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Real and imaginary parts of polynomial iterates

Julia A. Barnes, Clinton P. Curry and Lisbeth E. Schaubroeck

Abstract. Julia sets for complex-valued polynomials have been well-studied for years. However, the graphs of the polynomials themselves and their iterates are more difficult to visualize because they are four-dimensional. In this paper, we explore the dynamics of these functions by analyzing the behavior of the real and imaginary parts of the iterates. We also define two sets of points for which the real (respectively imaginary) parts of the iterates remain bounded, and prove how these sets relate to the corresponding filled Julia set. We end by applying our results to the well-known class of functions $f_c(z) = z^2 + c$.

Contents

1. Introduction

The subject of this paper is the real and imaginary parts of the iterates of complex polynomials. By the n^{th} *iterate* of a polynomial P, we mean the function

$$P^n = \overbrace{P \circ \cdots \circ P}^{n \text{ times}}.$$

In general, one way to gain insight into the properties of a function is to examine its graph. This tactic is not available to us in this case since the graph of a function $f : \mathbb{C} \to \mathbb{C}$ lives in $\mathbb{C}^2$, which cannot be easily visualized

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in our three (real) dimensions. Therefore, we study instead the graphs of the real and imaginary parts of the iterates; these are objects which live in three real dimensions, and therefore can be visualized and studied. In particular, we consider the graphs of the real and imaginary parts of $P^n(z)$ as n tends to infinity.

In Section 2, we provide background information, examples, and definitions used throughout our discussion. Part of the background information includes a formal definition for the filled Julia set for P, denoted $K(P)$. There are also formal definitions for two sets, $U(P)$ and $V(P)$, that are related to the real and imaginary parts of the iterates of P. Basically, a point z is in $U(P)$ if the set of its iterates under P has bounded real part. Similarly, a point z is in $V(P)$ if the set of its iterates under P has bounded imaginary part. We also include a theorem of Böttcher and definitions associated with that theorem.

In Section 3 we prove some technical results used later in the paper. In Section 4 we precisely prove the relationships between $U(P), V(P)$, and $K(P)$, including under what conditions $U(P)$ or $V(P)$ is equal to $K(P)$. In particular, when $P(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0$, we have that $K(P) \neq V(P)$ if and only if the image of the positive real line is "parallel at infinity" to a horizontal line in the complex plane and $K(P) \neq U(P)$ if and only if the image of the imaginary axis is "parallel at infinity" to a vertical line in the complex plane. (All terms will be made precise later.) For quadratic polynomials of the form $f_c(z) = z^2 + c$, this means that $U(f_c) = K(f_c)$ and $V(f_c) \neq U(f_c)$ for all $c \in \mathbb{C}$.

Finally, in Section 5 we discuss some qualitative differences between $U(P)$ and $V(P)$, as well as show the contour plots and corresponding filled Julia sets for some of the surfaces of well-known polynomials to illustrate our results.

2. Basic definitions

Here we discuss some general properties about the graphs of complex-analytic functions. By using calculus and some complex analysis, we can observe a few things about the graphs of their real and imaginary parts.

Theorem 1. *Let f be a nonconstant complex-analytic function with real and imaginary parts u and v. Then the critical points of f are the critical points of both u and v. Also, each critical point is a saddle point for both u and v.*

Proof. Since f is complex analytic, the functions u and v are differentiable, and indeed harmonic. As such, u and v satisfy a maximum modulus property: if u (respectively v) has a maximum value on an open disk Ω, then u (respectively v) is constant on Ω. Since f is nonconstant and analytic, neither u nor v is constant on any disk. Hence, any critical point of u or v cannot be the location of a local extremum, i.e., it is a saddle point. To

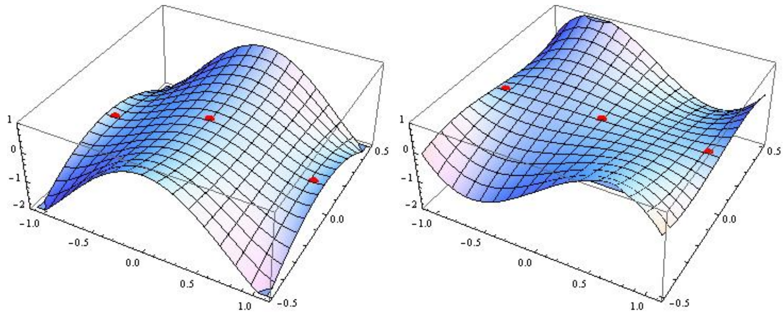

FIGURE 1. The real and imaginary parts of f_{-1}^2 with critical points marked.

finish, note that every critical point of u or v is a critical point of f by the Cauchy–Riemann equations. □

Example 2. As an illustration, consider the function $f_{-1}(z) = z^2 - 1$. The only critical point for f_{-1} is 0, while the critical points for f_{-1}^2 are $0, 1$ and -1. See Figure 1 for graphs of $\mathrm{Re}(f_{-1}^2)$ and $\mathrm{Im}(f_{-1}^2)$ along with their saddle points.

