On strongly summable ultrafilters

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ABSTRACT. We present some new results on strongly summable ultra-filters. As the main result, we extend a theorem by N. Hindman and D. Strauss on writing strongly summable ultrafilters as sums.

CONTENTS

Introduction

The equivalent notions of strongly summable and union ultrafilters have been important examples of idempotent ultrafilters ever since they were first conceived in [Hin72], [Bla87] respectively. Their unique properties have been applied in set theory, algebra in the Stone–Čech compactification and set theoretic topology. For example, strongly summable ultrafilters were, in a manner of speaking, the first idempotent ultrafilters known, cf. [Hin72] and [HS98, notes to Chapter 5]; they were the first strongly right maximal idempotents known and they are the only known class of idempotents with a maximal group isomorphic to $\mathbb{Z}$. Their existence is independent of *ZFC*, since it implies the existence of (rapid) *P*-points, cf. [BH87].[1]

The first part of this paper will focus on union ultrafilters for which we prove a new property; in the second part, this property is applied to

[1]Also, as a strongly right maximal idempotent ultrafilter, the orbit closure of a strongly summable ultrafilter is an interesting example of a van Douwen space, cf. [HS02]; this will, however, not be important in what follows.

strengthen a theorem on writing strongly summable ultrafilters as sums due to N. Hindman and D. Strauss [HS95], [HS98, Chapter 12].

The presentation of the proofs is inspired by [Ler83] and [Lam95] splitting the proofs into different levels, at times adding [[in the elevator]] comments in between. Online discussion is possible through the author's website at http://peter.krautzberger.info/papers.

1. Preliminaries

Let us begin by giving a nonexhaustive selection of standard terminology in which we follow N. Hindman and D. Strauss [HS98]; for standard set theoretic notation we refer to T. Jech [Jec03], e.g., natural numbers are considered as ordinals, i.e., $n = \{0, \ldots, n-1\}$. We work in *ZFC* throughout. The main objects of this paper are *(ultra)filters* on an infinite set S, i.e., (maximal) proper subsets of the power set $\mathfrak{P}(S)$ closed under taking finite intersections and supersets. S carries the discrete topology in which case the set of ultrafilters is βS, its Stone–Čech compactification. The Stone topology on βS is generated by basic clopen sets induced by subsets $A \subseteq S$ in the form $\overline{A} := \{p \in \beta S \mid A \in p\}$. Filters are usually denoted by upper case Roman letters, mostly F, G, H, ultrafilters by lower case Roman letters, mostly p, q, r, u.

The set S is always assumed to be the domain of a *(partial) semigroup* $(S, \cdot)$, i.e., the (partial) operation $\cdot$ fulfills the associativity law $s \cdot (t \cdot v) = (s \cdot t) \cdot v$ (in the sense that if one side is defined, then so is the other and they are equal). For a partial semigroup S and $s \in S$ the set of elements compatible with s is denoted by $\sigma(s) := \{t \in S \mid s \cdot t \text{ is defined}\}$. A partial semigroup is also assumed to be *adequate*, i.e., $\{\sigma(s) \mid s \in S\}$ has the finite intersection property. We denote the generated filter by $\sigma(S)$ and the corresponding closed subset of βS by δS. For partial semigroups S, T a map $\varphi : S \to T$ is a *partial semigroup homomorphism* if $\varphi[\sigma(s)] \subseteq \sigma(\varphi(s))$ and

$$(\forall s \in S)(\forall s' \in \sigma(s)) \; \varphi(s \cdot s') = \varphi(s) \cdot \varphi(s').$$

To simplify notation in a partial semigroup, $s \cdot t$ is always meant to imply $t \in \sigma(s)$. For $s \in S$, the restricted multiplication to s from the left (right) is denoted by λ_s (ρ_s).

It is easy to see that the operation of a partial semigroup can always be extended to a full semigroup operation by adjoining a (multiplicative) zero which takes the value of all undefined products. One key advantage of partial semigroups is that partial subsemigroups are usually much more diverse than subsemigroups. Nevertheless, it is convenient to think about most theoretical aspects (such as extension to βS) with a full operation in mind.

The semigroups considered in this paper are $(\mathbb{N}, +)$ (with $\mathbb{N} := \omega \setminus \{0\}$), $(\mathbb{Z}, +)$ and the most important adequate partial semigroup $\mathbb{F}$.

Definition 1. On $\mathbb{F} := \{s \subseteq \omega \mid \emptyset \neq s \text{ finite}\}$ we define a partial semigroup structure by

$$s \cdot t := s \cup t \text{ if and only if } s \cap t = \emptyset.$$

The theory of the Stone–Čech compactification allows for the (somewhat unique) extension of any operation on S to its compactification, in particular a semigroup operation.

Definition 2. For a semigroup $(S, \cdot)$, $s \in S$ and $A \subseteq S$, $p, q \in \beta S$ we define the following.

- $s^{-1}A := \{t \in S \mid st \in A\}$.
- $A^{-q} := \{s \in S \mid s^{-1}A \in q\}$.
- $p \cdot q := \{A \subseteq S \mid A^{-q} \in p\}$. Equivalently, $p \cdot q$ is generated by sets $\bigcup_{v \in V} v \cdot W_v$ for $V \in p$ and each $W_v \in q$.
- $A^{\star} := A^{-q} \cap A$. This notation will only be used when there is no confusion regarding the chosen ultrafilter.

As is well known, this multiplication on βS is well defined and extends the operation on S. It is associative and right topological, i.e., the operation with fixed right hand side is continuous. For these and all other theoretical background we refer to [HS98].

In the case of a partial semigroup, ultrafilters in δS in a way multiply as if the partial operation was total. With the arguments from the following proposition it is a simple but useful exercise to check that if $(S, \cdot)$ is partial the above definitions still work just as well in the sense that

$$s^{-1}A := \{t \in \sigma(s) \mid st \in A\}$$

and $p \cdot q$ is only defined if it is an ultrafilter.

Proposition 1. *Let S be a partial subsemigroup of a semigroup T. Then δS is a subsemigroup of βT.*

Proof.   (1) Simply observe that for $a \in S$

$$\bigcup_{b \in \sigma(a)} b \cdot (\sigma(ab) \cap \sigma(b)) \subseteq \sigma(a).$$

(2) Therefore $\sigma(S) \subseteq p \cdot q$ whenever $p, q \in \delta S$. $\square$

It is easy to similarly check that partial semigroup homomorphisms extend to full semigroup homomorphisms on δS.

Since A^{-q} is not an established notation, the following useful observations present a good opportunity to test it. The proof of the following is straightforward.

Proposition 2. *Let $p, q \in \beta S$, $A \subseteq S$ and $s, t \in S$.*

- $t^{-1}s^{-1}A = (st)^{-1}A$.
- $s^{-1}A^{-q} = (s^{-1}A)^{-q}$.
- $(A \cap B)^{-q} = A^{-q} \cap B^{-q}$.

- $(s^{-1}A)^\star = s^{-1}A^\star$ *(with respect to the same ultrafilter).*
- $(A^{-q})^{-p} = A^{-(p\cdot q)}$.

The proverbial big bang for the theory of ultrafilters on semigroups is the following theorem.

Theorem 1 (Ellis–Numakura Lemma). *If $(S, \cdot)$ is a compact, right topological semigroup then there exists an* idempotent *element in S, i.e., an element $p \in S$ such that $p \cdot p = p$.*

Proof. See, e.g., [HS98, notes to Chapter 2]. $\qquad\qquad\qquad\qquad\qquad\square$

Therefore the following classical fact is meaningful.

