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The A-polynomial of the $(-2, 3, 3 + 2n)$ pretzel knots

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Abstract. We show that the A-polynomial A_n of the 1-parameter family of pretzel knots $K_n = (-2, 3, 3+2n)$ satisfies a linear recursion relation of order 4 with explicit constant coefficients and initial conditions. Our proof combines results of Tamura–Yokota and the second author. As a corollary, we show that the A-polynomial of K_n and the mirror of K_{-n} are related by an explicit $\mathrm{GL}(2, \mathbb{Z})$ action. We leave open the question of whether or not this action lifts to the quantum level.

Contents

1. Introduction

1.1. The behavior of the A-polynomial under filling. In [CCGLS94], the authors introduced the A-polynomial A_W of a hyperbolic 3-manifold W with one cusp. It is a 2-variable polynomial which describes the dependence

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of the eigenvalues of a meridian and longitude under any representation of $\pi_1(W)$ into $\mathrm{SL}(2,\mathbb{C})$. The A-polynomial plays a key role in two problems:

- the deformation of the hyperbolic structure of W,
- the problem of exceptional (i.e., nonhyperbolic) fillings of W.

Knowledge of the A-polynomial (and often, of its Newton polygon) is translated directly into information about the above problems, and vice-versa. In particular, as demonstrated by Boyer and Zhang [BZ01], the Newton polygon is dual to the fundamental polygon of the Culler–Shalen seminorm [CGLS87] and, therefore, can be used to classify cyclic and finite exceptional surgeries.

In [Gar10], the first author observed a pattern in the behavior of the A-polynomial (and its Newton polygon) of a 1-parameter family of 3-manifolds obtained by fillings of a 2-cusped manifold. To state the pattern, we need to introduce some notation. Let $K = \mathbb{Q}(x_1, \dots, x_r)$ denote the field of rational functions in r variables $x_1, \dots, x_r$.

Definition 1.1. We say that a sequence of rational functions $R_n \in K$ (defined for all integers n) is *holonomic* if it satisfies a linear recursion with constant coefficients. In other words, there exists a natural number d and $c_k \in K$ for $k = 0, \dots, d$ with $c_d c_0 \neq 0$ such that for all integers n we have:

$$\sum_{k=0}^{d} c_k R_{n+k} = 0. \tag{1}$$

Depending on the circumstances, one can restrict attention to sequences indexed by the natural numbers (rather than the integers).

Consider a hyperbolic manifold W with two cusps C_1 and C_2. Let (μ_i, λ_i) for $i = 1, 2$ be pairs of meridian-longitude curves, and let W_n denote the result of $-1/n$ filling on C_2. Let $A_n(M_1, L_1)$ denote the A-polynomial of W_n with the meridian-longitude pair inherited from W.

Theorem 1.1 ([Gar10])**.** *With the above conventions, there exists a holonomic sequence $R_n(M_1, L_1) \in \mathbb{Q}(M_1, L_1)$ such that for all but finitely many integers n, $A_n(M_1, L_1)$ divides the numerator of $R_n(M_1, L_1)$. In addition, a recursion for R_n can be computed explicitly via elimination, from an ideal triangulation of W.*

1.2. The Newton polytope of a holonomic sequence. Theorem 1.1 motivates us to study the Newton polytope of a holonomic sequence of Laurent polynomials. To state our result, we need some definitions. Recall that the *Newton polytope* of a Laurent polynomial in n variables $x_1, \dots, x_n$ is the convex hull of the points whose coordinates are the exponents of its monomials. Recall that a *quasi-polynomial* is a function $p : \mathbb{N} \longrightarrow \mathbb{Q}$ of the form $p(n) = \sum_{k=0}^{d} c_k(n) n^k$ where $c_k : \mathbb{N} \longrightarrow \mathbb{Q}$ are periodic functions. When $c_d \neq 0$, we call d the *degree* of $p(n)$. We will call quasi-polynomials of degree at most one (resp. two) *quasi-linear* (resp. *quasi-quadratic*). Quasi-polynomials appear in lattice point counting problems (see [Ehr62, CW10]), in the Slope Conjecture in quantum topology (see [Gar11b]), in enumerative combinatorics (see [Gar11a]) and also in the A-polynomial of filling families of 3-manifolds (see [Gar10]).