The *filled Julia set* of a complex polynomial $P(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_1 z + a_0$ is defined by

$$K(P) = \{z \in \mathbb{C} \,:\, P^n(z) \not\to \infty\}.$$

Note that the boundary of $K(P)$ is the well studied Julia set $J(P)$, the locus of chaos of P. Its complement is known as the Fatou set. The complement of $K(P)$, denoted B_∞, is the attractive basin of infinity and is a subset of the Fatou set for P. $K(P)$ and B_∞ are well studied and have a rich theory; good references for the background are [1] and [3].

Example 3. The filled Julia set, $K(f_0)$, for $f_0(z) = z^2 + 0$ can be easily computed. Consider z in polar form, i.e. $z = re^{i\theta}$ with $r = |z|$, and θ the angle between the x-axis and the ray from 0 to z. When we evaluate $f_0(z)$, we see that $f_0(z) = z^2 = (re^{i\theta})^2 = r^2 e^{i2\theta}$. If we continue in this fashion, $f_0^n(z) = r^{2^n} e^{i2^n\theta}$, and $|f_0^n(z)| = |r^{2^n}||e^{i2^n\theta}| = r^{2^n}$. Note that

$$\lim_{n\to\infty} r^{2^n} = \begin{cases} 0 & \text{if } r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1. \end{cases}$$

Therefore, $\{f_0^n(z) : n \text{ is a natural number}\}$ is bounded if and only if $|z| = r \le 1$. Hence, the filled Julia set, $K(f_0)$, is the closed unit disk.

For other polynomials $P(z)$, it is generally not possible to determine the filled Julia set by hand, although it is not difficult to have a computer

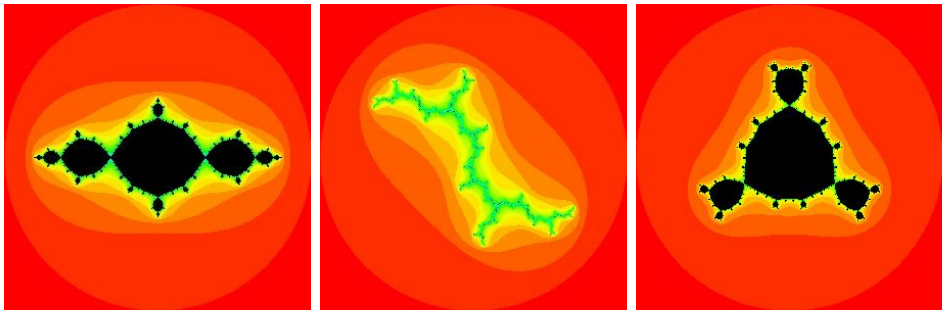

FIGURE 2. The filled Julia sets for z^2-1, z^2+i, and z^3+i.

generate approximate images. See Figure 2 for Julia sets for $f_{-1}=z^2-1$, $f_i=z^2+i$, and z^3+i.

Now we want to explore the similarities and differences of $\mathrm{Re}(P^n)$ and $\mathrm{Im}(P^n)$ and their corresponding filled Julia sets. To this end, we define two new sets.

Definition 4. For a polynomial P, let

$$U(P)=\{z:\mathrm{Re}(P^n(z))\not\to\infty\}$$

and

$$V(P)=\{z:\mathrm{Im}(P^n(z))\not\to\infty\}.$$

Note that we are not iterating $\mathrm{Re}(P^n)$ and $\mathrm{Im}(P^n)$; we are taking the real and imaginary parts of the iterates of P.

The proofs of our major theorems rely heavily on the following theorem of Böttcher. (See [3] for proofs and related discussion.)

