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# Homotopy equivalence of isospectral graphs

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Abstract. In previous work we defined a Quillen model structure, determined by cycles, on the category Gph of directed graphs. In this paper we give a complete description of the homotopy category of graphs associated to our model structure. We endow the categories of N-sets and Z-sets with related model structures, and show that their homotopy categories are Quillen equivalent to the homotopy category Ho(Gph). This enables us to show that Ho(Gph) is equivalent to the category cZSet of periodic Z-sets, and to show that two finite directed graphs are almost-isospectral if and only if they are homotopy-equivalent in our sense.

Contents



## 1. Introduction

Mathematicians often study complicated categories by means of *invariants* (which are equal for isomorphic objects in the category). Sometimes a complicated category can be replaced by a (perhaps simpler) homotopy category which is better related to the various invariants used to study it. In topology, this was first achieved by keeping track of when one continuous mapping could be continuously deformed into another. But it was eventually realized that most of the important features of this analysis are determined by the class of homotopy equivalences in the category.

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Quillen [Q67] presented an abstraction of this method that applies to many categories. A Quillen model structure on a category $\mathcal{E}$ works with three classes of morphisms in the category, which are assumed to satisfy the axioms described in Section 3 here. Quillen described the associated homotopy category $\mathrm{Ho}(\mathcal{E})$ as a *localization* or *category of fractions* with respect to the class of *weak equivalences* for the model structure; he defined the morphism sets for this homotopy category by using the classes of *fibrations* and *cofibrations* for the model structure.

In Bisson and Tsemo [BT08] we gave a model structure for the category Gph of directed and possibly infinite graphs, with loops and multiple arcs allowed (we give a precise definition of Gph in Section 2 here). We focussed on invariants in Gph defined in terms of cycles, and defined the weak equivalences for our model structure to be the Acyclics (graph morphisms which preserve cycles). The cofibrations and fibrations for the model are determined from the class of Whiskerings (graph morphisms produced by grafting trees). We review this model structure in Section 3 here.

The main goal of the present paper is to prove that the homotopy category Ho(Gph) for our model structure is equivalent to the category cZSet of periodic Z-sets. The proof is in Section 5 here. This result is applied in Section 6 to show that isospectral and almost-isospectral finite graphs are homotopy equivalent for our model structure.

We use the fact that whiskered cycles (disjoint unions of cycles with trees attached to them) are cofibrant objects in our model structure. These graphs can also be described as Cayley graphs of N-sets, where N is the monoid of the natural numbers under addition (with 1 as generator). Among these are the disjoint union of cycles, the Cayley graphs of Z-sets. This is explained in Sections 2 and 4.

Each of the categories Gph, NSet, and ZSet is a presheaf topos, and there are adjoint functors relating them. By selecting appropriate adjoint functors, we transport our model structure from Gph to the categories NSet and ZSet. It turns out that Ho(ZSet) and Ho(NSet) are both equivalent to Ho(Gph). We show this in Section 5 by using a further adjunction between ZSet and the category cZSet of periodic Z-sets. In fact, we exhibit Quillen equivalences between cZSet and ZSet, NSet, and Gph, where we use a trivial model structure on cZSet.

Here is a more detailed outline of the sections of this paper.

In Section 2 we define the category Gph, which is a presheaf category, and thus a topos. We define a subcategory NGph of Gph, which is equivalent to the category NSet of actions of the additive monoid of natural numbers. It follows that NGph is also a topos. Then we give a similar discussion of ZGph (which is equivalent to the topos ZSet of actions of the additive group of integers), and TGph (which is equivalent to the topos Set). We observe, in passing, that these equivalences provide (very simple) examples of Grothendieck's version of Galois theory. We show that these subcategories are reflective and coreflective subcategories of Gph. The functors we need for this are arising as "adjoint triples" $(F, G, H)$ of functors between presheaf categories. We establish our conventions for these functors at the end of Section 2; these functors are used almost everywhere in the paper.

In Section 3 we recall the definitions of model structure and cofibrantly-generated model structure. We recall our definitions of Surjecting, Whiskering, and Acyclic graph morphisms from Bisson and Tsemo [BT08], and the definition of our model structure on Gph. We show that our model structure is cofibrantly-generated, and use a general theorem on the transport of cofibrantly-generated model structures to define model structures on NSet and ZSet.

In Section 4, we analyze these new model structures on NSet and ZSet, with especial attention to fibrant objects and cofibrant objects. Motivated by this analysis, we develop a cofibrant replacement functor for the category Gph. Our construction uses the coreflection functor $H$ for ZGph as a subcategory of Gph.

In Section 5 we give some background on homotopy functors and on Quillen's construction of the homotopy category $\mathrm{Ho}(\mathcal{E})$ associated to a model structure on $\mathcal{E}$, and on his construction of derived functors (for adjoint functors satisfying appropriate conditions). We discuss examples of homotopy functors on Gph, and give examples of functors which satisfy the Quillen adjunction conditions. Most important for us is a particular adjunction relating Gph and ZSet; its left adjoint is the functor $H$ which assigns to each graph the set of all its bi-infinite paths. We use this to show that Ho(Gph) is equivalent to the category cZSet of periodic Z-sets.

In Section 6 we use the functor $H$ to associate a zeta series $Z_X(u)$ to each almost-finite graph $X$. This fits very well with work of Dress and Siebeneicher [DS88] on the Burnside ring of the category of almost-finite Z-sets. As a consequence of our calculation of Ho(Gph), we show that finite graphs are almost-isospectral if and only if they are homotopy equivalent (that is, isomorphic in the homotopy category).

## 2. Some subcategories of graphs

This paper is about Gph, a convenient category of graphs, precisely described in the paragraph below. In Bisson and Tsemo [BT08] we introduced a Quillen model structure on Gph. In this paper we show how to study that structure, and the resulting homotopy category, by means of some of its subcategories.

We define a *graph* to be a data-structure $X = (X_0, X_1, s, t)$ with a set $X_0$ of *nodes*, a set $X_1$ of *arcs*, and a pair of functions $s, t : X_1 \to X_0$ which specify the *source* and *target* node of each arc. We may say that $a \in X_1$ is an arc which *leaves node* $s(a)$ and *enters node* $t(a)$; and that a *loop* is an arc $a$ with $s(a) = t(a)$. A *graph morphism* $f : X \to Y$ is a pair of functions

$f_1 : X_1 \to Y_1$ and $f_0 : X_0 \to Y_0$ such that $s \circ f_1 = f_0 \circ s$ and $t \circ f_1 = f_0 \circ t$. This defines the particular category Gph that we study here.

In fact, Gph is the category of presheafs on a small category; see Lawvere [L89] or Lawvere and Schanuel [LS97] for fascinating discussions. It follows that Gph is a topos, and thus a category with many nice geometric and algebraic and logical properties; see Mac Lane and Moerdijk [MM94], for instance.

In this paper we want to consider some very special kinds of graphs, as follows.

**Definition.** A graph $X$ is:

a) an *N-graph* when each node of $X$ has exactly one arc entering;
b) a *Z-graph* when each node of $X$ has exactly one arc entering and exactly one arc leaving;
c) a *T-graph* when each node of $X$ has exactly one loop, and $X$ has no other arcs.

A T-graph might be called a *terminal* graph (or *graph of loops*), and a Z-graph might be called a *graph of cycles*. An N-graph might be called a *graph of whiskered cycles*.

Let NGph denote the *full subcategory* of Gph whose objects are the N-graphs; this means that we take all graph morphisms between N-graphs as the morphisms in NGph. Similarly, let ZGph denote the full subcategory of Gph whose objects are the Z-graphs, and let TGph denote the full subcategory of Gph whose objects are the terminal graphs. We have a chain of subcategories

$$\mathrm{TGph} \subset \mathrm{ZGph} \subset \mathrm{NGph} \subset \mathrm{Gph}.$$

In fact, these Gph subcategories are equivalent to some well-known categories. Let $G$ be a monoid, with associative binary operation $G \times G \to G : (g, h) \mapsto g * h$ and with neutral element $e$; a $G$-set is a set $S$ together with an action $\mu : G \times S \to S$ such that $\mu(e, x) = x$ and $\mu(g, \mu(h, x)) = \mu(g * h, x)$.

Consider the monoid $N$ of natural numbers under addition, and the group $Z$ of integers under addition. A set $S$ together with an arbitrary function $\sigma : S \to S$ defines an action of $N$ by $\mu(n, x) = \sigma^n(x)$ for $n \in N$. A set $S$ together with an arbitrary *invertible* function $\sigma : S \to S$ defines an action of $Z$ by $\mu(n, x) = \sigma^n(x)$ for $n \in Z$. This justifies the following description of the categories of N-sets and Z-sets.

**Definition.** Let Set denote the category of sets. Let NSet denote the category of N-sets; here an N-set is a pair $(S, \sigma)$ with $\sigma$ a function from $S$ to $S$, and a map of N-sets from $(S, \sigma)$ to $(S', \sigma')$ is a function $f : S \to S'$ such that $\sigma' \circ f = f \circ \sigma$. Let ZSet denote the full subcategory of NSet with objects $(S, \sigma)$ where $\sigma$ is a bijection.

For any N-set $(S, \sigma)$, we define a graph $X = G(S, \sigma)$ with nodes $X_0 = S$ and arcs $X_1 = S$, and with $s, t : X_1 \to X_0$ given by $s(x) = \sigma(x)$ and $t(x) = x$

for each $x \in S$. Thus the elements in the N-set $S$ give the nodes and the arcs in the graph $X$, and each arc $x$ has source $\sigma(x)$ and target $x$. In the N-set $S$ we think of $\sigma(x)$ as telling the unique "source" or "parent" of each element $x$.

Note that we are directing our arcs opposite to the way that seems natural in graphical representation of dynamical systems (see Lawvere and Schanuel [LS97], for instance). But our convention is designed to fit the "whiskered cycles" which are important in our model structure (see Sections 3 and 4 here).

**Proposition 1.** *$G$ is a functor from* NSet *to* Gph; *moreover:*

a) *the functor $G$ gives an equivalence from the category* NSet *to the subcategory* NGph *in* Gph;
b) *the restriction of $G$ gives an equivalence from the category* ZSet *to the subcategory* ZGph *in* Gph;
c) *the restriction of $G$ gives an equivalence from the category Set to the subcategory* TGph *in* Gph.