Definition 1.2. We say that a sequence N_n of polytopes is linear (resp. quasi-linear) if the coordinates of the vertices of N_n are polynomials (resp. quasi-polynomials) in n of degree at most one. Likewise, we say that a sequence N_n of polytopes is quadratic (resp. quasi-quadratic) if the coordinates of the vertices of N_n are polynomials (resp. quasi-polynomials) of degree at most two.

Theorem 1.2 ([Gar10]). *Let N_n be the Newton polytope of a holonomic sequence $R_n \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$. Then, for all but finitely many integers n, N_n is quasi-linear.*

1.3. Do favorable links exist? Theorems 1.1 and 1.2 are general, but in favorable circumstances more is true. Namely, consider a family of knot complements K_n, obtained by $-1/n$ filling on a cusp of a 2-component hyperbolic link J. Let f denote the linking number of the two components of J, and let A_n denote the A-polynomial of K_n with respect to its canonical meridian and longitude (M, L). By definition, A_n contains all components of irreducible representations, but *not* the component $L-1$ of abelian representations.

Definition 1.3. We say that J, a 2-component link in 3-space, with linking number f is *favorable* if $A_n(M, LM^{-f^2 n}) \in \mathbb{Q}[M^{\pm 1}, L^{\pm 1}]$ is holonomic.

The shift of coordinates, $LM^{-f^2 n}$, above is due to the canonical meridian-longitude pair of K_n differing from the corresponding pair for the unfilled component of J as a result of the nonzero linking number. Theorem 1.2 combined with the above shift implies that, for a favorable link, the Newton polygon of K_n is quasi-quadratic.

Hoste–Shanahan studied the first examples of a favorable link, the *Whitehead link* and its *half-twisted version* (see Figure 1), and consequently gave an explicit recursion relation for the 1-parameter families of A-polynomials of twist knots $K_{2,n}$ and $K_{3,n}$ respectively; see [HS04].

The goal of our paper is to give another example of a favorable link J (see Figure 1), whose 1-parameter filling gives rise to the family of $(-2,3,3+2n)$ *pretzel knots*. Our paper is a concrete illustration of the general Theorems 1.1 and 1.2 above. Aside from this, the 1-parameter family of knots K_n, where K_n is the $(-2,3,3+2n)$ pretzel knot, is well-studied in hyperbolic geometry (where K_n and the mirror of K_{-n} are pairs of geometrically similar knots; see [BH96, MM08]), in exceptional Dehn surgery (where for instance $K_2 = (-2,3,7)$ has three Lens space fillings 1/0, 18/1 and 19/1; see [CGLS87]) and in Quantum Topology (where K_n and the mirror of K_{-n} have different Kashaev invariant, equal volume, and different subleading corrections to the volume, see [GZ]).

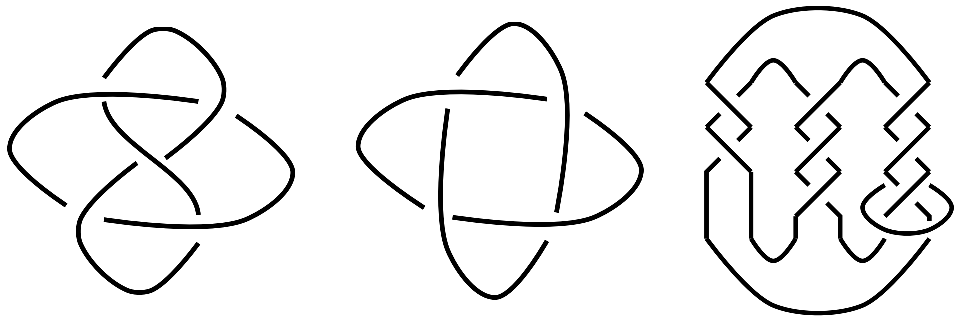

FIGURE 1. The Whitehead link on the left, the half-twisted Whitehead link in the middle and our seed link J at right.

The success of Theorems 1.3 and 1.4 below hinges on two independent results of Tamura–Yokota and the second author [TY04, Mat02], and an additional lucky coincidence. Tamura–Yokota compute an explicit recursion relation, as in Theorem 1.3, by elimination, using the gluing equations of the decomposition of the complement of J into six ideal tetrahedra; see [TY04]. The second author computes the Newton polygon N_n of the A-polynomial of the family K_n of pretzel knots; see [Mat02]. This part is considerably more difficult, and requires:

(a) The set of boundary slopes of K_n, which are available by applying the Hatcher–Oertel algorithm [HO89, Dun01] to the 1-parameter family K_n of Montesinos knots. The four slopes given by the algorithm are candidates for the slopes of the sides of N_n. Similarly, the fundamental polygon of the Culler–Shalen seminorm of K_n has vertices in rays which are the multiples of the slopes of N_n. Taking advantage of the duality of the fundamental polygon and Newton polygon, in order to describe N_n it is enough to determine the vertices of the Culler–Shalen polygon.