Theorem 5 (Böttcher Theorem)**.** *Let P be a polynomial of degree d. There exists a neighborhood A of ∞ and a univalent map $\varphi:A\to\mathbb{C}$ such that:*

(1) *$B:=\varphi(A)$ is a neighborhood of ∞.*
(2) *$P(\varphi(z))=\varphi(z^d)$, i.e., φ conjugates P to $g(z)=z^d$ as in the commutative diagram*

$$\begin{tikzcd} A \arrow[r, "g(z)=z^d"] \arrow[d, "\varphi"'] & A \arrow[d, "\varphi"] \\ \mathbb{C} \arrow[r, "P"'] & \mathbb{C}. \end{tikzcd}$$

(3) *The map φ is unique up to multiplication by a $(d-1)^{st}$ root of unity, and in the case that the leading coefficient of P is one we can choose φ to satisfy*

$$\lim_{z\to\infty}\frac{\varphi(z)}{z}=1.$$

We will call the plane in which P acts (i.e., the plane containing the domain of φ) the *polynomial plane* for P, whereas the plane in which $g(z) = z^d$ acts (i.e., the plane containing the range of φ) is called the *uniformized plane*. We define the *dynamic ray* $R_\theta(P)$ at $\theta \in \mathbb{R}/\mathbb{Z}$ for a polynomial P in the following way:

$$R_\theta(P) = \varphi(\{re^{2\pi i\theta} : r > a\}),$$

for some constant a depending on the polynomial P. Also, θ corresponds to an angle in the *uniformized plane*. When the context is clear, we will suppress the dependence on P in our notation. Hence, if we define $\sigma_d(\theta) = d \cdot \theta \mod 1$, iterating $P(z)$ on dynamic rays is analogous to iterating σ_d on the corresponding angles. Observe that Theorem 5 implies that the image of $R_\theta(P)$ under P is contained in $R_{d\cdot\theta}(P)$.

We say θ is a *fixed direction* if $P(R_\theta) \subset R_\theta \cup R_{\theta+1/2}$, and θ is an *invariant direction* if $P(R_\theta) \subset R_\theta$. As an example, for a cubic polynomial, $1/2$ is an invariant direction (since $3 \cdot 1/2 = 1/2 \mod 1$), while $1/4$ is a fixed direction but not an invariant direction (since $3 \cdot 1/4 = 3/4 \mod 1$).

We say that two paths $\gamma_1(t)$ and $\gamma_2(t)$ tending to infinity are *parallel at infinity* if $\lim_{t\to\infty} \arg(\gamma_1(t))$ and $\lim_{t\to\infty} \arg(\gamma_2(t))$ exist and are equal. Note that there will be exactly one ray with an endpoint at the origin that is parallel at infinity to R_θ for any θ.

3. Foundational theorems

In this section we prove results that will be used later to analyze the relationships between $U(P)$,$V(P)$, and $K(P)$. Throughout this section, ρ denotes the Euclidean metric, and we will often use it to denote the shortest distance between a point and a set. We prove the following theorems.

Theorem 6. *Let P be a polynomial of degree d, and let ψ be a fixed direction under P. If $z_0 \in B_\infty$ and $P^n(z_0) \notin R_\psi(P)$ for all natural numbers n, then $\limsup_{n\to\infty} \rho(P^n(z_0), R_\psi(P)) = \infty$.*

Proof. Without loss of generality, we may assume that the leading coefficient of $P(z)$ is one so that Condition (3) of Theorem 5 applies. For if $P(z)$ did not have leading coefficient one, we could compose with an affine transformation $\mu(z) = az + b$ so that the polynomial $Q = \mu \circ P \circ \mu^{-1}$ has leading coefficient one and prove the result for Q.

Notice that there is a neighborhood A of ∞ and a univalent map $\varphi : A \mapsto \mathbb{C}$ with the properties of Theorem 5. Since $z_0 \in B_\infty$, we may pass to an appropriate iterate and assume $z_0 \in A$ and hence contained on some external ray $R_\theta(P)$, with $\theta \neq \psi$.

Let L_ψ denote the straight line at angle $2\pi\psi$ with the positive real axis and let γ_θ denote the ray with one endpoint at the origin at angle $2\pi\theta$ with the positive real axis. Let $\zeta_0 = \varphi^{-1}(z_0)$, and $\zeta_n = g^n(\zeta_0) = \zeta_{n-1}^d$. Then $\zeta_n \in \gamma_{\sigma_d^n(\theta)} = \gamma_{d^n\theta}$. Define $I = (\psi - 1/(4d), \psi + 1/(4d))$ and $W = \{z = re^{2\pi it} : t \in I \cup -I \text{ and } r \geq 0\}$. Notice the only elements which remain in

$I \cup -I$ under iteration of σ_d are ψ and $-\psi$. Therefore there is an infinite sequence $(n_k)_{k=1}^{\infty}$ such that $d^{n_k}\theta \notin I \cup -I$ for all k, so $\zeta_{n_k} \notin W$. Consider the angles between L_ψ and the rays from the origin through each ζ_{n_k}. Notice that these angles are bounded below because $\zeta_{n_k} \notin W$. Combine this with the fact that ζ_{n_k} tends to ∞ to see that $\rho(\zeta_{n_k}, L_\psi)$ tends to infinity as k approaches infinity.