**Proof.** If $f : (S, \sigma) \to (S', \sigma')$ is a map of N-sets, we define a graph morphism $G(f) : G(S, \sigma) \to G(S', \sigma')$ by $G(f)_0(x) = G(f)_1(x) = f(x)$ for $x$ in $S$. This preserves composition and gives a functor from NSet to Gph. We note that $G(S, \sigma)$ is an N-graph, and every N-graph has a unique isomorphism to a graph $X$ in the image of $G$ (where $X_0 = X_1$ and $t$ is the identity). If $X$ is in NGph we define an N-set $H(X) = (X_0, \sigma)$ by $\sigma(x) = s(a)$ where $a$ is the unique arc entering the node $x$, and a graph morphism $g : X \to Y$ gives $H(g) : H(X) \to H(Y)$ by $H(g)(x) = g_0(x)$ for $x \in X_0$. This preserves composition and gives a functor from NGph to NSet. Thus we have

$$G : \mathrm{NSet} \to \mathrm{NGph} \quad \text{and} \quad H : \mathrm{NGph} \to \mathrm{NSet}.$$

Note that $H(G(S, \sigma)) = (S, \sigma)$ and $G(H(X)) = X$. In fact, $G$ and $H$ give inverse bijections between the set of N-set maps $(S, \sigma)$ to $(S', \sigma')$ and the set of graph morphisms from $G(S, \sigma)$ to $G(S', \sigma')$. This says that the functor $G$ is full and faithful, with image equivalent to the category NGph. We can carry out a similar analysis for the restriction of $G$ to the subcategory ZSet, and the restriction of $H$ to the subcategory ZGph. The analysis for Set and TGph is also similar (and rather trivial). □

Each part of the above proof exhibits an "adjoint pair" of functors $(G, H)$ (as discussed below), and shows that it gives an equivalence of categories. The proof can also be understood as an example of (the representable case of) Grothendieck's Galois theory. The paper by Dubuc and de la Vega [DV00] gives a self-contained exposition which seems relevant to our examples here.

In Section 3 we will use the fact that TGph and NGph and ZGph are "reflective and coreflective subcategories" of Gph. Roughly speaking, a reflection from a category $\mathcal{E}$ into a subcategory $\mathcal{E}'$ assigns to each object $X$

in $\mathcal{E}$ an object $X'$ in $\mathcal{E}'$, and a morphism $X \to X'$ in $\mathcal{E}$ which is universal in that any $X \to X''$ with $X''$ in $\mathcal{E}'$ factors through a unique morphism $X' \to X''$ in $\mathcal{E}'$. Dually, a coreflection is a morphism $X' \to X$ in which is couniversal in that any $X'' \to X$ with $X''$ in $\mathcal{E}'$ factors through a unique morphism $X'' \to X'$ in $\mathcal{E}'$.

These notations are best made precise in the language of adjoint functors. Here is a quick review of these standard definitions (see Mac Lane [M71], for instance).

**Definition.** An *adjunction* between categories $\mathcal{X}$ and $\mathcal{Y}$ is a pair $(L, R)$ of functors $L : \mathcal{X} \to \mathcal{Y}$ and $R : \mathcal{Y} \to \mathcal{X}$ together with a natural bijection of morphism sets $\mathcal{Y}(L(X), Y) \to \mathcal{X}(X, R(Y))$. In this case, we may say that $(L, R)$ is an *adjoint pair*, with $L$ as the *left adjoint* and $R$ as the *right adjoint*, and denote this by

$$L : \mathcal{X} \rightleftharpoons \mathcal{Y} : R.$$

For any adjoint pair $(L, R)$ we have a natural transformation $X \to RL(X)$, called the *unit* of the adjunction; dually, there is a natural transformation $LR(Y) \to Y$, called the *counit* of the adjunction. In particular, a subcategory $\mathcal{E}'$ of $\mathcal{E}$ is a *reflective subcategory* when the inclusion functor $G : \mathcal{E}' \to \mathcal{E}$ has a left adjoint functor $F$, with adjunction $(F, G)$; then $F$ is the *reflection* functor. Dually, $\mathcal{E}'$ is a *coreflective subcategory* of $\mathcal{E}$ when the inclusion functor $G : \mathcal{E}' \to \mathcal{E}$ has a right adjoint functor $H$, with adjunction $(G, H)$; then $H$ is the *coreflection* functor.

**Proposition 2.** TGph *and* NGph *and* ZGph *are reflective and coreflective subcategories of* Gph*, with*

$$\mathrm{TGph} \subset \mathrm{ZGph} \subset \mathrm{NGph} \subset \mathrm{Gph}.$$

**Proof.** Let us start with the full subcategory TGph of terminal graphs, those graphs which are disjoint unions of 1. This part is especially simple, and sets the tone for the other parts of the proof.

The reflection is equivalent to an adjunction

$$F : \mathrm{Gph} \rightleftharpoons \mathrm{Set} : G.$$

Here $G$ is the functor from Set to Gph which assigns to set $S$ the terminal graph with one loop for each element of $S$. The functor $F$ assigns to each graph $X$ its *set of components*; this is the set of equivalence classes of nodes of $X$, with respect to the equivalence relation generated by the relation $s(a) \sim t(a)$ for each arc $a$ in $X$ (the coequalizer of the source and target functions $s, t : X_1 \to X_0$). We may use the notation $F(X) = \pi_0(X)$ and $G(S) = \sum_S 1$. The unit of the adjunction $X \to GF(X)$ is universal among graph morphisms from $X$ to terminal graphs, as mentioned above. This shows the desired adjunction. Note that the counit $FG(S) \to S$ of the reflection is a bijection for every set $S$ (as must happen for any full reflective subcategory). The image of the functor $GF : \mathrm{Gph} \to \mathrm{Gph}$ is equivalent to the subcategory TGph of Gph; in fact, any terminal graph has a unique isomorphism to a graph $X$ with $X_0 = X_1$ and $s$ and $t$ as the identity.

The coreflection is equivalent to an adjunction

$$G : \mathrm{Set} \rightleftharpoons \mathrm{Gph} : H$$

where $H(X)$ is the set of graph morphisms from 1 to $X$, the set of those arcs of $X$ which are loops. The counit of the adjunction $GH(X) \to X$ is universal among morphisms from terminal graphs to $X$. This shows the coreflection. Note that the unit $S \to HG(S)$ of the coreflection is a bijection for every set $S$ (as must happen for any full coreflective subcategory).

Now we give a similar treatment of NGph. Recall the definition of $G$ : NSet $\to$ Gph from the previous proposition. The reflection comes from an adjunction

$$F : \mathrm{Gph} \rightleftharpoons \mathrm{NSet} : G$$

where we can describe the functor $F$ as follows. Let $\mathbf{P}$ denote the unending path graph; its nodes are the natural numbers, and there is one arc $n \to n{+}1$ for each $n \geq 0$. Let $\sigma : \mathbf{P} \to \mathbf{P}$ be the graph morphism given on nodes by $\sigma(n) = n + 1$. For any graph $X$, consider the set of connected components of the graph $\mathbf{P} \times X$. Let $F(X)$ denote the N-set $(\pi_0(\mathbf{P} \times X), \sigma)$, with $\sigma([n, x]) = [n + 1, x]$, induced by the graph morphism $\sigma \times \mathrm{id}$ from $\mathbf{P} \times X$ to itself. Let us sketch the bijection between graph morphisms $f : X \to G(S, \sigma)$ and N-set maps $g : F(X) \to (S, \sigma)$. For any morphism $f : X \to G(S)$ in Gph, we define a function from the nodes of $\mathbf{P} \times X$ to $S$ by $(n, x) \mapsto \sigma^n(f(x))$. This is well defined on connected components of $\mathbf{P} \times X$ since any arc $(n, x') \to (n + 1, x)$ in $\mathbf{P} \times X$ implies the existence of an arc $x' \to x$ in $X$, which implies $f(x') = \sigma(f(x))$ in $S$, so that $\sigma^n(f(x')) = \sigma^{n+1}(f(x))$. This gives an N-set map $g : (\pi_0(\mathbf{P} \times X), \sigma) \to (S, \sigma)$, since $g(\sigma[n, x]) = g([n + 1, x]) = \sigma^{n+1}(f(x)) = \sigma(g([n, x])$. This is bijective, and establishes the adjunction.

The coreflection comes from an adjunction

$$G : \mathrm{NSet} \rightleftharpoons \mathrm{Gph} : H$$

where we can describe the functor $H$ as follows. Let $\mathbf{P}^{op}$ denote the graph $G(N, +1)$, and let $\sigma : \mathbf{P}^{op} \to \mathbf{P}^{op}$ be the graph morphism given on nodes by $\sigma(n) = n + 1$. Let $H(X)$ be the set of graph morphisms from $\mathbf{P}^{op}$ to $X$, viewed as an N-set with $\sigma(f) = f \circ \sigma$. Then $G$ is left adjoint to $H$.

It is easy to see ZSet as a reflective and coreflective subcategory of NSet. Let $Z$ denote the integers. The right adjoint to the inclusion assigns to an N-set $(S, \sigma)$ the set of functions $f : Z \to S$ such that $f(n + 1) = \sigma(f(n))$ for all $n \in Z$; then $\sigma(f) = \sigma \circ f$ is invertible. The left adjoint to the inclusion assigns to an N-set $(S, \sigma)$ the set $(Z \times S)/\sim$, for the equivalence relation generated by $(n + 1, x) \sim (n, \sigma(x))$ for all $n \in Z$ and $x \in S$; then $\sigma(n, x) = (n, \sigma(x))$ is invertible.

It follows that ZGph is a reflective and coreflective subcategory of NGph; informally, the reflection combs any whiskers down along their source cycle, while the coreflection builds a ZGph from biinfinite walks. Since the composition of adjuctions is an adjunction (see Mac Lane [M71], for instance), this also gives the reflection from Gph to ZGph. □

For any monoid $G$, the category of $G$-sets is a presheaf category, and thus a Grothendieck topos, with all limits (products, pullbacks, etc) and colimits (coproducts, pushouts, etc); see Mac Lane and Moerdijk [MM94], for instance. In fact, since Gph, NSet, and ZSet are presheaf categories, these categorical constructions can be performed "elementwise". Here are the simplest examples. The N-set $(1, \mathrm{id})$ (the one point set with its identity function) is a *terminal object* in NSet; this means that for every N-set there is a unique N-set map $(S, \sigma) \to (1, \mathrm{id})$. The empty set with its identity function is an *initial object* in NSet; this means that for every N-set there is a unique N-set map $(0, \mathrm{id}) \to (S, \sigma)$. These objects 0 and 1 also provide the initial and terminal objects for ZSet.

