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Seiberg–Witten equations on certain manifolds with cusps

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Abstract. We study the Seiberg–Witten equations on noncompact manifolds diffeomorphic to the product of two hyperbolic Riemann surfaces. First, we show how to construct irreducible solutions of the Seiberg–Witten equations for any metric of finite volume which has a "nice" behavior at infinity. Then we compute the infimum of the L^2-norm of scalar curvature on these spaces and give nonexistence results for Einstein metrics on blow-ups. This generalizes to the finite volume setting some well-known results of LeBrun.

Contents

1. Introduction

In this paper we study the Seiberg–Witten equations on product manifolds $M = \Sigma \times \Sigma_g$, where Σ is a finite volume hyperbolic Riemann surface and Σ_g a compact Riemann surface of genus g.

The main problem with Seiberg–Witten theory on noncompact manifold is the lack of a satisfactory existence theory. Following Biquard [4], we solve the SW equations on M by working on the compactification $\overline{M}$. Here the compactification $\overline{M}$ is the obvious one coming from the compactification of Σ. More precisely, we produce an irreducible solution of the unperturbed

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SW equations on M as limit of solutions of the perturbed SW equations on $\overline{M}$. From the metric point of view, starting with (M, g) where g is assumed to be of finite volume and with a "nice" behavior at infinity, one has to construct a sequence $(\overline{M}, g_j)$ of metric compactifications that approximate (M, g) as j goes to infinity. The irreducible solution of the SW equations on (M, g) is then constructed by a bootstrap argument with the solutions of the SW equations on $(\overline{M}, g_j)$ with suitably constructed perturbations.

When $M = \Sigma \times \mathbb{C}P^1$ this construction was carried out by Rollin in [21].

An outline of the paper follows. Section 2 describes explicitly the metric compactifications $(\overline{M}, g_j)$. These metrics are completely analogous to the one used by Rollin and Biquard in [21] and [4]. Furthermore, few results concerning the scalar curvatures and volumes of the spaces $(\overline{M}, g_j)$ are given.

In Section 3 we recall some basic facts about the L^2 cohomology of complete noncompact manifolds. Moreover, a scalar curvature estimate for finite volume manifolds which admits irreducible solutions of the unperturbed SW equations is given.

In Section 4 we compute the L^2-cohomology of $(\Sigma \times \Sigma_g, g)$ when g is a metric C^0 asymptotic to a product metric $g_{-1}+g_2$, where g_{-1} is a hyperbolic metric on Σ and g_2 any metric on Σ_g.

Sections 5 and 6 contain the uniform Poincaré inequalities on functions and 1-forms needed for the bootstrap argument. Moreover the convergence, as j goes to infinity, of the harmonic forms on $(\overline{M}, g_j)$ is studied in detail.

In Section 7 the bootstrap argument is worked out. The existence result so obtained is summarized in Theorem A.

In Section 8, Theorem A is applied to derive several geometrical consequences. First, we give the *sharp* minimization of the Riemannian functional $\int s_g^2 d\mu_g$ on M, where by s we denote the scalar curvature. Second, an obstruction to the existence of Einstein metrics on blow-ups of M is given. These results are summarized in Theorem B and Theorem C. These theorems are the finite volume generalization of some well-known results of LeBrun for closed four-manifolds, see for example [16] and the bibliography therein.

2. The metric compactifications

Let Σ be a finite volume hyperbolic Riemann surface and denote with Σ_g a compact Riemann surface of genus g. In this chapter, we study the Seiberg–Witten equations on manifolds that topologically are products of the form $\Sigma \times \Sigma_g$. Recall that Σ is conformally equivalent to a compact Riemann surface $\overline{\Sigma}$ with a finite number of points removed, say $\{p_1, \ldots, p_l\}$, satisfying the condition that $2g(\overline{\Sigma}) - 2 + l > 0$. Conversely, given a compact Riemann surface $\overline{\Sigma}$ and points $\{p_1, \ldots, p_l\}$ such that $2g(\overline{\Sigma}) - 2 + l > 0$, the open Riemann surface $\Sigma = \overline{\Sigma} \backslash \{p_1, \ldots, p_l\}$ admits a finite volume real hyperbolic metric. In summary, a finite volume hyperbolic Riemann surface (Σ, g_{-1}) is a manifold with finitely many cusps corresponding to the marked points of the associated compactification $\overline{\Sigma}$. Our hyperbolic cusps are modeled on $\mathbb{R}^+ \times S^1$ with the metric $g_{-1} = dt^2 + e^{-2t}d\theta^2$. We can now fix a metric g_2 on the compact Riemann surface Σ_g and consider the Riemannian product $(\Sigma \times \Sigma_g, g_{-1} + g_2)$. For simplicity we define $M = \Sigma \times \Sigma_g$. It is then clear that M is a complete finite volume manifold with cusp ends modeled on $\mathbb{R}^+ \times S^1 \times \Sigma_g$ with the metric $g = dt^2 + e^{-2t}d\theta^2 + g_2$.

Definition 1. A metric $\tilde{g}$ on M of the form $g_{-1}+g_2$ will be called a *standard model.*

We now want to study the natural compactification of M. It is clear that each of the cusp end of M can be closed topologically as a manifold by adding a compact genus g Riemann surface. Let us denote by N the disjoint union of these embedded curves. Denoted with $\overline{M}$ the compactification of M, we then have $\overline{M}\backslash N \simeq M$. If we know consider Σ and Σ_g as complex manifolds, it is clear that M can be compactified as a complex manifold by adding a finite number of genus g divisors with trivial self intersection.

Let us now consider a standard model $\tilde{g}$ on M. We want to construct a sequence of metrics $\{\tilde{g}_j\}$ on $\overline{M}$ that approximate $(M, \tilde{g})$. More precisely, choose coordinates on the cusp ends of M such that the metric $\tilde{g}$ is given by $\tilde{g} = dt^2 + e^{-2t}d\theta^2 + g_2$ for $t > 0$. Then define

$$\tilde{g}_j = dt^2 + \varphi_j^2(t)d\theta^2 + g_2$$

where $\varphi_j(t)$ is a smooth warping function such that:

(1) $\varphi_j(t) = e^{-t}$ for $t \in [0, j+1]$,
(2) $\varphi_j(t) = T_j - t$ for $t \in [j+1+\epsilon, T_j]$,

where ϵ is a fixed number that can be chosen to be small, and T_j is an appropriate number bigger than $j + 1 + \epsilon$. Because of the second condition above, $\tilde{g}_j$ is a smooth metric on $\overline{M}$ for any j. Moreover the metrics $\{\tilde{g}_j\}$ are by construction isometric to $\tilde{g}$ on bigger and bigger compact sets of M. For later convenience we want to prescribe in more details the behavior of $\varphi_j(t)$ in the interval $t \in [j+1, j+1+\epsilon]$. We require that $\partial_t^2\varphi_j(t)$ decreases from $e^{-(j+1)}$ to 0 in the interval $[j+1, j+1+\delta_j]$ where δ_j is a positive number less than ϵ. Then for $t \in [j+1+\delta_j, \epsilon]$, we make $\partial_t^2\varphi_j$ very negative in order to decrease $\partial_t\varphi_j$ to -1 and smoothly glue $\varphi_j(t)$ to the function $T_j - t$. Moreover, by eventually letting the parameters δ_j go to zero as j goes to infinity, we require $\frac{|\partial_t\varphi_j|}{\varphi_j}$ to be increasing in the interval $[j+1, j+1+\delta_j]$. Finally, we require $\frac{|\partial_t\varphi_j|}{\varphi_j}$ to be bounded from above uniformly in j.