Now, because $\lim_{z\to\infty} \frac{\varphi(z)}{z} = 1$, for any angle $2\pi\nu$

$$\lim_{z\to\infty} \arg(R_\nu(P)) = \lim_{r\to\infty} \arg \varphi(re^{2\pi i\nu}) = \lim_{r\to\infty} \arg(re^{2\pi i\nu}) = 2\pi\nu.$$

Therefore R_ν is parallel at infinity to L_ν.

Let $z_{n_k} = \varphi(\zeta_{n_k}) = P^{n_k}(z_0)$. Note that z_{n_k} must lie outside of $\varphi(W)$. Consider the angles between L_ψ and the ray from the origin through each z_{n_k} in the polynomial plane. As in the above discussion for the uniformized plane, these angles are bounded below. And, since R_ψ and L_ψ are parallel at infinity, it follows that $\lim_{k\to\infty} \rho(z_{n_k}, R_\psi) = \infty$. Thus,

$$\limsup_{n\to\infty} \rho(P^n(z_0), R_\psi(P)) = \infty. \qquad \square$$

The following proof will make use of the "little-oh" notation. By $G(t) = o(H(t))$, we mean

$$\lim_{t\to\infty} \frac{G(t)}{H(t)} = 0.$$

Theorem 7. *Let P be a polynomial, and let ψ be an invariant direction under P. Let L_ψ denote the line through the origin which is parallel at infinity to $R_\psi(P)$. If $z_0 \in R_\psi(P)$, then $\limsup_{n\to\infty} \rho(P^n(z_0), L_\psi) < \infty$.*

Proof. Without loss of generality, we will assume $P(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0$. Therefore Condition (3) of Theorem 5 applies, giving us a neighborhood A of ∞ and a univalent map φ from A to $\mathbb{C}$ with $\lim_{z\to\infty} \frac{\varphi(z)}{z} = 1$.

We first prove the result in the case where $\psi = 0$, so that the real axis is the invariant direction. We define a parametrization $\gamma : [a, \infty) \to \mathbb{C}$ of the dynamic ray $R_0(P)$ by $\gamma(t) = \varphi^{-1}(t)$, and we call its real and imaginary parts $x(t)$ and $y(t)$ respectively. Note, by the Böttcher Theorem, that

$$P(\gamma(t)) = \gamma(t^d).$$

We will show that $\lim_{t\to\infty} y(t) = 0$. To do so, we first introduce the argument function $\alpha : [a, \infty)$ by

$$\alpha(t) = \frac{\mathrm{Im}(\gamma(t))}{\mathrm{Re}(\gamma(t))}.$$

Since $\lim_{z\to\infty} \frac{\varphi(z)}{z} = 1$, we find that

$$\frac{x(t)}{t} \to 1 \text{ and } \frac{y(t)}{t} \to 0 \text{ as } t \to \infty.$$

Therefore,

$$\alpha(t) \to 0 \text{ as } t \to \infty. \tag{1}$$

Choose $b > a$ large enough so that for all $t > b$ we have that $x(t^d) > x(t) > |y(t)| \geq 0$.

By way of contradiction, assume that $y(t) \not\to 0$ as $t \to \infty$. Then there exists $\varepsilon > 0$ so that $D = \{t > b : |y(t)| > \varepsilon\}$ is unbounded. In the following, we only consider $t \in D$. We see that

$$\begin{aligned}\gamma(t^d) &= P(\gamma(t)) \\ &= (x(t) + iy(t))^d + a_{d-2}(x(t) + iy(t))^{d-2} + \ldots + a_0 \\ &= (x(t) + iy(t))^d + o(t^{d-1}).\end{aligned}$$

Expanding $(x(t) + iy(t))^d$, and using $\lim_{t\to\infty} \frac{x^{d-2}y}{t^{d-1}} = \lim_{t\to\infty} \frac{x^{d-2}}{t^{d-2}} \cdot \frac{y}{t} = 0$,

$$\begin{aligned}\gamma(t^d) &= (x(t))^d + id(x(t))^{d-1}y(t) + y \cdot o(t^{d-1}) + o(t^{d-1}) \\ &= (x(t))^d + id(x(t))^{d-1}y(t) + (y + 1)o(t^{d-1}).\end{aligned}$$

