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Wandering subspaces and the Beurling type theorem. II

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Abstract. Let H^2 be the Hardy space over the bidisk. Let $\varphi(w)$ be a nonconstant inner function. We denote by $[z - \varphi(w)]$ the smallest invariant subspace for both operators T_z and T_w containing the function $z - \varphi(w)$. Aleman, Richter and Sundberg showed that the Beurling type theorem holds for the Bergman shift on the Bergman space. It is known that the compression operator S_z on $H^2 \ominus [z - w]$ is unitarily equivalent to the Bergman shift, so the Beurling type theorem holds for S_z on $H^2 \ominus [z - w]$. As a generalization, we shall show that the Beurling type theorem holds for S_z on $H^2 \ominus [z - \varphi(w)]$. Also we shall prove that the Beurling type theorem holds for the fringe operator F_w on $[z - w] \ominus z[z - w]$ and for F_z on $[z - \varphi(w)] \ominus w[z - \varphi(w)]$ if $\varphi(0) = 0$.

Contents

1. Introduction

Let T be a bounded linear operator on a Hilbert space H. For a subset E of H, we denote by $[E]$ the smallest invariant subspace for T containing E. Let $M \subset H$ be an invariant subspace for T. We denote by $M \ominus TM$ the orthogonal complement of TM in M. The space $M \ominus TM$ is called a *wandering subspace* of M for the operator T. We have $[M \ominus TM] \subset M$. We say that the Beurling type theorem holds for T if $[M \ominus TM] = M$ for all invariant subspaces M of H for T. Our basic problem is to find operators for which the Beurling type theorem holds.

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Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. We denote by $H^2(z)$ the Hardy space on $\mathbb{D}$ with variable z. Let T_z be the multiplication operator on $H^2(z)$ by the coordinate function z. The Beurling theorem [3] says that $M = [M \ominus T_z M]$ holds for all invariant subspaces M of $H^2(z)$ for T_z. Let $L^2_a(z)$, the Bergman space, be the Hilbert space consisting of square integrable analytic functions on $\mathbb{D}$ with respect to the normalized Lebesgue measure on $\mathbb{D}$. Let B be the Bergman shift on $L^2_a(z)$, that is, $Bf(z) = zf(z)$ for $f \in L^2_a(z)$. It is known that the dimension of wandering subspaces of invariant subspaces in $L^2_a(z)$ for B ranges from 1 to ∞ (see [2, 7, 9]). In [1], Aleman, Richter and Sundberg proved that the Beurling type theorem holds for the Bergman shift B. In [16], Shimorin showed that if $T : H \to H$ satisfies the following conditions:

(a) $\|Tx + y\|^2 \le 2(\|x\|^2 + \|Ty\|^2), \quad x, y \in H;$
(b) $\bigcap\{T^n H : n \ge 0\} = \{0\};$

then the Beurling type theorem holds for T. As an application of this theorem, Shimorin gave a simpler proof of the Aleman, Richter and Sundberg theorem. Later, different proofs of the the Beurling type theorem are given in [13, 14, 17]. Recently, the authors [10] proved the following.

Theorem A. *Suppose $T : H \to H$ satisfies the following conditions:*

(i) $\|Tx\|^2 + \|T^{*2}Tx\|^2 \le 2\|T^*Tx\|^2, \quad x \in H;$
(ii) *T is bounded below, i.e., there is $c > 0$ satisfying that $\|Tx\| \ge c\|x\|$ for every $x \in H$;*
(iii) *$\|T^{*n}x\| \to 0$ as $n \to \infty$ for every $x \in H$.*

Then the Beurling type theorem holds for T.

Also it was pointed out that conditions (i), (ii) and (iii) in Theorem A are equivalent to conditions (a) and (b) in Shimorin's theorem.

Let $H^2 = H^2(\mathbb{D}^2)$ be the Hardy space over the bidisk $\mathbb{D}^2$. We identify a function in H^2 with its boundary function on the distinguished boundary Γ^2 of $\mathbb{D}^2$, so we think of H^2 as a closed subspace of the Lebesgue space $L^2(\Gamma^2)$. We use z, w as variables in $\mathbb{D}^2$. We note that the Hardy space H^2 coincides with the closed tensor product $H^2(z) \otimes H^2(w)$. Let T_z and T_w be multiplication operators on H^2 by z and w. A closed subspace M of H^2 is called invariant if $T_z M \subset M$ and $T_w M \subset M$. For a subset E of H^2, we denote by $[E]$ the smallest invariant subspace of H^2 containing E. For a subspace E of H^2, we denote by P_E the orthogonal projection from $L^2(\Gamma^2)$ onto E. See books [4, 15] for the study of the Hardy space H^2.

Let M be an invariant subspace of H^2. Since T_z is an isometry on M, by the Wold decomposition theorem we have

$$M = \sum_{n=0}^{\infty} \oplus (M \ominus zM) z^n.$$

So many properties of the invariant subspace M are considered to be encoded in $M \ominus zM$. To study $M \ominus zM$, Yang defined the fringe operator F_w on

$M \ominus zM$ by

$$F_w f = P_{M\ominus zM} T_w f, \quad f \in M \ominus zM,$$

and studied the properties of F_w (see [21, 23, 24]). Similarly, we may define the fringe operator F_z on $M \ominus wM$.

Let $N = H^2 \ominus M$. Then $T_z^* N \subset N$ and $T_w^* N \subset N$. Let S_z and S_w be the compression operators on N defined by

$$S_z f = P_N T_z f \quad \text{and} \quad S_w f = P_N T_w f, \quad f \in N.$$

We note that $S_z^* = T_z^*|_N$ and $S_w^* = T_w^*|_N$.

One of the most interesting invariant subspaces of H^2 is $[z-w]$. It is known that $S_z = S_w$ on $H^2 \ominus [z-w]$ and S_z is unitarily equivalent to the Bergman shift on $L_a^2(\mathbb{D})$ (see [6, 12, 17, 18, 19, 20, 22]). So by the Aleman, Richter and Sundberg theorem, the Beurling type theorem holds for the operators S_z and S_w on $H^2 \ominus [z-w]$.