The calculation of products and coproducts is the same in each of the subcategories

$$\mathrm{TGph} \subset \mathrm{ZGph} \subset \mathrm{NGph} \subset \mathrm{Gph}.$$

In fact, the inclusion functor $G : \mathrm{NSet} \to \mathrm{Gph}$ preserves limits and colimits, since it has left adjoint $F$ and right adjoint $H$; and similarly for the other $G$ functors.

Note that in the above discussions we have actually used three functors $(F, G, H)$, made up of two overlapping adjunctions $(F, G)$ and $(G, H)$ between presheaf categories. These adjunctions are coming from functors between the sites for the presheaf categories, as follows. If $C$ is a small category then the topos of presheaves on $C$, which we may denote by $C\mathrm{Set}$, is the category of functors from $C^{op}$ to Set. If $\phi : C \to D$ is a functor, then there is an adjoint triple $(\phi_!, \phi^*, \phi_*)$ with

$$\phi_! : C\mathrm{Set} \rightleftharpoons D\mathrm{Set} : \phi^* \quad \text{and} \quad \phi^* : D\mathrm{Set} \rightleftharpoons C\mathrm{Set} : \phi_*.$$

See the analysis in Expose I.5 of Grothendieck [G72]. This concept is related to that of "essential geometric morphism" $\phi : C\mathrm{Set} \Rightarrow D\mathrm{Set}$ in topos theory; see Mac Lane and Moerdijk [MM94], for instance. Almost all the adjunctions used in this paper come from such adjoint triples $(F, G, H) = (\phi_!, \phi^*, \phi_*)$.

## 3. Quillen model structures

We review our model structure on the category Gph, and show that it is cofibrantly generated. Then we establish related model structures on the categories NSet and ZSet, for further analysis in the next two sections.

We start with some convenient notation, for our model category axioms.

**Definition.** Let $\ell : X \to Y$ and $r : A \to B$ be morphisms in a category $\mathcal{E}$. We say that $\ell$ is *weak orthogonal* to $r$ (abbreviated by $\ell \dagger r$) when, for all $f$

and $g$,

$$\text{if } \begin{array}{ccc} X & \xrightarrow{f} & A \\ {\scriptstyle \ell}\downarrow & & \downarrow{\scriptstyle r} \\ Y & \xrightarrow{g} & B \end{array} \quad \text{commutes, then } \begin{array}{ccc} X & \xrightarrow{f} & A \\ {\scriptstyle \ell}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle r} \\ Y & \xrightarrow{g} & B \end{array} \quad \text{commutes for some } h.$$

Given a class $\mathcal{F}$ of morphisms we define

$$\mathcal{F}^\dagger = \{r : f \dagger r,\ \forall f \in \mathcal{F}\} \quad \text{and} \quad {}^\dagger\mathcal{F} = \{\ell : \ell \dagger f,\ \forall f \in \mathcal{F}\}.$$

A *weak factorization system* in $\mathcal{E}$ is given by two classes $\mathcal{L}$ and $\mathcal{R}$, such that $\mathcal{L}^\dagger = \mathcal{R}$ and $\mathcal{L} = {}^\dagger\mathcal{R}$ and such that, for any morphism $c$ in $\mathcal{E}$, there exist $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$ with $c = r \circ \ell$.

Using the above, we may express Quillen's notion [Q67] of "model category" via the following axioms, which we learned from Section 7 of Joyal and Tierney [JT07].

**Definition.** Suppose that $\mathcal{E}$ is a category with finite limits and colimits. A *model structure* on $\mathcal{E}$ is a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes of morphisms in $\mathcal{E}$ that satisfies:

a) "three for two": if two of the three morphisms $a, b, a \circ b$ belong to $\mathcal{W}$ then so does the third;
b) the pair $(\underline{\mathcal{C}}, \mathcal{F})$ is a weak factorization system (where $\underline{\mathcal{C}} = \mathcal{C} \cap \mathcal{W}$);
c) the pair $(\mathcal{C}, \underline{\mathcal{F}})$ is a weak factorization system (where $\underline{\mathcal{F}} = \mathcal{W} \cap \mathcal{F}$).

The morphisms in $\mathcal{W}$ are called *weak equivalences*. The morphisms in $\mathcal{C}$ are called *cofibrations*, and the morphisms in $\underline{\mathcal{C}}$ are called *acyclic cofibrations*. The morphisms in $\mathcal{F}$ are called *fibrations*, and the morphisms in $\underline{\mathcal{F}}$ are called *acyclic fibrations*.

In Bisson and Tsemo [BT08] we introduced a Quillen model structure on Gph. Its description used three types of graph morphisms, which we called Surjectings, Whiskerings, and Acyclics. They can be defined as follows.

- A graph morphism $f : X \to Y$ is *Surjecting* when the induced function $f : X(x, *) \to Y(f(x), *)$ is surjective for all $x \in X_0$. Here, for any graph $Z$ and any node $z$, $Z(z, *)$ denotes the set of arcs in $Z$ which have source $z$.
- A graph morphism $f : X \to Y$ is *Acyclic* when $C_n(f) : C_n(X) \to C_n(Y)$ is bijective for all $n > 0$. Here $\mathbf{C}_n$ is the *(directed) cycle graph*, with the integers mod $n$ as its nodes and its arcs, and with $s(i) = i + 1$ and $t(i) = i$. Then $C_n(X)$ denotes the set of graph morphisms from $\mathbf{C}_n$ to $X$.
- A graph morphism $f : X \to Y$ is *Whiskering* when $Y$ is formed by attaching rooted trees to $X$. Here a *rooted tree* is a graph $T$ with a node $r$ (its *root*) such that, for each each node $x$ in $T$, there is a unique (directed) path in $T$ from $r$ to $x$. Then "attaching" the rooted tree $T$ to $X$ means identifying the root $r$ with a node of $X$;

this is forming the pushout of graph morphisms $r \to T$ and $r \to X$, where $r$ is considered as a graph with one node and no arcs.

Here we will interpret these morphism classes, and describe our model structure for Gph, in terms of "cofibrant generation" (see section 2.1 in Hovey [H99], for instance). A model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is *cofibrantly generated* when there are sets $I$ and $J$ of morphisms such that $J^\dagger = \mathcal{F}$ and $I^\dagger = \underline{\mathcal{F}}$, so that $\mathcal{C} = {}^\dagger(I^\dagger)$ and $\underline{\mathcal{C}} = {}^\dagger(J^\dagger)$. In short, a cofibrantly-generated model structure is a model structure given by weak factorization systems

$$(\mathcal{C}, \underline{\mathcal{F}}) = ({}^\dagger(I^\dagger), I^\dagger) \quad \text{and} \quad (\underline{\mathcal{C}}, \mathcal{F}) = ({}^\dagger(J^\dagger), J^\dagger).$$

We may say that $J$ generates the acyclic cofibrations, and that $I$ generates the cofibrations. There is usually a "smallness" assumption mentioned, but this smallness condition is vacuous in Gph: every object in Gph is small with respect to every set of morphisms in Gph, since Gph is a presheaf category on a small category. This follows from the proof at Example 2.1.5 in Hovey [H99], for instance.

Let us describe sets $I$ and $J$ which generate our model structure for Gph. Let $\mathbf{s} : \mathbf{D} \to \mathbf{A}$ be the "source" graph morphism, which exhibits the "dot" graph $\mathbf{D}$ as the source subgraph of the "arrow" graph $\mathbf{A}$. More precisely, $\mathbf{A}$ is the graph with two nodes, 0 and 1, and one arc $a$ from 0 to 1; and $\mathbf{s}$ is the inclusion of the subgraph $\mathbf{D}$ with one node 0 and no arcs. Let $\mathbf{i}_n : \mathbf{0} \to \mathbf{C}_n$ be the initial graph morphism, and let $\mathbf{j}_n : \mathbf{C}_n + \mathbf{C}_n \to \mathbf{C}_n$ be the coproduct graph morphism. Let $J = \{\mathbf{s}\}$; let $K = \{\mathbf{i_n}, \mathbf{j_n} : n > 0\}$; and let $I = J \cup K$.

**Theorem 3.** Gph *has a cofibrantly-generated model structure with acyclic cofibrations generated by $J$, and cofibrations generated by $I$, and with weak equivalences $\mathcal{W} = K^\dagger$.*

**Proof.** We show that the morphism classes given above agree with the model structure on Gph given in Bisson and Tsemo [BT08]; see the details there. Consider $\mathcal{F} = J^\dagger$ and $\mathcal{C} = {}^\dagger(I^\dagger)$. The following three observations follow directly from the definition of weak orthogonality:

a) $K^\dagger$ is precisely the Acyclic graph morphisms;
b) $J^\dagger$ is precisely the Surjecting graph morphisms;
c) $I^\dagger = (J \cup K)^\dagger$ is precisely the Acyclic Surjecting graph morphisms.

The Surjectings form the class $\mathcal{F}$ of fibrations for our model structure. The Acyclics form the class $\mathcal{W}$ of weak equivalences for our model structure. The Acyclic Surjectings form the class $\mathcal{W} \cap \mathcal{F} = \underline{\mathcal{F}}$ of acyclic fibrations for our model structure. Then ${}^\dagger(I^\dagger) = {}^\dagger(\mathcal{W} \cap \mathcal{F})$. So ${}^\dagger(I^\dagger) = \mathcal{C}$, the cofibrations for our model structure. □

The following standard notions will help us state a general theorem about "transporting" Quillen model structures to related categories. See section 2.1 in Hovey [H99] for discussion of pushouts, transfinite compositions, retracts in the morphism category, etc. Let $\mathcal{E}$ be a category with all limits and colimits. For a set $H$ of morphisms in $\mathcal{E}$, let cell$(H)$ denote the class of all transfinite compositions of pushouts of elements in $H$. Morphisms in cell($H$) are called *relative $H$-cell complexes*; a graph $X$ is called an *$H$-cell complex* if $0 \to X$ is a relative $H$-cell complex. All this is suggested by the notion in topology of building up a space by attaching cells.

Here are some examples in the category Gph, where we take $J = \{\mathbf{s}\}$ and $K = \{\mathbf{i_n}, \mathbf{j_n} : n > 0\}$ and $I = J \cup K$.

**Proposition 4.** *The morphisms in* cell($J$) *are the Whiskerings, the acyclic cofibrations for our model structure. Moreover,* $\mathrm{cell}(J) = {}^{\dagger}(J^{\dagger})$.