Lemma 1 (Galvin Fixpoint Lemma). *For idempotent $p \in \beta S$, $A \in p$ implies $A^\star \in p$ and $(A^\star)^\star = A^\star$.*

Proof. $(A^\star)^\star = A^\star \cap (A^\star)^{-p} = A^\star \cap (A \cap A^{-p})^{-p} = A^\star \cap A^{-p} \cap A^{-p\cdot p} = A^\star \cap A^{-p} = A^\star$. $\qquad\qquad\qquad\qquad\qquad\square$

The following definitions are central in what follows. Even though we mostly work in $\mathbb{N}$ and $\mathbb{F}$ we formulate them for a general setting.

Definition 3. Let $\mathbf{x} = (x_n)_{n<N}$ (with $N \leq \omega$) be a sequence in a partial semigroup $(S, \cdot)$ and let $K \leq \omega$.

- The set of finite products (the FP-*set*) is defined as

$$\mathrm{FP}(\mathbf{x}) := \left\{ \prod_{i \in v} x_i \mid v \in \mathbb{F} \right\},$$

  where products are in increasing order of the indices. In this case, all products are assumed to be defined.[2]
- $\mathbf{x}$ has *unique representations* if for $v, w \in \mathbb{F}$,

$$\prod_{i \in v} x_i = \prod_{j \in w} x_j \quad \Rightarrow \quad v = w.$$

- If $\mathbf{x}$ has unique representations and $z \in \mathrm{FP}(\mathbf{x})$ we can define the $\mathbf{x}$-*support of* z, $\mathbf{x}$-supp(z), by the equation $z = \prod_{j \in \mathbf{x}\text{-supp}(z)} x_j$. We can then also define $\mathbf{x}$-min $:= \min \circ \mathbf{x}$-supp, $\mathbf{x}$-max $:= \max \circ \mathbf{x}$-supp.
- A sequence $\mathbf{y} = (y_j)_{j<K}$ is called a *condensation* of $\mathbf{x}$, in short $\mathbf{y} \sqsubseteq \mathbf{x}$, if

$$\mathrm{FP}(\mathbf{y}) \subseteq \mathrm{FP}(\mathbf{x}).$$

  In particular, $\{y_i \mid i < K\} \subseteq \mathrm{FP}(\mathbf{x})$. For convenience,

$$\mathbf{x}\text{-supp}(\mathbf{y}) := \mathbf{x}\text{-supp}[\{y_i \mid i \in \omega\}].$$

- Define $\mathrm{FP}_k(\mathbf{x}) := \mathrm{FP}(\mathbf{x}')$ where $x'_n = x_{n+k}$ for all n.

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[2]Note that we will mostly deal with commutative semigroups so the order of indices is not too important in what follows.

- FP-sets have a natural partial subsemigroup structure induced by $\mathbb{F}$. Here $(\prod_{i \in s} x_i) \cdot (\prod_{i \in t} x_i)$ is defined as in S but only if $\max(s) < \min(t)$. With respect to this restricted operation define

$$\mathrm{FP}^\infty(\mathbf{x}) := \delta\mathrm{FP}(\mathbf{x}) = \bigcap_{k \in \omega} \overline{\mathrm{FP}_k(\mathbf{x})}.$$

- If the semigroup is written additively, we write $\mathrm{FS}(\mathbf{x})$, etc. accordingly (for finite sums); for $\mathbb{F}$ we write $\mathrm{FU}(\mathbf{x})$, etc. (for finite unions).

Instead of saying that a sequence has certain properties it is often convenient to say that the generated FP-set does.

The following classical result is the starting point for most applications of algebra in the Stone–Čech compactification. We formulate it for partial semigroups.

Theorem 2 (Galvin–Glazer Theorem)**.** *Let $(S, \cdot)$ be a partial semigroup, $p \in \delta S$ idempotent and $A \in p$. Then there exists $\mathbf{x} = (x_i)_{i \in \omega}$ in A such that*

$$\mathrm{FP}(\mathbf{x}) \subseteq A.$$

Proof. This can be proved essentially just like the the original theorem, cf. [HS98, Theorem 5.8], using the fact that $\sigma(S) \subseteq p$ to guarantee all products are defined. □

An immediate corollary is, of course, the following classical theorem, originally proved combinatorially for $\mathbb{N}$ in [Hin74].

Theorem 3 (Hindman's Theorem)**.** *Let $S = A_0 \cup A_1$. Then there exists $i \in \{0, 1\}$ and a sequence $\mathbf{x}$ such that $\mathrm{FP}(\mathbf{x}) \subseteq A_i$.*

2. Union and strongly summable ultrafilters

The first part of this paper deals primarily with ultrafilters on the partial semigroup $\mathbb{F}$. The following three kinds of ultrafilters were first described in [Bla87].

Definition 4 (Ordered, stable, union ultrafilters)**.** An ultrafilter u on $\mathbb{F}$ is called:

- *union* if it has a base of FU-sets (from disjoint sequences);
- *ordered union* if it has a base of FU-sets from ordered sequences, i.e., sequences $\mathbf{s}$ such that $\max(s_i) < \min(s_{i+1})$ (for all $i \in \omega$);
- *stable union* if it is union and whenever

$$\mathbb{F}^2_< := \{(v, w) \in \mathbb{F}^2 \mid \max(v) < \min(w)\}$$

is partitioned into finitely many pieces, there exists homogeneous $A \in u$, i.e., $A^2_<$ is included in one part.

The original definition of stability is similar to that of a P-point (or δ-stable ultrafilter) which we discuss later. For their equivalence see [Bla87, Theorem 4.2] and [Kra09, Theorem 4.13]

It is clear yet important to note that FU-sets always have unique representations and that all products are defined. At this point it might be useful to check the following. Union ultrafilters are elements of $\delta\mathbb{F}$ and they are idempotent since for each included FU-set they contain all FU_k-sets. It is also worth while to check that if our operation on $\mathbb{F}$ was not restricted to disjoint but ordered unions then $\sigma(\mathbb{F})$ and hence $\delta\mathbb{F}$ would remain the same.

The following notion was introduced in [BH87] to help differentiate union ultrafilters; it is a special case of isomorphism, but arguably the natural notion for union ultrafilters.

Definition 5 (Additive isomorphism). Given partial semigroups S, T, call two ultrafilters $p \in \beta S$, $q \in \beta T$ *additively isomorphic* if there exist $\mathrm{FP}(\mathbf{x}) \in p$, $\mathrm{FP}(\mathbf{y}) \in q$ both with unique products such that the following map maps p to q

$$\varphi : \mathrm{FP}(\mathbf{x}) \to \mathrm{FP}(\mathbf{y}), \quad \prod_{i \in s} x_i \mapsto \prod_{i \in s} y_i.$$

We call such a map a *natural (partial semigroup) isomorphism*. It extends to a homomorphism (in fact, isomorphism) between $\mathrm{FP}^{\infty}(\mathbf{x})$ and $\mathrm{FP}^{\infty}(\mathbf{y})$.

In the semigroup $(\mathbb{N}, +)$, our interest lies in strongly summable ultrafilters.

Definition 6 (Strongly summable ultrafilters). An ultrafilter p on $\mathbb{N}$ is called *(strongly) summable* if it has a base of FS-sets.

The following properties are well known and necessary to switch between summable and union ultrafilters; they are the basic tools for handling strongly summable ultrafilters, cf. [BH87], [HS98, Chapter 12].

