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Feynman's operational calculi: using Cauchy's integral formula

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Abstract. In this paper we express the disentangling, or the formation of a function of several noncommuting operators using Cauchy's Integral Formula in several complex variables. It is seen that the disentangling of a given function f can be expressed as a contour integral around the boundary of a polydisk where the standard Cauchy kernel is replaced by the disentangling of the Cauchy kernel expressed as an element of the disentangling algebra. This approach to the operational calculus allows for us to develop a "differential calculus" with disentanglings.

Contents

1. Introduction

The primary topic for this paper is, as stated in the title, Feynman's operational calculus. The approach to the operational calculus considered in this paper is that which was originated in and elaborated on by Jefferies and Johnson in the papers [5, 6, 7, 8]. Jefferies and Johnson constructed a "commutative world", specifically a commutative Banach algebra, in which the time ordering calculations required for the operational calculus are carried out in a mathematically rigorous way. The result of these calculations — the *disentangling* — is then mapped into the noncommutative world of $\mathcal{L}(X)$.

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In order to carry out the disentangling calculations, that is, in order to form the desired function of a (finite) set of not necessarily commuting operators, the Taylor series for the function is first written down and then the disentangling is carried out term-by-term. The current paper changes this point of view and looks to obtain the disentangling of a given function, say f, by avoiding the use of its Taylor series. Indeed, in the approach outlined below, only one Taylor series is ultimately needed and that is the series for the Cauchy kernel. Once the disentangling for the Cauchy kernel is determined (and this is very easily done), the disentangling for any desired (and allowed) function f is calculated via the use of Cauchy's integral formula. In fact, the end result is a contour integral of the complex-valued function $f(\xi_1, \ldots, \xi_n)$ against the disentangling of the Cauchy kernel; see Theorem 3.4 below. This representation makes the operational calculus much easier to deal with, in the opinion of the present author, as the Cauchy kernel is the *only* object that requires explicit disentangling; i.e., a term-by-term disentangling of a power series. An illustration of this idea is elaborated in the fourth section of this paper, where a "calculus" of disentanglings is defined and some examples are considered. Without the representation of the operational calculus using Cauchy's integral formula, this "calculus" would not be at all clear.

Before proceeding further, it may be helpful to present some background on the operational calculus. Feynman's operational calculus originated with the 1951 paper [3] and concerns itself with the formation of functions of noncommuting operators. Indeed, even with functions as simple as $f(x, y) = xy$ it is not clear how to define $f(A, B)$ if A and B do not commute — does one let $f(A, B) = AB$, $f(A, B) = BA$, $f(A, B) = \frac{1}{2}AB + \frac{1}{2}BA$, or some other expression involving products of A and B? One has to decide, then, usually with a particular problem in mind, how to form a given function of noncommuting operators. As mentioned above, one approach to this problem (the approach used in this paper) was developed by Jefferies and Johnson in the papers [5, 6, 7, 8] . This approach is expanded on in the papers [10], [11], [16], [13], [9], [12] and others. It is important to note that, in the setting of the Jefferies–Johnson approach, measures on intervals $[0, T]$ are used to determine when a given operator will act in products and the measures used are continuous measures. However, Johnson and the current author extended the operational calculus to measures with both continuous and discrete parts in the aforementioned [16].

The discussion just above, then, begs the question of how measures can be used to determine the order of operators in products. Feynman's heuristic rules for the formation of functions of noncommuting operators give us a starting point.

(1) Attach time indices to the operators to specify the order of operators in products.
(2) With time indices attached, form functions of these operators by treating them as though they were commuting.

(3) Finally, "disentangle" the resulting expressions; i.e., restore the conventional ordering of the operators.

As is well known, the central problem of the operational calculus is the disentangling process. Indeed in his 1951 paper, [3], Feynman points out that "The process is not always easy to perform and, in fact, is the central problem of this operator calculus."

We first address rule (1) above. It is in the use of this rule that we will see measures used to track the action of operators in products. First, it may be that the operators involved may come with time indices naturally attached. For example, we might have operators of multiplication by time dependent potentials. However, it is also commonly the case that the operators used are independent of time. Given such an operator A, we can (as Feynman most often did) attach time indices according to Lebesgue measure as follows:

$$A = \frac{1}{t}\int_0^t A(s)\,ds$$

where $A(s) := A$ for $0 \le s \le t$. This device does appear a bit artificial but does turn out to be extremely useful in many situations. We also note that mathematical or physical considerations may dictate that one use a measure different from Lebesgue measure. For example, if μ is a probability measure on the interval $[0, T]$, and if A is a linear operator, we can write

$$A = \int_{[0,T]} A(s)\,\mu(ds)$$

where once again $A(s) := A$ for $0 \le s \le T$. When we write A in this fashion, we are able to use the time variable to keep track of when the operator A acts. Indeed, if we have two operators A and B, consider the product $A(s)B(t)$ (here, time indices have been attached). If $t < s$, then we have $A(s)B(t) = AB$ since here we want B to act first (on the right). If, on the other hand, $s < t$, then $A(s)B(t) = BA$ since A has the earlier time index. In other words, the operator with the smaller (or earlier) time index, acts to the right of (or before) an operator with a larger (or later) time index. (It needs to be kept in mind that these equalities are heuristic in nature.) For a much more detailed discussion of using measures to attach time indices, see Chapter 14 of the book [14] (and Chapter 2 of the forthcoming book [15]) and the references contained therein.

Concerning the rules (2) and (3) above, we mention that, once we have attached time indices to the operators involved, we calculate functions of the noncommuting operators as if they actually do commute. These calculations are, of course, heuristic in nature but the idea is that with time indices attached, one carries out the necessary calculations giving no thought to the operator ordering problem; the time indices will enable us to restore the desired ordering of the operators once the calculations are finished; this is the *disentangling* process and is typically the most difficult part of any given problem. More details of the process are to be found below, in Section 2.

2. Definitions and basic properties of Feynman's operational calculus

2.1. The Banach algebras $\mathbb{A}$ and $\mathbb{D}$. We now move on to a discussion of the disentangling map. Before defining the map, however, we need some preliminary definitions and notation (see [5], [10], [15] and others). We begin by introducing two commutative Banach algebras $\mathbb{A}$ and $\mathbb{D}$. These algebras are closely related and play an important role in the rigorous development of the operational calculus.

Given $n \in \mathbb{N}$ and n positive real numbers $r_1, \ldots, r_n$, let $\mathbb{A}(r_1, \ldots, r_n)$ or, more briefly $\mathbb{A}$, be the space of complex-valued functions $(z_1, \ldots, z_n) \mapsto f(z_1, \ldots, z_n)$ of n complex variables that are analytic at the origin and are such that their power series expansion

$$f(z_1, \ldots, z_n) = \sum_{m_1, \ldots, m_n = 0}^{\infty} a_{m_1, \ldots, m_n} z_1^{m_1} \cdots z_n^{m_n} \tag{2.1}$$

converges absolutely at least in the closed polydisk

$$\{(z_1, \ldots, z_n) : |z_1| \le r_1, \ldots, |z_n| \le r_n\}.$$

All of these functions are analytic at least in the open polydisk

$$\{(z_1, \ldots, z_n) : |z_1| < r_1, \ldots, |z_n| < r_n\}.$$

Of course, all elements of $\mathbb{A}$ are continuous on the boundary of the polydisk. We remark that the entire functions of $(z_1, \ldots, z_n)$ are in $\mathbb{A}(r_1, \ldots, r_n)$ for any n–tuple $(r_1, \ldots, r_n)$ of positive real numbers.

For $f \in \mathbb{A}$ given by Equation (2.1) above, we let

$$\|f\|_{\mathbb{A}(r_1, \ldots, r_n)} = \|f\|_{\mathbb{A}} := \sum_{m_1, \ldots, m_n = 0}^{\infty} |a_{m_1, \ldots, m_n}| r_1^{m_1} \cdots r_n^{m_n}. \tag{2.2}$$

This expression is a norm on $\mathbb{A}$ and turns $\mathbb{A}$ into a commutative Banach algebra. (See Proposition 1.1 of [5].) (In fact $\mathbb{A}$ is a weighted ℓ^1-space.)

We now turn to the construction of the Banach algebra $\mathbb{D}$. Let X be a Banach space and let $A_1, \ldots, A_n \in \mathcal{L}(X)$. Construct the commutative Banach algebra $\mathbb{A}(\|A_1\|, \ldots, \|A_n\|)$ as in the previous paragraph. We associate to each of the operators A_j, $j = 1, \ldots, n$, the formal object $\tilde{A}_j$, $j = 1, \ldots, n$, by discarding all operator properties of A_j other than its operator norm $\|A_j\|_{\mathcal{L}(X)}$ and assuming that $\tilde{A}_i \tilde{A}_j = \tilde{A}_j \tilde{A}_i$ for any $1 \le i, j \le n$. (We remark that we will assume that the objects — "formal commuting objects" — $\tilde{A}_1, \ldots, \tilde{A}_n$ are all distinct even if two or more of the operators $A_1, \ldots, A_n$ are identical or are linear combinations of some or all of the $A_1, \ldots, A_n$.) In order to define $\mathbb{D}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right)$ we replace the indeterminates $z_1, \ldots, z_n$ in functions $f(z_1, \ldots, z_n) \in \mathbb{A}(\|A_1\|, \ldots, \|A_n\|)$ by $\tilde{A}_1, \ldots, \tilde{A}_n$, respectively.

That is, we take $\mathbb{D}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right)$ to be the collection of all expressions

$$f\left(\tilde{A}_1, \ldots, \tilde{A}_n\right) = \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} \left(\tilde{A}_1\right)^{m_1} \cdots \left(\tilde{A}_n\right)^{m_n} \tag{2.3}$$

for which

$$\begin{aligned} \|f\|_{\mathbb{D}\left(\tilde{A}_1,\ldots,\tilde{A}_n\right)} &= \|f\|_{\mathbb{D}} \\ &= \sum_{m_1,\ldots,m_n=0}^{\infty} |a_{m_1,\ldots,m_n}| \, \|A_1\|^{m_1} \cdots \|A_n\|^{m_n} < \infty. \end{aligned} \tag{2.4}$$

Clearly $f \in \mathbb{A}\left(\|A_1\|, \ldots, \|A_n\|\right)$ determines an element of $\mathbb{D}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right)$. Equation (2.4) defines a norm on $\mathbb{D}$ and $\mathbb{D}$ is a commutative Banach algebra with respect to this norm and point-wise operations of the elements of $\mathbb{D}$. Further, $\mathbb{A}$ and $\mathbb{D}$ can be identified — they are in fact isometrically isomorphic. For proofs of these statements, see [5] or [15]. The commutative Banach algebra $\mathbb{D}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right)$ is called the *disentangling algebra.*

Consider the elements $p^{\ell}_{\tilde{A}_j}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right) := \left(\tilde{A}_j\right)^{\ell}$ of $\mathbb{D}$ for $j = 1, \ldots, n$ and $\ell \in \mathbb{N}$. We note that

$$\left\| p^{\ell}_{\tilde{A}_j} \right\|_{\mathbb{D}} = \|A_j\|^{\ell}_{\mathcal{L}(X)} \tag{2.5}$$

for each $j = 1, \ldots, n$ and $\ell \in \mathbb{N}$. We will below suppress the notation $p^{\ell}_{\tilde{A}_j}$ and simply write $\left(\tilde{A}_j\right)^{\ell}$ and consider $\left(\tilde{A}_j\right)^{\ell}$ as an element of the disentangling algebra $\mathbb{D}$.