As generalized spaces of $[z-w]$, we have invariant subspaces $M_\varphi := [z - \varphi(w)]$ for nonconstant inner functions $\varphi(w)$. We put $N_\varphi = H^2 \ominus M_\varphi$. The space N_φ has been studied by Yang and the first author in [11, 12]. In Section 2, as an application of Theorem A we shall prove that the Beurling type theorem holds for some other unilateral operators. And we give a sufficient condition on unilateral weighted shifts $W_{\mathbf{c}}$ for which $\dim(M \ominus W_{\mathbf{c}} M) = 1$ for every invariant subspace for $W_{\mathbf{c}}$. In Section 3, as an application of Section 2 we shall prove that the Beurling type theorem holds for the operator S_z on N_φ. In Section 4, we prove that the Beurling type theorem holds for the fringe operator F_w on $[z-w] \ominus z[z-w]$. And also the Beurling type theorem holds for the fringe operator F_z on $M_\varphi \ominus wM_\varphi$ for every inner function $\varphi(w)$ with $\varphi(0) = 0$. In this case, we have $\dim(M \ominus F_z M) = 1$ for every invariant subspace M of $M_\varphi \ominus wM_\varphi$ for F_z.

2. Wandering subspaces

Let B be the Bergman shift on $L_a^2(z)$. We put

$$e_n(z) = \sqrt{n+1}\, z^n, \quad n \geq 0.$$

Then $\{e_n(z)\}_{n\geq 0}$ is an orthonormal basis of $L_a^2(z)$. We have $B^* e_0(z) = 0$,

$$Be_n(z) = \frac{\sqrt{n+1}}{\sqrt{n+2}} e_{n+1}(z) \quad \text{and} \quad B^* e_n(z) = \frac{\sqrt{n}}{\sqrt{n+1}} e_{n-1}(z), \quad n \geq 1.$$

Hence

$$B^* B e_n(z) = \frac{n+1}{n+2} e_n(z),$$

and

$$B^{*2} B e_n(z) = \frac{\sqrt{n}\sqrt{n+1}}{n+2} e_{n-1}(z), \quad n \geq 1.$$

By these equalities, we have

$$\|Bf\|^2 + \|B^{*2} B f\|^2 = 2\|B^* B f\|^2$$

for every $f(z) \in L_a^2(z)$ (see [10]). Books [5, 8] are nice references for the study of the Bergman space.

Let H be a separable Hilbert space with an orthonormal basis $\{\tau_n\}_{n\geq 0}$. Let $\mathbf{c} = \{c_n\}_{n\geq 0}$ be a sequence of positive numbers with $\sup_n c_n < \infty$. Let $W_{\mathbf{c}}$ be a unilateral weighted shift on H defined by $W_{\mathbf{c}}\tau_n = c_n\tau_{n+1}$ for $n \geq 0$. We have $W_{\mathbf{c}}^*\tau_0 = 0$ and $W_{\mathbf{c}}^*\tau_n = c_{n-1}\tau_{n-1}$ for $n \geq 1$. We note that $\{W_{\mathbf{c}}\tau_n : n \geq 0\}$ and $\{W_{\mathbf{c}}^*\tau_n : n \geq 1\}$ are orthogonal systems. For $x \in H$ and $x = \sum_{n=0}^\infty a_n\tau_n$, we have

$$\|W_{\mathbf{c}}x\|^2 = \left\|\sum_{n=0}^\infty a_n c_n \tau_{n+1}\right\|^2 = \sum_{n=0}^\infty |a_n|^2 c_n^2.$$

Then $W_{\mathbf{c}}$ is a bounded linear operator on H and $W_{\mathbf{c}}$ is bounded below if and only if $\inf_n c_n > 0$.

Theorem 2.1. *For another Hilbert space E, let $E \otimes H$ be the tensor product of E and H. We define a bounded linear operator $T = I \otimes W_{\mathbf{c}}$ on $E \otimes H$ by $T(x \otimes \tau_n) = x \otimes W_{\mathbf{c}}\tau_n$ for $x \in E$ and $n \geq 0$. If $1/\sqrt{2} \leq c_0 \leq 1$ and $1 \leq c_n^2(2 - c_{n-1}^2)$ for every $n \geq 1$, then the Beurling type theorem holds for T.*

Proof. First, we prove that $c_n \leq 1$ for every $n \geq 0$. To prove this, suppose that $c_m > 1$ for some $m \geq 1$. Since $1 \leq c_{m+1}^2(2 - c_m^2)$, we have $c_m^2 < 2$. Since $0 < c_m^4 - 2c_m^2 + 1$, we have $c_m^2 < 1/(2 - c_m^2) \leq c_{m+1}^2$. Thus we get $c_m < c_{m+1} < c_{m+2} < \cdots$. Since $\sup_n c_n < \infty$, $c_n \to \alpha$ as $n \to \infty$ for some $0 < \alpha < \infty$. Then $1/(2 - \alpha^2) = \alpha^2$, so $\alpha = 1$. This contradicts with $1 < c_m < \alpha$.

Since $1 \leq c_n^2(2 - c_{n-1}^2)$, we have $1/\sqrt{2} \leq c_n$ for every $n \geq 0$. Let $f \in E \otimes H$. We may write $f = \sum_{n=0}^\infty x_n \otimes \tau_n$ for some $x_n \in E$ with $\|f\|^2 = \sum_{n=0}^\infty \|x_n\|^2 < \infty$. Since $W_{\mathbf{c}}\tau_n \perp W_{\mathbf{c}}\tau_k$ for $n \neq k$, we have $\|Tf\|^2 = \sum_{n=0}^\infty \|x_n\|^2\|W_{\mathbf{c}}\tau_n\|^2$, so $\|f\|^2/2 \leq \|Tf\|^2 \leq \|f\|^2$. Then T is bounded below. We have

$$\begin{aligned}
\|T^{*k}f\|^2 &= \left\|\sum_{n=k}^\infty x_n \otimes W_{\mathbf{c}}^{*k}\tau_n\right\|^2 \\
&= \left\|\sum_{n=k}^\infty x_n \otimes (c_{n-1}c_{n-2}\cdots c_{n-k})\tau_{n-k}\right\|^2 \\
&\leq \sum_{n=k}^\infty \|x_n\|^2 \\
&\to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

We have also

$$Tf = \sum_{n=0}^\infty c_n(x_n \otimes \tau_{n+1}), \quad T^*Tf = \sum_{n=0}^\infty c_n^2(x_n \otimes \tau_n),$$

and

$$T^{*2}Tf = \sum_{n=1}^{\infty} c_n^2 c_{n-1}(x_n \otimes \tau_{n-1}).$$

Hence

$$\|Tf\|^2 + \|T^{*2}Tf\|^2 = c_0^2\|x_0\|^2 + \sum_{n=1}^{\infty} c_n^2(1 + c_n^2 c_{n-1}^2)\|x_n\|^2$$

and

$$2\|T^*Tf\|^2 = \sum_{n=0}^{\infty} 2c_n^4\|x_n\|^2.$$

Therefore

$$\begin{aligned} &2\|T^*Tf\|^2 - \big(\|Tf\|^2 + \|T^{*2}Tf\|^2\big) \\ &= c_0^2(2c_0^2 - 1)\|x_0\|^2 + \sum_{n=1}^{\infty} c_n^2\big(c_n^2(2 - c_{n-1}^2) - 1\big)\|x_n\|^2 \\ &\geq 0 \qquad \text{by the assumption.} \end{aligned}$$