Proposition 3 (and Definition). *Every strongly summable ultrafilter has a base of* $\mathrm{FS}(\mathbf{x})$*-sets with the property*

$$(\forall n < \omega) \ x_n > 4 \cdot \sum_{i < n} x_i.$$

In this case $\mathbf{x}$ *is said to have* sufficient growth *which implies the following:*
   (1) $\sum_{i \in s} x_i = \sum_{i \in t} x_i$ *iff* $s = t$ *(unique represenations).*
   (2) $\sum_{i \in s} x_i + \sum_{i \in t} x_i \in \mathrm{FS}(\mathbf{x})$ *iff* $s \cap t = \emptyset$ *(unique sums).*
        In particular, condensations of $\mathbf{x}$ *have pairwise disjoint* $\mathbf{x}$*-support and the map* $\sum_{i \in s} x_i \mapsto s$ *maps the strongly summable to a union ultrafilter.*
   (3) *To have sufficient growth is* hereditary for condensations*, i.e., if* $\mathbf{x}$ *has sufficient growth, so does* $\mathbf{y} \sqsubseteq \mathbf{x}$ *(assuming that* $\mathbf{y}$ *is increasing).*

Proof. This follows (in order) from [HS98, Lemma 12.20, Lemma 12.34, Lemma 12.32, Theorem 12.36]. The last observation follows easily from the

second bullet and the growth of $\mathbf{x}$ since the growth of $\mathbf{x}$ implies that to be increasing means to be $\mathbf{x}$-max-increasing. □

Maybe the most important aspect to remember is this: whenever we have a condensation of a sequence with sufficient growth, its elements have pairwise disjoint $\mathbf{x}$-support (by point (2)) and we can apply the much less messy intuition about FU-sets to understand the structure of the FS-set. In particular, whenever a sequence $\mathbf{x}$ in $\mathbb{N}$ has sufficient growth we can apply the terminology of $\mathbf{x}$-supp, $\mathbf{x}$-max and $\mathbf{x}$-min as introduced in the preliminaries.

Although it is not relevant in our setting note that on the one hand growth by a factor 2 (instead of 4) already implies the above properties (with identical proofs as in the references). On the other hand the proof of [HS98, Lemma 12.20] can easily be enhanced to show that for any $k \in \mathbb{N}$ every strongly summable ultrafilter will have a base with growth factor k which leads to other interesting properties such as [HS98, Lemma 12.40].

3. Strongly summable ultrafilters are special

Recall that we aim to extend a theorem by N. Hindman and D. Strauss on writing strongly summable ultrafilters as sums originally published in [HS95], cf. [HS98, Theorem 12.45]. The original result was shown for a certain class of strongly summable ultrafilters, the so-called special strongly summable ultrafilters. Our main result will extend this to a wider class of strongly summable ultrafilters. The proof will require one new observation, which we prove in this section, as well as a series of modifications of the original proof as presented in, e.g., [HS98, Chapter 12].

To investigate special strongly summable ultrafilters as described in [HS95] and [HS98, 12.24], it is useful to switch to union ultrafilters. However, the notion introduced below is strictly weaker than the original one used by N. Hindman and D. Strauss.

Definition 7. Let $\mathbf{x}, \mathbf{y}$ be sequences in $\mathbb{N}$.

- A strongly summable ultrafilter $p \in \beta\mathbb{N}$ is *special* if there exists $\mathrm{FS}(\mathbf{x}) \in p$ with sufficient growth such that

  $(\forall L \in [\omega]^\omega)(\exists \mathbf{y} \sqsubseteq \mathbf{x})\ \mathrm{FS}(\mathbf{y}) \in p$ and $|L \setminus \mathbf{x}\text{-supp}(\mathbf{y})| = \omega$.

  Given the sequence $\mathbf{x}$ we say that p is special *with respect to* $\mathbf{x}$.
- A union ultrafilter $u \in \beta\mathbb{F}$ is *special* if

$$(\forall L \in [\omega]^\omega)(\exists X \in u)\left|L \setminus \bigcup X\right| = \omega.$$

In [HS95] and [HS98, Chapter 12], the notion of "special" is in this terminology "special with respect to $(n!)_{n \in \omega}$ and additionally *divisible*", i.e., there is a base of sets $\mathrm{FS}(\mathbf{x})$ with $x_n | x_{n+1}$ for all $n \in \omega$. However, [HS95, Theorem 5.8] gives an example of a strongly summable ultrafilter that is not additively isomorphic to a divisible ultrafilter so our notion is consistently weaker.

It is not surprising yet very useful that to be the witness for specialness is hereditary for condensations.

Proposition 4. *If a strongly summable ultrafilter p is special with respect to $\mathbf{x}$ and $\mathbf{y} \sqsubseteq \mathbf{x}$ with $\mathrm{FS}(\mathbf{y}) \in p$, then p is special with respect to $\mathbf{y}$.*

> *Summary of argument.* The uniqueness of $\mathbf{x}$-support allows us to link the elements of the $\mathbf{y}$-support to the $\mathbf{x}$-support. Hence, for a common condensation, missing elements in the $\mathbf{x}$-support will imply missing elements in the $\mathbf{y}$-support.

Proof.   (1) Take any $L \in [\omega]^\omega$.
  (2) Then define $L' := \{i \in \omega \mid (\exists k \in L)\ i \in \mathbf{x}\text{-supp}(y_k)\}$. L' is obviously infinite.
      [[Note that the k's are unique thus linking the two kinds of support.]]
  (3) Since p is special there exists a condensation $\mathbf{z} \sqsubseteq \mathbf{x}$ with $\mathrm{FS}(\mathbf{z}) \in p$ and
      $$|L' \setminus \mathbf{x}\text{-supp}(\mathbf{z})| = \omega.$$
  (4) For a common condensation $\mathbf{v} \sqsubseteq \mathbf{y}, \mathbf{z}$ with $\mathrm{FS}(\mathbf{v}) \in p$, naturally
      $$|L' \setminus \mathbf{x}\text{-supp}(\mathbf{v})| = \omega.$$
  (5) $|L \setminus \mathbf{y}\text{-supp}(\mathbf{v})| = \omega$.
      (a) If $i \in L' \setminus \mathbf{x}\text{-supp}(\mathbf{v})$, then there exists (by definition of L') some $k_i \in L$ with $i \in \mathbf{x}\text{-supp}(y_{k_i})$.
      (b) But then no v_j can have $k_i \in \mathbf{y}\text{-supp}(v_j)$ (or else $x_i \in \mathbf{x}\text{-supp}(v_j)$ which is impossible due to the previous proposition).
      (c) In other words, $k_i \in L \setminus \mathbf{y}\text{-supp}(\mathbf{v})$.
      (d) Since $|L' \setminus \mathbf{x}\text{-supp}(\mathbf{v})| = \omega$ and the map $i \mapsto k_i$ is finite-to-one, $|L \setminus \mathbf{y}\text{-supp}(\mathbf{v})| = \omega$. $\square$

The second observation is that the notions of special summable and special union ultrafilters are in fact equivalent.

Proposition 5. *Let p be a strongly summable ultrafilter additively isomorphic to a union ultrafilter u. Then p is special if and only if u is.*

Proof.   (1) Assume that p and u are as above and additively isomorphic via
      $$\varphi : \mathrm{FS}(\mathbf{x}) \to \mathrm{FU}(\mathbf{s}), \quad \sum_{i \in F} x_i \mapsto F,$$
      for suitable sequences $\mathbf{x}, \mathbf{s}$ in $\mathbb{N}$ and $\mathbb{F}$ respectively.
  (2) By switching to a condensation we may assume that $\mathbf{x}$ has sufficient growth.
  (3) If u is special, then φ clearly guarantees that $\mathbf{x}$ is a witness for p being special.
  (4) If p is special, we can assume that $\mathbf{x}$ is a witness of specialness thanks to the preceding proposition.

(5) Then again φ will guarantee that u is special. $\square$

The key fact is that all union ultrafilters are special.

Theorem 4 (Union ultrafilters are special). *Every union ultrafilter is special. Accordingly, all strongly summable ultrafilters are special.*

> *Summary of argument.* Assuming that some set covers all of L, a simple parity argument on pairs of the form $(i, i+1)$ in the support will yield a condensation that misses a lot of L.