(b) Use of the exceptional $1/0$ filling and two fortunate exceptional Seifert fillings of K_n with slopes $4n + 10$ and $4n + 11$ to determine exactly the vertices of the Culler–Shalen polygon and consequently N_n. In particular, the boundary slope 0 is not a side of N_n (unless $n = -3$) and the Newton polygon is a hexagon for all hyperbolic K_n.

Given the work of [TY04] and [Mat02], if one is lucky enough to match N_n of [Mat02] with the Newton polygon of the solution of the recursion relation of [TY04] (and also match a leading coefficient), then Theorem 1.3 below follows; i.e., J is a favorable link.

1.4. Our results for the pretzel knots K_n. Let $A_n(M, L)$ denote the A-polynomial of the pretzel knot K_n, using the canonical meridian-longitude coordinates. Consider the sequences of Laurent polynomials $P_n(M, L)$ and

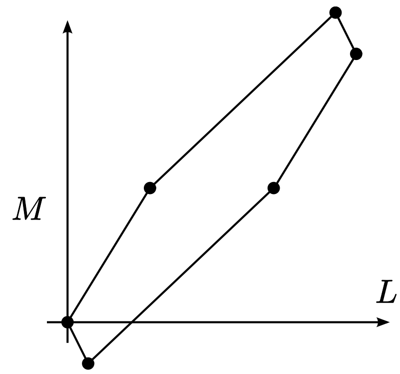

FIGURE 2. The Newton polygon NP_n.

$Q_n(M,L)$ defined by:

$$P_n(M,L) = A_n(M, LM^{-4n}) \tag{2}$$

for $n > 1$ and

$$Q_n(M,L) = A_n(M, LM^{-4n})M^{-4(3n^2+11n+4)} \tag{3}$$

for $n < -2$ and $Q_{-2}(M,L) = A_{-2}(M, LM^{-8})M^{-20}$. In the remaining cases $n = -1, 0, 1$, the knot K_n is not hyperbolic (it is the torus knot 5_1, 8_{19} and 10_{124} respectively), and one expects exceptional behavior. This is reflected in the fact that P_n for $n = 0, 1$ and Q_n for $n = -1, 0$ can be defined to be suitable rational functions (rather than polynomials) of M, L. Let NP_n and NQ_n denote the Newton polygons of P_n and Q_n respectively.

Theorem 1.3.

(a) *P_n and Q_n satisfy linear recursion relations*

$$\sum_{k=0}^{4} c_k P_{n+k} = 0, \qquad n \geq 0 \tag{4}$$

and

$$\sum_{k=0}^{4} c_k Q_{n-k} = 0, \qquad n \leq 0 \tag{5}$$

where the coefficients c_k and the initial conditions P_n for $n = 0, \dots, 3$ and Q_n for $n = -3, \dots, 0$ are given in Appendix A.

(b) *In (L, M) coordinates, NP_n and NQ_n are hexagons with vertices*

$$\{\{0,0\}, \{1, -4n+16\}, \{n-1, 12n-12\}, \{2n+1, 16n+18\}, \{3n-1, 32n-10\}, \{3n, 28n+6\}\} \tag{6}$$

for P_n with $n > 1$ and

(7) $$\{\{0, 4n+28\}, \{1, 38\}, \{-n, -12n+26\}, \{-2n-3, -16n-4\}, \{-3n-4, -28n-16\}, \{-3n-3, -32n-6\}\}$$

for Q_n with $n < -1$.