For simplicity of notation, we will now use $x = x(t)$ and $y = y(t)$. By definition of α,

$$\begin{aligned}|\alpha(t^d)| &= \left| \frac{dx^{d-1}y + (y+1)o(t^{d-1})}{x^d + (y+1)o(t^{d-1})} \right| \\ &= \left| \frac{y}{x} \left(\frac{dx^{d-1} + (1 + \frac{1}{y})o(t^{d-1})}{x^{d-1} + (\frac{y}{x} + \frac{1}{x})o(t^{d-1})} \right) \right|.\end{aligned}$$

Since $t \in D$, we have $1 + \frac{1}{y} \leq 1 + \frac{1}{\varepsilon}$ and $\lim_{t\to\infty} \frac{y+1}{x} = \lim_{t\to\infty} \alpha(t) + \frac{1}{x(t)} = 0$, so

$$\begin{aligned}|\alpha(t^d)| &= \left| \frac{y}{x} \left(\frac{dx^{d-1} + o(t^{d-1})}{x^{d-1} + o(t^{d-1})} \right) \right| \\ &= \left| \frac{y}{x} \left(\frac{d + \frac{o(t^{d-1})}{x^{d-1}}}{1 + \frac{o(t^{d-1})}{x^{d-1}}} \right) \right|.\end{aligned}$$

Since $\lim_{t\to\infty} \frac{x^{d-1}}{t^{d-1}} = 1$ and $\frac{o(t^{d-1})}{x^{d-1}} = \frac{o(t^{d-1})}{t^{d-1}} \cdot \frac{t^{d-1}}{x^{d-1}}$,

$$|\alpha(t^d)| = \left| \frac{y}{x} \left(\frac{d + o(1)}{1 + o(1)} \right) \right|,$$

and for $t \in D$ sufficiently large,

$$|\alpha(t^d)| > \left|\frac{y}{x}\right| = |\alpha(t)|.$$

However, if $|\alpha(t^d)| > |\alpha(t)|$ and $x(t^d) > x(t)$, then $|y(t^d)| > \varepsilon$ (since $|y(t^d)| = |\alpha(t^d)|x(t^d) > |\alpha(t)|x(t) = |y(t)|$). An inductive argument shows that $|\alpha(t)| < |\alpha(t^d)| < \cdots < |\alpha(t^{d^n})| < \cdots$, contradicting that $\alpha(t) \to 0$. Therefore, we conclude that $y(t) \to 0$ as $t \to \infty$. Thus, although R_0 is not necessarily equal to the real axis, it is parallel at infinity to the real axis $= L_0$. Therefore, the function $z \mapsto \rho(z, L_0)$ is bounded for all $z \in R_0$. Since $P^n(z_0) \in R_0$ whenever $z_0 \in R_0$, this means that $\rho(P^n(z_0), L_0)$ is bounded for all n.

If $\psi \neq 0$, consider instead the function $Q(z) = e^{-i\psi}P(e^{i\psi/d}z)$ which has a leading coefficient of one. The proof above shows that $\operatorname{Im}(Q(t)) \to 0$ as $t \to \infty$, which is equivalent to saying that $\arg(P(e^{i\psi/d}t)) \to e^{i\psi}$. Furthermore, since ψ is an invariant direction under P, $\arg(P(t)) \to e^{i\psi}$ as $t \to \infty$. Thus $\rho(P^n(z_0), L_\psi)$ is bounded for all n, and $\limsup_{n\to\infty} \rho(P^n(z_0), L_\psi)$ is also bounded. $\square$

Corollary 8. *Let P be a polynomial, and let ψ be a fixed direction under P. Let L_ψ denote the line through the origin which is parallel at infinity to R_ψ. If $z_0 \in R_\psi(P)$, then $\limsup_{n\to\infty} \rho(P^n(z_0), L_\psi) < \infty$.*

Proof. Since ψ is a fixed direction under P, the dynamic rays $R_\psi(P)$ and $R_{\psi+1/2}(P)$ are invariant under P^2. By the Theorem 7, if $z_0 \in R_\psi$ then

$$\limsup_{n\to\infty} \rho(P^{2n}(z_0), L_\psi) < \infty.$$

Further, $P(z_0) \in R_\psi \cup R_{\psi+1/2}$, so

$$\limsup_{n\to\infty} \rho(P^{2n}(P(z_0)), L_\psi) < \infty.$$

Combining these bounds, we see that $\limsup_{n\to\infty} \rho(P^n(z_0), L_\psi) < \infty$. $\square$

4. Relationship between $U(P)$, $V(P)$ and $K(P)$

In this section, we describe the sets $U(P)$ and $V(P)$, as well as how they relate to $K(P)$.