Proof.   (1) Let $L \in [\omega]^\omega$.
> [[Remember that $\bigcup \mathrm{FU}(\mathbf{s}) = \bigcup\{s_i \mid i \in \omega\}$. For the mental picture of the arguments it is helpful (though not necessary) to enumerate sequences according to the maximum.]]

(2) We may assume that $\{s \in \mathbb{F} \mid s \cap L \neq \emptyset\} \in u$.
   (a) Otherwise its complement, call it X, has $L \setminus \bigcup X = L$ infinite, as desired.
(3) Since u is a union ultrafilter, we find $\mathrm{FU}(\mathbf{s}) \in u$ included in this set.
(4) If $L \setminus \bigcup \mathrm{FU}(\mathbf{s})$ is infinite, we are done.
(5) So assume it is finite; without loss it is empty.
> [[In the following sense we can now think as if $L = \omega$. If $\mathbf{t} \sqsubseteq \mathbf{s}$ and $i \notin \mathbf{s}\text{-supp}(\mathbf{t})$, then $s_i \cap L \neq \emptyset$ but $s_i \cap \bigcup \mathrm{FU}(\mathbf{t}) = \emptyset$. So dropping elements in the $\mathbf{s}$-support means dropping elements in L (and vice versa). We can concentrate on $\mathbf{s}\text{-supp}(\mathbf{s}) = \omega$.]]

(6) Consider $\pi : \mathrm{FU}(\mathbf{s}) \to \omega$, $t \mapsto \{i : s_i, s_{i+1} \subseteq t\}$. We're interested in whether $\pi(t)$ is even or odd.
(7) Since u is a union ultrafilter, we can find $\mathrm{FU}(\mathbf{t}) \in u$ such that the elements of $\pi[\mathrm{FU}(\mathbf{t})]$ all have the same parity.
(8) But the elements of $\pi[\mathrm{FU}(\mathbf{t})]$ can only be of even size.
   (a) For any $x \in \mathrm{FU}(\mathbf{t})$, there exists $i, j \in \omega$ such that

$$\mathbf{s}\text{-max}(x) < i < \mathbf{s}\text{-min}(t_j).$$

   (b) In that case $\pi(x \cup t_j) = \pi(x) + \pi(t_j)$ — which is even since $\pi(x) = \pi(t_j)$.
(9) Then $L \setminus \bigcup \mathrm{FU}(\mathbf{t})$ is infinite.
   (a) Assume towards a contradiction that it is finite.
> [[We will study the gaps in the $\mathbf{s}$-support of elements in $\mathrm{FU}(\mathbf{t})$ since they correspond to elements in $L \setminus \mathrm{FU}(\mathbf{t})$.]]
   (b) The set $\mathbf{s}\text{-supp}(\mathbf{t})$ must be cofinite since $\mathbf{s}$ covers all of L and every $s_i \cap L \neq \emptyset$.
   (c) In other words, there exists $b \in \omega$ such that $(\forall i \geq b)(\exists j_i) s_i \subseteq t_{j_i}$.
> [[Consider for a moment t_{j_b}, the t_j containing s_b. Since $\mathbf{t}$ covers all later s_i, some t_j contains $\mathbf{s}\text{-max}(t_{j_b})+1$. Therefore their union "gains" a pair of adjacent indices, i.e., $\pi(t_{j_b} \cup t_j) \geq \pi(t_{j_b}) + \pi(t_j) + 1$. Since $\pi(t_{j_b} \cup t_j)$ is even it must "gain" even more. If $\mathbf{t}$ was ordered, this would be impossible. For the unordered case, we need to argue more subtly.]]

(d) We define $x := \bigcup_{i \leq b} t_{j_i} \in \mathrm{FU}(\mathbf{t})$, adding to t_{j_b} everything "below" it.

> [[x is our initial piece. It contains the $\mathbf{s}$-supp($\mathbf{t}$) up to b. This ensures that any t_j disjoint from x must have $\mathbf{s}$-support beyond b.]]

(e) Next we define $b_1 := \mathbf{s}\text{-}\max(x)$, i.e., the index of the last $s_i \subseteq x$.

(f) Of course, $b_1 \geq b$ by choice of $t_{j_b} \subseteq x$.

> [[We will derive the contradiction from the fact that we can fill the entire interval $[b, b_1 + 1]$ by choice of b.]]

(g) Then we define $y := t_{j_{b_1}+1}$, i.e., the t_j that contains the next element of the $\mathbf{s}$-support.

(h) Finally, let $z := (\bigcup \{t_{j_i} : i < b_1\}) \setminus (x \cup y)$.

> [[y follows on where x ends, z fills all the gaps in the $\mathbf{s}$-support of $x \cup y$ between b and b_1 (and, of course, the support of z lies only beyond b). We will now analyze how gaps in $\mathbf{s}$-supp(x) are actually filled.]]

(i) On the one hand, we can compare $\pi(x)$ and $\pi(x \cup y)$.

(j) By definition,

$$\pi(x \cup y) = \pi(x) \dot{\cup} \pi(y) \dot{\cup} \{i \mid s_i \subseteq x, s_{i+1} \subseteq y \text{ or vice versa}\}.$$

Let us call elements in the third set *emerged* indices.

(k) We know that $\pi(x \cup y)$ contains one emerged index, namely b_1.

(l) But $\pi(x \cup y)$ is even and x has no support past b_1.

(m) Therefore $\pi(x \cup y)$ must have an odd number of emerged indices below b_1.

(n) In particular, y has $\mathbf{s}$-support below b_1 (sitting inside the gaps of the $\mathbf{s}$-support of x).

(o) There are four ways how those $i \in \mathbf{s}\text{-supp}(y)$ with $i < b_1$ can be found within the gaps of $\mathbf{s}\text{-supp}(x)$: only at the beginning of a gap, only at the end of a gap, both at the beginning and end of a gap and finally at neither beginning nor end of a gap.

(p) The latter two cases do not change the parity of $\pi(x \cup y)$ since they account for two and zero emerged indices respectively.

(q) So to make up for b_1 there must be an odd number of cases where $\mathbf{s}\text{-supp}(y)$ fills only the beginning or only the end of a gap in $\mathbf{s}\text{-supp}(x)$.

(r) On the other hand, we can similarly compare $\pi(x)$ and $\pi(x \cup z)$.

(s) We know that $\mathbf{s}\text{-supp}(x \cup y \cup z)$ contains the entire interval $[b, b_1]$.

(t) In particular, $\mathbf{s}\text{-supp}(z)$ fills the beginning or end of any gap of $\mathbf{s}\text{-supp}(x)$ that was not filled by $\mathbf{s}\text{-supp}(y)$.

(u) By the above analysis of $\pi(x \cup y)$ and $\pi(x)$ this gives an odd number of emerged indices in $\pi(x \cup z)$ below b_1.

(v) But then $\pi(x \cup z)$ is odd since z has no support below b, x has no support above b_1 and neither contains $b_1 + 1$ – a contradiction.

$\square$

I am very grateful for Andreas Blass's help in closing a gap in the final step of the above proof.

4. Disjoint support and trivial sums

There is need for another notion of support before formulating the main result. Every divisible sequence $\mathbf{a} = (a_n)_{n \in \omega}$, i.e., with $a_n | a_{n+1}$ for $n \in \omega$, with $a_0 = 1$ induces a unique representation of the natural numbers; the easiest case to keep in mind would be $a_n = 2^n$, i.e., the binary representation. We will work with an arbitrary divisible sequence but it might be best to always think of the binary case.

Definition 8. For the rest of this section we fix some divisible sequence $\mathbf{a} = (a_n)_{n \in \omega}$, i.e., with $a_n | a_{n+1}$ for $n \in \omega$, with $a_0 = 1$.