2.2. The disentangling map. Let $A_1, \ldots, A_n$ be operators from $\mathcal{L}(X)$ (taken to be nonzero) and let $\mu_1, \ldots, \mu_n$ be continuous probability measures (*time-ordering measures*) defined at least on $\mathcal{B}([0,T])$, the Borel class of $[0,T]$, $T > 0$. (Recall that the measure μ is said to be *continuous* if $\mu(\{s\}) = 0$ for all $s \in [0,T]$.) We wish to define the disentangling map

$$\mathcal{T}_{\mu_1,\ldots,\mu_n} : \mathbb{D}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right) \longrightarrow \mathcal{L}(X) \tag{2.6}$$

according to the time-ordering directions given by the measures $\mu_1, \ldots, \mu_n$. Said another way, given any analytic function $f \in \mathbb{A}\left(\|A_1\|, \ldots, \|A_n\|\right)$, we wish to form the function $f_{\mu_1,\ldots,\mu_n}\left(A_1, \ldots, A_n\right)$ of the not necessarily commuting operators $A_1, \ldots, A_n$ as directed by $\mu_1, \ldots, \mu_n$. The measures $\mu_1, \ldots, \mu_n$ serve to tell us when (or in what order) operators act in products.

Given nonnegative integers $m_1, \ldots, m_n$, we let

$$P^{m_1,\ldots,m_n}(z_1, \ldots, z_n) = z_1^{m_1} \cdots z_n^{m_n}, \tag{2.7}$$

so that

$$P^{m_1,\dots,m_n}\left(\tilde{A}_1,\dots,\tilde{A}_n\right)=\tilde{A}_1^{m_1}\cdots\tilde{A}_n^{m_n}. \tag{2.8}$$

We will begin shortly by doing calculations in the setting of the *commutative* Banach algebra $\mathbb{D}\left(\tilde{A}_1,\dots,\tilde{A}_n\right)$, which will end by showing us, following Feynman's ideas, how to define $\mathcal{T}_{\mu_1,\dots,\mu_n}P^{m_1,\dots,m_n}\left(\tilde{A}_1,\dots,\tilde{A}_n\right)$. Since we want $\mathcal{T}_{\mu_1,\dots,\mu_n}$ to be linear and continuous, it will then be clear from (2.3) how to define the operator $\mathcal{T}_{\mu_1,\dots,\mu_n}f\left(\tilde{A}_1,\dots,\tilde{A}_n\right)$, for any element $f\left(\tilde{A}_1,\dots,\tilde{A}_n\right)$ of $\mathbb{D}\left(\tilde{A}_1,\dots,\tilde{A}_n\right)$.

It is worthwhile at this point to briefly remind the reader of Feynman's heuristic rules, since we will follow them explicitly, but in a mathematically rigorous way. The first of Feynman's "rules" was to attach time indices to the operators in question, in order to specify the order of operation in products. (Operators sometimes come with indices attached, especially in evolution problems. However, this situation will not concern us in in this paper.) As mentioned previously, the measures will determine the ordering of the operators in products and can do this in a variety of ways. Feynman did not use measures to order operators in products but his choice was nearly always Lebesgue measure when attaching time indices to operators.

Feynman's next "rule" was to form the desired function of the operators, just as if they were commuting and then "disentangle" the result, that is, bring the expression to a sum of time-ordered expressions. This disentangling will be done in our next proposition by working in $\mathbb{D}\left(\tilde{A}_1,\dots,\tilde{A}_n\right)$. Once we have the time-ordering, Feynman says to simply return from the commutative framework to the operators themselves. This is the point at which we will define

$$\mathcal{T}_{\mu_1,\dots,\mu_n}P^{m_1,\dots,m_n}\left(\tilde{A}_1,\dots,\tilde{A}_n\right)\quad\text{and then}$$
$$\mathcal{T}_{\mu_1,\dots,\mu_n}f\left(\tilde{A}_1,\dots,\tilde{A}_n\right),$$

for $f\left(\tilde{A}_1,\dots,\tilde{A}_n\right)\in\mathbb{D}\left(\tilde{A}_1,\dots,\tilde{A}_n\right)$.

For each $m=0,1,\dots$, let S_m denote the set of all permutations of the integers $\{1,\dots,m\}$, and given $\pi\in S_m$, we let

$$\Delta_m(\pi)=\left\{(s_1,\dots,s_m)\in[0,T]^m:0<s_{\pi(1)}<\cdots<s_{\pi(m)}<T\right\}. \tag{2.9}$$

For $j=1,\dots,n$, and all $s\in[0,T]$, we let

$$\tilde{A}_j(s)\equiv\tilde{A}_j. \tag{2.10}$$

Now, for nonnegative integers $m_1, \ldots, m_n$ and $m = m_1 + \cdots + m_n$, we define

$$\tilde{C}_i(s) = \begin{cases} \tilde{A}_1(s) & \text{if } i \in \{1, \ldots, m_1\}, \\ \tilde{A}_2(s) & \text{if } i \in \{m_1 + 1, \ldots, m_1 + m_2\}, \\ \vdots & \\ \tilde{A}_n(s) & \text{if } i \in \{m_1 + \cdots + m_{n-1} + 1, \ldots, m\}, \end{cases} \tag{2.11}$$

for $i = 1, \ldots, m$ and $0 \le s \le T$.

Although $\tilde{C}_i(s)$ depends on the nonnegative integers $m_1, \ldots, m_n$, we will suppress this dependence for clarity of notation. The following proposition is critical for the definition of the disentangling map. Its proof can be found in [5] and with more detail in [15].

Proposition 2.1.

$$P^{m_1,\ldots,m_n}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right) = \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} \tilde{C}_{\pi(m)}(s_{\pi(m)}) \cdots \tilde{C}_{\pi(1)}(s_{\pi(1)}) \left(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}\right)(ds_1, \ldots, ds_m). \tag{2.12}$$

We see from (2.9) that the right-hand side of (2.12) is the sum of time-ordered expressions. Following Feynman's ideas, we now define the map $\mathcal{T}_{\mu_1,\ldots,\mu_n}$ which will return us from our commutative framework to the non-commutative setting of $\mathcal{L}(X)$. We need notation as in (2.11), but involving the operators $A_1, \ldots, A_n$ instead of the indeterminates $\tilde{A}_1, \ldots, \tilde{A}_n$. Accordingly, for $j = 1, \ldots, n$ and all $s \in [0, T]$, we set

$$C_i(s) = \begin{cases} A_1(s) & \text{if } i \in \{1, \ldots, m_1\}, \\ A_2(s) & \text{if } i \in \{m_1 + 1, \ldots, m_1 + m_2\}, \\ \vdots & \\ A_n(s) & \text{if } i \in \{m_1 + \cdots + m_{n-1} + 1, \ldots, m\}. \end{cases} \tag{2.13}$$

Definition 2.2.

$$\mathcal{T}_{\mu_1,\ldots,\mu_n}\left(P^{m_1,\ldots,m_n}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right)\right) := \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \left(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}\right)(ds_1, \ldots, ds_m).$$

Then, for $f\left(\tilde{A}_1, \ldots, \tilde{A}_n\right) \in \mathbb{D}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right)$ given by

$$f\left(\tilde{A}_1, \ldots, \tilde{A}_n\right) = \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} \tilde{A}_1^{m_1} \cdots \tilde{A}_n^{m_n},$$

we define the action of the disentangling map on

$$f\left(\tilde{A}_1,\ldots,\tilde{A}_n\right)\in\mathbb{D}\left(\tilde{A}_1,\ldots,\tilde{A}_n\right)$$

to be

$$\mathcal{T}_{\mu_1,\ldots,\mu_n}f\left(\tilde{A}_1,\ldots,\tilde{A}_n\right) = \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n}\mathcal{T}_{\mu_1,\ldots,\mu_n}\left(P^{m_1,\ldots,m_n}\left(\tilde{A}_1,\ldots,\tilde{A}_n\right)\right). \tag{2.14}$$

As is customary, we shall write $\mathcal{T}_{\mu_1,\ldots,\mu_n}f$ in place of $\mathcal{T}_{\mu_1,\ldots,\mu_n}(f)$ for an element f of $\mathbb{D}\left(\tilde{A}_1,\ldots,\tilde{A}_n\right)$. We will also, in the sequel, use the notation $\overrightarrow{\mu}$ for the n-tuple $(\mu_1,\ldots,\mu_n)$ of measures and we will use $\mathcal{T}_{\overrightarrow{\mu}}f$ or $f_{\overrightarrow{\mu}}(A_1,\ldots,A_n)$ to denote $\mathcal{T}_{\mu_1,\ldots,\mu_n}f$. The next proposition will assure us that the sum (2.14) makes sense. We will state the proposition without proof and refer the reader to [5] or [15] for its proof.

Proposition 2.3. (1) *The series* (2.14) *converges absolutely in the uniform operator topology of* $\mathcal{L}(X)$, *for all*

$$f\left(\tilde{A}_1,\ldots,\tilde{A}_n\right)\in\mathbb{D}\left(\tilde{A}_1,\ldots,\tilde{A}_n\right).$$

(2) $\mathcal{T}_{\mu_1,\ldots,\mu_n}$ *is a linear map from* $\mathbb{D}\left(\tilde{A}_1,\ldots,\tilde{A}_n\right)$ *into* $\mathcal{L}(X)$.

(3) *For all* $f\left(\tilde{A}_1,\ldots,\tilde{A}_n\right)\in\mathbb{D}\left(\tilde{A}_1,\ldots,\tilde{A}_n\right)$, *we have*

$$\left\|\mathcal{T}_{\mu_1,\ldots,\mu_n}f\left(\tilde{A}_1,\ldots,\tilde{A}_n\right)\right\|_{\mathcal{L}(X)}\le\left\|f\left(\tilde{A}_1,\ldots,\tilde{A}_n\right)\right\|_{\mathbb{D}\left(\tilde{A}_1,\ldots,\tilde{A}_n\right)}. \tag{2.15}$$

In fact,

$$\|\mathcal{T}_{\mu_1,\ldots,\mu_n}\|=1. \tag{2.16}$$

In the following section we will develop an integral representation of the disentangling of a function $f\left(\tilde{A}_1,\ldots,\tilde{A}_n\right)$ by using the classical Cauchy Integral Formula from several complex variables. It will be seen that the disentangling of f can be represented by a contour integral of f against the disentangling of the Cauchy kernel.

3. Using Cauchy's integral formula to re-express disentanglings

Let $A_1,\ldots,A_n\in\mathcal{L}(X)$ be nonzero linear operators on the Banach space X. Associate to each operator A_j, $j=1,\ldots,n$, a continuous Borel probability measure μ_j on the interval $[0,T]$, $T>0$. Construct, as in Section 2.1, the commutative Banach algebras $\mathbb{A}\left(\|A_1\|,\ldots,\|A_n\|\right)$ and $\mathbb{D}\left(\tilde{A}_1,\ldots,\tilde{A}_n\right)$.