In summary, given a standard model $\tilde{g}$ for M we can always generate a sequence of metrics $\{\tilde{g}_j\}$ on $\overline{M}$ approximating $(M, \tilde{g})$. A similar argument shows that this is indeed the case for any metric g on M, that is asymptotic to a standard model. For later convenience, we restrict ourself to metrics that are asymptotic to a standard model at least in the C^2 topology. More precisely, if g is such a metric we set

$$g_j = (1-\chi_j)g + \chi_j \tilde{g}_j$$

where $\chi_j(t)$ is a sequence of smooth increasing functions defined on the cusps of M such that $\chi_j(t) = 0$ if $t \leq j$ and $\chi_j(t) = 1$ if $t \geq j+1$.

Proposition 2.1. *The scalar curvature of the metrics $\{g_j\}$ can be expressed as*

$$s_{g_j} = s^b_{g_j} - 2\chi_j \frac{\partial_t^2 \varphi_j}{\varphi_j}$$

where $s^b_{g_j}$ is a smooth function on $\overline{M}$ that can be bounded uniformly in j.

Proof. For $t \leq j$, the metrics g_j and g are isometric and therefore $s_{g_j} = s_g$. If $t \in [j, j+1]$, the metric g_j is close in the C^2 topology to g and then $s_{g_j} \approx s_g$. Finally if $t \geq j+1$, the scalar curvature function is explicitly given by

$$s_{g_j} = s_{\tilde{g}_j} = s_{g_2} - 2\frac{\partial_t^2 \varphi_j}{\varphi_j}. \qquad \square$$

We conclude this section with a proposition regarding the volumes of the Riemannian manifolds $(\overline{M}, g_j)$.

Proposition 2.2. *There exists a constant $K > 0$ such that*

$$\mathrm{Vol}_{g_j}(\overline{M}) \leq K$$

for any j.

3. L^2 Bochner lemma

We start with a review of some facts about L^2-cohomology and its relation to the space of L^2-harmonic forms. For further details we refer to [1] and the bibliography therein. Given a orientable noncompact manifold (M, g) we have, when the differential d is restricted to an appropriate dense subset, a Hilbert complex

$$\cdots \longrightarrow L^2\Omega_g^{k-1}(M) \longrightarrow L^2\Omega_g^k(M) \longrightarrow L^2\Omega_g^{k+1}(M) \longrightarrow \cdots$$

where the inner products on the exterior bundles are induced by g. Define the maximal domain of d, at the k-th level, to be

$$\mathrm{Dom}^k(d) = \left\{\alpha \in L^2\Omega_g^k(M), d\alpha \in L^2\Omega_g^{k+1}(M)\right\}$$

where $d\alpha \in L^2\Omega_g^{k+1}(M)$ is to be understood in the distributional sense. The (reduced) L^2-cohomology groups are then defined to be

$$H_2^k(M) = Z_g^k(M)/\overline{d\,\mathrm{Dom}^{k-1}(d)},$$

where

$$Z_2^k(M) = \left\{\alpha \in L^2\Omega_g^k(M), d\alpha = 0\right\}.$$

On (M,g) there is a Hodge–Kodaira decomposition

$$L^2\Omega^k_g(M) = \mathcal{H}^k_g(M) \oplus \overline{dC^\infty_c\Omega^{k-1}} \oplus \overline{d^*C^\infty_c\Omega^{k+1}},$$

where

$$\mathcal{H}^k_g(M) = \big\{\alpha \in L^2\Omega^k_g(M), d\alpha = 0, d^*\alpha = 0\big\}.$$

Moreover, if we assume (M,g) to be complete the maximal and minimal domain of d coincide. In other words

$$\overline{d\,\mathrm{Dom}^{k-1}(d)} = \overline{dC^\infty_c\Omega^{k-1}},$$

which implies

$$H^k_2(M) = \mathcal{H}^k_g(M).$$

Here the completeness assumption is crucial in showing that if $\alpha \in L^2\Omega^k_g(M)$ with $d\alpha \in L^2\Omega^{k+1}_g(M)$, we can generate a sequence $\{\alpha_n\} \in C^\infty_c\Omega^k(M)$ such that $\|\alpha - \alpha_n\|_{L^2} + \|d\alpha - d\alpha_n\|_{L^2} \to 0$.

Summarizing, if the manifold is complete, the harmonic L^2-forms compute the reduced L^2-cohomology. Moreover, in this case the L^2 harmonic forms can be characterized as follows:

$$\mathcal{H}^k_g(M) = \big\{\alpha \in L^2\Omega^k_2(M), (dd^* + d^*d)\alpha = 0\big\}.$$

Finally, the orientability of M gives a duality isomorphism via the Hodge $*$ operator

$$\mathcal{H}^k_g(M) \simeq \mathcal{H}^{n-k}_g(M).$$

If the manifold M has dimension $4n$ it then makes sense to talk about L^2 self-dual and anti-self-dual forms on $L^2\Omega^{2n}_g(M)$. If $\mathcal{H}^{2n}_g(M)$ is finite-dimensional, the concept of L^2-signature is well-defined.

Let (M,g) be a complete finite-volume 4-manifold. Let $\mathcal{L}$ be a complex line bundle on M. By extending the Chern–Weil theory for compact manifolds, we can define the L^2-Chern class of $\mathcal{L}$. More precisely, given a connection A on $\mathcal{L}$ such that $F_A \in L^2\Omega^2_g(M)$, we may define

$$c_1(\mathcal{L}) = \frac{i}{2\pi}[F_A]_{L^2}$$

where with F_A we indicate the curvature of the given connection. It is an interesting corollary of the L^2-cohomology theory that, on complete manifolds, such an L^2-cohomology element is connection independent as long as we allow connections that differ by a 1-form in the maximal domain of the d operator. More precisely, let $A^{'}$ be a connection on $\mathcal{L}$ such that $A^{'} = A + \alpha$ with $\alpha \in L^2_1\Omega^1_g(M)$. We then have $F_{A'} = F_A + d\alpha$ and therefore by the Hodge–Kodaira decomposition we conclude that $\frac{i}{2\pi}[F_A]_{L^2} = \frac{i}{2\pi}[F_{A'}]_{L^2}$.

The associated L^2-Chern number $c_1^2(\mathcal{L})$ is also well-defined. In fact, $\alpha \in \mathrm{Dom}^1(d)$ and then we can find a sequence $\{\alpha_n\} \in C^\infty_c\Omega^k(M)$ such that

$\|\alpha-\alpha_n\|_{L^2}+\|d\alpha-d\alpha_n\|_{L^2}\to 0$. This implies that

$$\int_M F_{A'}\wedge F_{A'}d\mu_g=\lim_{n\to\infty}\int_M (F_A+d\alpha_n)\wedge(F_A+d\alpha_n)d\mu_g$$
$$=\int_M F_A\wedge F_A d\mu_g.$$

The following lemma is an easy consequence of the Hodge–Kodaira decomposition.