- We consider $\prod_{i \in \omega} \frac{a_{i+1}}{a_i} = \prod_{i \in \omega} \{0, \ldots, \frac{a_{i+1}}{a_i} - 1\}$ as a compact, Hausdorff space (with the product topology, each coordinate discrete).
- We can then define $\alpha : \mathbb{N} \to \prod_{i \in \omega} \frac{a_{i+1}}{a_i}$ by the (unique) relation

$$n = \sum_{i \in \omega} \alpha(n)(i) \cdot a_i.$$

  In other words, $\alpha(n)$ yields the unique representation of n with respect to $\mathbf{a}$. Note that $\alpha(n)$ has only finitely many nonzero entries for any n but for $p \in \beta\mathbb{N}$ its continuation $\alpha(p)$ might not.
- The *α-support of n*, α-supp(n), is the (finite) set of indices i with $\alpha(n)(i) \neq 0$; similarly we define α-max(n), α-min to be its maximum and minimum respectively.
- A sequence $\mathbf{x} = (x_n)_{n \in \omega}$ has *disjoint α-support* if its elements do; allowing confusion, FS$(\mathbf{x})$ is said to have disjoint support.
- A strongly summable ultrafilter has *disjoint α-support* if it contains an FS-set with disjoint α-support and sufficient growth.
- An idempotent ultrafilter p can be *written as a sum only trivially* if

$$(\forall q, r \in \beta\mathbb{N}) \ q + r = p \quad \Rightarrow \quad q, r \in (\mathbb{Z} + p).$$

- For $(2^n)_{n \in \omega}$, the binary support is abbreviated bsupp; its maximum and minimum by bmax and bmin respectively.

For the "trivial sums" property we should note that it is an easy exercise to show that $\beta\mathbb{N} \setminus \mathbb{N}$ is a left ideal of $(\beta\mathbb{Z}, +)$; in particular $\mathbb{Z} + p \subseteq \beta\mathbb{N}$.

So far we have always been interested in the finite sums of a sequence. It might therefore cause confusion as to why we chose the α-support when we have so far only studied the $\mathbf{a}$-support (which only coincides on FS$(\mathbf{a})$). Why not just assume that FS$(\mathbf{a})$ is in our strongly summable ultrafilter? From a certain point of view, this is what happens in the original result by Hindman and Strauss, cf. [HS98, 12.24] and in [HS95]. The advantage of our notion of disjoint α-support lies precisely in dropping this requirement — we won't need (a suitable condensation of) FS$(\mathbf{a})$ in the strongly summable ultrafilter.

In this spirit, there hopefully won't be a lot of confusion between α-support and **a**-support. Nevertheless we will see that the reasoning with α-support is quite similar when considering sequences with disjoint α-support.

Since we will be concerned with $\bigcap_{n \in \mathbb{N}} \overline{a_n \mathbb{N}}$ it is worthwhile to point out that by divisibility, $a_n \mathbb{N} \supseteq a_{n+1} \mathbb{N}$. Therefore an ultrafilter containing infinitely many such sets already contains all of them. Also, it is well known that any idempotent ultrafilter contains the set of multiples for any number. The following will be the main result.

Theorem 5 (Strongly summable ultrafilters as sums). *Every strongly summable ultrafilter with disjoint α-support can be written as a sum only trivially.*

The proof requires a series of technical propositions, but the following convenient corollary is immediate.

Corollary 1. *Every strongly summable ultrafilter is additively isomorphic to a strongly summable ultrafilter that can only be written as a sum trivially.*

Proof.   (1) For any strongly summable ultrafilter p, pick $\mathrm{FS}(\mathbf{x}) \in p$ with sufficient growth.

    (2) Then, e.g., the natural additive isomorphism φ between $\mathrm{FS}(\mathbf{x})$ and $\mathrm{FS}((2^n)_{n \in \omega})$ maps p to a strongly summable ultrafilter with disjoint binary support.

        (a) Let p' be the image of p; clearly, p' is a strongly summable ultrafilter.

        (b) Fix some $\mathrm{FS}(\mathbf{y}) \in p'$ with sufficient growth.

        (c) Then $\mathrm{FS}(\mathbf{y}) = \varphi[\mathrm{FS}(\mathbf{z})] = \mathrm{FS}(\varphi[\mathbf{z}])$ for some $\mathbf{z} \sqsubseteq \mathbf{x}$.

        (d) The growth of $\mathbf{x}$ guarantees that each z_i is a disjoint union of elements from $\mathbf{x}$.

        (e) Hence each y_i is a disjoint union of $\varphi[\mathbf{x}] = (2^n)_{n \in \omega}$.

        (f) In other words, $\mathbf{y}$ has disjoint binary support, as desired.   □

In [HS95] it is shown that strongly summable ultrafilters that are divisible and special with respect to $(n!)_{n \in \omega}$ can only be written as a sum trivially; however, by [HS95, Theorem 5.8], there consistently exist strongly summable ultrafilters that are not additively isomorphic to a divisible strongly summable ultrafilter.[3] In so far, this is an improvement.

To begin the series of technical observations, note one additional detail concerning the hereditary nature of specialness.

Lemma 2. *A strongly summable ultrafilter p with disjoint α-support is also α-special in the sense that there exists $\mathrm{FS}(\mathbf{x}) \in p$*

$$(\forall L \in [\omega]^{\omega})(\exists \mathbf{y} \sqsubseteq \mathbf{x}) \ \mathrm{FS}(\mathbf{y}) \in p \ and \ |L \setminus \alpha\text{-}\mathrm{supp}(\mathbf{y})| = \omega.$$

---

[3]cf. the comment after Definition 7. We could summarize our approach as replacing $(n!)_{n \in \omega}$ with **a** and divisibility with disjoint α-support.

Summary of argument. We argue as for the heredity of specialness using a common condensation of witnesses for disjoint α-support and specialness.

Proof. (1) Pick $\mathbf{x}$ as a witness for the disjoint α-support of a strongly summable ultrafilter p.

(2) We may assume that $\mathbf{x}$ also witnesses that p is special.

    (a) By Proposition 4, to be the witness for specialness is hereditary.

    (b) By Proposition 3, any condensation of $\mathbf{x}$ has pairwise disjoint support $\mathbf{x}$-support, hence pairwise disjoint α-support; in other words, to have disjoint α-support is hereditary.

    (c) Therefore a common condensation of the respective witnesses will have both properties.

(3) Given $L \in [\omega]^\omega$; if $L \setminus \alpha\text{-supp}(\mathbf{x})$ is infinite, we are done.

(4) If not we can consider the (infinite) set

$$L' := \{n \mid (\exists i \in L)\ i \in \alpha\text{-supp}(x_n)\}.$$

(5) By specialness there exists $\mathbf{y} \sqsubseteq \mathbf{x}$ with $\mathrm{FS}(\mathbf{y}) \in p$ and $L' \setminus \mathbf{x}\text{-supp}(\mathbf{y})$ infinite.

(6) But this implies $L \setminus \alpha\text{-supp}(\mathbf{y})$ is infinite by choice of L' and the disjoint $\mathbf{x}$-supp of members of $\mathbf{y}$. $\qquad\square$

The following well known theorem proves, in a manner of speaking, half the theorem.

Theorem 6. *Every strongly summable ultrafilter p is a* strongly right maximal idempotent, *i.e., the equation $q + p = p$ has the unique solution $q = p$.*

Proof. This is, e.g., [HS98, Theorem 12.39]. $\qquad\square$

The next result is also well known and easily checked.

