define as usual $\sqrt{I} = \{a \in R$ such that $a^n \in I$ for some $n \in \mathbb{Z}$ with $n \geq 1\}$.

Lemma 2.10. *Let R be a commutative, Noetherian ring and let σ be an automorphism of R. Then there exists a positive integer m such that, for all $P \in \text{Ass}(R_R)$:*

(1) $\sigma^m(P) = P$.

(2) $\sigma^m(P^{(k)}) = P^{(k)}$ *for all* $k \geq 0$.

Proof. (1) Since $\text{Ass}(R_R)$ is a finite set and $\sigma^j(P) \in \text{Ass}(R_R)$ for any integer $j \geq 1$ whenever $P \in \text{Ass}(R_R)$, there exists an integer $m \geq 1$ such that $\sigma^m(P) = P$.

(2) Denote σ^m by θ. We have $\theta(P) = P$. Let $a \in P^{(k)}$. Then there exists some $d \in R$, $d \notin P$ such that $da \in P^k$. Therefore $\theta(da) \in \theta(P^k)$; i.e., $\theta(d)\theta(a) \in (\theta(P))^k = P^k$. Now $\theta(d) \notin P$ implies that $\theta(a) \in P^{(k)}$. Therefore $\theta(P^{(k)}) \subseteq P^{(k)}$. Hence $\theta(P^{(k)}) = P^{(k)}$. □

Lemma 2.11. *Let R be a commutative Noetherian $\mathbb{Q}$-algebra. Let σ be an automorphism of R and δ a σ-derivation of R. Let P be a prime ideal of R such that $\sigma(P) = P$ and $\delta(P) \subseteq P$. Then $\delta(P^{(k)}) \subseteq P^{(k)}$, for any integer $k \geq 1$.*

Proof. Let $a \in P^{(k)}$. Then there exists $d \notin P$ such that $da \in P^k$. Let $da = p_1 p_2 \dots p_k$, $p_i \in P$. Then

$$\begin{aligned}\delta(da) &= \delta(p_1 p_2 \dots p_{k-1})\sigma(p_k) + p_1 p_2 \dots p_{k-1}\delta(p_k)\\ &= \delta(p_1 p_2 \dots p_{k-2})\sigma(p_{k-1})\sigma(p_k) + p_1 p_2 \dots p_{k-2}\delta(p_{k-1})\sigma(p_k)\\ &\quad + p_1 p_2 \dots p_{k-1}\delta(p_k)\\ &\ \vdots\\ &= \delta(p_1)\sigma(p_2 \dots p_k) + \dots + p_1 p_2 \dots p_{k-2}\delta(p_{k-1})\sigma(p_k)\\ &\quad + p_1 p_2 \dots p_{k-1}\delta(p_k).\end{aligned}$$

This lies in P^k as $\sigma(P) = P$ and $\delta(P) \subseteq P$; i.e., $\sigma(d)\delta(a) + \delta(d)a \in P^k$. Now $a \in P^{(k)}$, and, therefore $\sigma(d)\delta(a) \in P^{(k)}$, which implies that there exists $d_1 \notin P$ such that $d_1\sigma(d)\delta(a) \in P^k$. Now $d_1\sigma(d)\delta(a) + d_1\delta(d)a \in P^k$, which implies that $d_1\sigma(d)\delta(a) \in P^k$ and since $d_1\sigma(d) \notin P$, we have $\delta(a) \in P^{(k)}$. □

3. Main result

In this section we prove the main result in the form of Theorem 3.4. We begin with the following remark:

Remark 3.1. If $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$, then σ can be extended to an automorphism of $O(R)$ such that $\sigma(x) = x$ and δ can be extended to a σ-derivation of $O(R)$ such that $\delta(x) = 0$, that is $\sigma(xa) = x\sigma(a)$ and $\delta(xa) = x\delta(a)$.