Applying Theorem A, we get the assertion. □

Remark 2.2. Let $E = \mathbb{C}$. We shall consider the extremal case of conditions $1/\sqrt{2} \leq c_0 \leq 1$ and $1 \leq c_n^2(2 - c_{n-1}^2)$. Take $c_0 = 1$ and inductively we take c_n such that $1 = c_n^2(2 - c_{n-1}^2)$. Then we have $c_n = 1$ for every $n \geq 0$. In this case, we may think that $H = H^2(z)$, $W_{\mathbf{c}} = T_z$, and $\prod_{i=0}^{\infty} c_i = 1 > 0$.

Take $c_0 = 1/\sqrt{2}$ and inductively we take c_n such that $1 = c_n^2(2 - c_{n-1}^2)$. We have $c_n = \sqrt{n+1}/\sqrt{n+2}$ for every $n \geq 0$. In this case, we may think that $H = L_a^2(z)$, $W_{\mathbf{c}} = B$, and $\prod_{i=0}^{n} c_i = 1/\sqrt{n+2} \to 0$ as $n \to \infty$.

Corollary 2.3. *Let E be a Hilbert space. Then the Beurling type theorem holds for $I \otimes B$ on $E \otimes L_a^2(z)$.*

We shall give a sufficient condition on $\mathbf{c} = \{c_n\}_{n\geq 0}$ for which

$$\dim(M \ominus W_{\mathbf{c}}M) = 1$$

for every invariant subspace M of H for $W_{\mathbf{c}}$. Let $\{\alpha_n\}_{n\geq 0}$ be a sequence of positive numbers and $\alpha_0 = 1$. We define a linear map

$$V : \operatorname{span}\{z^n : n \geq 0\} \to H$$

by $Vz^n = \alpha_n \tau_n$ for every $n \geq 0$.

Lemma 2.4. *We have that $VT_z = W_{\mathbf{c}}V$ on $\operatorname{span}\{z^n : n \geq 0\}$ if and only if $\alpha_{n+1} = \prod_{i=0}^{n} c_i$ for every $n \geq 0$. In this case, if $0 < \prod_{i=0}^{\infty} c_i < \infty$, then V has a bounded linear extension $\widetilde{V} : H^2(z) \to H$ satisfying that $\widetilde{V}$ is invertible and $\widetilde{V}T_z = W_{\mathbf{c}}\widetilde{V}$.*

Proof. We have $VT_z z^n = W_{\mathbf{c}} V z^n$ if and only if $\alpha_{n+1}\tau_{n+1} = \alpha_n c_n \tau_{n+1}$. Hence $VT_z = W_{\mathbf{c}} V$ on $\operatorname{span}\{z^n : n \geq 0\}$ if and only if $\alpha_{n+1} = \prod_{i=0}^{n} c_i$ for every $n \geq 0$. In this case, moreover suppose that $0 < \prod_{i=0}^{\infty} c_i < \infty$. Then V is bounded and bounded below on $\operatorname{span}\{z^n : n \geq 0\}$. Hence V has a bounded linear extension $\widetilde{V} : H^2(z) \to H$. It is easy to see that $\widetilde{V}$ is invertible and $\widetilde{V} T_z = W_{\mathbf{c}} \widetilde{V}$. □

We denote by $\operatorname{Lat}(W_{\mathbf{c}})$ and $\operatorname{Lat}(T_z)$ the lattice of invariant subspaces for $W_{\mathbf{c}}$ on H and T_z on $H^2(z)$, respectively. We write $\operatorname{Lat}(W_{\mathbf{c}}) \cong \operatorname{Lat}(T_z)$ if $\operatorname{Lat}(W_{\mathbf{c}})$ and $\operatorname{Lat}(T_z)$ have the same lattice structure.

Theorem 2.5. *If* $0 < \prod_{i=0}^{\infty} c_i < \infty$, *then* $\dim(M \ominus W_{\mathbf{c}} M) = 1$ *for every invariant subspace* M *for* $W_{\mathbf{c}}$. *Moreover we have* $\operatorname{Lat}(W_{\mathbf{c}}) \cong \operatorname{Lat}(T_z)$.

Proof. Let M be a nonzero invariant subspace M for $W_{\mathbf{c}}$. Let $\alpha_0 = 1$ and $\alpha_n = \prod_{i=0}^{n-1} c_i$ for $n \geq 1$. By Lemma 2.4, there is a bounded linear operator $\widetilde{V} : H^2(z) \to H$ satisfying $\widetilde{V} z^n = \alpha_n \tau_n$ for every $n \geq 0$, $\widetilde{V}$ is invertible and $\widetilde{V} T_z = W_{\mathbf{c}} \widetilde{V}$. Then we have

$$T_z \widetilde{V}^{-1} M = \widetilde{V}^{-1} W_{\mathbf{c}} M \subset \widetilde{V}^{-1} M.$$

Hence $\widetilde{V}^{-1} M$ is an invariant subspace for T_z. By the Beurling theorem, $\widetilde{V}^{-1} M = \theta(z) H^2(z)$ for an inner function $\theta(z)$, so $M = \widetilde{V}\theta(z) H^2(z)$. Since $\widetilde{V} T_z = W_{\mathbf{c}} \widetilde{V}$, M is an invariant subspace for $W_{\mathbf{c}}$ generated by $\widetilde{V}\theta(z)$. Therefore we get $\dim(M \ominus W_{\mathbf{c}} M) = 1$.