Proposition 6. *For $n \in \mathbb{N}$, $q, r \in \beta\mathbb{N}$ the following holds:*

- *If $q + r \in \overline{n\mathbb{N}}$, then either both $q, r \in \overline{n\mathbb{N}}$ or neither is.*
- *Similarly we can replace $\overline{n\mathbb{N}}$ by $\bigcap_{n \in \mathbb{N}} \overline{a_n\mathbb{N}}$ and $\mathbb{Z} + \bigcap_{n \in \mathbb{N}} \overline{a_n\mathbb{N}}$.*

Proof. This is, e.g., [HS95, Lemma 2.6]. $\qquad\square$

As mentioned earlier, our proof follows the same strategy as the proof in [HS95] and [HS98, Chapter 12]; the proof for the right summand consists of two parts. The first part proves that if one of the summands is close to the strongly summable ultrafilter, i.e., in $\bigcap_{n \in \omega} \overline{a_n\mathbb{N}}$, it is already equal. The second part shows that writing a strongly summable ultrafilter with disjoint support as a sum can only be done with the summands "close enough" to it.

For the first part, a technical lemma reflects the desired property: under restrictions typical for ultrafilter arguments, elements of an FS-set with disjoint α-support can be written as sums only trivially.

Lemma 3 (Trivial sums for FS-sets)**.** *Let* $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ *be a sequence with disjoint α-support and enumerated with increasing α-min, $a \in \mathbb{N}$ and*

$$m := \min\{i \mid \alpha\text{-max}(a) < \alpha\text{-min}(x_i)\}.$$

Then for every $b \in \mathbb{N}$ with $\alpha\text{-max}(x_m) < \alpha\text{-min}(b)$

$$a + b \in \mathrm{FS}(\mathbf{x}) \quad \Rightarrow \quad a, b \in \mathrm{FS}(\mathbf{x}).$$

Summary of argument. The simple idea is that neither the sums of the x_i nor the sum $a + b$ will have any carrying over in the α-support. Hence, the $\mathbf{x}$-support of $a + b$ splits into $\mathbf{x}$-support of a and b.

Proof.　　(1) Assume $\mathbf{x}$, a and b are given as in the lemma.
　　(2) Since $a + b \in \mathrm{FS}(\mathbf{x})$, there exists some finite, nonempty $H \subseteq \mathbb{N}$ with

⊛
$$a + b = \sum_{i \in H} x_i.$$

　　(3) Define

$$H_a := \{j \in H \mid \alpha\text{-supp}(x_j) \cap \alpha\text{-supp}(a) \neq \emptyset\}$$

　　　and H_b similarly.
　　(4) $H = H_a \dot\cup H_b$.
　　　　(a) On the one hand $\alpha\text{-supp}(a) \cap \alpha\text{-supp}(b) = \emptyset$ by assumptions on b; also $\mathbf{x}$ has disjoint α-support.
　　　　(b) So there is no carrying over (in the α-support) on either side of Equation ⊛, i.e.,

$$H = H_a \cup H_b.$$

　　　　(c) On the other hand, if $\alpha\text{-supp}(x_i) \cap \alpha\text{-supp}(a) \neq \emptyset$, then $i \leq m$ by choice of m.
　　　　(d) This in turn implies $\alpha\text{-supp}(x_i) \cap \alpha\text{-supp}(b) = \emptyset$ by the choice of b.
　　　　(e) In other words, $H_a \cap H_b = \emptyset$.
　　(5) Then $\sum_{i \in H_a} x_i = a$ and $\sum_{i \in H_b} x_i = b$, as desired.　　□

The next lemma takes the proof nearly all the way, i.e., if the second summand is "close enough" to the strongly summable ultrafilter, both are equal to it.

Lemma 4 (Trivial sums for $\bigcap_{n \in \omega} \overline{a_n \mathbb{N}}$)**.** *For any strongly summable ultrafilter p with disjoint α-support*

$$(\forall q \in \beta\mathbb{N})\left(\forall r \in \bigcap_{n \in \omega} \overline{a_n \mathbb{N}}\right) q + r = p \quad \Rightarrow \quad q = r = p.$$

Summary of argument. The proof is basically a reflection argument. Arguing indirectly, the addition on $\beta\mathbb{N}$ reflects to elements in the sets of the ultrafilters in such a way that nontrivial sums of ultrafilters lead to nontrivial sums of an FS-set, contradicting Lemma 3.

Proof. (1) By Theorem 6, any strongly summable ultrafilter is strongly right maximal, so it suffices to show that $r = p$. Assume to the contrary that $r \neq p$.

(2) Pick a witness for p, i.e., $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ with sufficient growth and disjoint α-support; without loss $\mathrm{FS}(\mathbf{x}) \in p \setminus r$.

(3) Since $q + r = p$, $\mathrm{FS}(\mathbf{x})^{-r} \in q$; so pick a such that $-a + \mathrm{FS}(\mathbf{x}) \in r$.

(4) Pick m as for Lemma 3, i.e., such that all $(x_n)_{n>m}$ have $\alpha\text{-max}(a) < \alpha\text{-min}(x_n)$ (which is possible since $\mathbf{x}$ has disjoint α-support).

(5) Define $M := \alpha\text{-max}(x_m) + 1$; note that the multiples of a_M have α-support beyond the support of both x_m and a.

(6) Now

$$(-a + \mathrm{FS}(\mathbf{x})) \cap (\mathbb{N} \setminus \mathrm{FS}(\mathbf{x})) \cap a_M \mathbb{N} \in r.$$

So pick b from this intersection.

(7) Then $a + b \in \mathrm{FS}(\mathbf{x})$. But applying Lemma 3 both $a, b \in \mathrm{FS}(\mathbf{x})$, contradicting $b \notin \mathrm{FS}(\mathbf{x})$. □

In the final and main lemma, it remains to show that if a strongly summable ultrafilter is written as a sum, then the summands are already "close enough".

Lemma 5 (Nearly trivial sums). *For any strongly summable ultrafilter p with disjoint α support*

$$(\forall q, r \in \beta\mathbb{N}) \; q + r = p \quad \Rightarrow \quad q, r \in \mathbb{Z} + \bigcap_{n \in \omega} \overline{a_n \mathbb{N}}.$$

Summary of argument. We follow the strategy of the proof of [HS98, Theorem 12.38] The argument is similar to the previous lemma, i.e., if $q \notin \mathbb{Z} + \bigcap_{n \in \omega} \overline{a_n \mathbb{N}}$, there will always be a sum $a + b$ that cannot end up in a certain FS-set. For this, the image of q under (the continuous extension of) α is analyzed. Using the fact that strongly summable ultrafilters are special, it turns out that there cannot be enough carrying over available to always end up in the FS-set.

Proof. (1) By Proposition 6 it suffices to show that $q \in \mathbb{Z} + \bigcap_{n \in \omega} \overline{a_n \mathbb{N}}$.

(2) Define the following subsets of ω.

$$Q_0 := \left\{ i \in \omega \mid \alpha(q)(i) < \frac{a_{i+1}}{a_i} - 1 \right\}$$
$$Q_1 := \{ i \in \omega \mid \alpha(q)(i) > 0 \}.$$

[[In other words, Q_0 counts where the α-support does not have a maximal entry, Q_1 counts where it does not have a minimal entry, i.e., Q_1 is just the support of the function $\alpha(q)$ in the usual sense.]]

(3) If either Q_0 or Q_1 is finite, then $q \in \mathbb{Z} + \bigcap_{n\in\omega} \overline{a_n\mathbb{N}}$.

    (a) *Case* 1. Q_1 is finite.

    (b) Pick $k \in \omega$ such that $\alpha(q)(n) = 0$ for $n > k$.