**Proof.** Each pushout of $\mathbf{s}$ attaches a single arc as a Whisker. Attaching a rooted tree corresponds to a composition of these; and the Whiskerings are exactly the class of all transfinite compositions of pushouts of elements in $J$. We showed in Bisson and Tsemo [BT08] that the Whiskerings are closed with respect to retracts in the morphism category of Gph. To prove the second statement, we use this, together with some general facts from Hovey. We always have $H \subseteq \mathrm{cell}(H) \subseteq {}^{\dagger}(H^{\dagger})$. Suppose that "the domains of morphisms in $H$ are small with respect to cell($H$)". Then Hovey uses a general version of the small object argument (based on Lemma 3 of Chapter II.3 in Quillen [Q67] ) to show that any morphism in ${}^{\dagger}(H^{\dagger})$ is the retract, in the category of morphisms of $\mathcal{E}$, of some morphism in cell($H$). But every object in Gph is small with respect to every set of morphisms in Gph, as mentioned above, so the smallness condition here is vacuous in Gph. □

**Proposition 5.** *If $C$ is a disjoint union of cycle graphs, then every inclusion $X \to X + C$ is in* cell($K$). *Every graph morphism between disjoint unions of cycle graphs is in* cell($K$).

**Proof.** For any graph $X$, the morphism $X \to X + C_n$ is a pushout of $\mathbf{i}_n$; if $C$ is any disjoint union of cycle graphs, then $X \to X + C$ is a transfinite composition of pushouts of $\mathbf{i}_n$ for $n > 0$. For the second statement, let $\pi_{n,k} : \mathbf{C}_{nk} \to \mathbf{C}_n$ (for $n > 0$ and $k > 0$) denote the graph morphism given on nodes by $\pi_{n,k}(i) = i \bmod n$. We can exhibit $\pi_{n,k}$ as a pushout of $\mathbf{j}_{nk}$, as follows. Consider the graph morphism $f : \mathbf{C}_{nk} + \mathbf{C}_{nk} \to \mathbf{C}_{nk}$ given on nodes by $f(i, 0) = i + n$ and $f(i, 1) = i$, where we think of graph $\mathbf{C}_{nk} + \mathbf{C}_{nk}$ as the product of $\mathbf{C}_{nk}$ and the set $\{0, 1\}$. Then $\pi_{n,k}$ is the pushout of $f$ and $\mathbf{j}_{nk}$. Any graph morphism between disjoint unions of cycle graphs is a pushout of such maps (up to isomorphisms). □

Since $\mathrm{cell}(K) \subseteq \mathrm{cell}(I) \subseteq \mathcal{C}$, these propositions are describing some of the cofibrations for our model structure on Gph. But here are some examples of morphisms which are **not** cofibrations for our model structure. We let $\mathbf{Z}$ denote the graph with the integers as its nodes and arcs, and $s(n) = n$ and $t(n) = n + 1$.

a) The graph morphism $\pi_n : \mathbf{Z} \to \mathbf{C}_n$, given by reduction mod $n$ on nodes, is not a cofibration. We may show this by constructing an explicit Acyclic Surjecting graph morphism $g : X \to Y$ such that the weak orthogonality

$\pi_n \dagger g$ fails. Let $Y = \mathbf{C}_n$ and let $X = \mathbf{C}_n\mathbf{P}$, the graph formed by attaching the root of the unending path $\mathbf{P}$ at the 0 node in $\mathbf{C}_n$ (recall that we defined $\mathbf{P}$ in Section 2 here). Let $f : \mathbf{Z} \to \mathbf{C}_n\mathbf{P}$ be the graph morphism given on nodes by $m \mapsto m \bmod n$ for $m \geq 0$ and $m \mapsto -m$ for $m \leq 0$. Then the commutative square with horizontal arrows $f : \mathbf{Z} \to \mathbf{C}_n\mathbf{P}$ and $\mathrm{id} : \mathbf{C}_n \to \mathbf{C}_n$ has no lifting. It follows that $\pi_n$ is not a cofibration for our model structure.

b) Also, $0 \to \mathbf{Z}$ is not a cofibration, since it is an Acyclic Surjecting graph morphism, and it is not weakly orthogonal to itself.

c) Also, $\mathbf{Z}+\mathbf{Z} \to \mathbf{Z}$ is not a cofibration, since it is is an Acyclic Surjecting graph morphism which is not weakly orthogonal to itself.

We want to use our model structure on Gph to define model structures on NSet and ZSet. We will use a general result, referred to as "creating model structures along a right adjoint" (by Hirschhorn, Hopkins, Beke, etc), or as "transferring model structures along adjoint functors" (by Crans, etc). According to Berger and Moerdijk [BM03]: "Cofibrantly generated model structures may be transferred along the left adjoint functor of an adjunction. The first general statement of such a transfer in the literature is due to Crans." The reference is to Crans [C95]. Here is a statement of this "transfer principle" (from Berger and Moerdijk [BM03]).

**Transport Theorem.** *Let $\mathcal{E}$ be a model category which is cofibrantly generated, with cofibrations generated by $I$ and acyclic cofibrations generated by $J$. Let $\mathcal{E}'$ be a category with all limits and colimits, and suppose that we have an adjunction*

$$L : \mathcal{E} \rightleftharpoons \mathcal{E}' : R \quad \text{with} \quad R(\text{cell } L(J)) \subseteq \mathcal{W}.$$

*Also, assume that the sets $L(I)$ and $L(J)$ each permit the small object argument. Then there is a cofibrantly generated model structure on $\mathcal{E}'$ with generating cofibrations $L(I)$ and generating acyclic cofibrations $L(J)$. Moreover, the model structure $(\mathcal{C}', \mathcal{W}', \mathcal{F}')$ satisfies $f \in \mathcal{W}'$ iff $R(f) \in \mathcal{W}$, and $f \in \mathcal{F}'$ iff $R(f) \in \mathcal{F}$.*

As mentioned before, the smallness conditions are automatically satisfied in our presheaf categories. So, in our examples, the main hypothesis for the theorem is: $f \in \text{cell } L(J)$ implies $R(f) \in \mathcal{W}$.

Let us translate some definitions from Gph into NSet. For an NSet map $f : (S, \sigma) \to (T, \sigma)$ we say that:

a) $f$ is *Acyclic* when $C_n(f) : C_n(S, \sigma) \to C_n(T, \sigma)$ is a bijection for every $n > 0$. Here $C_n(S, \sigma) = \{x \in S : \sigma^n(x) = x\}$ (we could call these the $n$-periodic points).

b) $f$ is *Surjecting* when $f : \sigma^{-1}(x) \to \sigma^{-1}(f(x))$ is a surjection for every $x$ in $S$.

c) $f$ is *Whiskering* when $f$ is an injective function such that for every $x \in T$ has $\sigma^n(x) \in f(S)$ for some natural number $n$.

**Proposition 6.** *There is a Quillen model structure on* NSet *with*

a) *weak equivalences* $\mathcal{W}$ *given by the Acyclic* NSet *maps,*
b) *fibrations* $\mathcal{F}$ *given by the Surjecting* NSet *maps,*
c) *cofibrations* $\mathcal{C} = {}^{\dagger}\underline{\mathcal{F}}$*, where* $\underline{\mathcal{F}} = \mathcal{W} \cap \mathcal{F}$.

*Moreover, the acyclic cofibrations* $\underline{\mathcal{C}}$ *are given by the Whiskering* NSet *maps.*

**Proof.** We create a model structure on NSet by applying the Transport Theorem to the adjunction

$$F : \text{Gph} \rightleftharpoons \text{NSet} : G.$$

We must check that $G(\text{cell}(FJ)) \subseteq \mathcal{W}$. But $J$ contains just the single graph morphism $\mathbf{s} : \mathbf{D} \to \mathbf{A}$. We calculate that $F(\mathbf{D}) = (\pi_0(\mathbf{P} \times \mathbf{D}), \sigma) = (N, \sigma)$ and $F(\mathbf{A}) = (\pi_0(\mathbf{P} \times \mathbf{A}), \sigma) = (N, \sigma)$; and the NSet map $F(\mathbf{s}) : (N, \sigma) \to (N, \sigma)$ is given by the successor function $\sigma : N \to N$. Here are the details. The graph $\mathbf{P} \times \mathbf{D}$ has no arcs and nodes $n$ for $n \geq 0$; the graph $\mathbf{P} \times \mathbf{A}$ has nodes $(n, 0)$ and $(n, 1)$ for $n \geq 0$, and an arc $(n, 0) \to (n + 1, 1)$ for each $n \geq 0$; and $\mathbf{s} : \mathbf{P} \times \mathbf{D} \to \mathbf{P} \times \mathbf{A}$ is given on nodes by $\mathbf{s}(n) = (n, 0)$. Then $\pi_0(\mathbf{P} \times \mathbf{D})$ has elements $n$ for $n \geq 0$ and $\pi_0(\mathbf{P} \times \mathbf{A})$ has elements $[m, 1]$ for $m \geq 0$; but on components we have $\mathbf{s}(n) = [n, 0] = [n + 1, 1]$.

Let us show that cell$(FJ)$ is given by the Whiskering NSet maps. The functor $G$, which establishes the equivalence between NSet and NGph, preserves limits and colimits, so that $G(\text{cell}(FJ)) = \text{cell}(GFJ)$. Recall that Whiskerings in Gph are attaching rooted trees, with arcs leaving the root; but a rooted tree is not in NGph, because the root is a node with no arcs entering. To stay within NGph, we attach a *taprooted tree*, which is a rooted tree with a copy of the N-graph $\mathbf{P}^{op}$ (an infinite sequence of arcs and nodes leading into the node 0) attached to it by identifying its root with the 0 node. The graph morphisms in $G(\text{cell}(FJ))$ all come from attaching “taprooted forests” in NGph; via the functor $G$, these correspond to the Whiskering NSet maps. Since every graph morphism in $G(\text{cell}(FJ))$ is thus an Acyclic, the hypothesis of the Transport Theorem is met. It follows that $\mathcal{F}$ and $\underline{\mathcal{F}}$ in NSet are defined in terms of the functor $G$, which is the inclusion of NSet as the full subcategory NGph in Gph. If $X = G(S, \sigma)$ then $C_n(S, \sigma) = C_n(X)$. Thus the weak equivalences in NSet are the Acyclic NSet maps. Also, we have $\sigma^{-1}(x) = X(x, *)$, the arcs leaving node $x$, so $G(f) : G(S, \sigma) \to G(S', \sigma)$ is a Surjecting graph morphism if and only if $f$ is a Surjecting NSet map. Thus the fibrations in NSet are the Surjecting NSet maps. □

We define the Acyclic ZSet maps and the Surjecting ZSet maps by exactly copying the definitions used for NSet. But these definitions simplify quite a bit in the category ZSet.