For an inner function $\theta_1(z)$, $\widetilde{V}\theta_1(z) H^2(z)$ is an invariant subspace for $W_{\mathbf{c}}$. Thus $\operatorname{Lat}(W_{\mathbf{c}}) \cong \operatorname{Lat}(T_z)$. □

3. The Beurling type theorem for S_z

Let $\varphi(w)$ be a nonconstant inner function,

$$M_\varphi = [z - \varphi(w)] \quad \text{and} \quad N_\varphi = H^2 \ominus M_\varphi.$$

Let T_φ be the multiplication operator on $H^2(w)$ by $\varphi(w)$. Its adjoint operator T_φ^* is represented by $T_\varphi^* f = P_{H^2(w)} \overline{\varphi} f$, $f \in H^2(w)$. In [11], Yang and the first author showed that

$$N_\varphi = \left\{ \sum_{n=0}^{\infty} \oplus (T_\varphi^{*n} f(w)) z^n : f \in H^2(w), \sum_{n=0}^{\infty} \|T_\varphi^{*n} f\|^2 < \infty \right\}.$$

Let

$$\sigma_n(z, w) = \frac{\sum_{i=0}^{n} z^i w^{n-i}}{\sqrt{n+1}}, \quad n \geq 0.$$

We note that $\sigma_0(z, w) = 1$. It is known that $\{\sigma_n\}_{n \geq 0}$ is an orthonormal basis of $N_w = H^2 \ominus [z - w]$, the special case $\varphi(w) = w$. If we define the operator $V : N_w \to L_a^2(z)$ by $V\sigma_n = \sigma_n(z, z)$, then V is a unitary operator and $S_z = S_w = V^* B V$.

Since T_φ is an isomerty on $H^2(w)$, by the Wold decomposition theorem we have

$$H^2(w) = \sum_{n=0}^{\infty} \oplus \varphi(w)^n \big(H^2(w) \ominus \varphi(w)H^2(w)\big).$$

Let $\{\lambda_k(w)\}_{k=0}^m$ be an orthonormal basis of $H^2(w) \ominus \varphi(w)H^2(w)$, where $0 \le m \le \infty$. Also let

$$E_{k,n}(z,w) = \lambda_k(w)\sigma_n(z,\varphi(w)) \in H^2, \quad 0 \le k \le m, n \ge 0.$$

In [12], Yang and the first author proved the following.

Lemma 3.1. *The set $\{E_{k,n} : 0 \le k \le m, n \ge 0\}$ is an orthonormal basis of N_φ and*

$$S_z E_{k,n} = \frac{\sqrt{n+1}}{\sqrt{n+2}} E_{k,n+1}.$$

We define the operator

$$U : N_\varphi \to \big(H^2(w) \ominus \varphi(w)H^2(w)\big) \otimes L_a^2(z)$$

by

$$UE_{k,n} = \lambda_k(w) \otimes e_n(z).$$

Then U is clearly a unitary operator, and by Lemma 3.1 one easily checks that

$$S_z = U^*(I \otimes B)U \quad \text{and} \quad S_z^* = U^*(I \otimes B^*)U.$$

By Corollary 2.3, we have the following theorem.

Theorem 3.2. *The Beurling type theorem holds for the operator S_z on N_φ for every nonconstant inner function $\varphi(w)$.*

Let $S_\varphi = P_{N_\varphi} T_\varphi|_{N_\varphi}$. Then $S_\varphi^* = P_{N_\varphi} T_\varphi^*|_{N_\varphi}$. Since $T_z^* = T_\varphi^*$ on N_φ, we have $S_z^* = S_\varphi^*$, so $S_z = S_\varphi$. By Theorem 3.2, we have the following.

Corollary 3.3. *The Beurling type theorem holds for S_φ on N_φ for every nonconstant inner function $\varphi(w)$.*

If $\varphi(w) \ne aw$, $|a| = 1$, then $S_z \ne S_w$. There are some differences between the operators S_z and S_w on N_φ.

Proposition 3.4. *Let $\varphi(w) = w^2\varphi_0(w)$ for an inner function $\varphi_0(w)$. Then*

$$\|S_w f\|^2 + \|S_w^{*2} S_w f\|^2 > 2\|S_w^* S_w f\|^2$$

for some $f \in N_\varphi$.

Proof. The set $\{1, \varphi_0(w), w\varphi_0(w)\}$ is contained in $H^2(w) \ominus \varphi(w)H^2(w)$. Let $f(w) = w\varphi_0(w) \in N_\varphi$. Then $wf(w) = \varphi(w)$. Let

$$r(w) \in H^2(w) \ominus \varphi(w)H^2(w) \quad \text{with } r(w) \perp 1.$$

Then $\varphi(w) \perp r(w)\varphi(w)^n$, and by Lemma 3.1 $r(w)\sigma_n(z,\varphi(w)) \in N_\varphi$ for every $n \geq 0$. We have

$$\sigma_n(z,\varphi(w)) = \frac{\sum_{i=0}^n z^i \varphi(w)^{n-i}}{\sqrt{n+1}} \in N_\varphi.$$

For every $n \geq 0$, we have

$$\begin{aligned}\big\langle wf(w), r(w)\sigma_n(z,\varphi(w))\big\rangle &= \frac{1}{\sqrt{n+1}}\Big\langle \varphi(w), r(w)\sum_{i=0}^n z^i\varphi(w)^{n-i}\Big\rangle \\ &= \frac{1}{\sqrt{n+1}}\langle \varphi(w), r(w)\varphi(w)^n\rangle \\ &= 0.\end{aligned}$$

By Lemma 3.1, $\sigma_n(z,\varphi(w))$ and $\varphi_0(w)\sigma_0(z,\varphi(w)) = \varphi_0(w)$ are contained in N_φ. For $j \neq 1$, since $\varphi(0) = 0$ we have also

$$\langle wf(w), \sigma_j(z,\varphi(w))\rangle = \frac{1}{\sqrt{j+1}}\langle \varphi(w), \varphi(w)^j\rangle = 0.$$

Hence

$$\begin{aligned}S_w f(w) &= \langle wf(w), \sigma_1(z,\varphi(w))\rangle \sigma_1(z,\varphi(w)) \\ &= \Big\langle \varphi(w), \frac{\varphi(w)+z}{\sqrt{2}}\Big\rangle \sigma_1(z,\varphi(w)) \\ &= \frac{1}{\sqrt{2}}\sigma_1(z,\varphi(w)).\end{aligned}$$

We have

$$T_w^* S_w f(w) = \frac{1}{\sqrt{2}} T_w^*\Big(\frac{\varphi(w)+z}{\sqrt{2}}\Big) = \frac{1}{2} w\varphi_0(w) \in N_\varphi.$$

Hence $S_w^* S_w f(w) = \frac{1}{2} w\varphi_0(w)$, so $S_w^{*2} S_w f(w) = \frac{1}{2}\varphi_0(w)$. Therefore

$$\|S_w f(w)\|^2 + \|S_w^{*2} S_w f(w)\|^2 = \frac{1}{2} + \frac{1}{4} > \frac{1}{2} = 2\|S_w^* S_w f(w)\|^2. \qquad \square$$

By Proposition 3.4, we may not apply Theorem A for S_w on N_φ. So we do not know whether or not the Beurling type theorem holds for the operator S_w on N_φ.