Theorem 9. *Let P be a polynomial. Then $K(P) = U(P) \cap V(P)$.*

Proof. By the triangle inequality,

$$|P^n(z)| \leq |\operatorname{Re}(P^n(z))| + |\operatorname{Im}(P^n(z))| \leq 2|P^n(z)|$$

so $(P^n(z))_{n=1}^\infty$ is bounded if and only if $(\operatorname{Re}(P^n(z)))_{n=1}^\infty$ and $(\operatorname{Im}(P^n(z)))_{n=1}^\infty$ are both bounded. $\square$

Lemma 10. *Let P be a polynomial. If ψ is a fixed direction for P and R_ψ is parallel at infinity to the real (or imaginary) axis, then $R_\psi \subset V(P)$ (or $U(P)$ respectively).*

Proof. If $z \in R_\psi$, then Corollary 8 implies that $\limsup_{n\to\infty} \rho(P^n(z), \mathbb{R}) < \infty$; this is exactly that $\mathrm{Im}(P^n(z))$ is bounded for all n. The proof for $U(P)$ is analogous. □

Theorem 11. *Let P be a polynomial. Let R_ψ denote the dynamic ray parallel at infinity to the positive real axis and R_θ denote the dynamic ray parallel at infinity to the positive imaginary axis.*

(1) *If θ is not a fixed direction, then $U(P) = K(P)$.*
(2) *If θ is a fixed direction, then*

$$U(P) = K(P) \cup \bigcup_{n=0}^{\infty} P^{-n}(R_\theta \cup R_{\theta+1/2}).$$

(3) *If ψ is not a fixed direction, then $V(P) = K(P)$.*
(4) *If ψ is a fixed direction, then*

$$V(P) = K(P) \cup \bigcup_{n=0}^{\infty} P^{-n}(R_\psi \cup R_{\psi+1/2}).$$

Proof. We will prove the claims relevant to $U(P)$, since the arguments for $V(P)$ are analogous.

Suppose first that θ is not a fixed direction, and let $z_0 \notin K(P)$. By way of contradiction, suppose that $\mathrm{Re}(P^n(z_0))$ is bounded as $n \to \infty$, i.e., $\{P^n(z_0) : n \in \mathbb{N}\}$ lies in some vertical strip $S = \{\mathrm{Re}(z) < M_1\}$ for some $M_1 > 0$. Let $I = [\alpha, \beta] \cup [\alpha + 1/2, \beta + 1/2]$ be a neighborhood of $\{\theta, \theta + 1/2\}$ such that $\sigma_d(I) \cap I = \emptyset$. Notice that every dynamic ray in S converging to infinity is parallel at infinity to the imaginary axis. Let $\nu \notin \{\theta, \theta + 1/2\}$. Then R_ν cannot be parallel at infinity to the imaginary axis because dynamic rays parallel at infinity are unique. Thus $R_\nu \cap S$ is bounded for all such ν, and the only dynamic rays intersecting S in an unbounded set are R_θ and $R_{\theta+1/2}$.

We define the set $A = \cup_{\nu \notin I} R_\nu$. Note that the boundary of A is $\partial A = R_\alpha \cup R_\beta \cup R_{\alpha+1/2} \cup R_{\beta+1/2}$. Therefore $\partial A \cap S$ must be bounded. Thus there is some M_2 such that $\partial A \cap S$ is contained in the disk of radius M_2 centered at the origin, $D(0, M_2)$. Suppose that $(\mathbb{C} \backslash D(0, M_2)) \cap S \cap A$ is nonempty. Then ∂A must have a nonempty intersection with $(\mathbb{C} \backslash D(0, M_2))$ and S. But this contradicts the definition of M_2. Thus $(\mathbb{C} \backslash D(0, M_2)) \cap S \cap A$ is empty. Since the orbit of z_0 is unbounded and by assumption contained in S, there is some k such that $|P^k(z_0)| > M_2$. Since $P^k(z_0) \in S$, it is not in A, so is on a dynamic ray R_ξ for some $\xi \in I$. Therefore, $P^{k+1}(z_0)$ lies on the dynamic ray $R_{d \cdot \xi}$. By definition of I, we know that $d \cdot \xi$ is not in I, thus $P^{k+1}(z_0) \in A$. Additionally, $|P^{k+1}(z_0)| > M_2$, so $P^{k+1}(z_0)$ is not in the strip S, contradicting the assumption that $\mathrm{Re}(P^n(z_0))$ is bounded. Therefore, $U(P) = K(P)$.