We begin with some simple observations. First, we recall from (2.5) that

$$\left\|\left(\tilde{A}_j\right)^{\ell}\right\|_{\mathbb{D}} = \|A_j\|_{\mathcal{L}(X)}^{\ell}$$

for $\ell \in \mathbb{N}$ and $j = 1, \ldots, n$. This tells us that the spectral radius of $p_j^1\left(\tilde{A}_j\right) = \tilde{A}_j \in \mathbb{D}$ is

$$\operatorname{spr}\left(p_j^1\right) = \operatorname{spr}\left(\tilde{A}_j\right) = \lim_{\ell\to\infty}\left\|\left(\tilde{A}_j\right)^{\ell}\right\|_{\mathbb{D}}^{1/\ell} = \lim_{\ell\to\infty}\|A_j\|_{\mathcal{L}(X)} = \|A_j\|_{\mathcal{L}(X)}.$$

It follows that

$$\operatorname{spr}\left(\tilde{A}_j\right) \geq \operatorname{spr}\left(A_j\right)$$

for each $j = 1, \ldots, n$ and so $\sigma\left(A_j\right) \subseteq \sigma\left(\tilde{A}_j\right)$ or $\rho\left(\tilde{A}_j\right) \subseteq \rho\left(A_j\right)$ (where, for an operator B, $\sigma(B)$ denotes its spectrum and $\rho(B)$ its resolvent set) for each $j = 1, \ldots, n$. (Here and below we suppress the use of p_j^{ℓ} and will instead write $\tilde{A}_j^{\ell}$.)

We will be interested in the function

$$h\left(\xi_1, \ldots, \xi_n; \tilde{A}_1, \ldots, \tilde{A}_n\right) := \xi_1 \cdots \xi_n\left(\xi_1 - \tilde{A}_1\right)^{-1} \cdots \left(\xi_n - \tilde{A}_n\right)^{-1}. \tag{3.1}$$

In particular, we wish to determine the set of $(\xi_1, \ldots, \xi_n)$ in $\mathbb{C}^n$ for which $h \in \mathbb{D}$. In view of the previous paragraph we must choose $\xi_1, \ldots, \xi_n \in \mathbb{C}$ such that

$$|\xi_j| > \left\|\tilde{A}_j\right\|_{\mathbb{D}} = \|A_j\|_{\mathcal{L}(X)} \tag{3.2}$$

for $j = 1, \ldots, n$. For such $\xi_1, \ldots, \xi_n$ we have

$$\begin{aligned} h\left(\xi_1, \ldots, \xi_n; \tilde{A}_1, \ldots, \tilde{A}_n\right) &= \xi_1 \cdots \xi_n\left(\xi_1 - \tilde{A}_1\right)^{-1} \cdots \left(\xi_n - \tilde{A}_n\right)^{-1} \\ &= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{\left(\tilde{A}_1\right)^{m_1} \cdots \left(\tilde{A}_n\right)^{m_n}}{\xi_1^{m_1} \cdots \xi_n^{m_n}} \in \mathbb{D} \end{aligned}$$

since

$$\left\|\frac{\tilde{A}_j}{\xi_j}\right\|_{\mathbb{D}} = \frac{\left\|\tilde{A}_j\right\|_{\mathbb{D}}}{|\xi_j|} < 1 \tag{3.3}$$

for $j = 1, \ldots, n$. Hence, given any n-tuple $(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ with $|\xi_j| > \|A_j\|_{\mathcal{L}(X)}$, we have $h\left(\xi_1, \ldots, \xi_n; \tilde{A}_1, \ldots, \tilde{A}_n\right) \in \mathbb{D}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right)$. Said differently,

$$h\left(\cdot\,; \tilde{A}_1, \ldots, \tilde{A}_n\right) : \rho\left(\tilde{A}_1\right) \times \cdots \times \rho\left(\tilde{A}_n\right) \to \mathbb{D}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right).$$

As a function on $\rho\left(\tilde{A}_1\right) \times \cdots \times \rho\left(\tilde{A}_n\right)$, h is clearly continuous.

To accommodate the use of $h\left(\xi_1,\ldots,\xi_n;\tilde{A}_1,\ldots,\tilde{A}_n\right)$ in the operational calculus, we need a construction that allows us to use functions analytic on any polydisk containing

$$P_0 := \left\{(z_1,\ldots,z_n) : |z_j| \le \|A_j\|_{\mathcal{L}(X)},\, j=1,\ldots,n\right\}. \tag{3.4}$$

We start by choosing sequences $\{\epsilon_{j,k}\}_{k=1}^{\infty}$, $j=1,\ldots,n$, of strictly decreasing positive real numbers for which

$$\lim_{k\to\infty}\epsilon_{j,k} = 0. \tag{3.5}$$

Using these sequences, define positive real numbers $r_{1,k},\ldots,r_{n,k}$ by

$$r_{j,k} := \|A_j\|_{\mathcal{L}(X)} + \epsilon_{j,k} \tag{3.6}$$

for $j=1,\ldots,n$. Use these numbers to define closed polydisks P_k, $k\in\mathbb{N}$:

$$P_k := \{(z_1,\ldots,z_n)\in\mathbb{C}^n : |z_j| \le r_{j,k},\, j=1,\ldots,n\}. \tag{3.7}$$

Since the sequences $\{\epsilon_{j,k}\}_{k=1}^{\infty}$ are decreasing for each j, the closed polydisks P_k form a decreasing family, $P_{k+1}\subseteq P_k$ for all k, and so the corresponding commutative Banach algebras

$$\mathbb{A}_k := \mathbb{A}\left(r_{1,k},\ldots,r_{n,k}\right) \tag{3.8}$$

form an increasing family:

$$\mathbb{A}_1 \subseteq \mathbb{A}_2 \subseteq \cdots \subseteq \mathbb{A}_k \subseteq \mathbb{A}_{k+1} \subseteq \cdots \subseteq \mathbb{A}\left(\|A_1\|,\ldots,\|A_n\|\right). \tag{3.9}$$

Now that we have our family $\{\mathbb{A}_k\}_{k=1}^{\infty}$ of commutative Banach algebras, we can easily obtain the corresponding family $\{\mathbb{D}_k\}_{k=1}^{\infty}$ of disentangling algebras. This is done by replacing the indeterminates $z_1,\ldots,z_n$ for $f\in\mathbb{A}_k$ by the formal objects $\tilde{A}_1,\ldots,\tilde{A}_n$, respectively. We have, of course, $\mathbb{D}_k\cong\mathbb{A}_k$ (they are isometrically isomorphic) for all $k\in\mathbb{N}$.

Now, given $k,l\in\mathbb{N}$, with $k<l$, we define $g_{lk}:\mathbb{A}_k\to\mathbb{A}_l$ by $g_{lk}(f) := f|_{P_l}$. Clearly g_{lk} is linear and because our sequences $\{r_{j,k}\}_{k=1}^{\infty}$ of weights are decreasing, it follows that

$$\|g_{lk}(f)\|_{\mathbb{A}_k} = \|f|_{P_l}\|_{\mathbb{A}_l} \le \|f\|_{\mathbb{A}_k}. \tag{3.10}$$

Hence g_{lk} is continuous for $k,l\in\mathbb{N}$, $k<l$. We therefore have a inductive system $(\mathbb{A}_k, g_{kl})$ and we define

$$\mathbb{A}_\infty := \varinjlim g_{lk}(\mathbb{A}_k); \tag{3.11}$$

the inductive limit of the increasing family $\{\mathbb{A}_k\}$ of Banach algebras. It is well-known that $\mathbb{A}_\infty$ is an algebra and it is clear that for $f\in\mathbb{A}_\infty$, f is analytic on a polydisk containing the closed polydisk P_0. Further, for $f\in\mathbb{A}_\infty$, there is obviously a least positive integer k_0 such that $f\in\mathbb{A}_{k_0}$ and, of course, $f\in\mathbb{A}_k$ for all $k\ge k_0$.

As for the disentangling algebras $\mathbb{D}_1,\mathbb{D}_2,\ldots$, we repeat the construction above to obtain an increasing family $\{\mathbb{D}_k\}_{k=1}^{\infty}$ of commutative Banach algebras, the maps $g_{lk}:\mathbb{D}_k\longrightarrow\mathbb{D}_l$ and the inductive limit algebra $\mathbb{D}_\infty$.

We now define the disentangling map on the inductive limit algebra $\mathbb{D}_\infty$. Given $f \in \mathbb{D}_\infty$, there is a least positive integer k_0 such that $f \in \mathbb{D}_{k_0}$. As observed above in regard to the algebras $\mathbb{A}_k$, it is clear that $f \in \mathbb{D}_k$ for all $k \geq k_0$. Given any $k \in \mathbb{N}$, we denote by $\mathcal{T}^{(k)}_{\mu_1,\ldots,\mu_n}$ the "standard" disentangling map from $\mathbb{D}_k$ into $\mathcal{L}(X)$ (defined in Definition 2.2 above).

Definition 3.1. Given $f \in \mathbb{D}_\infty$, let $k_0 \in \mathbb{N}$ be the least integer for which $f \in \mathbb{D}_{k_0}$. Define $\mathcal{T}_{\overrightarrow{\mu}} : \mathbb{D}_\infty \longrightarrow \mathcal{L}(X)$ by

$$\mathcal{T}_{\overrightarrow{\mu}} f := \mathcal{T}^{(k_0)}_{\mu_1,\ldots,\mu_n} f =: \mathcal{T}^{(k_0)}_{\overrightarrow{\mu}} f. \tag{3.12}$$

Remark 3.2. This definition gives the same operator for every $k \geq k_0$ as f is analytic on every P_k with $k \geq k_0$ and so the disentangling series is the same for every $k \geq k_0$.

It follows at once from Proposition 5.6 of [1] that, because $\mathcal{T}^{(k)}_{\mu_1,\ldots,\mu_n}$ is continuous from each $\mathbb{D}_k$ into $\mathcal{L}(X)$, $\mathcal{T}_{\overrightarrow{\mu}}$ is continuous from $\mathbb{D}_\infty$ into $\mathcal{L}(X)$. It is obvious that $\mathcal{T}_{\overrightarrow{\mu}}$ is linear.