4. The fringe operators

Let M be a nonzero invariant subspace of the Hardy space H^2 and $N = H^2 \ominus M$. One easily checks the following.

Lemma 4.1. *For $f \in M$, $f \in M \ominus zM$ if and only if $T_z^* f \in N$.*

We define the fringe operators F_w on $M \ominus zM$ by

$$F_w = P_{M\ominus zM} T_w|_{M\ominus zM}$$

and F_z on $M \ominus wM$ by $F_z = P_{M\ominus wM} T_z|_{M\ominus wM}$. Let $\varphi(w)$ be a nonconstant inner function. We use the same notations as the ones given in Section 3. Let $\{\lambda_k(w)\}_{k=0}^m$ be an orthonormal basis of $H^2(w) \ominus \varphi(w)H^2(w)$. Let

$$E_n = \frac{z\sigma_n(z,\varphi(w)) - \sqrt{n+1}\varphi(w)^{n+1}}{\sqrt{n+2}}, \quad n \geq 0.$$

Then we may verify the following lemma (see [12]).

Lemma 4.2. *The set $\{\lambda_k(w)E_n : 0 \leq k \leq m, n \geq 0\}$ is an orthonormal basis of $M_\varphi \ominus zM_\varphi$.*

Theorem 4.3. *The Beurling type theorem holds for the fringe operator F_w on $[z-w] \ominus z[z-w]$. Moreover, $\dim(M \ominus F_w M) = 1$ for every invariant subspace M for F_w.*

Proof. Let

$$X_n = \frac{1}{\sqrt{n+2}}\left(\frac{\sum_{i=0}^n z^{i+1}w^{n-i}}{\sqrt{n+1}} - \sqrt{n+1}w^{n+1}\right)$$

for every $n \geq 0$. By Lemma 4.2, $\{X_n\}_{n\geq 0}$ is an orthonormal basis of $[z-w] \ominus z[z-w]$ (see also [6, 17, 18]). It is not difficult to see that $wX_n \perp X_j$ for $j \neq n+1$. Hence

$$\begin{aligned}
F_w X_n &= \langle wX_n, X_{n+1}\rangle X_{n+1} \\
&= \Bigg\langle \frac{1}{\sqrt{n+2}}\left(\frac{\sum_{i=0}^n z^{i+1}w^{n+1-i}}{\sqrt{n+1}} - \sqrt{n+1}w^{n+2}\right), \\
&\qquad \frac{1}{\sqrt{n+3}}\left(\frac{\sum_{i=0}^{n+1} z^{i+1}w^{n+1-i}}{\sqrt{n+2}} - \sqrt{n+2}w^{n+2}\right)\Bigg\rangle X_{n+1} \\
&= \frac{1}{\sqrt{n+2}\sqrt{n+3}}\left(\frac{n+1}{\sqrt{n+1}\sqrt{n+2}} + \sqrt{n+1}\sqrt{n+2}\right) X_{n+1} \\
&= \frac{\sqrt{n+1}\sqrt{n+3}}{n+2} X_{n+1}.
\end{aligned}$$

Let

$$c_n = \frac{\sqrt{n+1}\sqrt{n+3}}{n+2}.$$

Then $c_0 = \sqrt{3}/2$, so $1/\sqrt{2} < c_0$, and $c_n < 1$ for every $n \geq 0$. It is not difficult to check $c_n^2(2 - c_{n-1}^2) \geq 1$. By Theorem 2.1, we get the first assertion.

We have

$$\prod_{n=0}^k c_n = \frac{\sqrt{3}}{2}\frac{\sqrt{2}\sqrt{4}}{3}\frac{\sqrt{3}\sqrt{5}}{4}\cdots\frac{\sqrt{k+1}\sqrt{k+3}}{k+2} = \frac{1}{\sqrt{2}}\frac{\sqrt{k+3}}{\sqrt{k+2}}.$$

Hence $\prod_{n=0}^\infty c_n = 1/\sqrt{2}$. By Theorem 2.5, we get the second assertion. □

Since

$$T_z^* X_n = \frac{1}{\sqrt{n+2}}\sigma_n(z,w),$$

$T_z^*([z-w] \ominus z[z-w])$ is dense in $H^2 \ominus [z-w]$. As mentioned in the introduction, S_w on $H^2 \ominus [z-w]$ is unitary equivalent to the Bergman shift B on $L_a^2(\mathbb{D})$. We note that the dimension of wandering subspaces of invariant subspaces in $L_a^2(z)$ for B ranges from 1 to ∞.

Proposition 4.4. *Let $\varphi(w) = w^2\varphi_0(w)$ for an inner function $\varphi_0(w)$. Then*

$$\|F_w f\|^2 + \|F_w^{*2} F_w f\|^2 > 2\|F_w^* F_w f\|^2$$

for some $f \in M_\varphi \ominus zM_\varphi$.

Proof. We have

$$\{1, \varphi_0(w), w\varphi_0(w)\} \subset H^2(w) \ominus \varphi(w)H^2(w)$$

By Lemma 4.2, $E_n, \varphi_0(w)E_n, w\varphi_0(w)E_n$ are contained in $M_\varphi \ominus zM_\varphi$ for every $n \geq 0$. Let $f = w\varphi_0(w)E_0$. Then

$$wf = \varphi(w)E_0 = \frac{\varphi(w)z - \varphi(w)^2}{\sqrt{2}}.$$

Let

$$r(w) \in H^2(w) \ominus \varphi(w)H^2(w) \quad \text{with } r(w) \perp 1.$$

Then $\varphi(w) \perp r(w)\varphi(w)^n$, and by Lemma 4.2 we have $r(w)E_n \in M_\varphi \ominus zM_\varphi$ for $n \geq 0$. Hence for every $n \geq 0$, we have

$$\begin{aligned}
\langle wf, r(w)E_n\rangle &= \frac{1}{\sqrt{2}\sqrt{n+2}}\Big\langle \varphi(w)z - \varphi(w)^2, \\
&\qquad r(w)\frac{\sum_{i=0}^n z^{i+1}\varphi(w)^{n-i}}{\sqrt{n+1}} - \sqrt{n+1}\, r(w)\varphi(w)^{n+1}\Big\rangle \\
&= \frac{1}{\sqrt{2}\sqrt{n+2}}\bigg(\frac{\langle\varphi(w), r(w)\varphi(w)^n\rangle}{\sqrt{n+1}} \\
&\qquad + \sqrt{n+1}\langle\varphi(w)^2, r(w)\varphi(w)^{n+1}\rangle\bigg) \\
&= 0.
\end{aligned}$$