Now suppose that θ is a fixed direction. By Theorem 6, the orbit of any point which does not eventually map into R_θ or $R_{\theta+1/2}$ has unbounded real

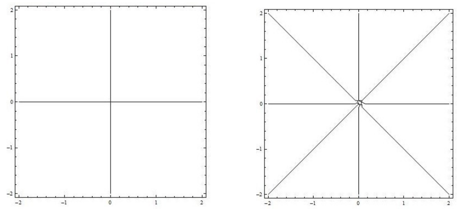

FIGURE 3. The zero level curves for $\mathrm{Im}(f_0)$ and $\mathrm{Im}(f_0^2)$.

part. However, by Corollary 8, the orbit of any point which maps into R_θ or $R_{\theta+1/2}$ has bounded real part. Therefore, $U(P)$ is the union of $K(P)$ together with all points eventually mapping into $R_\theta \cup R_{\theta+1/2}$. □

Example 12. We consider the well-known family of functions $f_c(z) = z^2+c$. The functions $f_c(z)$ are conjugate to z^2 by the Böttcher Theorem. Under z^2, the real axis is a fixed direction, and the imaginary axis is not. After conjugating back to z^2+c by $\varphi(z)$, the real axis is parallel at infinity to R_0, and is therefore a fixed direction for $f_c(z)$ as well. Similarly, the imaginary axis is parallel at infinity to $R_{1/4}$, which is not a fixed direction because it maps to $R_{1/2}$ under $z^2 + c$. Thus, by Theorem 11, $U(f_c) = K(f_c)$ and $V(f_c) \neq K(f_c)$.

Example 13. In particular, a direct application of Theorem 11 to the function $f_0(z) = z^2$ shows that

$$V(f_0) = K(f_0) \cup \{re^{2\pi i\theta} : \theta = k/2^n \text{ for } k, n \in \mathbb{N}\}.$$

Note that V_0 is dense in $\mathbb{C}_\infty$. One way to see this description of $V(f_0)$ is that the zero level curves for $\mathrm{Im}(f_0^n)$ correspond to the points which eventually land on the real axis. The level curves for $\mathrm{Im}(f_0)$ and $\mathrm{Im}(f_0^2)$ are illustrated in Figure 3.

Example 14. For monic centered polynomials, that is, polynomials of the form

$$P(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0,$$

we have the following results about $U(P)$ and $V(P)$. For all values of d, $V(P) \neq K(P)$ since the real axis is a fixed direction as described in Example 12. For even values of d, $U(P) = K(P)$, again by the same reasoning used in Example 12. For odd values of d, the imaginary axis is a fixed direction, so $U(P) \neq K(P)$.

5. Illustrations and conclusions

In this section, we compare the graphs of $\mathrm{Re}(P^k(z))$ and $\mathrm{Im}(P^k(z))$ to the filled Julia set $K(P)$ for some familiar functions P. We also outline the differences between $\mathrm{Re}(f_c^n)$ and $\mathrm{Im}(f_c^n)$.

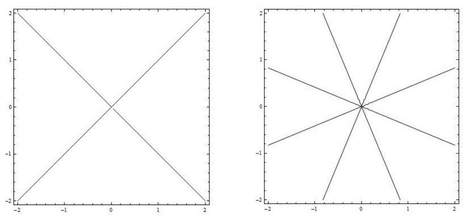

FIGURE 4. The zero level curves for $\mathrm{Re}(f_0)$ and $\mathrm{Re}(f_0^2)$.

The contour diagrams of $\mathrm{Re}(P^k(z))$ and $\mathrm{Im}(P^k(z))$, along with the corresponding filled Julia set $K(P)$, are presented in Figure 5 for a few common polynomials. The contour plots show the contour lines for function values of -2 and 2, with the shading indicating relative height of the regions in between the curves. The alternating shading indicates that the height of the surface oscillates away from the filled Julia set. Notice that the filled Julia sets are beginning to appear in the center of these images, even after a small number of iterations. Though they look very similar, the contour plots of the real and imaginary parts of the iterates are not identical.

Comparing Figures 3 and 4, we see the zero level curves behave differently for the real and imaginary parts of the iterates of f_0. We noted in Example 13 that the zero level curves of $\mathrm{Im}(f_0^n)$ contain those of $\mathrm{Im}(f_0^{n-1})$, giving us an infinite number of external rays that are contained in $V(f_c) \setminus K(f_c)$. Figure 4 illustrates that this is not the case for the real part. In particular, points along the zero level curves of $\mathrm{Re}(f_0)$ are not on the zero level curves of $\mathrm{Re}(f_0^2)$, indicating that, when $\mathrm{Re}(f_0(z))$ is zero, $\mathrm{Re}(f_0^2(z))$ is not zero. Extending this reasoning to further iterates, we see that if $z \notin K(f_0)$ then $\mathrm{Re}(f_0^n(z))$ does not remain bounded.