Remark 1.4. We can give a single recursion relation valid for $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$ as follows. Define

$$R_n(M, L) = A_n(M, LM^{-4n}) b^{|n|} \epsilon_n(M), \tag{8}$$

where

$$b = \frac{1}{LM^8(1-M^2)(1+LM^{10})}, \tag{9}$$

$$c = \frac{L^3M^{12}(1-M^2)^3}{(1+LM^{10})^3},$$

$$\epsilon_n(M) = \begin{cases} 1 & \text{if } n > 1 \\ cM^{-4(3+n)(2+3n)} & \text{if } n < -2 \\ cM^{-28} & \text{if } n = -2. \end{cases}$$

Then, R_n satisfies the palindromic fourth order linear recursion

$$\sum_{k=0}^{4} \gamma_k R_{n+k} = 0 \tag{10}$$

where the coefficients γ_k and the initial conditions R_n for $n = 0, \ldots, 3$ are given in Appendix B. Moreover, R_n is related to P_n and Q_n by:

$$R_n = \begin{cases} P_n b^{|n|} & \text{if } n \geq 0 \\ Q_n b^{|n|} c M^{-8} & \text{if } n \leq 0. \end{cases} \tag{11}$$

Remark 1.5. The computation of the Culler–Shalen seminorm of the pretzel knots K_n has an additional application, namely it determines the number of components (containing the character of an irreducible representation) of the $\mathrm{SL}(2, \mathbb{C})$ character variety of the knot, and consequently the number of factors of its A-polynomial. In the case of K_n, (after translating the results of [Mat02] for the pretzel knots $(-2, 3, n)$ to the pretzel knots $(-2, 3, 3+2n)$) it was shown by the second author [Mat02, Theorem 1.6] that the character variety of K_n has one (resp. two) components when 3 does not divide n (resp. divides n). The nongeometric factor of A_n is given by

$$\begin{cases} 1 - LM^{4(n+3)} & n \geq 3 \\ L - M^{-4(n+3)} & n \leq -3 \end{cases}$$

for $n \neq 0$ a multiple of 3.

Since the A-polynomial has even powers of M, we can define the B-polynomial by

$$B(M^2, L) = A(M, L).$$

Our next result relates the A-polynomials of the geometrically similar pair $(K_n, -K_{-n})$ by an explicit $\mathrm{GL}(2,\mathbb{Z})$ transformation.

Theorem 1.4. *For $n > 1$ we have:*

$$B_{-n}(M, LM^{2n-5}) = (-L)^n M^{3(2n^2-7n+7)} B_n(-L^{-1}, L^{2n+5}M^{-1})\eta_n \tag{12}$$

where $\eta_n = 1$ (resp. M^{22}) when $n > 2$ (resp. $n = 2$).

2. Proofs

2.1. The equivalence of Theorem 1.3 and Remark 1.4. In this subsection we will show the equivalence of Theorem 1.3 and Remark 1.4. Let $\gamma_k = c_k/b^k$ for $k = 0, \dots, 4$ where b is given by (9). It is easy to see that the γ_k are given explicitly by Appendix B, and moreover, they are palindromic. Since $R_n = P_n b^n$ for $n = 0, \dots, 3$ it follows that R_n and $P_n b^n$ satisfy the same recursion relation (10) for $n \geq 0$ with the same initial conditions. It follows that $R_n = P_n b^n$ for $n \geq 0$.

Solving (10) backwards, we can check by an explicit calculation that $R_n = Q_n b^{|n|} c M^{-8}$ for $n = -3, \dots, 0$ where b and c are given by (9). Moreover, R_n and $Q_n b^{|n|} c M^{-8}$ satisfy the same recursion relation (10) for $n < 0$. It follows that $R_n = Q_n b^{|n|} c M^{-8}$ for $n < 0$. This concludes the proof of Equations (10) and (11).

2.2. Proof of Theorem 1.3. Let us consider first the case of $n \geq 0$, and denote by P'_n for $n \geq 0$ the unique solution to the linear recursion relation (4) with the initial conditions as in Theorem 1.3. Let $R'_n = P'_n b^n$ be defined according to Equation (11) for $n \geq 0$.

Remark 1.4 implies that R'_n satisfies the recursion relation of [TY04, Thm. 1]. By [TY04, Thm. 1], $A_n(M, LM^{-4n})$ divides $P'_n(M, L)$ when $n > 1$.

Next, we claim that the Newton polygon NP'_n of $P'_n(M, L)$ is given by (6). This can be verified easily by induction on n.