With the inductive limit constructions done and the proper definition of the analogue of the disentangling map defined, we turn to the calculations that will lead us to our representation of the disentangling using the classical Cauchy Integral Formula from several complex variables. Given $f \in \mathbb{D}_\infty$, let k_0 be the smallest positive integer for which $f \in \mathbb{D}_{k_0}$. Write

$$f\left(\tilde{A}_1, \ldots, \tilde{A}_n\right) = \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} \left(\tilde{A}_1\right)^{m_1} \cdots \left(\tilde{A}_n\right)^{m_n} \tag{3.13}$$

on the polydisk P_{k_0}. Via Cauchy's Integral Formula for derivatives we may write

$$a_{m_1,\ldots,m_n} = (2\pi i)^{-n} \int_{|\xi_1|=r_{1,k_0}} \cdots \int_{|\xi_n|=r_{n,k_0}} f(\xi_1,\ldots,\xi_n)\, \xi_1^{-m_1-1} \cdots \xi_n^{-m_n-1} d\xi_1 \cdots d\xi_n.$$

We can therefore write

$$\mathcal{T}_{\overrightarrow{\mu}} f = \sum_{m_1,\ldots,m_n=0}^{\infty} \Bigg[(2\pi i)^{-n} \int_{|\xi_1|=r_{1,k_0}} \cdots \int_{|\xi_n|=r_{n,k_0}} f(\xi_1,\ldots,\xi_n)\, \xi_1^{-m_1-1} \cdots \xi_n^{-m_n-1} d\xi_1 \cdots d\xi_n \Bigg] \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \cdot (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1,\ldots,ds_m).$$

Provided that we can establish the validity of interchanging the sum over $m_1,\ldots,m_n \in \mathbb{N} \cup \{0\}$ and the integral over the boundary of the polydisk P_{k_0}, we may rewrite $\mathcal{T}_{\overrightarrow{\mu}} f$ as

$$\mathcal{T}_{\overrightarrow{\mu}} f = (2\pi i)^{-n} \int\limits_{|\xi_1|=r_{1,k_0}} \cdots \int\limits_{|\xi_n|=r_{n,k_0}} f(\xi_1,\ldots,\xi_n)\,\xi_1^{-1}\cdots\xi_n^{-1}$$
$$\sum_{m_1,\ldots,m_n=0}^{\infty} \xi_1^{-m_1}\cdots\xi_n^{-m_n} \sum_{\pi\in S_m} \int\limits_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)})\cdots C_{\pi(1)}(s_{\pi(1)})$$
$$(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n})\,(ds_1,\ldots,ds_m)d\xi_1\cdots d\xi_n.$$

To continue further, we define symbols $\tilde{C}_j^{\xi}$ and C_j^{ξ} by modifying (2.11) and (2.13), respectively, to

$$\tilde{C}_j^{\xi}(s) := \begin{cases} \frac{\tilde{A}_1(s)}{\xi_1} & \text{if } j\in\{1,\ldots,m_1\} \\ \frac{\tilde{A}_2(s)}{\xi_2} & \text{if } j\in\{m_1+1,\ldots,m_1+m_2\} \\ \vdots & \\ \frac{\tilde{A}_n(s)}{\xi_n} & \text{if } j\in\{m_1+\cdots+m_{n-1}+1,\ldots,m\} \end{cases} \tag{3.14}$$

and

$$C_j^{\xi}(s) := \begin{cases} \frac{A_1(s)}{\xi_1} & \text{if } j\in\{1,\ldots,m_1\} \\ \frac{A_2(s)}{\xi_2} & \text{if } j\in\{m_1+1,\ldots,m_1+m_2\} \\ \vdots & \\ \frac{A_n(s)}{\xi_n} & \text{if } j\in\{m_1+\cdots+m_{n-1}+1,\ldots,m\}. \end{cases} \tag{3.15}$$

Of course, we must have $\xi_1,\ldots,\xi_n\in\mathbb{C}\setminus\{0\}$. We can, using the C_j^{ξ} just defined, write

$$\mathcal{T}_{\overrightarrow{\mu}} f = (2\pi i)^{-n} \int\limits_{|\xi_1|=r_{1,k_0}} \cdots \int\limits_{|\xi_n|=r_{n,k_0}} f(\xi_1,\ldots,\xi_n)\,\xi_1^{-1}\cdots\xi_n^{-1} \tag{3.16}$$
$$\sum_{m_1,\ldots,m_n=0}^{\infty} \sum_{\pi\in S_m} \int\limits_{\Delta_m(\pi)} C^{\xi}_{\pi(m)}(s_{\pi(m)})\cdots C^{\xi}_{\pi(1)}(s_{\pi(1)})$$
$$(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n})\,(ds_1,\ldots,ds_m)d\xi_1\cdots d\xi_n.$$

At this point we ask ourselves what function has a disentangling equal to

$$\sum_{m_1,\ldots,m_n=0}^{\infty} \sum_{\pi\in S_m} \int\limits_{\Delta_m(\pi)} C^{\xi}_{\pi(m)}(s_{\pi(m)})\cdots C^{\xi}_{\pi(1)}(s_{\pi(1)}) \tag{3.17}$$
$$(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n})\,(ds_1,\ldots,ds_m)$$

and is such that the contour integral in (3.16) makes sense. Consider the function $h\left(\xi_1,\ldots,\xi_n;\tilde{A}_1,\ldots,\tilde{A}_n\right)$ defined by

$$h\left(\xi_1,\ldots,\xi_n;\tilde{A}_1,\ldots,\tilde{A}_n\right) = \xi_1\cdots\xi_n\left(\xi_1-\tilde{A}_1\right)^{-1}\cdots\left(\xi_n-\tilde{A}_n\right)^{-1} \tag{3.18}$$

for $|\xi_j| = r_{j,k_0}$, $j = 1,\dots,n$. In view of the discussion at the start of this section, h as defined is analytic on any polydisk containing P_0 because $\xi_j \in \rho(\tilde{A}_j)$ for $j = 1,\dots,n$. It is routine to calculate the disentangling of the function h; however, to ensure convergence of the contour integral, we apply the disentangling map $\mathcal{T}_{\vec{\mu}}^{(k_0+1)}$ to h to obtain

$$\mathcal{T}_{\vec{\mu}}^{(k_0+1)} h = \sum_{m_1,\dots,m_n=0}^{\infty} \sum_{\pi\in S_m} \int_{\Delta_m(\pi)} C^{\xi}_{\pi(m)}(s_{\pi(m)}) \cdots C^{\xi}_{\pi(1)}(s_{\pi(1)}) \cdot \left(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}\right)(ds_1,\dots,ds_m).$$

Hence

$$\mathcal{T}_{\vec{\mu}} f = (2\pi i)^{-n} \int_{|\xi_1|=r_{1,k_0}} \cdots \int_{|\xi_n|=r_{n,k_0}} f(\xi_1,\dots,\xi_n)\, \mathcal{T}_{\vec{\mu}}^{(k_0+1)} \left(\left(\xi_1 - \tilde{A}_1\right)^{-1} \cdots \left(\xi_n - \tilde{A}_n\right)^{-1}\right) d\xi_1 \cdots d\xi_n. \tag{3.19}$$

Note that for any $k, l \in \mathbb{N}$, we have

$$\mathcal{T}_{\vec{\mu}}^{(k)} \left(\left(\xi_1 - \tilde{A}_1\right)^{-1} \cdots \left(\xi_n - \tilde{A}_n\right)^{-1}\right) = \mathcal{T}_{\vec{\mu}}^{(l)} \left(\left(\xi_1 - \tilde{A}_1\right)^{-1} \cdots \left(\xi_n - \tilde{A}_n\right)^{-1}\right)$$

for $\xi_j \in \rho\left(\tilde{A}_j\right)$, $j = 1,\dots,n$.

Remark 3.3. The reason that we apply the disentangling map

$$\mathcal{T}_{\vec{\mu}}^{(k_0+1)} : \mathbb{D}_{k_0+1} \to \mathcal{L}(X)$$

to the Cauchy kernel is due to the norm on the disentangling algebras. Indeed, on $\mathbb{D}_k$, we have

$$\begin{aligned}
&\left\| \left(\xi_1 - \tilde{A}_1\right)^{-1} \cdots \left(\xi_n - \tilde{A}_n\right)^{-1} \right\|_{\mathbb{D}_k} \\
&= \left\| \xi_1^{-1} \cdots \xi_n^{-1} \sum_{m_1,\dots,m_n=0}^{\infty} \left(\frac{\tilde{A}_1}{\xi_1}\right)^{m_1} \cdots \left(\frac{\tilde{A}_n}{\xi_n}\right)^{m_n} \right\|_{\mathbb{D}_k} \\
&= |\xi_1|^{-1} \cdots |\xi_n|^{-1} \sum_{m_1,\dots,m_n=0}^{\infty} \frac{r_{1,k}^{m_1} \cdots r_{n,k}^{m_n}}{|\xi_1|^{m_1} \cdots |\xi_n|^{m_n}}.
\end{aligned}$$

If $|\xi_1| = r_{1,k},\dots,|\xi_n| = r_{n,k}$, i.e., if we're on the boundary of the polydisk P_k, then this series fails to converge. Further, by disentangling with the index k_0+1, we obtain the inequality $\|\mathcal{T}_{\vec{\mu}}^{(k_0)} f\| \le \|f\|_{\mathbb{D}_{k_0}}$. (See Proposition 3.5 below.) Finally, using the index k_0+1 on the disentangling of the Cauchy kernel does not effect the disentangling of the kernel and, as long as the index is larger than k_0, there is no effect on the conclusion of Proposition 3.5.

All that remains to be done in order to show that (3.19) is the disentangling of $f \in \mathbb{D}_\infty$ is to verify that we can indeed interchange the sum over $m_1, \ldots, m_n \in \mathbb{N} \cup \{0\}$ and the contour integral around the boundary of the polydisk in question (P_{k_0}). The tool we use for this task is the vector version of Corollary 12.33 of [4]. The scalar version is stated here.

Corollary (Corollary 12.33 of [4]). *Let $\{f_n\}$ be a sequence of $\mathbb{C}$-valued measurable functions on the measure space $(\Omega, \mathcal{A}, \mu)$ such that*

$$\sum_{n=1}^{\infty} |f_n| \in L^1(\Omega, \mu)$$

or, equivalently,

$$\sum_{n=1}^{\infty} \int_\Omega |f_n| \, d\mu < \infty.$$

Then $\sum_{n=1}^{\infty} f_n \in L^1(\Omega, \mu)$ and

$$\int_\Omega \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \int_\Omega f_n \, d\mu.$$

In our setting we take, for the sequence $\{f_n\}$,

$$g_{m_1,\ldots,m_n}(\xi_1, \ldots, \xi_n) = f(\xi_1, \ldots, \xi_n) \xi_1^{-1} \cdots \xi_n^{-1} \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C^{\xi}_{\pi(m)}(s_{\pi(m)}) \cdots C^{\xi}_{\pi(1)}(s_{\pi(1)}) (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_m).$$

It is clear that $g_{m_1,\ldots,m_n}$ is continuous on the boundary

$$\{(\xi_1, \ldots, \xi_n) : |\xi_j| = r_{j,k_0}, \ j = 1, \ldots, n\}$$

of the polydisk P_{k_0} for all $m_1, \ldots, m_n$. Since

$$\sum_{m_1,\ldots,m_n=0}^{\infty} \int_{|\xi_1|=r_{1,k_0}} \cdots \int_{|\xi_n|=r_{n,k_0}} g_{m_1,\ldots,m_n}(\xi_1, \ldots, \xi_n) \, d\xi_1 \cdots d\xi_n$$

is the disentangling series for $f \in \mathbb{D}_{k_0}$, it is norm convergent in $\mathcal{L}(X)$. The sum–integral interchange follows at once from the (obvious) vector-valued version of the corollary stated above.

The discussion above is summarized in the following theorem.