As we have seen in this paper, the filled Julia sets for polynomials are closely related to the sets $U(P)$ and $V(P)$, which correspond to the real and imaginary parts of the iterates of P. For polynomials of odd degree with leading coefficient of one, $U(P) \neq K(P)$ and $V(P) \neq K(P)$. For polynomials of even degree with leading coefficient of one, $U(P) = K(P)$ and $V(P) = K(P)$. In particular, this means that for the well-known family of functions $f_c(z) = z^2 + c, U(f_c) = K(f_c)$ and $V(f_c) \neq K(f_c)$ even though the contour plots for the real and imaginary parts of the iterates of $f_c(z)$ seem to be the same and the filled Julia set seems to appear in the centers of the contour plots.

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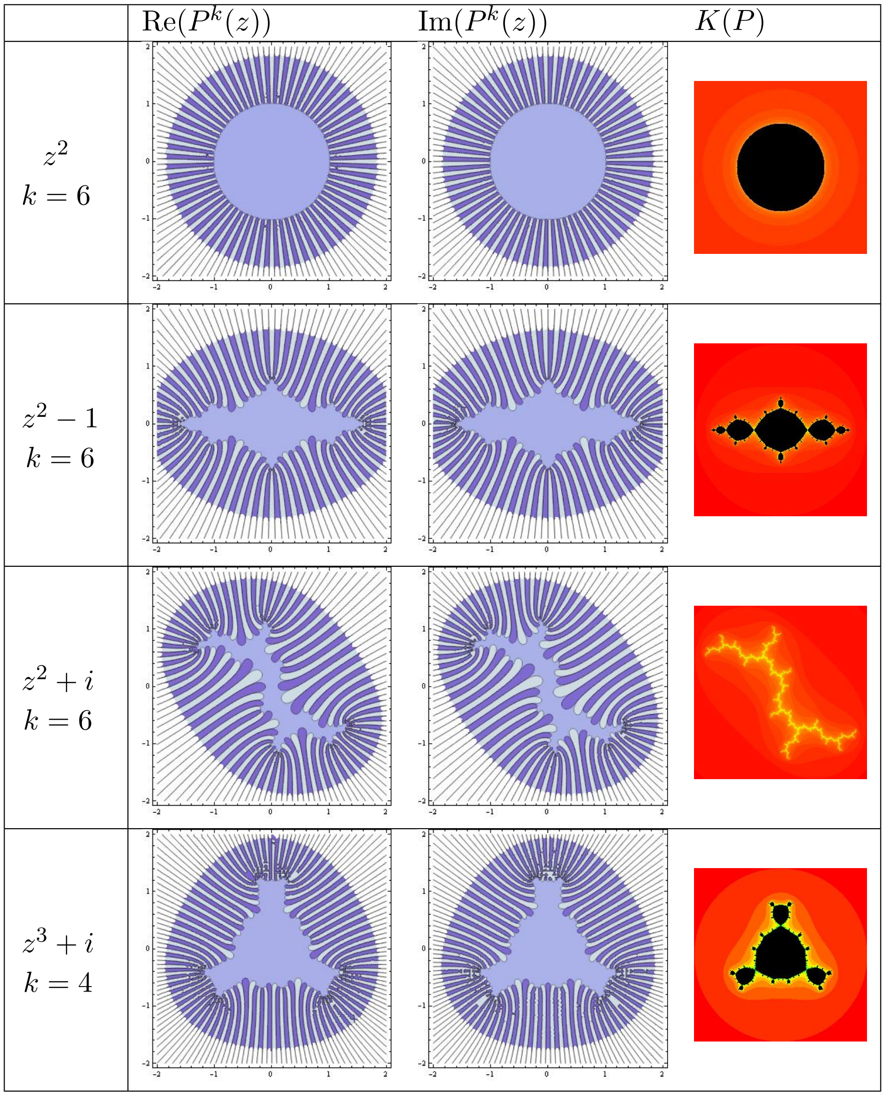

FIGURE 5. *Mathematica* images of contour plots for $\mathrm{Re}(P^k(z))$ and $\mathrm{Im}(P^k(z))$; *FracTool* images of filled Julia sets for the corresponding functions

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Western Carolina University
jbarnes@email.wcu.edu

Stony Brook University
clintonc@math.sunysb.edu

United States Air Force Academy
Beth.Schaubroeck@usafa.edu

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