Next, in [Mat02, p. 1286], the second author computes the Newton polygon N_n of the $A_n(M, L)$. It is a hexagon given in (L, M) coordinates by

$$\{\{0,0\},\{1,16\},\{n-1,4(n^2+2n-3)\},\{2n+1,2(4n^2+10n+9)\},$$
$$\{3n-1,2(6n^2+14n-5)\},\{3n,2(6n^2+14n+3)\}\}$$

when $n > 1$,

$$\{\{-3n-4,0\},\{-3(1+n),10\},\{-3-2n,4(3+4n+n^2)\},$$
$$\{-n,2(4n^2+16n+21)\},\{0,4(3n^2+12n+11)\},\{1,6(2n^2+8n+9)\}\}$$

when $n < -2$ and

$$\{\{0,0\},\{1,0\},\{2,4\},\{1,10\},\{2,14\},\{3,14\}\}$$

when $n = -2$. Notice that the above 1-parameter families of Newton polygons are quadratic. It follows by explicit calculation that the Newton polygon of $A_n(M, LM^{-4n})$ is quadratic and exactly agrees with NP'_n for all $n > 1$.

By the above discussion, $P_n(M, L)$ is a rational mutiple of $A_n(M, LM^{-4n})$. Since their leading coefficients (with respect to L) agree, they are equal. This proves Theorem 1.3 for $n > 1$. The case of $n < -1$ is similar. □

2.3. Proof of Theorem 1.4. Using Equations (2) and (3), convert Equation (12) into

$$Q_{-n}(\sqrt{M}, L/M^5) = (-L)^n M^{n+13} P_n(i\sqrt{L}, L^5/M). \tag{13}$$

Note that, under the substitution $(M, L) \mapsto (i/\sqrt{L}, L^{2n+5}/M)$, LM^{4n} becomes L^5/M. Similarly, LM^{-4n} becomes L/M^5 under the substitution $(M, L) \mapsto (\sqrt{M}, LM^{2n-5})$.

It is straightforward to verify equation (13) for $n = 2, 3, 4, 5$. For $n \geq 6$, we use induction. Let c_k^- denote the result of applying the substitutions $(M, L) \mapsto (\sqrt{M}, L/M^5)$ to the c_k coefficients in the recursions (4) and (5). For example,

$$c_0^- = \frac{L^4(1+L)^4(1-M)^4}{M^2}.$$

Similarly, define c_k^+ to be the result of the substitution

$$(M, L) \mapsto (i/\sqrt{L}, L^5/M)$$

to c_k. It is easy to verify that for $k = 0, 1, 2, 3$,

$$\frac{c_k^-}{c_4^-}(-LM)^{k-4} = \frac{c_k^+}{c_4^+}.$$

Then,

$$\begin{aligned}
Q_{-n}(\sqrt{M}, L/M^5) &= -\frac{1}{c_4^-}\sum_{k=0}^{3} c_k^- Q_{-n+4-k}(\sqrt{M}, L/M^5) \\
&= -\frac{1}{c_4^-}\sum_{k=0}^{3} c_k^- (-L)^{n-4+k} M^{n-4+k+13} P_{n-4+k}(i\sqrt{L}, L^5/M) \\
&= -(-L)^n M^{n+13} \sum_{k=0}^{3} \frac{c_k^-}{c_4^-}(-LM)^{k-4} P_{n-4+k}(i\sqrt{L}, L^5/M) \\
&= -(-L)^n M^{n+13} \sum_{k=0}^{3} \frac{c_k^+}{c_4^+} P_{n-4+k}(i\sqrt{L}, L^5/M) \\
&= (-L)^n M^{n+13} P_n(i\sqrt{L}, L^5/M).
\end{aligned}$$

By induction, Equation (13) holds for all $n > 1$ proving Theorem 1.4. □

Appendix A. The coefficients c_k and the initial conditions for P_n and Q_n

$$
\begin{aligned}
c_4 &= M^4\\
c_3 &= 1+M^4+2LM^{12}+LM^{14}-LM^{16}+L^2M^{20}-L^2M^{22}-2L^2M^{24}\\
&\quad -L^3M^{32}-L^3M^{36}\\
c_2 &= (-1+LM^{12})(-1-2LM^{10}-3LM^{12}+2LM^{14}-L^2M^{16}+2L^2M^{18}\\
&\quad -4L^2M^{20}-2L^2M^{22}+3L^2M^{24}-3L^3M^{28}+2L^3M^{30}+4L^3M^{32}\\
&\quad -2L^3M^{34}+L^3M^{36}-2L^4M^{38}+3L^4M^{40}+2L^4M^{42}+L^5M^{52})\\
c_1 &= -L^2(-1+M)^2M^{16}(1+M)^2(1+LM^{10})^2(-1-M^4-2LM^{12}-LM^{14}\\
&\quad +LM^{16}-L^2M^{20}+L^2M^{22}+2L^2M^{24}+L^3M^{32}+L^3M^{36})\\
c_0 &= L^4(-1+M)^4M^{36}(1+M)^4(1+LM^{10})^4
\end{aligned}
$$