Theorem 3.4. *Let $\mathbb{D}_\infty$ be as constructed above. For $f \in \mathbb{D}_\infty$, let $k_0 \in \mathbb{N}$ be the least integer such that $f \in \mathbb{D}_{k_0}$. Then*

$$f(\xi_1, \ldots, \xi_n) \mathcal{T}^{(k_0+1)}_{\vec{\mu}} \left(\left(\xi_1 - \tilde{A}_1 \right)^{-1} \cdots \left(\xi_n - \tilde{A}_n \right)^{-1} \right)$$

is Bochner integrable on the boundary of P_{k_0} and we have

$$\mathcal{T}_{\overrightarrow{\mu}} f = \mathcal{T}^{(k_0)}_{\mu_1,\dots,\mu_n} f = \mathcal{T}^{(k_0)}_{\overrightarrow{\mu}} f = f_{\mu_1,\dots,\mu_n}(A_1,\dots,A_n)$$
$$= (2\pi i)^{-n} \int\limits_{|\xi_1|=r_{1,k_0}} \cdots \int\limits_{|\xi_n|=r_{n,k_0}} f(\xi_1,\dots,\xi_n)\, \mathcal{T}^{(k_0+1)}_{\overrightarrow{\mu}} \left(\left(\xi_1 - \tilde{A}_1\right)^{-1} \cdots \left(\xi_n - \tilde{A}_n\right)^{-1} \right) d\xi_1 \cdots d\xi_n.$$

It is clear that the same result obtains for any $k \geq k_0$.

Proof. The only part of the theorem that needs proof is the assertion that $f(\xi_1,\dots,\xi_n)\cdot\ \mathcal{T}^{(k_0+1)}_{\overrightarrow{\mu}}\left(\left(\xi_1 - \tilde{A}_1\right)^{-1} \cdots \left(\xi_n - \tilde{A}_n\right)^{-1}\right)$ is Bochner integrable. But this is clear since f is continuous on the boundary of P_{k_0} and $\mathcal{T}^{(k_0+1)}_{\overrightarrow{\mu}}\left(\left(\xi_1 - \tilde{A}_1\right)^{-1} \cdots \left(\xi_n - \tilde{A}_n\right)^{-1}\right)$ is a bounded operator. Hence the contour integral of the scalar function

$$\left\| f(\xi_1,\dots,\xi_n) \cdot \mathcal{T}^{(k_0+1)}_{\overrightarrow{\mu}}\left(\left(\xi_1 - \tilde{A}_1\right)^{-1} \cdots \left(\xi_n - \tilde{A}_n\right)^{-1}\right) \right\|_{\mathcal{L}(X)}$$

around the boundary of P_{k_0} is finite and so

$$f(\xi_1,\dots,\xi_n)\mathcal{T}^{(k_0+1)}_{\overrightarrow{\mu}}\left(\left(\xi_1 - \tilde{A}_1\right)^{-1} \cdots \left(\xi_n - \tilde{A}_n\right)^{-1}\right)$$

is Bochner integrable as claimed. (See Theorem 2, page 45 of [2].) □

It is natural to ask if representing the disentangling using the disentangling of the Cauchy kernel causes the disentangling map to be other than a contraction. The answer is "no" as is seen in the following proposition.

Proposition 3.5. *For $f \in \mathbb{D}_\infty$, $\left\|\mathcal{T}_{\overrightarrow{\mu}} f\right\|_{\mathcal{L}(X)} \leq \|f\|_{\mathbb{D}_{k_0}} = \|f\|_{\mathbb{A}_{k_0}}$.*

Proof. Let $f \in \mathbb{D}_\infty$. Let $k_0 \in \mathbb{N}$ be the least integer such that $f \in \mathbb{D}_{k_0}$. We calculate as follows:

$$\left\| f_{\overrightarrow{\mu}}(A_1,\dots,A_n) \right\|_{\mathcal{L}(X)}$$
$$= \left\| (2\pi i)^{-n} \int\limits_{|\xi_1|=r_{1,k_0}} \cdots \int\limits_{|\xi_n|=r_{n,k_0}} f(\xi_1,\dots,\xi_n) \cdot \mathcal{T}^{(k_0+1)}_{\overrightarrow{\mu}} \left(\left(\xi_1 - \tilde{A}_1\right)^{-1} \cdots \left(\xi_n - \tilde{A}_n\right)^{-1} \right) d\xi_1 \cdots d\xi_n \right\|_{\mathcal{L}(X)}$$
$$\overset{(a)}{=} \Big\|_{\mathcal{L}(X)} (2\pi i)^{-n} \int\limits_{|\xi_1|=r_{1,k_0}} \cdots \int\limits_{|\xi_n|=r_{n,k_0}} \mathcal{T}^{(k_0+1)}_{\overrightarrow{\mu}}$$

$$\left(f\left(\xi_1,\ldots,\xi_n\right)\left(\xi_1-\tilde{A}_1\right)^{-1}\cdots\left(\xi_n-\tilde{A}_n\right)^{-1}\right)d\xi_1\cdots d\xi_n\Bigg\|_{\mathcal{L}(X)}$$

$$\stackrel{(b)}{=}\Bigg\|\mathcal{T}_{\overrightarrow{\mu}}^{(k_0+1)}\Bigg((2\pi i)^{-n}\int\limits_{|\xi_1|=r_{1,k_0}}\cdots\int\limits_{|\xi_n|=r_{n,k_0}}f\left(\xi_1,\ldots,\xi_n\right)$$

$$\left(\xi_1-\tilde{A}_1\right)^{-1}\cdots\left(\xi_n-\tilde{A}_n\right)^{-1}d\xi_1\cdots d\xi_n\Bigg)\Bigg\|_{\mathcal{L}(X)}$$

$$\stackrel{(c)}{=}\left\|\mathcal{T}_{\overrightarrow{\mu}}^{(k_0+1)}f\left(\tilde{A}_1,\ldots,\tilde{A}_n\right)\right\|_{\mathcal{L}(X)}\le\|f\|_{\mathbb{D}_{k_0+1}}\le\|f\|_{\mathbb{D}_{k_0}}.$$

Equality (a) follows from the linearity of the disentangling map. Equality (b) follows from the argument above that shows that we can interchange the contour integral and the sum over $m_1,\ldots,m_n$. Also, note that we can apply $\mathcal{T}_{\overrightarrow{\mu}}^{(k_0+1)}$ to the contour integral representing f at this point since $\mathbb{D}_{k_0}\subseteq\mathbb{D}_{k_0+1}$. Equality (c) follows from Cauchy's Integral Formula. The second to the last inequality is due to the fact that $\mathcal{T}_{\overrightarrow{\mu}}^{(k)}$ is a linear contraction for every $k\in\mathbb{N}$ and the last inequality is due to the fact that the sequences $\{r_{j,k}\}_{k\in\mathbb{N}}$ of weights are (strictly) decreasing. □

It is striking that the representation obtained in Theorem 3.4 tells us that we can obtain the disentangling of a function $f\in\mathbb{D}_\infty$ by integrating this function against the disentangling of the standard Cauchy kernel around the boundary of a polydisk. Upon some reflection, this type of representation seems very natural in view of the Cauchy representation of analytic functions.

4. An application of the Cauchy representation of Feynman's operational calculus

We can use the representation of Feynman's operational calculus (in the time-independent setting) obtained above in Theorem 3.4 to obtain a "differential calculus" for the operational calculus. The motivation for the "partial derivative" introduced below is the Cauchy representation for the partial derivative of an analytic function of several variables. Indeed, given a function $g(z_1,\ldots,z_n)$ that is analytic on the open polydisk P with radii $r_1,\ldots,r_n$ and continuous on its boundary, we can write, for $(c_1,\ldots,c_n)\in P$,

$$\frac{\partial g}{\partial z_j}(c_1,\ldots,c_n)=\frac{1}{(2\pi i)^n}\int\limits_{|\xi_1|=r_1}\cdots\int\limits_{|\xi_n|=r_n}g(\xi_1,\ldots,\xi_n)\left(\xi_1-c_1\right)^{-1}\cdots$$

$$\left(\xi_{j-1}-c_{j-1}\right)^{-1}\left(\xi_j-c_j\right)^{-2}\left(\xi_{j+1}-c_{j+1}\right)^{-1}\cdots\left(\xi_n-c_n\right)^{-1}d\xi_1\cdots d\xi_n.$$

We use this idea to define the "derivative" $\delta_{A_j}f_{\overrightarrow{\mu}}\left(A_1,\ldots,A_n\right)$ of the disentangled operator $f_{\overrightarrow{\mu}}\left(A_1,\ldots,A_n\right)$.

Definition 4.1. Given $f \in \mathbb{D}_\infty$, let k_0 be the smallest positive integer such that $f \in \mathbb{D}_{k_0}$. Define, for each $j = 1, \ldots, n$, $\delta_{A_j} f_{\overrightarrow{\mu}} (A_1, \ldots, A_n)$ by

$$\begin{aligned}(4.1)\quad \delta_{A_j} f_{\overrightarrow{\mu}} (A_1, \ldots, A_n) := (2\pi i)^{-n} \int\limits_{|\xi_1| = r_{1,k_0}} \cdots \int\limits_{|\xi_n| = r_{n,k_0}} f (\xi_1, \ldots, \xi_n) \cdot \\ \mathcal{T}_{\overrightarrow{\mu}}^{(k_0+1)} \left(\left(\xi_1 - \tilde{A}_1\right)^{-1} \cdots \left(\xi_{j-1} - \tilde{A}_{j-1}\right)^{-1} \left(\xi_j - \tilde{A}_j\right)^{-2} \right. \\ \left. \left(\xi_{j+1} - \tilde{A}_{j+1}\right)^{-1} \cdots \left(\xi_n - \tilde{A}_n\right)^{-1} \right) d\xi_1 \cdots d\xi_n\end{aligned}$$

Proceeding exactly as in the proof of Proposition 3.5, we have:

Proposition 4.2. *For $f \in \mathbb{D}_\infty$ and for any $j = 1, \ldots, n$,*

$$\left\| \delta_{A_j} f_{\overrightarrow{\mu}} (A_1, \ldots, A_n) \right\|_{\mathcal{L}(X)} \le \left\| \frac{\partial f}{\partial z_j} \right\|_{\mathbb{D}_{k_0}}$$

where k_0 is the least integer for which $f \in \mathbb{D}_{k_0}$.

The basic properties of δ_{A_j} are given in the next proposition.

Proposition 4.3. *Let $f, g \in \mathbb{D}_\infty$ and let k_0 be the smallest positive integer for which $f, g \in \mathbb{D}_{k_0}$. Let $\alpha, \beta \in \mathbb{C}$.*

(1) δ_{A_j} is linear:

$$\begin{aligned}(4.2)\quad \delta_{A_j} \left[\alpha f_{\overrightarrow{\mu}} (A_1, \ldots, A_n) + \beta g_{\overrightarrow{\mu}} (A_1, \ldots, A_n) \right] \\ = \alpha \delta_{A_j} f_{\overrightarrow{\mu}} (A_1, \ldots, A_n) + \beta \delta_{A_j} g_{\overrightarrow{\mu}} (A_1, \ldots, A_n).\end{aligned}$$

for $j = 1, \ldots, n$.