For $j \neq 1$, since $\varphi(0) = 0$ we have also

$$\begin{aligned}
\langle wf, E_j\rangle &= \frac{1}{\sqrt{2}\sqrt{j+2}}\bigg(\frac{\langle\varphi(w), \varphi(w)^j\rangle}{\sqrt{j+1}} \\
&\qquad\qquad + \sqrt{j+1}\langle\varphi(w)^2, \varphi(w)^{j+1}\rangle\bigg) \\
&= 0.
\end{aligned}$$

Hence

$$F_w f = \langle wf, E_1\rangle E_1 = \frac{1}{\sqrt{2}\sqrt{3}}\left(\frac{1}{\sqrt{2}} + \sqrt{2}\right)E_1 = \frac{\sqrt{3}}{2}E_1.$$

We have

$$\begin{aligned}T_w^* F_w f &= \frac{\sqrt{3}}{2} T_w^* \left(\frac{1}{\sqrt{3}} \left(\frac{\varphi(w)z + z^2}{\sqrt{2}} - \sqrt{2}\varphi(w)^2 \right) \right) \\ &= \frac{1}{2} \left(\frac{w\varphi_0(w)z}{\sqrt{2}} - \sqrt{2} w\varphi_0(w)\varphi(w) \right).\end{aligned}$$

Let

$$r_1(w) \in H^2(w) \ominus \varphi(w) H^2(w) \quad \text{with } r_1(w) \perp w\varphi_0(w).$$

Then $w\varphi_0(w) \perp r_1(w)\varphi(w)^n$ for $n \geq 0$. Hence for every $n \geq 0$, we have

$$\begin{aligned}\langle T_w^* F_w f, r_1(w) E_n \rangle &= \frac{1}{2\sqrt{n+2}} \Big\langle \frac{w\varphi_0(w)z}{\sqrt{2}} - \sqrt{2} w\varphi_0(w)\varphi(w), \\ &\qquad r_1(w) \frac{\sum_{i=0}^n z^{i+1}\varphi(w)^{n-i}}{\sqrt{n+1}} - \sqrt{n+1} r_1(w)\varphi(w)^{n+1} \Big\rangle \\ &= \frac{1}{2\sqrt{n+2}} \left(\frac{1}{\sqrt{2}\sqrt{n+1}} \langle w\varphi_0(w), r_1(w)\varphi(w)^n \rangle \right. \\ &\qquad \left. + \sqrt{2}\sqrt{n+1} \langle w\varphi_0(w), r_1(w)\varphi(w)^n \rangle \right) \\ &= 0.\end{aligned}$$

For $j > 0$, since $\varphi(0) = 0$ we have

$$\begin{aligned}\langle T_w^* F_w f, w\varphi_0(w) E_j \rangle &= \frac{1}{2\sqrt{j+2}} \left(\frac{1}{\sqrt{2}\sqrt{j+1}} \langle w\varphi_0(w), w\varphi_0(w)\varphi(w)^j \rangle \right. \\ &\qquad \left. + \sqrt{2}\sqrt{j+1} \langle w\varphi_0(w), w\varphi_0(w)\varphi(w)^j \rangle \right) \\ &= 0.\end{aligned}$$

Hence

$$\begin{aligned}F_w^* F_w f &= \langle T_w^* F_w f, w\varphi_0(w) E_0 \rangle w\varphi_0(w) E_0 \\ &= \frac{1}{2\sqrt{2}} \left(\frac{1}{\sqrt{2}} \langle w\varphi_0(w), w\varphi_0(w) \rangle \right. \\ &\qquad \left. + \sqrt{2} \langle w\varphi_0(w), w\varphi_0(w) \rangle \right) w\varphi_0(w) E_0 \\ &= \frac{1}{2\sqrt{2}} \left(\frac{1}{\sqrt{2}} + \sqrt{2} \right) w\varphi_0(w) E_0 \\ &= \frac{3}{4} w\varphi_0(w) E_0.\end{aligned}$$

Since

$$T_w^* F_w^* F_w f = \frac{3}{4} \varphi_0(w) E_0 \in M_\varphi \ominus z M_\varphi,$$

we have $F_w^{*2}F_wf = \frac{3}{4}\varphi_0(w)E_0$. Therefore

$$\|F_wf\|^2 + \|F_w^{*2}F_wf\|^2 = \frac{3}{4} + \Big(\frac{3}{4}\Big)^2 > 2\Big(\frac{3}{4}\Big)^2 = 2\|F_w^*F_wf\|^2. \qquad \square$$

By Proposition 4.4, we may not apply Theorem A for the operator F_w on $M_\varphi \ominus zM_\varphi$. So we do not know whether or not the Beurling type theorem holds for the operator F_w on $M_\varphi \ominus zM_\varphi$.

By the symmetry of variables in $[z-w]$ and Theorem 4.3, the Beurling type theorem holds for the operator F_z on $[z-w] \ominus w[z-w]$. We may generalize this fact as follows.

Theorem 4.5. *Let $\varphi(w)$ be an inner function with $\varphi(0) = 0$. Then the fringe operator F_z on $M_\varphi \ominus wM_\varphi$ is unitarily equivlent to the fringe operator F_w on $[z-w] \ominus z[z-w]$, and the Beurling type theorem holds for F_z and $\dim(M \ominus F_zM) = 1$ for every invariant subspace M of $M_\varphi \ominus wM_\varphi$ for F_z.*

To prove this, we need some lemmas. Let $\varphi(w)$ be an inner function with $\varphi(0) = 0$. One easily checks the following lemma.