$$
\begin{aligned}
P_0 &= \frac{\left(-1+LM^{12}\right)\left(1+LM^{12}\right)^2}{\left(1+LM^{10}\right)^3}\\
P_1 &= \frac{\left(-1+LM^{11}\right)^2\left(1+LM^{11}\right)^2}{1+LM^{10}}\\
P_2 &= -1+LM^8-2LM^{10}+LM^{12}+2L^2M^{20}+L^2M^{22}-L^4M^{40}\\
&\quad -2L^4M^{42}-L^5M^{50}+2L^5M^{52}-L^5M^{54}+L^6M^{62}\\
P_3 &= (-1+LM^{12})(-1+LM^4-LM^6+2LM^8-5LM^{10}+LM^{12}\\
&\quad +5L^2M^{16}-4L^2M^{18}+L^2M^{22}+L^3M^{26}+3L^3M^{30}+2L^3M^{32}\\
&\quad -2L^4M^{36}-3L^4M^{38}+3L^4M^{40}+2L^4M^{42}-2L^5M^{46}-3L^5M^{48}\\
&\quad -L^5M^{52}-L^6M^{56}+4L^6M^{60}-5L^6M^{62}-L^7M^{66}+5L^7M^{68}\\
&\quad -2L^7M^{70}+L^7M^{72}-L^7M^{74}+L^8M^{78})
\end{aligned}
$$

$$
\begin{aligned}
Q_0 &= -\frac{\left(-1+LM^{12}\right)\left(1+LM^{12}\right)^2}{L^3(-1+M)^3M^4(1+M)^3}\\
Q_{-1} &= -\frac{M^{12}\left(1+LM^{14}\right)^2}{L(-1+M)(1+M)}\\
Q_{-2} &= M^{20}(1-LM^8+2LM^{10}+2LM^{12}-LM^{16}+LM^{18}+L^2M^{20}\\
&\quad -L^2M^{22}+2L^2M^{26}+2L^2M^{28}-L^2M^{30}+L^3M^{38})\\
Q_{-3} &= M^{16}(-1+LM^{12})(1+LM^{10}+5LM^{12}-LM^{14}-2LM^{16}+2LM^{18}\\
&\quad -LM^{20}+2L^2M^{20}+LM^{22}+4L^2M^{22}+3L^2M^{26}-3L^2M^{28}
\end{aligned}
$$

$$- L^3M^{28} + 5L^3M^{30} + 5L^2M^{32} - L^2M^{34} - 3L^3M^{34} + 3L^3M^{36} + 4L^3M^{40} + L^4M^{40} + 2L^3M^{42} - L^4M^{42} + 2L^4M^{44} - 2L^4M^{46} - L^4M^{48} + 5L^4M^{50} + L^4M^{52} + L^5M^{62})$$

Appendix B. The coefficients γ_k and the initial conditions for R_n

$$\gamma_4 = L^4(-1+M)^4M^{36}(1+M)^4\left(1+LM^{10}\right)^4$$

$$\gamma_3 = L^3(-1+M)^3M^{24}(1+M)^3\left(1+LM^{10}\right)^3(-1-M^4-2LM^{12}-LM^{14} + LM^{16} - L^2M^{20} + L^2M^{22} + 2L^2M^{24} + L^3M^{32} + L^3M^{36})$$

$$\gamma_2 = L^2(-1+M)^2M^{16}(1+M)^2\left(1+LM^{10}\right)^2\left(-1+LM^{12}\right)(-1-2LM^{10} - 3LM^{12} + 2LM^{14} - L^2M^{16} + 2L^2M^{18} - 4L^2M^{20} - 2L^2M^{22} + 3L^2M^{24} - 3L^3M^{28} + 2L^3M^{30} + 4L^3M^{32} - 2L^3M^{34} + L^3M^{36} - 2L^4M^{38} + 3L^4M^{40} + 2L^4M^{42} + L^5M^{52})$$

$$\gamma_1 = \gamma_3$$

$$\gamma_0 = \gamma_4$$

Let P_n for $n = 0, \ldots, 3$ be as in Appendix A. Then,

$$R_n = P_n b^n \tag{14}$$

for $n = 0, \ldots, 3$ where b is given by Equation (9).

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