**Proposition 7.** *All* ZSet *maps are Surjecting.*

**Proof.** In any ZSet $(S, \sigma)$, the function $\sigma$ is invertible, so $\sigma^{-1}(x)$ has exactly one element for every $x \in S$. Consider any ZSet map $f : (S, \sigma) \to (T, \sigma)$. For every $x \in S$ we restrict $f$ to give $f : \sigma^{-1}(x) \to \sigma^{-1}(f(x))$, and any function between one element sets is surjective; thus every ZSet map is Surjecting. □

We can use the following definition to describe the Acyclic ZSet maps. For any ZSet $(S,\sigma)$, let $j(S,\sigma) = \{x \in S : \exists n > 0, \sigma^n(x) = x\}$.

We may call $j(S,\sigma)$ the *periodic part* of the Z-set $(S,\sigma)$. Any ZSet map $f : (S,\sigma) \to (T,\sigma)$ restricts to give $j(f) : j(S,\sigma) \to j(T,\sigma)$, since if $\sigma^n(x) = x$ for some $x \in S$, then $\sigma^n(f(x)) = f(x)$ in $T$.

**Proposition 8.** *A* ZSet *map $f$ is Acyclic if and only if $j(f)$ is a bijection.*

**Proof.** Suppose that $f : (S,\sigma) \to (T,\sigma)$ is Acyclic. We want to show that $j(f)$ is a bijection. Certainly, $j(f)$ is a surjection, since for every $y \in jT$ we have $y \in C_n(T)$ for some $n > 0$, and $C_n(S) \to C_n(T)$ is bijective by assumption. So there is a unique $x \in C_n(S)$ with $f(x) = y$. Suppose that $j(f)$ is not an injection; then there exists some $y \in jT$ with more than one preimage in $jS$. We know that $y \in C_n(T)$ for some $n > 0$; let $n$ be the smallest such. Then there is a unique $x \in C_n(S)$ with $f(x) = y$. We have assumed there is another element $x' \in jS$ with $f(x') = y$. So $x' \notin C_n(S)$, and $x' \in C_m(S)$ for some $m > 0$ with $m \neq n$. But then $f(x') = y$ must be in $C_m(T)$, and it follows that $m$ is a proper multiple of $n$. Then we have $x, x' \in C_m(S)$, both mapping to $y \in C_m(T)$. But $C_m(f)$ is a bijection. Contradiction. □

**Corollary 9.** *There is a Quillen model structure on* ZSet *with*

a) *the Acyclic* ZSet *maps as the weak equivalences $\mathcal{W}$,*
b) *all* ZSet *maps as the fibrations $\mathcal{F}$,*
c) *cofibrations $\mathcal{C} = {}^\dagger\underline{\mathcal{F}} = {}^\dagger\mathcal{W}$.*

**Proof.** Consider the adjoint functors

$$F : \mathrm{Gph} \rightleftharpoons \mathrm{ZSet} : G.$$

We use these to transport our model structure on Gph to ZSet. Note that $F(\mathbf{s}) : (Z,\sigma) \to (Z,\sigma)$ is the successor function $\sigma : Z \to Z$, which is an isomorphism. This implies that every morphism in $\mathrm{cell}(FJ)$ is an isomorphism, so the hypothesis for the Transport Theorem is satisfied. Note that $\underline{\mathcal{F}} = \mathcal{W} \cap \mathcal{F} = \mathcal{W}$, since $\mathcal{F}$ is all ZSet maps; thus $\mathcal{C} = {}^\dagger\mathcal{W}$. □

We may summarily describe the above model structure on ZSet by

$$(\underline{\mathcal{C}}, \mathcal{F}) = (\mathrm{iso}, \mathrm{all}) \quad \text{and} \quad (\mathcal{C}, \underline{\mathcal{F}}) = (\mathcal{C}, \mathcal{W}).$$

Note that not every Acyclic ZSet map is an isomorphism. For instance, let $Z = (Z, +1)$ denote the integers viewed as a Z-set; any set $A$ gives a ZSet $\sum_{a \in A} Z = A \times Z$ (viewing the set $A$ as a Z-set with trivial action, and taking the product of Z-sets). If $A$ and $B$ are sets then any function $f : A \to B$ gives an ZSet map $f \times Z : A \times Z \to B \times Z$, by $(f \times Z)(a,n) = (f(a), n)$ for $a \in A$ and $n \in Z$. Then $f \times Z$ is always Acyclic, but is usually not an isomorphism.

Let us say that an element $x$ in a Z-set $(S, \sigma)$ is *free* when $\sigma^n(x) = x$ implies $n = 0$. Let us say that a ZSet map $f : (S, \sigma) \to (T, \sigma)$ *maps the free elements bijectively* when $f$ maps the free elements of $S$ bijectively to the free elements of $T$. It turns out that $f$ is a cofibration for our model on ZSet if and only if $f$ maps the free elements bijectively; since we will not need this result, the proof is left to the reader.

## 4. Fibrant graphs and cofibrant graphs

A Quillen model structure on a category determines some important classes of objects: the fibrant objects, the cofibrant objects, and the fibrant-cofibrant objects. In Section 5 we will describe how these help to establish a well-behaved theory of homotopy classes of morphisms in the category. In this section we investigate these objects for our model structures on Gph, NSet, and ZSet. Then we describe a functor to "replace" any graph by a related cofibrant graph.

**Definition.** Let $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ be a model structure on a category $\mathcal{E}$, and let $X$ be an object in $\mathcal{E}$. We say that $X$ is *fibrant* when $X \to 1$ is in $\mathcal{F}$ (where 1 is a terminal object); we say that $X$ is *cofibrant* when $0 \to X$ is in $\mathcal{C}$ (where 0 is an initial object). We say that $X$ is *fibrant-cofibrant* when it is both fibrant and cofibrant.

Let us see how this works in our model structures on Gph, NSet, and ZSet. We start by introducing some terminology specialized to these different categories.

- For any graph $X$, a *dead-end* in $X$ is a node with no arc leaving it.
- For any N-set $(S, \sigma)$, the *trajectory* $N(x)$ of any element $x$ in $S$ is the set $\{\sigma^n(x) : n \geq 0\}$. We may say that an element $x$ is *periodic* when $\sigma^n(x) = x$ for some $n > 0$. We may say that $x$ is *eventually periodic* when $x$ has finite trajectory, since $x$ has finite trajectory if and only if $\sigma^k(x) = \sigma^{n+k}(x)$ for some $n$ and $k$ (so that $\sigma^k(x)$ is periodic).
- For any Z-set $(S, \sigma)$, the *orbit* $Z(x)$ for $x$ in $S$ is the set of elements $\sigma^n(x)$ as $n$ ranges over the integers.

Note that we define the trajectory of an element in any N-set, but the orbit of an element only makes sense in a Z-set, since the definition involves the inverse function of $\sigma$. An element in a Z-set is periodic if and only if it has a finite orbit. In a Z-set, every element is either periodic (finite orbit) or free (infinite orbit). But in an N-set, an element with a finite trajectory may fail to be periodic.

**Proposition 10** (Gph)**.** *A graph $X$ is fibrant if and only if $X$ has no dead-ends. A graph $X$ is cofibrant if and only if $X$ is a disjoint union of whiskered finite-cycle graphs. A graph $X$ is fibrant-cofibrant if and only if $X$ is a disjoint union of whiskered finite-cycle graphs in which the whiskers have no dead-ends.*

**Proof.** A graph $X$ is fibrant if and only if the morphism $X \to 1$ is Surjecting; but this is true if and only if $X$ has at least one arc leaving each of its nodes. A graph $X$ is cofibrant if and only if $0 \to X$ is in $\mathcal{C} = {}^{\dagger}(I^{\dagger})$, the class of retracts (in the morphisms category) of morphisms in cell$(I)$. If graph $X$ is a disjoint union of whiskered finite-cycle graphs, then $0 \to X$ is in cell$(I)$; and every cofibrant graph is of this form, because the only way to get the empty graph 0 as domain in cell$(I)$ is to use a transfinite composition of pushouts of the $\mathbf{i}_n$, and taking retracts can't introduce any new morphisms with domain an empty graph. The description of fibrant-cofibrant graphs follows. □

It follows that the fibrant graphs are exactly those in which every path can be continued forever; this fits well with the terminology "no dead-ends". This can also be expressed by saying that any path $\mathbf{P}_n \to X$ (for any length $n \geq 0$) can be extended to an infinite path $\mathbf{P} \to X$. Such graphs are convenient for the study of symbolic dynamics (see Lind and Marcus [LM95], for instance).

Note that every cofibrant graph is in NGph, but $\mathbf{Z}$ is a N-graph which is not a cofibrant graph. Also, not every fibrant graph is in NGph.

**Proposition 11** (NSet)**.** *An* N*-set* $(S, \sigma)$ *is fibrant if and only if* $\sigma$ *is surjective. An* N*-set* $(S, \sigma)$ *is cofibrant if and only if each element in* $S$ *has finite trajectory.*

**Proof.** Recall the adjunctions $F : \mathrm{Gph} \rightleftharpoons \mathrm{NSet} : G$. By the transport theorem used to define the model structure on NSet, $f$ is a fibration in NSet if and only if $G(f)$ is a fibration in Gph. The first statement of the proposition follows from this, since the graph corresponding to an N-set $(S, \sigma)$ has a dead-end if and only if $\sigma$ is not surjective. The transport theorem also says that the set $F(\{\mathbf{s}\} \cup K)$ generates the cofibrations of NSet, where $K = \{\mathbf{i}_n, \mathbf{j}_n : n > 0\}$. Note that $GF(\mathbf{i}_n) = \mathbf{i}_n$ and $GF(\mathbf{j}_n) = \mathbf{j}_n$. Also, $GF(\mathbf{s})$ (is a whiskering which) attaches an arc to the zero node of the N-graph $\mathbf{P}^{op}$; its pushouts are attaching taprooted forests, in terminology from the previous section. Since $G$ is a left adjoint, it preserves colimits, and the class of N-set maps generated by $F(K)$ corresponds to the class of graph morphisms generated by $K$. Since the graph corresponding to an N-set $(S, \sigma)$ is a disjoint union of whiskered finite-cycle graphs if and only if each element in $S$ is eventually periodic, in that each trajectory in $S$ is finite, the second statement of the proposition follows. □

**Proposition 12** (ZSet)**.** *Every* Z*-set is fibrant. A* Z*-set* $(S, \sigma)$ *is cofibrant if and only if each element in* $S$ *has finite orbit.*

**Proof.** Every ZSet map is a fibration. Also, the graph corresponding to a Z-set is cofibrant if and only if all of its connected components are finite-cycle graphs. The rest of the proof follows that of the previous proposition. □

**Definition.** A *cofibrant replacement* for an object $X$ in a model category $\mathcal{E}$ is a weak equivalence $f : X' \to X$ where $X'$ is cofibrant. We will say that $f$ is a *nice cofibrant replacement* when we also have $f \in \underline{\mathcal{F}}$. Dually, a *fibrant replacement* of an object $Y$ is a weak equivalence $f : Y \to Y'$ where $Y'$ is fibrant; and $f$ is a *nice fibrant replacement* if also $f \in \underline{\mathcal{C}}$.