Lemma 3.2. *Let R be a ring which is an order in an Artinian ring S. Let σ be an automorphism of R and δ a σ-derivation of R. Then σ can be extended to an automorphism (say) τ of S and δ can be extended to a τ-derivation (say) ρ of S.*

Proof. Define τ on S as for any $as^{-1} \in S$; $\tau(as^{-1}) = \sigma(a)(\sigma(s))^{-1}$, then it can be easily verified that τ is an automorphism of S. Now define ρ on S as for any $as^{-1} \in S$; $\rho(as^{-1}) = (\delta(a) - as^{-1}\delta(s))(\sigma(s))^{-1}$. Now it can be seen that ρ is a τ-derivation of S. The details are left to the reader. □

Theorem 3.3. *Let R be a ring which is an order in a right Artinian ring S. Then $O(R)$ is an order in a right Artinian ring.*

Proof. See Theorem 2.11 of Bhat [3]. □

We are now in a position to state and prove the main result in the form of the following theorem:

Theorem 3.4. *Let R be a commutative Noetherian $\mathbb{Q}$-algebra, σ be an automorphism of R. Then there exists an integer $m \geq 1$ such that the skew-polynomial ring $R[x; \alpha, \delta]$ is a transparent ring, where $\sigma^m = \alpha$ and δ is an α-derivation of R such that $\alpha(\delta(a)) = \delta(\alpha(a))$, for all $a \in R$.*

Proof. $R[x;\alpha,\delta]$ is Noetherian by Hilbert Basis Theorem, namely Theorem 1.12 of Goodearl and Warfield [8]. Now R is a commutative Noetherian $\mathbb{Q}$-algebra, therefore, the ideal (0) has a reduced primary decomposition. Let I_j, $1 \le j \le n$ be irreducible ideals of R such that $(0) = \cap_{j=1}^n I_j$. For this see Theorem 4 of Zariski and Samuel [17]. Let $\sqrt{I_j} = P_j$, where P_j is a prime ideal belonging to I_j. Now by Theorem 23 of Zariski and Samuel [17] there exists a positive integer k such that $P_j^{(k)} \subseteq I_j$, $1 \le j \le n$. Therefore we have $\cap_{j=1}^n P_j^{(k)} = 0$. Now $P_j \in \mathrm{Ass}(R_R)$, $1 \le j \le n$ by first uniqueness Theorem. Now every P_j contains a minimal prime ideal (say) U_j by Proposition 2.3 of Goodearl and Warfield [8], and since Min.Spec(R) is finite, there exists an integer $m \ge 1$ such that $\sigma^m(U_j) = U_j$. Denote σ^m by α. Now $\alpha(U_j) = U_j$, and therefore, $\alpha(U_j^{(k)}) = U_j^{(k)}$ by Lemma 2.10. Also $\delta(U_j) \subseteq U_j$ by Lemma 2.6 and therefore, $\delta(U_j^{(k)}) \subseteq U_j^{(k)}$ by Lemma 2.11. Thus $U_j^{(k)}[x;\alpha,\delta]$ is an ideal of $R[x;\alpha,\delta]$. Now $R/U_j^{(k)}$ has no embedded primes, therefore $R/U_j^{(k)}$ has an Artinian quotient ring by Theorem 2.11 of Robson [15]. Now by Theorem 3.3 $R[x;\alpha,\delta]/U_j^{(k)}[x;\alpha,\delta]$ has an Artinian quotient ring. More over $\cap_{j=1}^n U_j^{(k)}[x;\alpha,\delta] = 0$, therefore, Lemma 2.8 implies that $R[x;\alpha,\delta]$ is a transparent ring. □

Remark 3.5. (1) Let R be a Noetherian ring having an Artinian quotient ring. Let σ be an automorphism of R and δ a σ-derivation of R. Then $R[x;\sigma,\delta]$ is a transparent ring.

(2) Let R be a commutative Noetherian ring and σ an automorphism of R. Then the skew polynomial ring $R[x;\sigma]$ is a transparent ring.

(3) Let R be a commutative Noetherian ring and σ an automorphism of R. Then the skew Laurent polynomial ring $R[x;x^{-1},\sigma]$ is a transparent ring.

(4) Let R be a commutative Noetherian $\mathbb{Q}$-algebra and δ a derivation of R. Then the differential operator ring $R[x;\delta]$ is a transparent ring.

Question. If R is a commutative Noetherian $\mathbb{Q}$-algebra, σ is an automorphism of R and δ a σ-derivation of R. Is $R[x;\sigma,\delta]$ a transparent ring even if $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$? The main hurdle is that in such a situation $\delta(P) \subseteq P$ need not imply $\delta(P^{(k)}) \subseteq P^{(k)}$.

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