(2) δ_{A_j} satisfies the Leibniz rule:

$$\begin{aligned}(4.3)\quad \delta_{A_j} \left[f_{\overrightarrow{\mu}} (A_1, \ldots, A_n) \, g_{\overrightarrow{\mu}} (A_1, \ldots, A_n) \right] \\ = \left(\delta_{A_j} f_{\overrightarrow{\mu}} (A_1, \ldots, A_n) \right) g_{\overrightarrow{\mu}} (A_1, \ldots, A_n) \\ + f_{\overrightarrow{\mu}} (A_1, \ldots, A_n) \left(\delta_{A_j} g_{\overrightarrow{\mu}} (A_1, \ldots, A_n) \right)\end{aligned}$$

for $j = 1, \ldots, n$.

The proof of this proposition will be delayed so that we can establish some necessary machinery. We first note that, for any fixed j,

$$(4.4)\qquad \lim_{\omega \downarrow 0} \frac{\left(\xi_j - \left(\tilde{A}_j - \omega \tilde{I}\right)\right)^{-1} - \left(\xi_j - \tilde{A}_j\right)^{-1}}{\omega} = \left(\xi_j - \tilde{A}_j\right)^{-2}.$$

Remark 4.4. A one-sided limit is used above for reasons that will become clear as we proceed.

We would like to be able to write (where the limit is taken in the norm topology on $\mathcal{L}(X)$)

$$\begin{aligned}(4.5)\quad & \delta_{A_j} f_{\overrightarrow{\mu}}(A_1, \ldots, A_n) \\ &= \lim_{\omega \downarrow 0} \frac{1}{\omega}\big(f_{\overrightarrow{\mu}}(A_1, \ldots, A_{j-1}, A_j - \omega I, A_{j+1}, \ldots, A_n) \\ &\qquad - f_{\overrightarrow{\mu}}(A_1, \ldots, A_n)\big).\end{aligned}$$

Before we can establish this formula, we must make sure that, when choosing $f \in \mathbb{D}_\infty$, we are able to make mathematical sense of

$$f\big(\tilde{A}_1, \ldots, \tilde{A}_{j-1}, \tilde{A}_j - \omega \tilde{I}, \tilde{A}_{j+1}, \ldots, \tilde{A}_n\big)$$

for small ω (since we are letting ω tend to zero). We will work with the algebras $\mathbb{A}_{(\cdot)}$. As always, for $f \in \mathbb{A}_\infty$, there is a smallest integer k_0 such that $f \in \mathbb{A}_{k_0}$. We start by choosing $\omega_0 > 0$ so that

$$\begin{aligned}(4.6)\quad r_{j,k_0} > r_{j,k_0}^{\omega_0} &:= \|A_j - \omega_0 I\| + \epsilon_{j,k_0} \\ &\geq \|A_j\| - \omega_0 + \epsilon_{j,k_0} \equiv r_{j,k_0} - \omega_0 > r_{j,k_0+1}.\end{aligned}$$

It follows that

$$\begin{aligned}(4.7)\qquad \mathbb{A}_{k_0} &\equiv \mathbb{A}\left(r_{1,k_0}, \ldots, r_{n,k_0}\right) \\ &\subseteq \mathbb{A}\left(r_{1,k_0}, \ldots, r_{j-1,k_0}, r_{j,k_0}^{\omega_0}, r_{j+1,k_0}, \ldots, r_{n,k_0}\right) \\ &\subseteq \mathbb{A}\left(r_{1,k_0}, \ldots, r_{j-1,k_0}, r_{j,k_0} - \omega_0, r_{j+1,k_0}, \ldots, r_{n,k_0}\right) \\ &\subseteq \mathbb{A}_{k_0+1}.\end{aligned}$$

It is due to this chain of inclusions that we use the limit $\omega \downarrow 0$ above in (4.4). For any ω for which the preceding chain of inclusions holds, in particular for any $\omega \leq \omega_0$, we have

$$(4.8)\qquad f \in \mathbb{A}_{k_0} \subseteq \mathbb{A}\left(r_{1,k_0}, \ldots, r_{j-1,k_0}, r_{j,k_0}^{\omega_0}, r_{j+1,k_0}, \ldots, r_{n,k_0}\right).$$

Because of the identification of the algebras $\mathbb{A}_k$ and $\mathbb{D}_k$, it makes sense to consider, for any $\omega \leq \omega_0$, the element

$$f\left(\tilde{A}_1, \ldots, \tilde{A}_{j-1}, \tilde{A}_j - \omega \tilde{I}, \tilde{A}_{j+1}, \ldots, \tilde{A}_n\right) \in \mathbb{D}(\omega)$$

where $\mathbb{D}(\omega)$ denotes the disentangling algebra corresponding to

$$\mathbb{A}\big(r_{1,k_0}, \ldots, r_{j-1,k_0}, r_{j,k_0}^{\omega}, r_{j+1,k_0}, \ldots, r_{n,k_0}\big)$$

for $f \in \mathbb{D}_{k_0}$. The disentangling of $f\big(\tilde{A}_1, \ldots, \tilde{A}_{j-1}, \tilde{A}_j - \omega \tilde{I}, \tilde{A}_{j+1}, \ldots, \tilde{A}_n\big)$, $f \in \mathbb{D}_{k_0}$, is

$$\begin{aligned}(4.9)\quad & f_{\overrightarrow{\mu}}(A_1, \ldots, A_{j-1}, A_j - \omega I, A_{j+1}, \ldots, A_n) \\ &= (2\pi i)^{-n} \int_{|\xi_1| = r_{1,k_0}} \cdots \int_{|\xi_n| = r_{n,k_0}} f(\xi_1, \ldots, \xi_n)\, \mathcal{T}_{\overrightarrow{\mu}}^{(k_0+1)}\left(\left(\xi_1 - \tilde{A}_1\right)^{-1} \cdots\right.\end{aligned}$$

$$\left(\xi_{j-1}-\tilde{A}_{j-1}\right)^{-1}\left(\xi_j-\left[\tilde{A}_j-\omega\tilde{I}\right]\right)^{-1}\left(\xi_{j+1}-\tilde{A}_{j+1}\right)^{-1}\cdots\left(\xi_n-\tilde{A}_n\right)^{-1}\Big) \, d\xi_1\cdots d\xi_n.$$

Next, we want to establish that

$$\lim_{\omega\downarrow 0}\left\|f_{\overrightarrow{\mu}}\left(A_1,\ldots,A_{j-1},A_j-\omega I,A_{j+1},\ldots,A_n\right) - f_{\overrightarrow{\mu}}\left(A_1,\ldots,A_{j-1},A_j,A_{j+1},\ldots,A_n\right)\right\|_{\mathcal{L}(X)}=0. \tag{4.10}$$

All that we need to do to obtain this limit is to appeal to the definition of the disentangling map and use the time independence of the operators. Indeed, we can easily obtain

$$\begin{aligned}
&\Big\|\mathcal{T}_{\overrightarrow{\mu}}^{(k_0+1)}\Big(\left(\xi_1-\tilde{A}_1\right)^{-1}\cdots\left(\xi_{j-1}-\tilde{A}_{j-1}\right)^{-1} \\
&\qquad\left(\xi_j-\left[\tilde{A}_j-\omega\tilde{I}\right]\right)^{-1}\left(\xi_{j+1}-\tilde{A}_{j+1}\right)^{-1}\left(\xi_n-\tilde{A}_n\right)^{-1}\Big) \\
&\quad-\mathcal{T}_{\overrightarrow{\mu}}^{(k_0+1)}\Big(\left(\xi_1-\tilde{A}_1\right)^{-1}\cdots\left(\xi_{j-1}-\tilde{A}_{j-1}\right)^{-1}\left(\xi_j-\tilde{A}_j\right)^{-1} \\
&\qquad\left(\xi_{j+1}-\tilde{A}_{j+1}\right)^{-1}\cdots\left(\xi_n-\tilde{A}_n\right)^{-1}\Big)\Big\|_{\mathcal{L}(X)} \\
&\le\sum_{m_1,\ldots,m_n=0}^{\infty}\left|\xi_1^{-m_1-1}\cdots\xi_n^{-m_n-1}\right| \\
&\qquad\sum_{\pi\in S_m}\left\|C_{\pi(m)}^{(\omega)}\cdots C_{\pi(1)}^{(\omega)}-C_{\pi(m)}\cdots C_{\pi(1)}\right\|_{\mathcal{L}(X)} \\
&\qquad\qquad\left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)\left(\Delta_m(\pi)\right)
\end{aligned}$$

where $C_j^{(\omega)}$ denotes the operators in the disentangling of the version of the Cauchy kernel containing $\tilde{A}_j-\omega\tilde{I}$. It is clear that the norm difference in the sum over S_m that appears in the second to last line of the above display goes to zero as $\omega\downarrow 0$ since the norm on $\mathcal{L}(X)$ is continuous. Using the Dominated Convergence Theorem, we obtain

$$\lim_{\omega\downarrow 0}\sum_{m_1,\ldots,m_n=0}^{\infty}\left|\xi_1^{-m_1-1}\cdots\xi_n^{-m_n-1}\right| \sum_{\pi\in S_m}\left\|C_{\pi(m)}^{(\omega)}\cdots C_{\pi(1)}^{(\omega)}-C_{\pi(m)}\cdots C_{\pi(1)}\right\|_{\mathcal{L}(X)} \left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)\left(\Delta_m(\pi)\right)=0. \tag{4.11}$$

It therefore follows, again by the Dominated Convergence Theorem, that

$$\lim_{\omega\downarrow 0}\left\|f_{\overrightarrow{\mu}}\left(A_1,\ldots,A_{j-1},A_j-\omega I,A_{j+1},\ldots,A_n\right)- f_{\overrightarrow{\mu}}\left(A_1,\ldots,A_{j-1},A_j,A_{j+1},\ldots,A_n\right)\right\|_{\mathcal{L}(X)}=0. \tag{4.12}$$

With the discussion just above completed, we can verify (4.5) by calculating as follows, using the definition of δ_{A_j}.