Each object $X$ in $\mathcal{E}$ has at least one cofibrant replacement, since $0 \to X$ has a $(\mathcal{C}, \underline{\mathcal{F}})$ factorization with $0 \to X'$ in $\mathcal{C}$ and $f : X' \to X$ in $\underline{\mathcal{F}}$; so this is actually a nice cofibrant replacement. Dually, we have the existence of (nice) fibrant replacements. If $X'$ is cofibrant and $g : X' \to X''$ is a nice fibrant replacement, then $X''$ is fibrant-cofibrant, since $\mathcal{C} = {}^{\dagger}\underline{\mathcal{F}}$ is closed under composition ($0 \to X'$ in $\mathcal{C}$ and $g$ in $\underline{\mathcal{C}}$ implies $0 \to X''$ in $\mathcal{C}$). Thus any nice fibrant replacement of a cofibrant object is fibrant-cofibrant, Dually, a nice cofibrant replacement of a fibrant object is fibrant-cofibrant.

Let us define a special cofibrant replacement for our model structure on Gph. Recall, from the end of Section 2, the adjoint pair

$$G : \mathrm{ZSet} \rightleftharpoons \mathrm{Gph} : H.$$

Here $H$ is given by the natural $Z$-action on the set of graph morphisms from the line graph $\mathbf{Z}$ to the graph $X$; and $G$ can be thought of as a Cayley graph construction (this is equivalent to the inclusion of ZGph as a subcategory of Gph). The adjoint pair $(G, H)$ has counit $G(H(X)) \to X$. Recall that for any Z-set $(S, \sigma)$ we have defined $j(S, \sigma)$ as the set of all elements $x \in S$ such that $\sigma^n(x) = x$ for some $n > 0$. Since $\sigma$ carries $jS$ into itself, we have a functor $j : \mathrm{ZSet} \to \mathrm{ZSet}$, and we may define $c(X) = G(jH(X))$. We may refer to $c(X)$ as the *cycle resolution* of $X$. For example, $c(\mathbf{C}_n) = \mathbf{C}_n$, and $c(X) = 0$ if $X$ is an acyclic graph.

Applying $G$ to $jH(X) \subseteq H(X)$ gives a natural graph morphism

$$c(X) = G(jH(X)) \to G(H(X)) \to X.$$

We have the following:

**Proposition 13.** *For every graph $X$, the graph morphism $c(X) \to X$ is a cofibrant replacement (although not in general a nice cofibrant replacement).*

**Proof.** We must show that $c(X)$ is cofibrant and that $c(X) \to X$ is an Acyclic graph morphism. Note that $c(X)$ is always isomorphic to a disjoint union of finite cycle graphs, and is thus a cofibrant graph. Clearly $jH(X) \to H(X)$ is an Acyclic ZSet map, so applying $G$ to it gives an Acyclic graph morphism. It remains to show that $h : GH(X) \to X$ is an Acyclic graph morphism; in other words, that $C_*(h)$ is a bijection. Any $\mathbf{C}_n \to X$ is the image under $C_n(h)$ of the graph morphism $\mathbf{C}_n = G(H(\mathbf{C}_n) \to G(H(X))$, so $C_*(h)$ is surjective.

Conversely, for any $\alpha : \mathbf{C}_n \to GH(X)$ consider $h \circ \alpha : \mathbf{C}_n \to X$. Applying the functor $GH$ gives $GH(h \circ \alpha) = GH(h) \circ GH(\alpha)$. But $GH(\mathbf{C}_n) = \mathbf{C}_n$ and $GHGH(X) = X$ (since $HG$ is the identity on any Z-set). Making these identifications in $GH(\alpha) : GH(\mathbf{C}_n) \to GHGH(X)$, we have $GH(h \circ \alpha) =$

$\alpha$. It follows that $C_*(h)$ is injective, so $C_*(h)$ is a bijection. To see that $c(X) \to X$ is not in general a nice cofibrant replacement, we consider the graph $X$ with two nodes, 0 and 1, and two arcs, $\ell : 0 \to 0$ and $a : 0 \to 1$. Then $c(X) = 1$ and the cofibrant replacement $c(X) \to X$ is not Surjecting, and is thus not in $\underline{\mathcal{F}}$. □

**Definition.** Let cZSet denote the full subcategory of ZSet with objects those Z-sets with every element periodic.

Let $i$ denote the inclusion of cZSet as full subcategory of ZSet. We may reinterpret $j$ (described above) as the left adjoint in the adjoint pair of functors

$$i : \text{cZSet} \rightleftharpoons \text{ZSet} : j.$$

The functor $i \circ j$ from ZSet to ZSet (with image cZSet) is a "comonad" on ZSet, and cZset is isomorphic to the topos of "coactions" for this comonad. This exhibits cZSet as a "quotient topos" of ZSet. See Mac Lane and Moerdijk [MM94] for a discussion of these concepts.

Let $Gi = G \circ i$ and $jH = j \circ H$, with adjoint pair $Gi : \text{cZSet} \rightleftharpoons \text{Gph} : jH$, which results from the composition of the two adjoint pairs

$$i : \text{cZSet} \rightleftharpoons \text{ZSet} : j \quad \text{and} \quad G : \text{ZSet} \rightleftharpoons \text{Gph} : H.$$

We may interpret the cofibrant replacement functor $c : \text{Gph} \to \text{Gph}$ as $c = Gi \circ jH$, which is the counit of the adjoint pair $(Gi, jH)$, from the composition of adjoint functors

$$\text{cZSet} \rightleftharpoons \text{ZSet} \rightleftharpoons \text{NSet} \rightleftharpoons \text{Gph}.$$

## 5. Homotopy categories

Quillen [Q67] introduced model categories as a framework for defining and working with homotopy categories. We discuss homotopy functors in general, then Quillen's definition of the homotopy category as a category of fractions, and then Quillen adjunctions and equivalences. Finally, we put these ideas to work in showing that our homotopy category of graphs is equivalent to cZSet.

Suppose that we are given a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a category $\mathcal{E}$. Recall that the morphisms in $\mathcal{W}$ are called weak equivalences. In the homotopy category, these should all become isomorphisms. Let us sneak up on this idea. We will say that a functor with domain $\mathcal{E}$ is a *homotopy functor* when it takes every $f \in \mathcal{W}$ to an isomorphism.

We want to understand the homotopy functors for our model structure on Gph. Consider functors from Gph to Set, for example. Recall the adjoint triple $(F, G, H)$, with $F(X) = \pi_0(X)$ and $H(X) = C_1(X)$. Then:

**Proposition 14.** *$H$ is a homotopy functor, and $F$ is not a homotopy functor.*

**Proof.** The functor $H$ is clearly a homotopy functor, since $C_1(f)$ is a bijection for every graph morphism in $\mathcal{W}$. For the second part, the following example, where $\mathbf{D}$ is the "dot" with one node and no arcs, shows that $F$ is not a homotopy functor: the graph morphism $f : 0 \to \mathbf{D}$ is in $\mathcal{W}$ since $C_n(f)$ is the bijection $\emptyset \to \emptyset$ for all $n > 0$. But $\pi_0(f)$ is $\emptyset \to 1$, which is not a bijection of sets. □

Recall the related adjoint triples $(F, G, H)$ relating Gph with NSet and with ZSet. We will use subscripts to distinguish the cases.

**Proposition 15.** *The functors* $F_{\mathrm{N}}, H_{\mathrm{N}} : \mathrm{Gph} \to \mathrm{NSet}$ *and the functors* $F_{\mathrm{Z}}, H_{\mathrm{Z}} : \mathrm{Gph} \to \mathrm{ZSet}$ *are not homotopy functors.*

**Proof.** The functor $\pi_0$ is a composition of the reflection functors

$$F : \mathrm{Gph} \to \mathrm{NSet} \to \mathrm{ZSet} \to \mathrm{Set}$$

including $F_{\mathrm{N}}$ and $F_{\mathrm{Z}}$. It follows that neither $F_{\mathrm{N}}$ nor $F_{\mathrm{Z}}$ is a homotopy functor, since if $F_1 : \mathrm{Gph} \to \mathcal{A}$ is a homotopy functor, and $F_2 : \mathcal{A} \to \mathcal{B}$ is any functor, then $F_2 \circ F_1 : \mathrm{Gph} \to \mathcal{B}$ must be a homotopy functor. The following example shows that neither of the $H$ functors is a homotopy functor. Let $\mathbf{Z}$ denote the graph $G(Z, +1)$. the graph morphism $f : 0 \to \mathbf{Z}$ is in $\mathcal{W}$ since $C_n(f)$ is the bijection $\emptyset \to \emptyset$ for all $n > 0$. But $H_{\mathrm{N}}(\mathbf{Z}) = H_{\mathrm{Z}}(\mathbf{Z}) = Z$, and $H_{\mathrm{N}}(f) = H_{\mathrm{Z}}(f)$ is $\emptyset \to Z$, which is not a bijection. □

In the discussion below we will show that $jH : \mathrm{Gph} \to \mathrm{cZSet}$ is a homotopy functor. This result underlies our calculation of the homotopical algebra of graphs.

Quillen [Q67] showed how to use a model structure to avoid set theoretic difficulties in the construction of a "category of fractions", which universally inverts the morphisms in $\mathcal{W}$ so that they become isomorphisms.