Lemma 4.6. *We have $T_w^*\varphi(w) \in H^2(w) \ominus \varphi(w)H^2(w)$, and if $\lambda(w) \in H^2(w) \ominus \varphi(w)H^2(w)$ and $\lambda(w) \perp T_w^*\varphi(w)$, then*

$$T_w\lambda(w) \in H^2(w) \ominus \varphi(w)H^2(w).$$

By Lemma 3.1, N_φ coincides with the closed linear span of

$$\big\{\lambda(w)\sigma_n(z,\varphi(w)) : \lambda(w) \in H^2(w) \ominus \varphi(w)H^2(w), n \geq 0\big\}.$$

By Lemma 4.6, $(T_w\lambda(w))\sigma_n(z,\varphi(w)) \in N_\varphi$ for every

$$\lambda(w) \in \big(H^2(w) \ominus \varphi(w)H^2(w)\big) \ominus \mathbb{C}\cdot T_w^*\varphi(w)$$

and $n \geq 0$. Let

$$N_{\varphi,0} = \{f \in N_\varphi : T_wf \in N_\varphi\}.$$

Since $\varphi(0) = 0$, $T_w(T_w^*\varphi(w)) = \varphi(w)$ and $\varphi(w)\sigma_n(z,\varphi(w)) \notin N_\varphi$ for every $n \geq 0$. Hence the space $N_\varphi \ominus N_{\varphi,0}$ coincides with the closed linear span of $\big\{(T_w^*\varphi(w))\sigma_n(z,\varphi(w)) : n \geq 0\big\}$. By Lemma 3.1, we have that

$$(T_w^*\varphi(w))\sigma_n(z,\varphi(w)) \perp (T_w^*\varphi(w))\sigma_j(z,\varphi(w)) \quad \text{for } n \neq j,$$

and $\|(T_w^*\varphi(w))\sigma_n(z,\varphi(w))\| = 1$. So

$$\big\{(T_w^*\varphi(w))(w)\sigma_n(z,\varphi(w)) : n \geq 0\big\}$$

is an orthonormal basis of $N_\varphi \ominus N_{\varphi,0}$.

One easily sees that $T_w^*(M_\varphi \ominus wM_\varphi) \perp N_{\varphi,0}$. Therefore by Lemma 4.1, we have the following.

Lemma 4.7. *Let $g \in M_\varphi \ominus wM_\varphi$. Then we may write*

$$T_w^*g = \sum_{n=0}^\infty a_n(T_w^*\varphi(w))\sigma_n(z,\varphi(w)), \quad \sum_{n=0}^\infty |a_n|^2 < \infty.$$

Let

$$Y_n = \frac{1}{\sqrt{n+2}}\Big(\varphi(w)\sigma_n(z,\varphi(w)) - \sqrt{n+1}z^{n+1}\Big), \quad n \geq 0.$$

Lemma 4.8. *Let $\varphi(w)$ be an inner function with $\varphi(0)=0$. Then $\{Y_n\}_{n\geq 0}$ is an orthonormal basis of $M_\varphi \ominus wM_\varphi$.*

Proof. We have

$$\sqrt{n+1}\sqrt{n+2}Y_n = \varphi(w)\big(z^n + z^{n-1}\varphi(w) + \cdots + \varphi(w)^n\big) - (n+1)z^{n+1}.$$

Letting $n=0$, we have

$$\sqrt{2}Y_0 = \varphi(w) - z \in M_\varphi.$$

By induction, we shall show that $Y_n \in M_\varphi$ for every $n \geq 0$. Suppose that

$$\sqrt{k+1}\sqrt{k+2}Y_k = \varphi(w)\big(z^k + z^{k-1}\varphi(w) + \cdots + \varphi(w)^k\big) - (k+1)z^{k+1} \in M_\varphi.$$

We have

$$\sqrt{k+1}\sqrt{k+2}\varphi(w)Y_k = \varphi(w)^2\big(z^k + z^{k-1}\varphi(w) + \cdots + \varphi(w)^k\big) - (k+1)z^{k+1}\varphi(w) \in M_\varphi.$$

Then

$$\varphi(w)^{k+2} = \sqrt{k+1}\sqrt{k+2}\varphi(w)Y_k + (k+1)z^{k+1}\varphi(w) - \big(z^k\varphi(w)^2 + z^{k-1}\varphi(w)^3 + \cdots + z\varphi(w)^{k+1}\big).$$

Hence

$$\begin{aligned}
&\sqrt{k+2}\sqrt{k+3}Y_{k+1} \\
&\quad = \varphi(w)\big(z^{k+1} + z^k\varphi(w) + \cdots + \varphi(w)^{k+1}\big) - (k+2)z^{k+2} \\
&\quad = \sqrt{k+1}\sqrt{k+2}\varphi(w)Y_k + (k+1)z^{k+1}\varphi(w) \\
&\qquad - \big(z^k\varphi(w)^2 + z^{k-1}\varphi(w)^3 + \cdots + z\varphi(w)^{k+1}\big) \\
&\qquad + z^{k+1}\varphi(w) + z^k\varphi(w)^2 + \cdots + z\varphi(w)^{k+1} - (k+2)z^{k+2} \\
&\quad = \sqrt{k+1}\sqrt{k+2}\varphi(w)Y_k + (k+2)z^{k+1}(\varphi(w) - z) \in M_\varphi.
\end{aligned}$$

This completes the induction. Thus we get $Y_n \in M_\varphi$ for every $n \geq 0$.

We have also

$$\begin{aligned}
T_w^* Y_n &= \frac{1}{\sqrt{n+2}} T_w^*\big(\varphi(w)\sigma_n(z,\varphi(w))\big) \\
&= \frac{1}{\sqrt{n+2}}(T_w^*\varphi(w))\sigma_n(z,\varphi(w)) && \text{because } \varphi(0)=0 \\
&\in N_\varphi && \text{by Lemmas 3.1 and 4.6.}
\end{aligned}$$

Hence by Lemma 4.1, $Y_n \in M_\varphi \ominus wM_\varphi$ for $n \geq 0$. Since $\varphi(0) = 0$ and $\|\varphi(w)\sigma_n(z, \varphi(w))\| = 1$, it is not difficult to show that $\|Y_n\| = 1$ for $n \geq 0$.

Let $0 \leq n < j$. Then

$$\left\langle \varphi(w)\sigma_n(z, \varphi(w)) - \sqrt{n+1}z^{n+1}, z^{j+1} \right\rangle = 0$$

and $\left\langle z^n, \varphi(w)\sigma_j(z, \varphi(w)) \right\rangle = 0$. So

$$\begin{aligned}
\langle Y_n, Y_j \rangle &= \frac{1}{\sqrt{n+2}\sqrt{j+2}} \left\langle \varphi(w)\sigma_n(z, \varphi(w)), \varphi(w)\sigma_j(z, \varphi(w)) \right\rangle \\
&= \frac{1}{\sqrt{n+2}\sqrt{j+2}} \left\langle \sigma_n(z, \varphi(w)), \sigma_j(z, \varphi(w)) \right\rangle \\
&= \frac{1}{\sqrt{n+1}\sqrt{n+2}\sqrt{j+1}\sqrt{j+2}} \left\langle \sum_{i=0}^{n} z^i \varphi(w)^{n-i}, \sum_{\ell=0}^{j} z^\ell \varphi(w)^{j-\ell} \right\rangle \\
&= \frac{1}{\sqrt{n+1}\sqrt{n+2}\sqrt{j+1}\sqrt{j+2}} \sum_{i=0}^{n} \langle \varphi(w)^{n-i}, \varphi(w)^{j-i} \rangle \\
&= 0 \qquad \text{because } \varphi(0) = 0 \text{ and } n < j.
\end{aligned}$$

Hence $\{Y_n\}_{n \geq 0}$ is an orthonormal system in $M_\varphi \ominus wM_\varphi$.