    (c) Then show that $z := \sum_{i\leq k} \alpha(q)(i)a_i$ has

$$(\forall n > k) \ z + a_n\mathbb{N} \in q.$$

       (i) Given $n > k$ define

$$U_{z,n} := \left\{ s \in \prod_{i\in\omega} \frac{a_{i+1}}{a_i} \ \middle| \ s\!\restriction\! n = \alpha(q)\!\restriction\! n = \alpha(z)\!\restriction\! n \right\}.$$

      (ii) Obviously, $U_{z,n}$ is an open neighbourhood of $\alpha(q)$, hence $\alpha^{-1}[U_z] \in q$.

     (iii) But it is easily checked that $\alpha^{-1}[U_z] = z + a_n\mathbb{N}$.

    (d) Since **a** was divisible, $q \in z + \bigcap_{n\in\omega} \overline{a_n\mathbb{N}}$, as desired.

    (e) *Case* 2. Q_0 is finite.

    (f) Pick k such that $\alpha(q)(n) = \frac{a_{n+1}}{a_n}$, i.e., maximal, for $n > k$.

    (g) This time show that $z := a_{k+1} - \sum_{i<k} \alpha(q)(i)a_i$ has

$$(\forall n > k) \ -z + a_n\mathbb{N} \in q,$$

and therefore again $q \in -z + \bigcap_{n\in\omega} \overline{a_n\mathbb{N}}$.

       (i) Again, given $n > k$, consider $\alpha^{-1}[U_{z,n}]$. This time we check that $\alpha^{-1}[U_{z,n}] = -z + a_n\mathbb{N}$.

      (ii) Let $w \in \alpha^{-1}[U_{z,n}]$. Then for some $b \geq 0$

$$w = b \cdot a_{n+1} + \sum_{i>k}^{n} \left(\frac{a_{i+1}}{a_i} - 1 \right) a_i + \sum_{i\leq k} \alpha(q)(i)a_i,$$

since by assumption that Q_0 is finite, i.e., all of $\alpha(q)(i)$ beyond k is maximal.

     (iii) But this implies

$$w + z = b \cdot a_{n+1} + \sum_{i>k}^{n} \left(\frac{a_{i+1}}{a_i} - 1 \right) a_i + a_{k+1}$$

$$= b \cdot a_{n+1} + a_{n+1} = (b+1)a_{n+1},$$

as desired. This concludes case 2.

(4) So let us assume to the contrary that $q \notin \mathbb{Z} + \bigcap_{n\in\omega} \overline{a_n\mathbb{N}}$, i.e., both Q_0, Q_1 are infinite.

(5) Since u is strongly summable with disjoint α-support, pick a sequence $\mathbf{x} = (x_n)_{n\in\omega}$ with disjoint α-support, sufficient growth and $\mathrm{FS}(\mathbf{x}) \in u$.

(6) By Lemma 2, assume without loss that both $Q_0 \setminus \alpha\text{-supp}(\mathbf{x})$ and $Q_1 \setminus \alpha\text{-supp}(\mathbf{x})$ are infinite.

> [[Towards the final contradiction, it is now necessary to choose a couple of natural numbers; each choice will be followed by a short comment.]]

(7) By $q + r = p$ of course $\mathrm{FS}(\mathbf{x})^{-r} \in q$; so pick a with $-a + \mathrm{FS}(\mathbf{x}) \in r$.

> [[a can r-often be translated into $\mathrm{FS}(\mathbf{x})$ — which will be too often.]]

(8) Next, pick $s_1 \in Q_1 \setminus \alpha\text{-supp}(\mathbf{x})$ and $s_2 \in Q_0 \setminus \alpha\text{-supp}(\mathbf{x})$ with

$$s_2 > s_1 > \alpha\text{-max}(a)$$

> [[On the one hand, s_1 ensures $\sum_{i \leq s_2} \alpha(q)(i)a_i - a > 0$, but this difference has a nonmaximal entry at α-max since $s_2 \in Q_0$. On the other hand, $\alpha(q)(s_2)$ is not maximal, $\alpha(q)(s_1)$ is not minimal, but every $z \in \mathrm{FS}(\mathbf{x})$ has $\alpha(z)(s_2) = \alpha(z)(s_1) = 0$.]]

(9) By $q + r = p$ also $(a_{s_2+1}\mathbb{N})^{-r} \in q$, so pick b with

$$b \in (a_{s_2+1}\mathbb{N})^{-r} \cap \left(\sum_{i \leq s_2} \alpha(q)(i)a_i + a_{s_2+1}\mathbb{N} \right) \in q.$$

where the latter set is in q since it is $U_{q|(s_2+1),s_2+1}$; cf. step (3).

> [[So b has $\alpha(b)(s_i) = \alpha(q)(s_i)$ (for $i = 2, 1$), i.e., nonmaximal and nonminimal respectively. In particular, $b - a > a_{s_1} - a > 0$ but $\alpha(b - a)(s_2)$ is not maximal.]]

(10) Finally, choose $y \in (-b + a_{s_2+1}\mathbb{N}) \cap (-a + \mathrm{FS}(\mathbf{x})) \in r$.

> [[Note that since $s_2 \notin \alpha\text{-supp}(\mathbf{x})$ and $a + y \in \mathrm{FS}(\mathbf{x})$ we have $\alpha(a + y)(s_2) = 0$. But also $y + b \in a_{s_2+1}\mathbb{N}$.]]

(11) Recapitulating the choices so far:

    (a) $\alpha(q)(s_1) > 0, \alpha(q)(s_2) < \frac{a_{s_2+1}}{a_{s_2}} - 1$ (since $s_1 \in Q_1$, $s_2 \in Q_0$).

    (b) $\alpha\text{-max}((\sum_{i \leq s_2} \alpha(q)(i)a_i) - a) > 0$ (since $\alpha\text{-max}(a) < s_1 \in Q_1$).

    (c) $\alpha\text{-max}((\sum_{i \leq s_2} \alpha(q)(i)a_i) - a)$ is not maximal (since $\alpha(q)(s_2)$ not maximal and $s_2 > \alpha\text{-max}(a)$).

    (d) $\alpha\text{-min}(b + y) > s_2$ (since $b + y \in a_{s_2+1}\mathbb{N}$).

    (e) $s_2 \notin \alpha\text{-supp}(a + y)$ (since $a + y \in \mathrm{FS}(\mathbf{x})$).

> [[The lurking contradiction lies in the fact that since y translates such a small a into $\mathrm{FS}(\mathbf{x})$, it cannot simultaneously translate elements like b, i.e., elements that agree with $\alpha(q)$ up to s_2, to be divisible by a_{s_2+1}.
>
> This is due to the (nonmaximal) "hole" of both $(y + a)$ and $(b - a)$ at s_2 which simply does not allow for enough carrying over in the sum $(y + b)$ to get a multiple of 2^{s_2+1}.]]

(12) First calculate

$$\sum_{i > s_2} \alpha(b + y)(i)a_i = (a + y) + (b - a)$$

$$= \sum_{i \in \omega} \alpha(a + y)(i)a_i + \sum_{i \in \omega} \alpha(b - a)(i)a_i.$$

Recall that $b - a > 0$, so not all $\alpha(b-a)(i)$ are zero, but $\alpha(b-a)(s_2)$ is not maximal (as noted before).

(13) Rearranging this equation yields

$$\sum_{i>s_2}\alpha(b+y)(i)a_i - \sum_{i>s_2}\alpha(a+y)(i)a_i - \sum_{i>s_2}\alpha(b-a)(i)a_i$$
$$= \sum_{i\leq s_2}\alpha(a+y)(i)a_i + \sum_{i\leq s_2}\alpha(b-a)(i)a_i$$
$$= \sum_{i<s_2}\alpha(a+y)(i)a_i + \sum_{i\leq s_2}\alpha(b-a)(i)a_i,$$

since $s_2 \notin \alpha\text{-supp}(a+y)$.