More precisely, Quillen used a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ for a category $\mathcal{E}$ to describe a particular category $\mathrm{Ho}(\mathcal{E})$, together with a functor $\gamma : \mathcal{E} \to \mathrm{Ho}(\mathcal{E})$ which is *initial* among homotopy functors on $\mathcal{E}$. This means that $\gamma$ is a homotopy functor and that any homotopy functor $\Phi : \mathcal{E} \to \mathcal{D}$ factors uniquely through $\gamma$, in that $\Phi = \Phi' \circ \gamma$ for a unique functor $\Phi' : \mathrm{Ho}(\mathcal{E}) \to \mathcal{D}$.

For example, if we use the trivial model structure (all, iso, all) on $\mathcal{E}$, then $\mathrm{Ho}(\mathcal{E})$ is isomorphic to $\mathcal{E}$. This will apply to our model structure on cZSet, and will help us describe Ho(Gph).

In Quillen's description, the objects of the category $\mathrm{Ho}(\mathcal{E})$ are the objects of $\mathcal{E}$. Together with the universal definition, this determines $\mathrm{Ho}(\mathcal{E})$ up to isomorphism of categories. The category $\mathrm{Ho}(\mathcal{E})$ is called the *homotopy category* for the model structure.

The universal definition of $\mathrm{Ho}(\mathcal{E})$ does not involve the fibrations and cofibrations, but these are used in Quillen's description of the set of morphisms from $X$ to $Y$ in $\mathrm{Ho}(\mathcal{E})$, for objects $X$ and $Y$ in $\mathcal{E}$. We may denote this homotopy morphism set by $\mathrm{Ho}(X, Y)$.

Here is a sketch of Quillen's description of $\mathrm{Ho}(X, Y)$ for any objects $X$ and $Y$ in $\mathcal{E}$. It uses the fibrations $\mathcal{F}$ and the cofibrations $\mathcal{C}$ as a kind of "scaffolding". Suppose that $X' \to X$ and $Y' \to Y$ are nice cofibrant replacements, and that $X' \to X''$ and $Y' \to Y''$ are nice fibrant replacements. Note that $X''$ and $Y''$ are objects which are both fibrant and cofibrant. Then any $f : X \to Y$ can be factored by $f' : X' \to Y'$, which can be factored by some $f'' : X'' \to Y''$. Quillen defines a "homotopy" equivalence relation $\sim$ on $\mathcal{E}(X'', Y'')$, and uses the fibrant, cofibrant scaffolding to formally define $\mathrm{Ho}(X, Y) = \mathcal{E}(X'', Y'')/\sim$. Quillen shows that this definition supports a well-defined composition (which is independent of the choice of scaffolding), and that this gives the desired category $\mathrm{Ho}(\mathcal{E})$ with functor $\gamma : \mathcal{E} \to \mathrm{Ho}(\mathcal{E})$.

The functor $\gamma : \mathcal{E} \to \mathrm{Ho}(\mathcal{E})$ gives a function $\gamma : \mathcal{E}(X, Y) \to \mathrm{Ho}(X, Y)$; we may denote $\gamma(f)$ by $[f]$. However, the function $\gamma$ is not always surjective; in general morphisms in $\mathrm{Ho}(\mathcal{E})$ are zig-zag compositions of homotopy classes of morphisms in $\mathcal{E}$, as we see from the use of replacements in the above discussion.

We say that two objects in $\mathcal{E}$ are *homotopy-equivalent* when they become isomorphic in $\mathrm{Ho}(\mathcal{E})$. Suppose that $\mathcal{E}_1$ and $\mathcal{E}_2$ are Quillen model categories, with model structures $(\mathcal{C}_i, \mathcal{W}_i, \mathcal{F}_i)$ for $\mathcal{E}_i$ $(i = 1, 2)$. If $F : \mathcal{E}_1 \to \mathcal{E}_2$ satisfies $F(\mathcal{W}_1) \subseteq \mathcal{W}_2$, then $\gamma \circ F : \mathcal{E}_1 \to \mathrm{Ho}(\mathcal{E}_2)$ is a homotopy functor, and thus factors through a unique $\mathrm{Ho}(\mathcal{E}_1) \to \mathrm{Ho}(\mathcal{E}_2)$. Most functors we consider don't satisfy such a strong condition. But Quillen gave a notion of *derived functor* suitable for homotopical algebra, as follows.

Suppose that $(L, R)$ is an adjoint pair of functors

$$L : \mathcal{E}_1 \rightleftharpoons \mathcal{E}_2 : R$$

between Quillen model categories $\mathcal{E}_1$ and $\mathcal{E}_2$. We say that $(L, R)$ is a *Quillen adjunction* when we have $L(\mathcal{C}_1) \subseteq \mathcal{C}_2$ and $L(\underline{\mathcal{C}}_1) \subseteq \underline{\mathcal{C}}_2$. It turns out to be equivalent to have $R(\mathcal{F}_2) \subseteq \mathcal{F}_1$ and $R(\underline{\mathcal{F}}_2) \subseteq \underline{\mathcal{F}}_1$ (see Hovey [H99], for instance). A Quillen adjunction $L : \mathcal{E}_1 \rightleftharpoons \mathcal{E}_2 : R$ between model categories leads to an adjunction between the respective homotopy categories, by means of derived functors. More precisely, Quillen described a *(total) left derived functor* $L'$ associated to $L$, and a *(total) right derived functor* $R'$ associated to $R$, giving an adjunction

$$L' : \mathrm{Ho}(\mathcal{E}_1) \rightleftharpoons \mathrm{Ho}(\mathcal{E}_2) : R'.$$

Here is a sketch of $L'$. Suppose we choose a nice cofibrant replacement $X' \to X$ for each object $X$. We can define $L'(X) = L(X')$; and we can define $L'([f]) = [f']$ for any $f : X \to Y$, where $X' \to X$ and $Y' \to Y$ are nice cofibrant replacements and $f' : X' \to Y'$ is a lifting of $f$. This definition of $L'$ extends uniquely to all morphisms in $\mathrm{Ho}(\mathcal{E}_1)$. The functor $L'$ comes with a natural transformation $\epsilon : L' \circ \gamma \to \gamma \circ L$, and it is final (closest on the left) among all such factorizations through $\gamma$ (see Proposition 1 of Chapter I.4 in Quillen [Q67]). This means that there is a unique $L'' \to L'$ for any natural transformation $\epsilon' : L'' \circ \gamma \to \gamma \circ L$. This universal condition determines the left derived functor $L'$ up to natural isomorphism of functors. The description of $R'$ is dual to this, using nice fibrant replacements, and has a dual universal property (initial, or closest *on the right* among all factorizations through $\gamma$). See Chapter I.4 in Quillen [Q67] for more details.

Here are some examples of derived functors on Gph. Recall from Section 2 the adjoint pair

$$F : \mathrm{Gph} \rightleftharpoons \mathrm{Set} : G$$

with $F(X) = \pi_0(X)$ and $G(S) = \sum_S 1$. Consider our model structure on Gph and the trivial model structure on Set.

**Proposition 16.** *$(F, G)$ is a Quillen adjunction, with derived adjunction $F' : \mathrm{Ho(Gph)} \rightleftharpoons \mathrm{Ho(Set)} : G'$ having $G'(S) = G(S)$ and $F'(X) = \pi_0(c(X))$.*

**Proof.** We use the trivial model structure (all, iso, all) on Set, and our model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on Gph. It is easy to see that $G(f) \in \mathcal{C}$ for any function $f$, and $G(h) \in \mathcal{C}$ for any bijection $h$; so the adjunction $(F, G)$ satisfies the Quillen conditions. In fact, $G$ is clearly a homotopy functor, so we may take $G' \circ \gamma = G$. We claim that we can take $F'(X) = F(c(X))$ in the left derived functor, despite the fact that $c(X) \to X$ is not a *nice* cofibrant replacement. This will become easy to verify when we have finished our calculation of Ho(Gph) by the end of this section. □

It is not hard to show that the reflections

$$F : \mathrm{Gph} \rightleftharpoons \mathrm{NSet} : G \quad \text{and} \quad F : \mathrm{Gph} \rightleftharpoons \mathrm{ZSet} : G$$

are also Quillen adjunctions; but we get more complete results by working instead with the coreflection adjoint pairs

$$G_Z : \mathrm{ZSet} \rightleftharpoons \mathrm{Gph} : H_Z \quad \text{and} \quad i : \mathrm{cZSet} \rightleftharpoons \mathrm{ZSet} : j.$$

Their derived functors are actually equivalences of categories, and this leads to a complete description of Ho(Gph). Here is the idea. Recall that an equivalence of categories is just a special kind of adjunction. A Quillen adjunction $(L, R)$ for which $(L', R')$ is an equivalence is called a *Quillen equivalence*. We will use the following standard characterization of Quillen equivalence (see Hovey [H99], for instance).

**Quillen Equivalence Theorem.** *A Quillen adjunction $(L, R)$ is a Quillen equivalence if and only if for all cofibrant $X$ in $\mathcal{E}_1$ and all fibrant $Y$ in $\mathcal{E}_2$ we have $LX \to Y$ in $\mathcal{W}_2$ if and only if $X \to RY$ in $\mathcal{W}_1$.*

We apply this to the following situation. We have described a model structure for each of the categories Gph, ZSet, and cZSet. We have an adjoint pair $(i, j)$ relating cZSet and ZSet. We have adjoint pairs $(G, H)$ and $(Gi, jH)$ relating ZSet and cZSet with Gph.

**Theorem 17.** *The homotopy categories* Ho(Gph) *and* Ho(ZSet) *are each equivalent to* cZSet. *More precisely:*

a) *The adjunction* $G : \mathrm{ZSet} \rightleftharpoons \mathrm{Gph} : H$ *is a Quillen equivalence.*

b) *The adjunction* $i : \mathrm{cZSet} \rightleftharpoons \mathrm{ZSet} : j$ *is a Quillen equivalence.*
c) *The adjunction* $Gi : \mathrm{cZSet} \rightleftharpoons \mathrm{Gph} : jH$ *is a Quillen equivalence.*
d) *The homotopy map* $\gamma : \mathrm{Ho(cZSet)} \to \mathrm{cZSet}$ *is an isomorphism of categories.*

**Proof.** For a) we use the second characterization of Quillen adjunctions, and check the behavior of $H$ on fibrations and acyclic fibrations. But, every Z-set map is a fibration; and $f$ is an acyclic fibration in Gph implies $H(f)$ is an acyclic fibration in ZSet, since $C_n(X) = C_n(H(X)$ for any graph $X$. The Quillen equivalence condition is satisfied, since $S$ cofibrant in ZSet means $S$ and $G(S)$ are disjoint unions of cycles, so $G(S) \to X$ in Gph preserves cycles if and only if $S \to H(X)$ in ZSet preserves periodic elements.