$$\begin{aligned}
&\delta_{A_j} f_{\overrightarrow{\mu}}(A_1,\ldots,A_n) \\
&= (2\pi i)^{-n} \int\limits_{|\xi_1|=r_{1,k_0}} \cdots \int\limits_{|\xi_n|=r_{n,k_0}} f(\xi_1,\ldots,\xi_n)\, \mathcal{T}_{\overrightarrow{\mu}}^{(k_0+1)} \Big(\big(\xi_1-\tilde{A}_1\big)^{-1} \cdots \\
&\qquad \big(\xi_{j-1}-\tilde{A}_{j-1}\big)^{-1} \big(\xi_j-\tilde{A}_j\big)^{-2} \big(\xi_{j+1}-\tilde{A}_{j+1}\big)^{-1} \cdots \big(\xi_n-\tilde{A}_n\big)^{-1} \Big) \\
&\qquad\qquad d\xi_1\cdots d\xi_n \\
&= (2\pi i)^{-n} \int\limits_{|\xi_1|=r_{1,k_0}} \cdots \int\limits_{|\xi_n|=r_{n,k_0}} f(\xi_1,\ldots,\xi_n)\, \mathcal{T}_{\overrightarrow{\mu}}^{(k_0+1)} \\
&\qquad \Big(\big(\xi_1-\tilde{A}_1\big)^{-1} \cdots \big(\xi_{j-1}-\tilde{A}_{j-1}\big)^{-1} \\
&\qquad\quad \left[\lim_{\omega\downarrow 0} \frac{\big(\xi_j - \big(\tilde{A}_j - \omega\tilde{I}\big)\big)^{-1} - \big(\xi_j-\tilde{A}_j\big)^{-1}}{\omega} \right] \\
&\qquad\quad \big(\xi_{j+1}-\tilde{A}_{j+1}\big)^{-1} \cdots \big(\xi_n-\tilde{A}_n\big)^{-1} \Big) d\xi_1\cdots d\xi_n \\
&\overset{(\star)}{=} \lim_{\omega\downarrow 0} \frac{1}{\omega} \Bigg[(2\pi i)^{-n} \int\limits_{|\xi_1|=r_{1,k_0}} \cdots \int\limits_{|\xi_n|=r_{n,k_0}} f(\xi_1,\ldots,\xi_n)\, \mathcal{T}_{\overrightarrow{\mu}}^{(k_0+1)} \\
&\qquad \Big(\big(\xi_1-\tilde{A}_1\big)^{-1} \cdots \big(\xi_{j-1}-\tilde{A}_{j-1}\big)^{-1} \big(\xi_j - \big(\tilde{A}_j - \omega\tilde{I}\big)\big)^{-1} \\
&\qquad\quad \big(\xi_{j+1}-\tilde{A}_{j+1}\big)^{-1} \cdots \big(\xi_n-\tilde{A}_n\big)^{-1} \Big) d\xi_1\cdots d\xi_n \\
&\qquad - (2\pi i)^{-n} \int\limits_{|\xi_1|=r_{1,k_0}} \cdots \int\limits_{|\xi_n|=r_{n,k_0}} f(\xi_1,\ldots,\xi_n)\, \mathcal{T}_{\overrightarrow{\mu}}^{(k_0+1)} \\
&\qquad \Big(\big(\xi_1-\tilde{A}_1\big)^{-1} \cdots \big(\xi_{j-1}-\tilde{A}_{j-1}\big)^{-1} \big(\xi_j-\tilde{A}_j\big)^{-1} \\
&\qquad\quad \big(\xi_{j+1}-\tilde{A}_{j+1}\big)^{-1} \cdots \big(\xi_n-\tilde{A}_n\big)^{-1} \Big) d\xi_1\cdots d\xi_n \Bigg] \\
&= \lim_{\omega\downarrow 0} \frac{1}{\omega} \Big[f_{\overrightarrow{\mu}}(A_1,\ldots,A_{j-1},A_j-\omega I,A_{j+1},\ldots,A_n) \\
&\qquad\qquad - f_{\overrightarrow{\mu}}(A_1,\ldots,A_{j-1},A_j,A_{j+1},\ldots,A_n) \Big]
\end{aligned}$$

where equality $(\star)$ follows easily from the dominated convergence theorem and the linearity of the disentangling map. Finally, the last equality just above follows from (4.9). We see, then, that (4.5) holds as claimed.

The proof of Proposition 4.3 is now straightforward.

Proof of Proposition 4.3. The proof of item (1) is obvious. For the proof of item (2), the Leibniz property for δ_{A_j}, $j = 1, \ldots, n$, we proceed much as one does in elementary calculus. For simplicity of notation we execute the proof for $j = 1$ as the proof for arbitrary j will be identical except for the obvious changes in notation. Given $f, g \in \mathbb{D}_\infty$ let k_0 be the least positive integer such that $f, g \in \mathbb{D}_{k_0}$. (It should be noted here, however, that the notation we are using suppresses the role of $\mathbb{D}_{k_0}$ in the calculation below.) We calculate as shown below, appealing to (4.5) and (4.12) in the first and fourth equalities below:

$$\begin{aligned}
&\delta_{A_1}\left[f_{\overrightarrow{\mu}}\left(A_1, \ldots, A_n\right) g_{\overrightarrow{\mu}}\left(A_1, \ldots, A_n\right)\right] \\
&= \lim_{\omega \downarrow 0} \frac{1}{\omega}\left[f_{\overrightarrow{\mu}}\left(A_1 - \omega I, A_2, \ldots, A_n\right) g_{\overrightarrow{\mu}}\left(A_1 - \omega I, A_2, \ldots, A_n\right)\right. \\
&\qquad\qquad \left. - f_{\overrightarrow{\mu}}\left(A_1, A_2, \ldots, A_n\right) g_{\overrightarrow{\mu}}\left(A_1, A_2, \ldots, A_n\right)\right] \\
&= \lim_{\omega \downarrow 0} \frac{1}{\omega}\left[f_{\overrightarrow{\mu}}\left(A_1 - \omega I, A_2, \ldots, A_n\right) g_{\overrightarrow{\mu}}\left(A_1 - \omega I, A_2, \ldots, A_n\right)\right. \\
&\qquad\qquad - f_{\overrightarrow{\mu}}\left(A_1 - \omega I, A_2, \ldots, A_n\right) g_{\overrightarrow{\mu}}\left(A_1, A_2, \ldots, A_n\right) \\
&\qquad\qquad + f_{\overrightarrow{\mu}}\left(A_1 - \omega I, A_2, \ldots, A_n\right) g_{\overrightarrow{\mu}}\left(A_1, A_2, \ldots, A_n\right) \\
&\qquad\qquad \left. - f_{\overrightarrow{\mu}}\left(A_1, A_2, \ldots, A_n\right) g_{\overrightarrow{\mu}}\left(A_1, A_2, \ldots, A_n\right)\right] \\
&= \lim_{\omega \downarrow 0} f_{\overrightarrow{\mu}}\left(A_1 - \omega I, A_2, \ldots, A_n\right) \\
&\qquad \left\{\frac{1}{\omega}\left(g_{\overrightarrow{\mu}}\left(A_1 - \omega I, A_2, \ldots, A_n\right) - g_{\overrightarrow{\mu}}\left(A_1, A_2, \ldots, A_n\right)\right)\right\} + \\
&\qquad \lim_{\omega \downarrow 0}\left\{\frac{1}{\omega}\left(f_{\overrightarrow{\mu}}\left(A_1 - \omega I, A_2, \ldots, A_n\right) - f_{\overrightarrow{\mu}}\left(A_1, A_2, \ldots, A_n\right)\right)\right\} \\
&\qquad g_{\overrightarrow{\mu}}\left(A_1, A_2, \ldots, A_n\right) \\
&= f_{\overrightarrow{\mu}}\left(A_1, A_2, \ldots, A_n\right)\left[\delta_{A_1} g_{\overrightarrow{\mu}}\left(A_1, A_2, \ldots, A_n\right)\right] \\
&\qquad + \left[\delta_{A_1} f_{\overrightarrow{\mu}}\left(A_1, A_2, \ldots, A_n\right)\right] g_{\overrightarrow{\mu}}\left(A_1, A_2, \ldots, A_n\right) \qquad \square
\end{aligned}$$

With the basic properties of δ_{A_j} put down, we can define higher order "derivatives", or iterations of δ_{A_j}.

Definition 4.5. Given $f \in \mathbb{D}_\infty$, we define $\delta_{A_j}\left[\delta_{A_i} f_{\overrightarrow{\mu}}\left(A_1, \ldots, A_n\right)\right]$ by

$$\begin{aligned}
&\delta_{A_j}\left[\delta_{A_i} f_{\overrightarrow{\mu}}\left(A_1, \ldots, A_n\right)\right] \\
&= \delta_{A_j}\left\{(2\pi i)^{-n} \int\limits_{|\xi_1| = r_{1,k_0}} \cdots \int\limits_{|\xi_n| = r_{n,k_0}} f\left(\xi_1, \ldots, \xi_n\right) \mathcal{T}_{\overrightarrow{\mu}}^{(k_0+1)}\left(\left(\xi_1 - \tilde{A}_1\right)^{-1} \cdots\right.\right.
\end{aligned}$$

$$\left(\xi_{i-1}-\tilde{A}_{i-1}\right)^{-1}\left(\xi_i-\tilde{A}_i\right)^{-2}\left(\xi_{i+1}-\tilde{A}_{i+1}\right)^{-1}\cdots\left(\xi_n-\tilde{A}_n\right)^{-1}\Big)$$

$$d\xi_1\cdots d\xi_n\Big\}$$

$$= (2\pi i)^{-n}\int\limits_{|\xi_1|=r_{1,k_0}}\cdots\int\limits_{|\xi_n|=r_{n,k_0}} f(\xi_1,\ldots,\xi_n)\,\mathcal{T}_{\overrightarrow{\mu}}^{(k_0+1)}\Big(\left(\xi_1-\tilde{A}_1\right)^{-1}\cdots$$
$$\left(\xi_{i-1}-\tilde{A}_{i-1}\right)^{-1}\left(\xi_i-\tilde{A}_i\right)^{-2}\left(\xi_{i+1}-\tilde{A}_{i+1}\right)^{-1}\cdots\left(\xi_{j-1}-\tilde{A}_{j-1}\right)^{-1}$$
$$\left(\xi_j-\tilde{A}_j\right)^{-2}\left(\xi_{j+1}-\tilde{A}_{j+1}\right)^{-1}\cdots\left(\xi_n-\tilde{A}_n\right)^{-1}\Big)\,d\xi_1\cdots d\xi_n$$

where we've assumed that $j > i$ for notational simplicity. The corresponding definitions for $j < i$ and $j = i$ are obvious. Further, k_0 is, as usual, the least positive integer for which $f \in \mathbb{D}_{k_0}$.

Remark 4.6. It is clear from Definition 4.5 how to iterate δ_{A_j} to any order. In fact, we can state the following proposition concerning the commutativity of the iteration of δ_{A_j}.

Proposition 4.7. *Given $f \in \mathbb{D}_\infty$, we have*

$$\delta_{A_j}\left[\delta_{A_i} f_{\overrightarrow{\mu}}(A_1,\ldots,A_n)\right] = \delta_{A_i}\left[\delta_{A_j} f_{\overrightarrow{\mu}}(A_1,\ldots,A_n)\right] \tag{4.13}$$

for $i, j \in \{1,\ldots,n\}$.

Proof. The proof is accomplished by using Definition 4.5 on both sides of (4.13) and observing that the expressions obtained are identical. □

We now turn our attention to the exponential function; i.e., to the disentangling of the exponential function. From elementary calculus, we know that the derivative of the exponential function is itself. Such a statement is true for the disentangled exponential function as is seen from the following proposition.

Proposition 4.8. *For the entire function*

$$f(z_1,\ldots,z_n) = \exp(z_1+\cdots+z_n)$$

we have

$$\delta_{A_j}\left[f_{\overrightarrow{\mu}}(A_1,\ldots,A_n)\right] = f_{\overrightarrow{\mu}}(A_1,\ldots,A_n) \tag{4.14}$$

for any $j = 1,\ldots,n$. Further, any iterate of δ_{A_j} returns the same result.