(14) Clearly, a_{s_2+1} divides the first line, so the last line must add up to (a multiple of) a_{s_2+1}.

[[However, there is not enough carrying over.]]

(15) But

$$0 < \sum_{i<s_2}\alpha(a+y)(i)a_i + \sum_{i\leq s_2}\alpha(b-a)(i)a_i < a_{s_2} + \left(\frac{a_{s_2+1}}{a_{s_2}} - 1\right)a_{s_2} = a_{s_2+1}.$$

(a) Since $\alpha(b)$ agrees with $\alpha(q)$ up to s_2, step (11b) implies that both summands are positive.

(b) Also since $\alpha(b)$ agrees with $\alpha(q)$ up to s_2, $\alpha(b)(s_2)$ is not maximal, i.e., less than $\left(\frac{a_{s_2+1}}{a_{s_2}} - 1\right)$.

(c) Finally, by choice of $s_1 > \alpha\text{-max}(a)$, also $\alpha(b-a)(s_2)$ is not maximal. □

After this complicated proof, the main result follows almost immediately.

Theorem 7 (Trivial sums). *A strongly summable ultrafilter with disjoint α-support can only be written as a sum trivially.*

Proof. (1) Assume that p is a strongly summable ultrafilter with disjoint α-support and $q, r \in \beta\mathbb{N}$ with

$$q + r = p.$$

(2) The above Lemma 5 implies $r \in \mathbb{Z} + \bigcap_{n\in\omega}\overline{a_n\mathbb{N}}$.

(3) Therefore there exists $k \in \mathbb{Z}$ such that $-k + r \in \bigcap_{n\in\omega}\overline{a_n\mathbb{N}}$; in particular

$$(k + q) + (-k + r) = p.$$

(4) But now applying Lemma 4 with $k + q$ and $-k + r$ implies $k + q = -k + r = p$, as desired. □

This result, however, leaves some obvious questions open.

Question 1. • Does every strongly summable ultrafilter have the trivial sums property?

- Does every strongly summable ultrafilter have disjoint α-support for some **a**?
- Do other (idempotent) ultrafilters have the trivial sums property?

A slight progress on the first two is the following proposition.

Proposition 7. *Let p be a strongly summable ultrafilter additively isomorphic to a stable ordered union ultrafilter. Then p has disjoint binary support (hence trivial sums).*

> *Summary of argument.* Ordered unions guarantee ordered **x**-support for appropriate **x**. Since FS(**x**) always contains elements with ordered binary support, stability "enforces" this throughout a condensation.

Proof.   (1) Consider an additive isomorphism φ defined on a suitable FS(**x**) $\in p$ such that $\varphi(p)$ is stable ordered union.
  (2) Consider the following set
  $$\{(v, w) \in \varphi[\mathrm{FS}(\mathbf{x})]^2_< \mid \mathrm{bmax}(\varphi^{-1}(v)) < \mathrm{bmin}(\varphi^{-1}(w))\}.$$
  (3) Since $\varphi(p)$ is a stable ordered union ultrafilter, there exists ordered FU(**s**) $\in \varphi(p)$ such that FU(**s**)$^2_<$ is included or disjoint from the above set.
  (4) But FU(**s**)$^2_<$ cannot be disjoint.
    (a) For any FU(**s**) $\in \varphi(p)$ there is some $\mathbf{y} \sqsubseteq \mathbf{x}$ with $\varphi^{-1}[\mathrm{FU}(\mathbf{s})] = \mathrm{FS}(\mathbf{y})$.
    (b) But for any $z \in \mathrm{FS}(\mathbf{y})$ we can pick $z' \in \mathrm{FS}(\mathbf{y}) \cap 2^{\mathrm{bmax}(z)}\mathbb{N}(\in p)$.
    (c) Then the pair $(\varphi(z), \varphi(z'))$ is included in the above set.
  (5) The homogeneous FU(**s**) $\in \varphi(p)$ yields some $\varphi^{-1}[\mathrm{FU}(\mathbf{s})] = \mathrm{FS}(\mathbf{y}) \in p$.
  (6) Since **s** is ordered, y must have ordered, hence disjoint binary support. $\qquad\square$

So, as usual, the strongest notion of strongly summable ultrafilter has the desired trivial sums property. A negative answer to the first question would probably require the identification of a new kind of union ultrafilter.

The most natural answer to the second question would be to prove that bsupp maps strongly summables to union ultrafilters — after all, its inverse map maps union ultrafilters to strongly summable ultrafilters.

For the closing remark, recall the following two notions. An ultrafilter in $\mathbb{N}$ is a *P-point* if whenever we pick countably many of its elements $(A_n)_{n\in\omega}$, it includes a pseudo-intersection B, i.e., $A_n \setminus B$ is finite for all n. An ultrafilter is *rapid* if for every unbounded function $f : \mathbb{N} \to \mathbb{N}$ it contains an element B such that $|f^{-1}(n) \cap B| \leq n$ for all n. Since union ultrafilters map to rapid P-points under max, the following might suggest a positive answer.

Proposition 8. *Let p be strongly summable. Then $\mathrm{bmax}(p)$ is a rapid P-point.*

Summary of argument. The proof is a modification of the
proof of [BH87, Theorem 2].

Proof.   (1) Pick a sequence $\mathbf{x}$ with sufficient growth and $\mathrm{FS}(\mathbf{x}) \in p$.
  (2) Given $f \in \omega^\omega$ consider the set

$$A = \{a \in \mathrm{FS}(\mathbf{x}) \mid f(\mathrm{bmax}(a)) \leq \min(\mathbf{x}\text{-supp}(a))\}.$$

Then either A or its complement is in p.

  (3) If $A \in p$ then f is bounded (and therefore constant) on a set in
  $\mathrm{bmax}(p)$.
  - (a) Pick $a \in A$ and $\mathrm{FS}(\mathbf{y}) \subseteq (A \cap \mathrm{FS}_{>\mathbf{x}\text{-max}(a)}(\mathbf{x}) \cap 2^a\mathbb{N})$ in p.
  - (b) Then for $b \in \mathrm{FS}(\mathbf{y})$ calculate

$$f(\mathrm{bmax}(b)) = f(\mathrm{bmax}(a + b)) \leq \min(\mathbf{x}\text{-supp}(a + b))$$
$$= \min(\mathbf{x}\text{-supp}(a)).$$

  - (c) In other words, f is bounded on $\mathrm{bmax}[\mathrm{FS}(\mathbf{y})] \in \mathrm{bmax}(p)$.
  (4) If $\mathbb{N} \setminus A \in p$, then f has $|f^{-1}(n)| \leq n$ on a set in $\mathrm{bmax}(p)$.
  - (a) Pick $\mathbf{y} \sqsubseteq \mathbf{x}$ with $\mathrm{FS}(\mathbf{y}) \in p$, disjoint from A.
  - (b) Therefore, each $z \in \mathrm{FS}(\mathbf{y})$ with $n = f(\mathrm{bmax}(z))$ must have
    $n > x\text{-min}(z)$.
  - (c) Since $\mathbf{y}$ has sufficient growth, $\mathrm{bmax}[\mathrm{FS}(\mathbf{y})] = \mathrm{bmax}[\mathbf{y}]$.
  - (d) Due to the disjoint $\mathbf{x}$-support of the y_i, there are at most n
    indices i such that $n > x\text{-min}(y_i)$. $\square$

Thanks to the above proposition we might favor that all strongly summa-
ble ultrafilters have disjoint binary support. However, an answer remains
elusive. It seems, however, that further progress on writing strongly sum-
mable ultrafilters as sums might lead to a better understanding of the phe-
nomena in $\beta\mathbb{N}$ in general, just as it did with strongly right maximality.

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