Lemma 3.1. *Given $\mathcal{L}$ and A as above, we have*

$$\int_M |F_A^+|^2 d\mu_g \geq 4\pi^2(c_1^+(\mathcal{L}))^2$$

where $c_1^+(\mathcal{L})$ is the self-dual part of the g-harmonic L^2 representative of $[c_1(\mathcal{L})]$.

Proof. We have

$$\int_M |F_A^+|^2 d\mu_g = 2\pi^2\int_M c_1(\mathcal{L})\wedge c_1(\mathcal{L})d\mu_g+\frac{1}{2}\int_M |F_A|^2 d\mu_g$$
$$=2\pi^2 c_1^2(\mathcal{L})+\frac{1}{2}\int_M |F_A|^2 d\mu_g.$$

By Hodge–Kodaira decomposition, given any L^2-cohomology class, we have a unique harmonic representative that minimizes the L^2-norm. Thus, given $F_A\in L^2\Omega_g^2(M)$, let us denote by φ its harmonic representative. We then have

$$\frac{1}{2}\int_M |F_A|^2 d\mu_g \geq \frac{1}{2}\int_M |\varphi|^2 d\mu_g$$

which implies

$$\int_M |F_A^+|^2 d\mu_g \geq 2\pi^2 c_1(\mathcal{L})^2+\frac{1}{2}\int_M |\varphi|^2 d\mu_g$$
$$=\int_M |\varphi^+|^2 d\mu_g = 4\pi^2(c_1^+(\mathcal{L}))^2. \qquad \square$$

We can now formulate the L^2 analogue of the scalar curvature estimate discovered in [13] for compact manifolds.

Theorem 3.2. *Let (M^4,g) be a finite volume Riemannian manifold where g is C^2 asymptotic to a standard model. Let $(A,\psi)\in L_1^2(M,g)$ be an irreducible solution of the SW equations associated to a* Spin^c *structure $\mathfrak{c}$ with determinant line bundle $\mathcal{L}$. Then*

$$\int_M s_g^2 d\mu_g \geq 32\pi^2(c_1^+(\mathcal{L}))^2$$

with equality if and only if g has constant negative scalar curvature, and is Kähler with respect to a complex structure compatible with $\mathfrak{c}$.

Proof. Following the strategy outlined in [13], the proof reduces to an integration by parts using the completeness of g. □

4. L^2-cohomology of products

Let (Σ, g_{-1}) be a finite volume hyperbolic Riemann surface. Furthermore, let (Σ_g, g_2) be a genus g compact Riemann surface equipped with a fixed metric. Let us consider $(\Sigma \times \Sigma_g, g_{-1} + g_2)$, where by $g_{-1} + g_2$ we denote the product metric. We then want compute the L^2 cohomology of $(\Sigma \times \Sigma_g, g)$ when g is a metric "asymptotic" to the product metric $g_{-1} + g_2$. Following the definition of Section 2 a metric of the from $g_{-1}+g_2$ is referred as standard metric or model. For simplicity let us define $M := \Sigma \times \Sigma_g$. Let us start by computing the L^2-cohomology of M when equipped with a standard metric.

Regarding the L^2-cohomology of $(M, g_{-1} + g_2)$, an L^2-Künneth formula argument [23] reduces the problem to the computation of the L^2-cohomology of a hyperbolic Riemann surface of finite topological type. This computation can be achieved by using the following classical theorem.

Theorem 1 (Huber)**.** *Let (Σ, g) be a complete finite volume Riemann surface with bounded curvature. Then Σ is conformally equivalent to a compact Riemann surface $\overline{\Sigma}$ with a finite number of points removed.*

Proof. See [10]. □

Corollary 4.1. *Let (Σ, g_{-1}) be a complete finite volume hyperbolic Riemann surface. Then we have the isomorphism*

$$H_2^*(\Sigma, g_{-1}) \simeq H^*(\overline{\Sigma}).$$

Proof. We clearly just have to prove that $H_2^1(\Sigma) \simeq H^1(\overline{\Sigma})$. Since Σ is complete, the space of L^2 harmonic forms computes the L^2-cohomology. Let $(\overline{\Sigma}\backslash\{p_1, \dots, p_l\}, \overline{g})$ as in Theorem 1, where $\overline{g} = e^{2u}g$. Since the L^2-cohomology is conformally invariant in the middle dimension, we have that $\mathcal{H}^1_{\overline{g}}(\overline{\Sigma}\backslash\{p_1, \dots, p_l\}) \simeq \mathcal{H}^1_{g_{-1}}(\Sigma)$. But now one can show that any harmonic field in $\mathcal{H}^1_{\overline{g}}(\overline{\Sigma}\backslash\{p_1, \dots, p_l\})$ can be smoothly extended across the cusp points. For the proof of this simple analytical fact see [5]. We therefore have $\mathcal{H}^1_{\overline{g}}(\overline{\Sigma}\backslash\{p_1, \dots, p_l\}) \simeq \mathcal{H}^1_{\overline{g}}(\overline{\Sigma})$. The corollary is now a consequence of the classical Hodge theorem for closed manifolds. □

We can now formulate the main result of this section.

Proposition 4.2. *In the notation above, consider (M, g) where g is a Riemannian metric C^0 asymptotic a standard model. Then we have the isomorphism*

$$H_2^*(M) \simeq H^*(\overline{M}; \mathbb{R}).$$

Proof. The L^2-cohomology is a quasi-isometric invariant. □

5. Poincaré inequalities and convergence of 1-forms

We need to show that, given the sequence of metrics $\{g_j\}$, we can find a uniform Poincaré inequality on functions. We have the following lemma.

Lemma 5.1. *Consider the metric $g = dt^2 + g_t$ on the product $[0, \infty) \times N$, such that the mean curvature of the cross-section N is uniformly bounded from below by a positive constant h_0. Then, for any function f we have*

$$\int |\partial_t f|^2 d\mu_g \geq h_0^2 \int |f|^2 d\mu_g + h_0 \int_{t=T} |f|^2 d\mu_{g_t} - h_0 \int_{t=0} |f|^2 d\mu_{g_t}.$$

Proof. See Lemma 4.1 in [4]. □

Using this lemma, we can now derive the desired uniform Poincaré inequality.

Proposition 5.2. *There exists a positive constant c, independent of j, such that*

$$\int_{\overline{M}} |df|^2 d\mu_{g_j} \geq c \int_{\overline{M}} |f|^2 d\mu_{g_j}$$

for any function f on $\overline{M}$ such that $\int_{\overline{M}} f d\mu_{g_j} = 0$.

Proof. See Corollaire 4.3. in [4]. □

Next, we have to derive an uniform Poincaré inequality for 1-forms. Given a 1-form α the following lemma holds:

Lemma 5.3. *There exists $T > 0$ such that*

$$\int_N |\nabla \alpha|^2 + Ric^{g_j}(\alpha, \alpha) d\mu_{g_t} \geq \int_N |\nabla_{\partial_t} \alpha|^2 d\mu_{g_t}$$

for any $t \in [T, T_j)$.