Recall that cZSet is the full subcategory of Z-sets in which every element is periodic. We are using the trivial model structure on cZSet; this shows d), as discussed in Section 4.

For b), we compare this with our model structure on ZSet, as described in Section 3. The functor $j : \mathrm{ZSet} \to \mathrm{cZSet}$ is given by $j(S, \sigma) = (jS, \sigma)$, and the functor $i : \mathrm{cZSet} \to \mathrm{ZSet}$ is just the inclusion of the subcategory. In Section 4 we showed that they form an adjoint pair $(i, j)$. To check the behavior of $i$ on cofibrations (and acyclic cofibrations), it suffices to note that $i(f)$ is a cofibration in ZSet for every $f$ in cZSet (and use that the acyclic cofibrations in cZSet are the isomorphisms). The Quillen equivalence condition follows from the fact that $C$ in cZSet is a disjoint union of cycles, so that $C \to j(S)$ is acyclic if and only if $i(C) \to S$ is acyclic, for any $S$ in ZSet.

For c), just consider the composition of $(i, j)$ with the coreflection adjoint pair $(G, H)$ between ZSet and Gph. □

A similar argument shows that Ho(NSet) is equivalent to Ho(ZSet), but we do not need this here.

## 6. Isospectral graphs

It seems that our homotopy category of graphs fits well with algebraic graph theory and other parts of combinatorics. Let us illustrate this by connecting the treatment of zeta series in Bisson and Tsemo [BT08] with that in Dress and Siebeneicher [DS88] and [DS89].

There they work with Burnside rings of Z-sets and actions of profinite groups, and show how this algebra is mirrored in theories of zeta series and Witt vectors. Recall that a Z-set is a set $S$ together with an invertible function $\sigma : S \to S$. For example, the integers modulo $n$ form a Z-set by taking $\sigma(i) = i + 1 \bmod n$; let us denote this Z-set by $Z/n$. For Z-sets $S$ and $T$, let $[S, T]$ denote the set of Z-set maps from $S$ to $T$. Let $Z(x)$ denote the orbit of an element $x$ in a Z-set. Recall that we say that an element $x$ in a Z-set is periodic when $Z(x)$ is finite.

**Definition** (Dress and Siebeneicher)**.** A Z-set $S$ is *essentially-finite* when $[Z/n, S]$ is finite for all $n > 0$. The *zeta series* of an essentially-finite Z-set $S$ is defined by

$$Z_S(u) = \exp\left(\sum_{n=1}^{\infty} c_n \frac{u^n}{n}\right),$$

where $c_n$ is the cardinality of $[Z/n, S]$, for all $n > 0$. An *almost-finite* Z-set is an essentially-finite Z-set for which every element is periodic.

Recall that $j$ : ZSet $\to$ cZSet is given by taking the periodic part of a Z-set, so that $j(S)$ is the set of periodic elements in $S$. Note that if $S$ is essentially-finite then $j(S)$ is almost-finite. In fact, $S$ and $j(S)$ have the same zeta series: by definition, two essentially-finite Z-sets $S$ and $T$ have the same zeta series if and only if $[Z/n, S]$ and $[Z/n, T]$ have the same cardinality for all $n > 0$; but the following result is noted in Dress and Siebeneicher [DS89] (without proof).

**Proposition 18.** *Two almost-finite* Z*-sets* $S$ *and* $T$ *have the same zeta series if and only if* $S$ *and* $T$ *are isomorphic as* Z*-sets.*

**Proof.** Let $Z/(n)$ denote the finite cyclic group of order $n$. If $S$ is an almost finite Z-set, then let $S_n = \{x : |Z(x)| = n\}$. So $S$ is the disjoint union of the $S_n$, and each $S_n$ is a finite and free $Z/(n)$-set. Let $s_n Z/n$ denote the sum ( disjoint union) of $s_n$ copies of the Z-set $Z/n$, where $s_n = |S_n|/n$. Thus $S_n$ is isomorphic to $s_n Z/n$ as Z-sets. This shows that $S$ is isomorphic to $\sum_{n>0} s_n Z/n$. It suffices to show that the numbers $(c_n : n > 0)$ determine the numbers $(s_n : n > 0)$. This is true because the "triangular" system of equations $c_n = \sum_{k|n} k s_k$ has a unique solution. □

This result shows that assigning a zeta series to each almost-finite Z-set gives an isomorphism between the Burnside ring of almost-finite Z-sets and the universal Witt ring (with integer coefficients). See Dress and Siebeneicher [DS88] for details.

Let us lift some of these definitions to the category Gph, by using our functor $H$ : Gph $\to$ ZSet; recall that $H(X)$ is the Z-set given by the natural "shift" on the set of all graph morphisms from $\mathbf{Z}$ to $X$.

**Definition.** A graph $X$ is *essentially-finite* if $H(X)$ is an essentially-finite Z-set. The *zeta series* of an essentially-finite graph $X$ is the formal power series $Z_X(u) = Z_{H(X)}(u)$. A graph $X$ is *almost-finite* when $H(X)$ is an almost-finite Z-set.

For example, the graph $\mathbf{Z}$ is an essentially-finite graph which is not almost-finite. Note that a graph $X$ is essentially finite if and only if $C_n(X)$ is finite for all $n > 0$. Also, a graph $X$ is essentially-finite if and only if $jH(X)$ is an almost-finite Z-set.

Let us review a few concepts from algebraic graph theory, with terminology as in Bisson and Tsemo [BT08]. A *finite graph* $X$ is one with finitely many nodes and arcs. The *characteristic polynomial* of a finite graph $X$ is defined as $a(x) = \det(xI - A)$, the characteristic polynomial of the adjacency operator $A$ for $X$. If $X$ has $n$ nodes, then $a(x)$ is a monic polynomial of degree $n$, and the *reversed characteristic polynomial* of $X$ is defined to be $u^n a(u^{-1}) = \det(I - uA)$. The roots of the characteristic polynomial of $X$ form the *spectrum* of $X$ (the eigenvalues of the adjacency operator for $X$). This motivates the following.

**Definition.** Two finite graphs $X$ and $Y$ are *isospectral* if they have the same characteristic polynomial. Two finite graphs $X$ and $Y$ are *almost-isospectral* if they have the same reversed characteristic polynomial.

Loosely speaking, $X$ and $Y$ are almost-isospectral if and only if they have the same non-zero eigenvalues, since $u = z$ is a root of $\det(I - uA)$ if and only if $z \neq 0$ and $x = z^{-1}$ is a root of $\det(xI - A)$.

We give a proof in Bisson and Tsemo [BT08] of the following (folk) result on directed graphs.

**Proposition 19.** *Two finite graphs $X$ and $Y$ are almost-isospectral if and only if $Z_X(u) = Z_Y(u)$.*

Note that every finite graph is almost-finite. We say that two graphs are *homotopy equivalent* when they become isomorphic in Ho(Gph).

**Theorem 20.** *Two finite graphs $X$ and $Y$ are almost-isospectral if and only if they are homotopy-equivalent.*

**Proof.** If $X$ is a finite graph, then $Z_X(u) = Z_{H(X)}(u) = Z_{jH(X)}(u)$. In Section 5 we showed that $jH : \mathrm{Gph} \to \mathrm{cZSet}$ is a homotopy functor. So there is a unique functor $H'' : \mathrm{Ho(Gph)} \to \mathrm{cZSet}$ such that $H'' \circ \gamma_1 = jH$ where $\gamma_1 : \mathrm{Gph} \to \mathrm{Ho(Gph)}$. In fact, $\gamma \circ H''$ is the total left derived functor of the Quillen equivalence $jH$, where $\gamma_2 : \mathrm{cZSet} \to \mathrm{Ho(cZSet)}$ is an isomorphism of categories. So $H''$ is an equivalence of categories.

Suppose that $X$ and $Y$ are almost-isospectral. Then $Z_X(u) = Z_Y(u)$. So $Z_{jH(X)}(u) = Z_{jH(Y)}(u)$. But $jH(X)$ and $jH(Y)$ are almost-finite Z-sets. So by the Dress-Siebeneicher Proposition above, $jH(X)$ and $jH(Y)$ are isomorphic in cZSet. So $H''(\gamma_1(X))$ and $H''(\gamma_1(Y))$ are isomorphic in cZSet. But $H''$ is an equivalence of categories, so $\gamma_1(X)$ and $\gamma_1(Y)$ are isomorphic in Ho(Gph); so $X$ and $Y$ are homotopy equivalent.

Conversely, suppose that $X$ and $Y$ are homotopy equivalent. This means that $\gamma_1(X)$ and $\gamma_1(Y)$ are isomorphic. So $H''(\gamma_1(X)) = jH(X)$ is isomorphic to $H''(\gamma_1(Y)) = jH(Y)$ in cZSet. So $jH(X)$ and $jH(Y)$ have the same zeta series. So $X$ and $Y$ have the same zeta series, so $X$ and $Y$ are almost-isospectral. $\square$

**Corollary 21.** *Two finite graphs have the same zeta series if and only if they are homotopy-equivalent in our model stucture for* Gph.

**Example.** Consider the graph with vertices $0, 1, 2, 3, 4$ and arcs $(0, i)$ and $(i, 0)$ for $i = 1, 2, 3, 4$; we will call it the Cross. Let $U\mathbf{C}_4$ be the undirected cycle, with nodes the integers mod 4, with arcs $(i, i+1)$ and $(i, i-1)$ for all $i$ mod 4, and with source and target given by $s(i, j) = i$ and $t(i, j) = j$. The characteristic polynomial of $U\mathbf{C}_4$ is $x^4 - 4x^2$; the characteristic polynomial of the Cross is $x^5 - 4x^3$. So they have the same reversed characteristic polynomial $1 - 4u^2$, and thus the same zeta series

$$Z(u) = (1 - 4u^2)^{-1} = \sum_{n \geq 0} 2^{2n} u^{2n} = \exp\left(\sum_{n>0} 2^{2n+1} \frac{u^{2n}}{2n}\right).$$

So $U\mathbf{C}_4$ and the Cross are almost-isospectral, and thus must be homotopically equivalent for our model structure for Gph.

We note in passing that the formula for $Z(u)$ says that there are no graph morphisms from an odd cycle to either graph, and that there are exactly $2^{2n+1}$ graph morphisms from $\mathbf{C}_{2n}$ to each graph (since $c_{2n} = 2^{2n+1}$).

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