Proof. First, it is clear that, because f is an entire function, $f \in \mathbb{D}_\infty$. Indeed, $f \in \mathbb{D}_k$ for all $k \in \mathbb{N}$ and also $f \in \mathbb{D}\left(\tilde{A}_1,\ldots,\tilde{A}_n\right)$, the "base" algebra. We will carry out the proof of this proposition for $j = 1$; the proof for any other j is a simple and obvious change in notation. We have, by definition of δ_{A_1} and because $f \in \mathbb{D}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right)$,

(4.15)

$$\begin{aligned}
&\delta_{A_1}\left[f_{\overrightarrow{\mu}}\left(A_1, \ldots, A_n\right)\right] \\
&= (2\pi i)^{-n} \int\limits_{|\xi_1|=r_{1,k}} \cdots \int\limits_{|\xi_n|=r_{n,k}} \exp\left(\xi_1 + \cdots + \xi_n\right) \mathcal{T}_{\overrightarrow{\mu}}^{(k+1)} \\
&\quad \left(\left(\xi_1 - \tilde{A}_1\right)^{-2}\left(\xi_2 - \tilde{A}_2\right)^{-1} \cdots \left(\xi_n - \tilde{A}_n\right)^{-1}\right) d\xi_1 \cdots d\xi_n \\
&= (2\pi i)^{-n} \int\limits_{|\xi_1|=r_{1,k}} \cdots \int\limits_{|\xi_n|=r_{n,k}} \exp\left(\xi_1 + \cdots + \xi_n\right) \mathcal{T}_{\overrightarrow{\mu}}^{(k+1)} \\
&\quad \left(\sum_{m_1=1; m_2, \ldots, m_n=0}^{\infty} m_1 \xi_1^{-m_1-1} \cdots \xi_n^{-m_n-1}\left(\tilde{A}_1\right)^{m_1-1}\left(\tilde{A}_2\right)^{m_1} \cdots \left(\tilde{A}_n\right)^{m_n}\right) \\
&\qquad d\xi_1 \cdots d\xi_n \\
&= \sum_{m_1=1; m_2, \ldots, m_n=0}^{\infty} m_1 \left\{(2\pi i)^{-n} \int\limits_{|\xi_1|=r_{1,k}} \cdots \int\limits_{|\xi_n|=r_{n,k}} \exp\left(\xi_1 + \cdots + \xi_n\right)\right. \\
&\qquad \left. \xi_1^{-m_1-1} \cdots \xi_n^{-m_n-1} d\xi_1 \cdots d\xi_n\right\} \\
&\quad \cdot \sum_{\pi \in S_{m-1}} \int\limits_{\Delta_{m-1}(\pi)} C_{\pi(m-1)}(s_{\pi(m-1)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \left(\mu_1^{m_1-1} \times \cdots \times \mu_n^{m_n}\right) \\
&\qquad (ds_1, \ldots, ds_{m-1}) \\
&= \sum_{m_1=1; m_2, \ldots, m_n=0}^{\infty} \left(\frac{1}{(m_1-1)!\, m_2! \cdots m_n!}\right) \sum_{\pi \in S_{m_1-1+m_2+\cdots+m_n}} \\
&\quad \int\limits_{\Delta_{m_1-1+m_2+\cdots+m_n}(\pi)} C_{\pi(m_1-1+m_2+\cdots+m_n)}(s_{\pi(m_1-1+m_2+\cdots+m_n)}) \\
&\qquad \cdots C_{\pi(1)}(s_{\pi(1)}) \left(\mu_1^{m_1-1} \times \cdots \times \mu_n^{m_n}\right)(ds_1, \ldots, ds_{m_1-1+m_2+\cdots+m_n}) \\
&= \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{1}{m_1! \cdots m_n!} \sum_{\pi \in S_m} \int\limits_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \cdot \\
&\qquad \left(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}\right)(ds_1, \ldots, ds_m) \\
&= f_{\overrightarrow{\mu}}\left(A_1, \ldots, A_n\right).
\end{aligned}$$

Finally, the last comment of the proposition concerning the result of any iterate of δ_{A_j} is simply Proposition 4.7. $\square$

We conclude with some examples to show that δ_{A_j} acts as expected when applied to disentangled monomials/polynomials. These examples also illustrate how the Cauchy representation of the disentangling works for some simple, easy to calculate, disentanglings.

Example 4.9. We consider here the constant function $f(z_1,\ldots,z_n)=\alpha$, for $\alpha\in\mathbb{C}$. Of course, for *any* choices of the measures $\mu_1,\ldots,\mu_n$ we have $f_{\overrightarrow{\mu}}(A_1,\ldots,A_n)=\alpha$. We would expect that $\delta_{A_j}f_{\overrightarrow{\mu}}(A_1,\ldots,A_n)=0$. To see that this is indeed the case, we calculate $\delta_{A_1}f_{\overrightarrow{\mu}}(A_1,\ldots,A_n)$; the calculation for $\delta_{A_j}f_{\overrightarrow{\mu}}(A_1,\ldots,A_n)$ is the same. We have, for any $k\in\mathbb{N}$:

$$\begin{aligned}
&\delta_{A_1}f_{\overrightarrow{\mu}}(A_1,\ldots,A_n)\\
&=(2\pi i)^{-n}\int\limits_{|\xi_1|=r_{1,k}}\cdots\int\limits_{|\xi_n|=r_{n,k}}\alpha\mathcal{T}_{\overrightarrow{\mu}}^{(k+1)}\Big(\big(\xi_1-\tilde{A}_1\big)^{-2}\big(\xi_2-\tilde{A}_2\big)^{-1}\cdots\\
&\qquad\qquad\big(\xi_n-\tilde{A}_n\big)^{-1}\Big)d\xi_1\cdots d\xi_n\\
&=\sum_{m_1=1;m_2,\ldots,m_n=0}^{\infty}\left\{(2\pi i)^{-n}\int\limits_{|\xi_1|=r_{1,k}}\cdots\int\limits_{|\xi_n|=r_{n,k}}\alpha\xi_1^{-m_1-1}\cdots\xi_n^{-m_n-1}d\xi_1\cdots d\xi_n\right\}\\
&\quad\cdot m_1\sum_{\pi\in S_{m-1}}\int\limits_{\Delta_{m-1}(\pi)}C_{\pi(m-1)}(s_{\pi(m-1)})\cdots C_{\pi(1)}(s_{\pi(1)})\Big(\mu_1^{m_1-1}\times\cdots\times\mu_n^{m_n}\Big)\\
&\qquad\qquad(ds_1,\ldots,ds_{m-1})\\
&=0
\end{aligned}$$

since, for $m_1\geq 1$ and $m_2,\ldots,m_n\geq 0$ we have

$$(2\pi i)^{-n}\int\limits_{|\xi_1|=r_{1,k}}\cdots\int\limits_{|\xi_n|=r_{n,k}}\alpha\xi_1^{-m_1-1}\cdots\xi_n^{-m_n-1}d\xi_1\cdots d\xi_n=0.$$

Example 4.10. We consider here the case where $n=2$ and $f(z_1,z_2)=z_1^2z_2$. We let μ_1 and μ_2 be any continuous probability measures on $[0,T]$. We will calculate $\delta_{A_2}f_{\mu_1,\mu_2}(A_1,A_2)$. Because the disentangling $f_{\mu_1,\mu_2}(A_1,A_2)$ is linear in A_2, we would expect that the "derivative" $\delta_{A_2}f_{\mu_1,\mu_2}(A_1,A_2)$ will not involve A_2 at all. We have, for any $k\in\mathbb{N}$,

$$\begin{aligned}
&\delta_{A_2}f_{\mu_1,\mu_2}(A_1,A_2)\\
&=\sum_{m_1=0,m_2=1}^{\infty}\left\{(2\pi i)^{-2}\int\limits_{|\xi_1|=r_{1,k}}\int\limits_{|\xi_2|=r_{2,k}}\xi_1^{-m_1+1}\xi_2^{m_2}d\xi_1d\xi_2\right\}
\end{aligned}$$

$$\cdot\, m_2 \sum_{\pi\in S_{m_1+m_2-1}} \int_{\Delta_{m_1+m_2-1}(\pi)} C_{\pi(m_1+m_2-1)}(s_{\pi(m_1+m_2-1)})$$

$$\cdots C_{\pi(1)}(s_{\pi(1)}) \left(\mu_1^{m_1} \times \mu_2^{m_2-1}\right)(ds_1, \ldots, ds_{m_1+m_2-1})$$

$$= 2 \sum_{\pi\in S_2} \int_{\Delta_2(\pi)} C_{\pi(2)} C_{\pi(1)} \left(\mu_1^2\right)(ds_1, ds_2)$$

$$= 2\left(A_1^2 \mu_1^2 (\{s_1 < s_2\}) + A_1^2 \mu_1^2 (\{s_2 < s_1\})\right)$$

$$= 2A_1^2$$

since the only nonzero terms in the sum are for $m_1 = 1$ and $m_2 = 2$ and since μ_1 is a probability measure. Further, when $m_1 = 1$ and $m_2 = 2$, both operators $C_{\pi(2)}$ and $C_{\pi(1)}$ are equal to A_1, for both permutations in S_2.

Example 4.11. For this example we stay with the same function as that used in the preceding example but we calculate $\delta_{A_1} f_{\mu_1,\mu_2}(A_1, A_2)$. We would assume, since the function is quadratic in z_1, that $\delta_{A_1} f_{\mu_1,\mu_2}(A_1, A_2)$ will be linear in both A_1 and A_2. For any $k \in \mathbb{N}$ we have

$$\delta_{A_1} f_{\mu_1,\mu_2}(A_1, A_2)$$

$$= \sum_{m_1=1, m_2=0}^{\infty} \left\{ (2\pi i)^{-2} \int_{|\xi_1|=r_{1,k}} \int_{|\xi_2|=r_{2,k}} \xi_1^{-m_1+1} \xi_2^{-m_2} d\xi_1 d\xi_2 \right\}$$

$$\cdot\, m_1 \sum_{\pi\in S_{m_1-1+m_2}} \int_{\Delta_{m_1-1+m_2}(\pi)} C_{\pi(m_1-1+m_2)}(s_{\pi(m_1-1+m_2)})$$

$$\cdots C_{\pi(1)}(s_{\pi(1)}) \cdot \left(\mu_1^{m_1-1} \times \mu_2^{m_2}\right)(ds_1, \ldots, ds_{m_1-1+m_2})$$

$$= 2 \sum_{\pi\in S_2} \int_{\Delta_2(\pi)} C_{\pi(2)} C_{\pi(1)} (\mu_1 \times \mu_2)(ds_1, ds_2)$$

$$= 2\left(A_1 A_2 (\mu_1 \times \mu_2)(\{s_1 < s_2\}) + A_2 A_1 (\mu_1 \times \mu_2)(\{s_2 < s_1\})\right)$$

since the only nonzero terms of the sum are for $m_1 = 2$ and $m_2 = 1$. In this instance, one of the operators $C_{\pi(j)}$ is A_1 and the other is A_2 in both terms of the sum over S_2.

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