The proof of this lemma consists in a rather lengthy but elementary computation. This computation is based on an idea of Biquard [4], see also [21]. For the analytical details we refer to [6].

Observe now that for $[t_1, t_2] \subset [T, T_j]$

$$\begin{aligned}
\int_{\partial\{[t_1,t_2]\times N\}} |\alpha|^2 d\mu_{g_j} &= \int_{[t_1,t_2]\times N} \partial_t (|\alpha|^2 d\mu_{g_t}) dt \\
&= \int_{[t_1,t_2]\times N} \partial_t |\alpha|^2 d\mu_{g_t} dt + \int_{[t_1,t_2]\times N} |\alpha|^2 \partial_t d\mu_{g_t} dt \\
&= \int_{[t_1,t_2]\times N} \partial_t |\alpha|^2 d\mu_{g_j} - 2 \int_{[t_1,t_2]\times N} h |\alpha|^2 d\mu_{g_j}.
\end{aligned}$$

We then obtain

$$\int_{[t_1,t_2]\times N} \partial_t |\alpha|^2 d\mu_{g_j} \geq \int_{\partial\{[t_1,t_2]\times N\}} |\alpha|^2 d\mu_{g_j} + 2h_0 \int_{[t_1,t_2]\times N} |\alpha|^2 d\mu_{g_j}.$$

where h_0 is a uniform lower bound for the mean curvature. But now

$$\partial_t|\alpha|^2 = 2(\alpha, \nabla_{\partial_t}\alpha) \leq 2|\alpha||\nabla_{\partial_t}\alpha| \leq h_0|\alpha|^2 + \frac{1}{h_0}|\nabla_{\partial_t}\alpha|^2$$

which then implies

(1)
$$\int_{[t_1,t_2]\times N} |\nabla_{\partial_t}\alpha|^2 d\mu_{g_j} \geq h_0 \int_{\partial\{[t_1,t_2]\times N\}} |\alpha|^2 d\mu_{g_j} + h_0^2 \int_{[t_1,t_2]\times N} |\alpha|^2 d\mu_{g_j}.$$

We summarize the discussion above into the following lemma.

Lemma 5.4. *There exist positive numbers $c > 0$, $T > 0$ such that*

$$\int_{[t_1,t_2]\times N} |d\alpha|^2 + |d^{*g_j}\alpha|^2 d\mu_{g_j} \geq c \int_{[t_1,t_2]\times N} |\alpha|^2 d\mu_{g_j}$$

for any $[t_1, t_2] \subset [T, T_j)$ and α with support contained in $[t_1, t_2] \times N$.

Proof. Combining (1) and Lemma 5.3, the result follows from the well-known Bochner formula for 1-forms. □

The above lemma is almost the desired uniform Poincaré inequality. To conclude the proof we need few results concerning the convergence of harmonic 1-forms.

Proposition 5.5. *Let $[a] \in H^1_{dR}(\overline{M})$ and $\{\alpha_j\}$ be the sequence of harmonic representatives with respect the metrics $\{g_j\}$. Then $\{\alpha_j\}$ converges, with respect to the C^∞ topology on compact sets, to a harmonic 1-form $\alpha \in L^2\Omega^1_g(M)$.*

Proof. See Proposition 4.4. in [4]. □

It is now possible to refine Proposition 5.5 and analyze the convergence in more details. Notice that β can be chosen as follows:

$$\beta = \beta_c + \gamma$$

where β_c is a smooth closed 1-form with support not intersecting the cusp points $\{p_1, \ldots, p_l\}$ and $\gamma \in H^1(\Sigma_g; \mathbb{R})$. The metric g is C^2 asymptotic to a standard model, as a result

$$\lim_{t\to\infty} d^{*g}\gamma = 0$$

since γ can be chosen harmonic with respect to the metric g_2. Furthermore, given $\epsilon > 0$ we can find T big enough that $\lim_{j\to\infty} \|d^{*j}\gamma\|_{L^2_{g_j}(t\geq T)} \leq \epsilon$.

In other words we proved:

Lemma 5.6. *Given $\epsilon > 0$, there exists T big enough such that*

$$\int_{t\geq T} |d^*\beta|^2 d\mu_g \leq \epsilon, \quad \int_{t\geq T} |d^{*j}\beta|^2 d\mu_{g_j} \leq \epsilon.$$

We can now prove:

Lemma 5.7. *Given $\epsilon > 0$, there exists T big enough such that*

$$\int_{t\geq T} |\alpha|^2 d\mu_g \leq \epsilon, \quad \int_{t\geq T} |\alpha_j|^2 d\mu_{g_j} \leq \epsilon.$$

Proof. Recall that by construction $\alpha_j = \beta + df_j$, thus

$$\int_{t\geq T} |df_j|^2 d\mu_{g_j} = \int_{t=T} f_j \wedge *df_j - \int_{t\geq T} (d^* df_j, f_j) d\mu_{g_j}.$$

But now

$$d^{*_j} \alpha_j = d^{*_j} \beta + d^{*_j} df_j = 0 \Longrightarrow d^{*_j} df_j = -d^{*_j} \beta,$$

thus

$$\int_{t\geq T} |df_j|^2 d\mu_{g_j} = \int_{t=T} f_j \wedge *df_j + \int_{t\geq T} (d^* \beta, f_j) d\mu_{g_j}.$$

By the Cauchy inequality

$$\int_{t\geq T} (d^* \beta, f_j) d\mu_{g_j} \leq \|f_j\|_{L^2_{g_j}} \|d^{*_j} \beta\|_{L^2_{g_j}(t\geq T)}$$

and then this term can be made arbitrarily small. It remains to study the term $\int_{t=T} f_j \wedge *df_j$. Recall that $f_j \to f$ in the C^∞ topology on compact sets. Thus, for a fixed T

$$\int_{t=T} f_j \wedge *df_j \to \int_{t=T} f \wedge *df.$$

It remains to show that $\int_{t=T} f \wedge *df$ can be made arbitrarily small by taking T big enough. Define the function $F(s) = \int_{t=s} f * df$, since $f \in L^2_1$ we have $F(s) \in L^1(\mathbb{R}^+)$ and then we can find a sequence $\{s_k\} \to \infty$ such that $F(s_k) \to 0$. □

Proposition 5.8. *There exists $c > 0$ independent of j such that*

$$\int_{\overline{M}} |d\alpha|^2 + |d^{*_{g_j}} \alpha|^2 d\mu_{g_j} \geq c \int_{\overline{M}} |\alpha|^2 d\mu_{g_j}$$

for any $\alpha \perp \mathcal{H}^1_{g_j}$.