Let $g \in M_\varphi \ominus wM_\varphi$. By Lemma 4.7, we may write

$$T_w^* g = \sum_{n=0}^{\infty} a_n (T_w^* \varphi(w)) \sigma_n(z, \varphi(w))$$

for some $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. We have

$$\begin{aligned}
g(z, w) &= w \left(\sum_{n=0}^{\infty} a_n (T_w^* \varphi(w)) \sigma_n(z, \varphi(w)) \right) + g(z, 0) \\
&= \left(\sum_{n=0}^{\infty} a_n \varphi(w) \sigma_n(z, \varphi(w)) \right) + g(z, 0).
\end{aligned}$$

Since $g \in [z - \varphi(w)]$, $g(\varphi(\zeta), \zeta) = 0$ for every $\zeta \in \mathbb{D}$. Then

$$\begin{aligned}
g(\varphi(\zeta), 0) &= -\sum_{n=0}^{\infty} a_n \varphi(\zeta) \sigma_n(\varphi(\zeta), \varphi(\zeta)) \\
&= -\sum_{n=0}^{\infty} \sqrt{n+1} a_n \varphi(\zeta)^{n+1}.
\end{aligned}$$

Hence

$$g(z, 0) = -\sum_{n=0}^{\infty} \sqrt{n+1} a_n z^{n+1}, \quad z \in \mathbb{D}.$$

Therefore for $(z,w) \in \mathbb{D}^2$ we get

$$\begin{aligned} g(z,w) &= \sum_{n=0}^{\infty} a_n\big(\varphi(w)\sigma_n(z,\varphi(w)) - \sqrt{n+1}z^{n+1}\big) \\ &= \sum_{n=0}^{\infty} \sqrt{n+2}a_n Y_n \end{aligned}$$

and

$$\sum_{n=0}^{\infty}(n+2)|a_n|^2 < \infty.$$

Thus we get the assertion. □

Remark 4.9. By the last paragraph of the proof of Lemma 4.8, we have

$$T_w^*(M_\varphi \ominus wM_\varphi) = \left\{\sum_{n=0}^{\infty} a_n(T_w^*\varphi(w))\sigma_n(z,\varphi(w)) : \sum_{n=0}^{\infty}(n+2)|a_n|^2 < \infty\right\}.$$

Remark 4.10. If $\varphi(0) \neq 0$, we can prove that

$$Z_n := (\varphi(w) - \varphi(0))\sigma_n(z,\varphi(w)) - \sqrt{n+1}(z - \varphi(0))z^n \in M_\varphi \ominus wM_\varphi$$

for every $n \geq 0$. But in this case, $Z_n \not\perp Z_j$ for $n \neq j$.

Proof of Theorem 4.5. We note that

$$Y_n = \frac{1}{\sqrt{n+2}}\Big(\frac{\sum_{i=0}^n z^i\varphi(w)^{n+1-i}}{\sqrt{n+1}} - \sqrt{n+1}z^{n+1}\Big), \quad n \geq 0.$$

We have $T_zY_n \perp Y_j$ for $j \neq n+1$. For, we have

$$\begin{aligned} &\langle T_zY_n, Y_j\rangle \\ &\quad = \frac{1}{\sqrt{n+2}\sqrt{j+2}}\big\langle z\varphi(w)\sigma_n(z,\varphi(w)), \varphi(w)\sigma_j(z,\varphi(w))\big\rangle \end{aligned}$$

because $\varphi(0) = 0$

$$\begin{aligned} &\quad = \frac{1}{\sqrt{n+2}\sqrt{j+2}}\left\langle \frac{\sum_{i=0}^n z^{i+1}\varphi(w)^{n+1-i}}{\sqrt{n+1}}, \frac{\sum_{\ell=0}^j z^\ell\varphi(w)^{j+1-\ell}}{\sqrt{j+1}}\right\rangle \\ &\quad = \frac{1}{\sqrt{n+1}\sqrt{n+2}\sqrt{j+1}\sqrt{j+2}}\sum_{i=0}^n\sum_{\ell=0}^j \langle\varphi(w)^{n-i}, \varphi(w)^{j-l}\rangle\langle z^{i+1}, z^\ell\rangle. \end{aligned}$$

If either $n - i \neq j - \ell$ or $i + 1 \neq \ell$, then

$$\langle\varphi(w)^{n-i}, \varphi(w)^{j-l}\rangle\langle z^{i+1}, z^\ell\rangle = 0$$

because $\varphi(0) = 0$. If $n - i = j - \ell$ and $i + 1 = \ell$, then $j = n + 1$. Thus $T_zY_n \perp Y_j$ for $j \neq n+1$.

Hence we get

$$
\begin{aligned}
F_zY_n &= \langle T_zY_n, Y_{n+1}\rangle Y_{n+1} \\
&= \frac{1}{\sqrt{n+2}\sqrt{n+3}}\left(\frac{1}{\sqrt{n+1}\sqrt{n+2}}\left(\sum_{i=0}^{n}\sum_{\ell=0}^{n+1}\langle\varphi(w)^{n-i},\varphi(w)^{n+1-\ell}\rangle\right.\right. \\
&\qquad\qquad \left.\left.\langle z^{i+1}, z^{\ell}\rangle\right) + \sqrt{n+1}\sqrt{n+2}\right)Y_{n+1} \\
&= \frac{1}{\sqrt{n+2}\sqrt{n+3}}\left(\frac{\sqrt{n+1}}{\sqrt{n+2}} + \sqrt{n+1}\sqrt{n+2}\right)Y_{n+1} \\
&= \frac{\sqrt{n+1}}{\sqrt{n+3}}\left(\frac{1}{n+2}+1\right)Y_{n+1} \\
&= \frac{\sqrt{n+1}\sqrt{n+3}}{n+2}Y_{n+1}.
\end{aligned}
$$

By the proof of Theorem 4.3, F_z on $M_\varphi \ominus wM_\varphi$ is unitarily equivalent to F_w on $[z-w]\ominus z[z-w]$. By Theorem 4.3, we get the assertion. $\square$

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