Proof. Let us proceed by contradiction. Assume the existence of a sequence $\{\alpha_j\} \in (\mathcal{H}^1_{g_j})^\perp$ such that $\|\alpha_j\|_{L^2(g_j)} = 1$ and for which

$$\int_{\overline{M}} |d\alpha_j|^2 + |d^{*_{g_j}} \alpha_j|^2 d\mu_{g_j} \longrightarrow 0$$

as $j \to \infty$. By eventually passing to a subsequence, a diagonal argument shows that $\{\alpha_j\}$ converges, with respect to the C^∞ topology on compact sets, to a 1-form $\alpha \in L^2\Omega^1_g(M)$. By construction $\alpha \in \mathcal{H}^1_g(M)$. On the other hand, Lemma 5.7 combined with the isomorphism $H^1_2(M) \simeq H^1(\overline{M})$ gives that $\alpha \in (\mathcal{H}^1_g)^\perp$. We conclude that $\alpha = 0$. Lemma 5.4 can now be easily applied to derive a contradiction. □

6. Convergence of 2-forms

In this section we have to study the convergence of 2-forms. The first result is completely analogous to the case of 1-forms.

Proposition 6.1. *Let $[a] \in H^2_{dR}(\overline{M})$ and $\{\alpha_j\}$ be the sequence of harmonic representatives with respect the sequence of metrics $\{g_j\}$. Then $\{\alpha_j\}$ converges, with respect to the C^∞ topology on compact sets, to a harmonic 2-forms $\alpha \in L^2\Omega^2_g(M)$.*

Proof. Given an element $a \in H^2_{dR}(\overline{M})$, take a smooth representative of the form $\beta = \beta_c + \gamma$ where β_c is a closed 2-form with support not intersecting the cusp points and $\gamma \in H^2(\Sigma_g; \mathbb{R})$. Given g_j, let α_j be the harmonic representative of the cohomology class determined by a. By the Hodge decomposition theorem we can write $\alpha_j = \beta + d\sigma_j$ with $\sigma_j \in (\mathcal{H}^1_{g_j})^\perp$ such that $d^{*_j}\sigma_j = 0$. Thus

$$0 = d^{*_j}\beta + d^{*_j}d\sigma_j \Longrightarrow d^*d\sigma_j = -d^{*_j}\beta.$$

Taking the global L^2 inner product of $d^*d\sigma_j$ with σ_j we obtain the estimate

$$(2) \qquad \begin{aligned}(d^*d\sigma_j, \sigma_j)_{L^2(g_j)} = \|d\sigma_j\|^2_{L^2} &= -\int_{\overline{M}} (\sigma_j, d^*\beta) d\mu_{g_j} \\ &\le \|\sigma_j\|_{L^2(g_j)} \|d^*\beta\|_{L^2(g_j)}.\end{aligned}$$

By Proposition 5.8, we conclude that

$$(3) \qquad \|\sigma_j\|^2_{L^2(g_j)} \le c\|d\sigma_j\|^2_{L^2(g_j)}.$$

Combining (2) and (3) we then obtain

$$\|\sigma_j\|^2_{L^2(g_j)} \le c\|d\sigma_j\|^2_{L^2(g_j)} \le c\|\sigma_j\|_{L^2(g_j)} \|d^{*_j}\beta\|_{L^2(g_j)}.$$

Since $\|d^{*_j}\beta\|_{L^2(g_j)}$ is bounded independently of j, we conclude that the same is true for $\|\sigma_j\|_{L^2(g_j)}$ and $\|d\sigma_j\|_{L^2(g_j)}$. By the elliptic regularity, we conclude that $\|\sigma_j\|_{L^2_1(g_j)}$ is uniformly bounded. Now a standard diagonal argument allows us to conclude that, up to a subsequence, $\{\sigma_j\}$ weakly converges to an element $\sigma \in L^2_1$. Using the elliptic equation

$$\Delta^{g_j}_H \sigma_j = -d^{*_j}\beta$$

and a bootstrapping argument it is possible to show that $\sigma_j \to \sigma$ in the C^∞ topology on compact sets. This proves the proposition. □

We know want to obtain a refinement of Proposition 6.1. We begin with the following simple lemma.

Lemma 6.2. *Given $\epsilon > 0$, there exists T big enough such that*

$$\int_{t\ge T} |d^{*_g}\beta|^2 d\mu_g \le \epsilon, \quad \int_{t\ge T} |d^{*_j}\beta|^2 d\mu_{g_j} \le \epsilon.$$

Proof. Since $\beta = \beta_c + \gamma$ with γ a fixed element in $H^2(\Sigma_g;\mathbb{R})$, the lemma follows from the definition of the metrics $\{g_j\}$. $\square$

An analogous result holds for the 2-forms $\{d\sigma_j\}$.

Lemma 6.3. *Given $\epsilon > 0$, there exists T big enough such that*

$$\int_{t\geq T} |d\sigma|^2 d\mu_g \leq \epsilon, \quad \int_{t\geq T} |d\sigma_j|^2 d\mu_{g_j} \leq \epsilon.$$

Proof. The first inequality follows easily from the fact that $\alpha \in L^2\Omega^2_g(M)$. By Lemma 6.2, given $\epsilon > 0$ we can find T such that

$$\|\sigma_j\|_{L^2(g_j)} \left\{ \int_{t\geq T} |d^*\beta|^2 d\mu_{g_j} \right\}^{\frac{1}{2}} \leq \frac{\epsilon}{2}$$

independently of the index j. Now

$$\int_{t\geq T} |d\sigma_j|^2 d\mu_{g_j} = \int_{t=T} \sigma_j \wedge *_j d\sigma_j - \int_{t\geq T} (d^{*_j} d\sigma_j, \sigma_j) d\mu_{g_j}$$

but $d^{*_j} d\sigma_j = -d^{*_j}\beta$, thus

$$\int_{t\geq T} |d\sigma_j|^2 d\mu_{g_j} \leq \frac{\epsilon}{2} + \left| \int_{t=T} \sigma_j \wedge *_j d\sigma_j \right|.$$

Since $\sigma_j \to \sigma$ in the C^∞ topology on compact sets, we have that $\int_{t=T} \sigma_j \wedge *_j d\sigma_j \to \int_{t=T} \sigma \wedge *_g d\sigma$. But now $\sigma \in L^2_1(g)$ and therefore we can conclude the proof of the proposition. $\square$

Lemma 6.4. *σ is orthogonal to the harmonic 1-form on (M, g).*

Proof. By construction we have $\sigma_j \in (\mathcal{H}^1_{g_j})^\perp$. Recall that fixed a cohomology element $[a] \in H^1_{dR}(\overline{M})$, denoted by $\{\gamma_j\}$ the sequence of the harmonic representatives with respect to the $\{g_j\}$, given $\epsilon > 0$ we can chose T such that $\int_{t\geq T} |\gamma_j|^2 d\mu_{g_j} \leq \epsilon$. Now, given $\gamma \in \mathcal{H}^1_g$ we want to show that $(\sigma, \gamma)_{L^2(g)} = 0$. Since $H^1_{dR}(\overline{M}) = \mathcal{H}^1_g(M)$, we can find a sequence of harmonic 1-forms $\{\gamma_j\}$ such that $\gamma_j \to \gamma$ in the C^∞ topology on compact sets. Let K be a compact set in M, then

$$\left| \int_{\overline{M}\setminus K} (\sigma_j, \gamma_j) d\mu_j \right| \leq \|\sigma_j\|_{L^2_{g_j}} \|\gamma_j\|_{L^2_{g_j}(\overline{M}\setminus K)} \tag{4}$$

can be made arbitrarily small by choosing the compact K big enough. Since $(\sigma_j, \gamma_j)_{L^2(\overline{M}, g_j)} = 0$, we have

$$\int_K (\sigma_j, \gamma_j) d\mu_{g_j} = -\int_{\overline{M}\setminus K} (\sigma_j, \gamma_j) d\mu_{g_j}$$

and then the integral $\int_K (\sigma_j, \gamma_j) d\mu_{g_j}$ can be made arbitrarily small. On the other hand

$$\left| \int_M (\sigma, \gamma) d\mu_g \right| \leq \left| \int_K (\sigma, \gamma) d\mu_g \right| + \|\sigma\|_{L^2(M,g)} \|\gamma\|_{L^2_g(M\setminus K)}.$$

Since $\gamma \in L^2\Omega^1_g(M)$ we conclude that $\sigma \in (\mathcal{H}^1_g)^\perp$. □

We now want to study the intersection form of $(\overline{M}, g_j)$ and eventually show the convergence to the L^2 intersection form of (M, g). Recall the isomorphism $H^2_{dR}(\overline{M}) \simeq \mathcal{H}^2(M)$, moreover given $[a] \in H^2_{dR}(\overline{M})$ we can generate $\{\alpha_j\} \in \mathcal{H}^2_{g_j}(\overline{M})$ that converges in the C^∞ topology on compact sets to a $\alpha \in \mathcal{H}^2_g(M)$. We also have that, fixed a compact set K, then $*_j = *_g$ for j big enough. As a result

$$\mathcal{H}^{+g_j} \oplus \mathcal{H}^{-g_j} \to \mathcal{H}^{+g} \oplus \mathcal{H}^{-g}.$$

Indeed

$$\alpha_j = \alpha_j^{+_j} + \alpha_j^{-_j} = \frac{\alpha_j + *_j\alpha_j}{2} + \frac{\alpha_j - *_j\alpha_j}{2} \to \alpha^{+g} + \alpha^{-g} = \alpha_g.$$

7. Biquard's construction

In this section we show how to construct an irreducible solution of the Seiberg–Witten equations on (M, g), *for any* metric g asymptotic to a standard model $\tilde{g}$.

Fix a Spin^c structure on $\overline{M}$, with determinant line bundle L, and let g be a cuspidal metric on $\overline{M}\backslash\Sigma$ that is assumed to be C^2 asymptotic to a standard model. Let $\{g_j\}$ be the sequence of metrics on $\overline{M}$ approximating (M, g) constructed in Section 2. Let (A_j, ψ_j) be a solution of the perturbed Seiberg–Witten equations on $(\overline{M}, g_j)$

$$\begin{cases} \mathcal{D}_{A_j}\psi_j = 0 \\ F^+_{A_j} + i2\pi\omega^+_j = q(\psi_j) \end{cases}$$

where $\omega_j = \frac{i}{2\pi}F_{B_j}$ and B_j is the connection 1-form on the line bundle $\mathcal{O}_{\overline{M}}(\Sigma)$ given by

$$B_j = d - i\chi_j(\partial_t\varphi_j)d\theta.$$

The idea is to show that, up to gauge transformations, the (A_j, ψ_j) converge in the C^∞ topology on compact sets to a solution of the unperturbed Seiberg–Witten equations on (M, g),

$$\begin{cases} \mathcal{D}_A\psi = 0, \\ F^+_A = q(\psi), \end{cases}$$

where $A = C + a$ with C is a fixed smooth connection on $L \otimes \mathcal{O}(-\Sigma)$, and $a \in L^2_1(\Omega^1_g(M))$ with $d^*a = 0$.

Lemma 7.1. *We have the decomposition*

$$s_{g_j} = s^b_{g_j} - 2\chi_j\frac{\partial^2_t\varphi_j}{\varphi_j}$$

$$F_{B_j} = -i\chi_j\frac{\partial^2_t\varphi_j}{\varphi_j}dt \wedge \varphi_j d\theta + F^b_j$$

with $s^b_{g_j}$ *and* F^b_j *bounded independently of* j*.*

Proof. See Proposition 2.1. □

Since $i2\pi\omega_j = -F_{B_j}$, we can rewrite the perturbed Seiberg–Witten equations as follows:

$$\begin{cases} \mathcal{D}_{A_j}\psi_j = 0, \\ F^+_{A_j} - F^+_{B_j} = q(\psi_j). \end{cases}$$

Recall that in the case under consideration, the twisted Licherowicz formula [11] reads as follows

$$\mathcal{D}^2_{A_j}\psi_j = \nabla^*_{A_j}\nabla_{A_j}\psi_j + \frac{s_{g_j}}{4}\psi_j + \frac{1}{2}F^+_{A_j}\cdot\psi_j.$$

By using the SW equations we have

$$0 = \nabla^*_{A_j}\nabla_{A_j}\psi_j + \frac{s_{g_j}}{4}\psi_j + \frac{|\psi_j|^2}{4}\psi_j + \frac{1}{2}F^+_{B_j}\cdot\psi_j.$$

Keeping into account the decomposition given in Lemma 7.1 we obtain

$$0 = \nabla^*_{A_j}\nabla_{A_j}\psi_j + P_j\psi_j + P^b_j\psi_j + \frac{|\psi_j|^2}{4}\psi_j$$

where

$$P_j\psi_j = -\frac{1}{2}\chi_j\frac{\partial^2_t\varphi_j}{\varphi_j}\psi_j - \frac{i}{2}\chi_j\frac{\partial^2_t\varphi_j}{\varphi_j}(dt\wedge\varphi_j d\theta)^+\cdot\psi_j$$

with P^b_j uniformly bounded in j. Now, it can be explicitly checked that for a metric of the form $dt^2+\varphi^2_j d\theta^2+g_2$ the self-dual form $(dt\wedge\varphi_j d\theta)^+$ acts by Clifford multiplication with eigenvalues $\pm i$. The eigenvalues of the operator P_j are then given by 0 and $-\chi_j\frac{\partial^2_t\varphi_j}{\varphi_j}$.

Lemma 7.2. *There exists a constant* $K>0$ *such that*

$$|\psi_j(x)|^2 \leq K$$

for every j *and* $x\in\overline{M}$*.*

Proof. Given a point $x\in\overline{M}$ choose an orthonormal frame $\{e_i\}$ centered at x such that $\nabla_{e_j}e_{i_{|x}} = 0$. We then compute

$$\begin{aligned} &-\sum_i e_i(e_i\langle\psi_j,\psi_j\rangle)_x \\ &\qquad = -\sum_i\{\langle\nabla_{e_i}\nabla_{e_i}\psi_j,\psi_j\rangle + 2\langle\nabla_{e_i}\psi_j,\nabla_{e_i}\psi_j\rangle + \langle\psi_j,\nabla_{e_i}\nabla_{e_i}\psi_j\rangle\}. \end{aligned}$$

Since $\nabla^2_{e_i,e_i}\psi_j = \nabla_{e_i}\nabla_{e_i}\psi_j$ and $\nabla^*_{A_j}\nabla_{A_j} = -\sum_i\nabla^2_{e_i,e_i}$ we have

$$\Delta|\psi_j|^2 + 2|\nabla_{A_j}\psi_j|^2 = 2\mathrm{Re}\langle\nabla^*_{A_j}\nabla_{A_j}\psi_j,\psi_j\rangle.$$

Thus, if x_j is a maximum point for $|\psi_j|^2$ we have $\Delta|\psi_j|^2_{x_j} \geq 0$ and therefore $\mathrm{Re}\langle \nabla^*_{A_j}\nabla_{A_j}\psi_j, \psi_j\rangle \geq 0$. In conclusion

$$\begin{aligned} 0 &= \mathrm{Re}\langle \nabla^*_{A_j}\nabla_{A_j}\psi_j, \psi_j\rangle_{x_j} + \mathrm{Re}\langle \{P_j + P^b_j\}\psi_j, \psi_j\rangle_{x_j} + \frac{|\psi_j|^4_{x_j}}{4} \\ &\geq \mathrm{Re}\langle \{P_j + P^b_j\}\psi_j, \psi_j\rangle_{x_j} + \frac{|\psi_j|^4_{x_j}}{4}. \end{aligned}$$

By construction the operator $P_j + P^b_j$ is uniformly bounded from below, the proof is then complete. □

Since $F^+_{A_j} - F^+_{B_j} = q(\psi_j)$ and by Lemma 7.2 the norms of the ψ_j are uniformly bounded, a similar estimate holds for $F^+_{A_j} - F^+_{B_j}$.

Lemma 7.3. *There exists a constant $K > 0$ such that*

$$\|\nabla_{A_j}\psi_j\|_{L^2(\overline{M}, g_j)} \leq K$$

for any j.

Proof. We have

$$\begin{aligned} 0 &= \int_{\overline{M}} \mathrm{Re}\langle \nabla^*_{A_j}\nabla_{A_j}\psi_j, \psi_j\rangle d\mu_{g_j} + \int_{\overline{M}} \mathrm{Re}\langle \{P^b_j + P_j\}\psi_j, \psi_j\rangle d\mu_{g_j} \\ &\quad + \frac{1}{2}\int_{\overline{M}} \mathrm{Re}\langle q(\psi_j)\psi_j, \psi_j\rangle d\mu_{g_j} \\ &= \|\nabla_{A_j}\psi_j\|^2_{L^2(\overline{M}, g_j)} + \int_{\overline{M}} \mathrm{Re}\langle \{P^b_j + P_j\}\psi_j, \psi_j\rangle d\mu_{g_j} + \frac{1}{4}\int_{\overline{M}} |\psi_j|^4 d\mu_{g_j} \end{aligned}$$

but now

$$\int_{\overline{M}} \mathrm{Re}\langle \{P^b_j + P_j\}\psi_j, \psi_j\rangle d\mu_{g_j} \geq -k\|\psi_j\|^2_{L^2(\overline{M}, g_j)}$$

which then implies

$$\begin{aligned} \|\nabla_{A_j}\psi_j\|^2_{L^2(\overline{M}, g_j)} &\leq k\|\psi_j\|^2_{L^2(\overline{M}, g_j)} - \frac{1}{4}\|\psi_j\|^4_{L^2(\overline{M}, g_j)} \\ &\leq k\|\psi_j\|^2_{L^2(\overline{M}, g_j)}. \end{aligned}$$

Since by Proposition 2.2 the volumes of the Riemannian manifolds $(\overline{M}, g_j)$ are uniformly bounded, the lemma follows from Lemma 7.2. □

Define $C_j = A_j - B_j$ and let C be a fixed smooth connection on the line bundle $L \otimes \mathcal{O}(-\Sigma)$. By the Hodge decomposition theorem we can write

$$C_j = C + \eta_j + \beta_j$$

where η_j is g_j-harmonic and $\beta_j \in (\mathcal{H}^1_{g_j})^\perp$. Thus

$$F^+_{C_j} = q(\psi_j) = F^+_C + d^+\beta_j.$$

Since C is a fixed connection 1-form, $\|F_C\|_{L^2(\overline{M},g_j)}$ is uniformly bounded in the index j. As a result, there exists $K>0$ such that

$$\|d^+\beta_j\|_{L^2(\overline{M},g_j)} \leq K$$

for any j. By the Stokes' theorem

$$\|d^+\beta_j\|^2_{L^2(\overline{M},g_j)} = \|d^-\beta_j\|^2_{L^2(\overline{M},g_j)}$$

and we then obtain an uniform upper bound on $\|d\beta_j\|_{L^2(\overline{M},g_j)}$. By Gauge fixing we can always assume $d^*\beta_j = 0$. The Poincaré inequality given in Proposition 5.8 can then be used to conclude that

$$c\|\beta_j\|^2_{L^2(\overline{M},g_j)} \leq \|d\beta_j\|^2_{L^2(\overline{M},g_j)} \leq 2K.$$

By a diagonal argument we can now extract a weak limit

$$\beta_j \rightharpoonup \beta$$

with $\beta \in L^2_1(M,g)$. Similarly we extract a weak limit

$$\eta_j \rightharpoonup \eta$$

with $\eta \in L^2(M,g)$ and harmonic with respect to g, see Proposition 5.5.

Define $a_j = \eta_j + \beta_j$ that by construction satisfies $d^*a_j = 0$. If we fix a compact set $K \subset M$, there exists j_0 such that for any $j \geq j_0$ the connection 1-form B_j restricted to K is zero. Thus, for any $j \geq j_0$ we have $A_j = C_j$ and then $C = A_j - a_j$. We know that a_j is uniformly bounded in $L^2(\overline{M}, g_j)$, by using Lemma 7.3 we conclude that $\|\nabla_C\psi_j\|^2_{L^2(K,g_j)}$ is bounded independently of j. On this compact set K we can therefore extract a weak limit of the sequence $\{\psi_j\} \rightharpoonup \psi$. By using a diagonal argument and recalling that in a Hilbert space the norm is lower semicontinuous with respect the weak convergence, we obtain a weak limit $\psi \in L^2_1(M,g)$.

Recall that on any compact set K, for j big enough we have $F^+_{A_j} = q(\psi_j)$. Since

$$\nabla F^+_{A_j} = \nabla_{A_j}\psi_j \otimes \psi_j^* + \psi_j \otimes \nabla_{A_j}\psi_j^* - \mathrm{Re}\langle \nabla_{A_j}\psi_j, \psi_j\rangle \mathrm{Id}$$

we conclude that $\|\nabla F^+_{A_j}\|_{L^2(K,g_j)}$ is uniformly bounded. In summary we have an L^2_1 bound on $F^+_{A_j}$. Consider now the first order elliptic operator $d^+ \oplus d^*$. By the elliptic L^p estimates we obtain

$$\begin{aligned} c\|a_j\|_{L^2_2(K,g_j)} &\leq \|a_j\|_{L^2(K,g_j)} + \|(d^+ \oplus d^*)a_j\|_{L^2_1(K,g_j)} \\ &\leq \|a_j\|_{L^2(K,g_j)} + \|d^+\beta_j\|_{L^2_1(K,g_j)} \end{aligned}$$

which gives us an uniform L^2_2 bound on a_j. Since $C = A_j - a_j$ on K, we can write

$$0 = \mathcal{D}_{A_j}\psi_j = \mathcal{D}_{C+a_j}\psi_j = \mathcal{D}_C\psi_j + \frac{1}{2}a_j \cdot \psi_j,$$

in other words

$$\mathcal{D}_C\psi_j = -\frac{1}{2}a_j \cdot \psi_j. \tag{5}$$

Combining the L^2_2 bound on a_j and the L^∞ bound on ψ_j with the Sobolev multiplication $L^2_2 \otimes L^p \to L^4$, for p big enough, we obtain a L^4 bound on $-\frac{1}{2}a_j \cdot \psi_j$, that is exactly the forcing term in the first order elliptic equation given in 5. By the elliptic L^p estimates we then obtain

$$c\|\psi_j\|_{L^4_1} \leq \|\psi_j\|_{L^4} + \|f\|_4$$

where we define $f = -\frac{1}{2}a_j \cdot \psi_j$. This shows $\psi_j \in L^4_1$ that combined with the Sobolev multiplication $L^2_2 \otimes L^4_1 \to L^3_1$ can be used to obtain a L^3_1 estimate on f. By applying again the elliptic L^p estimate we obtain

$$c\|\psi_j\|_{L^3_2} \leq \|\psi_j\|_{L^2} + \|f\|_{L^3_1}.$$

Now the Sobolev multiplication $L^2_2 \otimes L^3_2 \to L^2_2$ combined with the fact that $\psi_j \in L^3_2$, we obtain a L^2_2 bound on f. Once more the L^p elliptic estimates gives us

$$c\|\psi_j\|_{L^2_3} \leq \|\psi_j\|_{L^2} + \|f\|_{L^2_2}.$$

By using the Sobolev multiplication $L^2_3 \otimes L^2_3 \to L^2_3$ we then obtain a L^2_3 bound on $q(\psi_j)$ and therefore by the Seiberg–Witten equations on $F^+_{A_j}$. But now a L^2_3 estimate on $F^+_{A_j}$ gives us a analogous estimate on d^+a_j. The argument can now be reiterated to obtain an estimate on $\|\psi_j\|_{L^2_k}$ for any k. Then by the Sobolev embedding $L^2_k \hookrightarrow C^{k-3}$ we conclude that the ψ_j are indeed smooth. A completely analogous argument can now be used to show the C^∞ on compact sets of the $\{\psi_j\}$.

Let us summarize the discussion above into a theorem.

Theorem A. *Fix a* Spin^c *structure on* $\overline{M}$ *with determinant line bundle* L. *Let* g *be a metric on* M *asymptotic to a standard model in the* C^2 *topology, and let* $\{g_j\}$ *the sequence of metrics on* $\overline{M}$ *that approximate* g. *Let* $\{(A_j, g_j)\}$ *be the sequence of solutions of the SW equations with perturbations* $\{F^+_{B_j}\}$ *on* $\{(\overline{M}, g_j)\}$. *Then, up to gauge transformations, the solutions* $\{(A_j, \psi_j)\}$ *converge, in the* C^∞ *topology on compact sets, to a solution* (A, ψ) *of the unperturbed SW equations on* (M, g) *such that:*

- $A{=}C{+}a$ *where* C *is a fixed smooth connection on* $L\otimes\mathcal{O}(-\Sigma)$, $d^*a = 0$ *and* $a \in L^2_1(\Omega^1_g(M))$.
- $\psi \in L^2_1(M, g)$ *and there exists* $K > 0$ *such that* $sup_{x\in M}|\psi(x)| \leq K$.

8. Geometric applications

For a compact oriented 4-manifold N, the Gauss–Bonnet and Hirzebruch theorems state that

$$\chi(N) = \int_N E(g)d\mu_g, \quad \sigma(N) = \int_N L(g)d\mu_g$$

where $E(g)$ and $L(g)$ are respectively the Euler and signature forms associated to the metric g.

For noncompact manifolds the above curvature integrals might be not defined or dependent on the choice of the metric. Nevertheless, if the manifold has finite volume and bounded curvature these curvature integrals are defined. In this case it remains to study their metric dependence. Here, we want to compute

$$\chi(M,g) = \int_M E(g)d\mu_g, \quad \sigma(M,g) = \int_M L(g)d\mu_g$$

when g is a metric C^2 asymptotic to a standard model for M. The idea is to approximate the metric g with the sequence of metrics $\{g_j\}$ on $\overline{M}$. We then have

$$\chi(M,g) = \lim_{j\to\infty}\int_{t\leq j+1} E(g_j)d\mu_{g_j}, \quad \sigma(M,g) = \int_{t\leq j+1} L(g_j)d\mu_{g_j}.$$

Thus

$$\chi(M,g) = \chi(\overline{M}) - \lim_{j\to\infty}\int_{t\geq j+1} E(\tilde{g}_j)d\mu_{\tilde{g}_j}$$

and

$$\sigma(M,g) = \sigma(\overline{M}) - \lim_{j\to\infty}\int_{t\geq j+1} L(\tilde{g}_j)d\mu_{\tilde{g}_j}.$$

In other words, the characteristic numbers of (M,g) are computed in terms of $\chi(\overline{M})$ and $\sigma(\overline{M})$ plus a contribution coming from the cusps. More precisely we have the following proposition.

Proposition 8.1. *Let M be equipped with a metric g asymptotic in the C^2 topology to a standard model. Then, we have the equalities*

$$\chi(M,g) = \chi(\overline{M}) - l\chi(\Sigma_g), \quad \sigma(M,g) = \sigma(\overline{M}) = 0,$$

where l is the number of cusp ends of M.

Proof. See Proposition 3.4. in [4]. □

A simple Mayer–Vietoris argument can now be used to show that

$$\chi(M) = \chi(\overline{M}) - l\chi(\Sigma_g).$$

We then conclude that $\chi(M,g) = \chi(M)$. This discussion can then be summarized into the following proposition.

Proposition 8.2. *The Gauss–Bonnet theorem is valid on (M, g) for any metric g asymptotic in the C^2 topology to a standard model.*

We can now study the Riemannian functional $\int_M s_g^2 d\mu_g$ restricted to the space of metrics asymptotic to a standard model.

Theorem B. *Let M be equipped with a metric g asymptotic to a standard model in the C^2 topology. Then*

$$\frac{1}{32\pi^2}\int_M s_g^2 d\mu_g \geq 2\chi(\Sigma)\cdot\chi(\Sigma_g)$$

with equality if and only if g is the product of two -1 hyperbolic metrics on Σ and Σ_g.

Proof. Let us consider the standard Spin^c structure associated to the complex structure of $\overline{M}$. Theorem A can be used to construct an irreducible solution of the SW equations on (M, g). Furthermore, by applying Theorem 3.2 we conclude that

$$\frac{1}{32\pi^2}\int_M s_g^2 d\mu_g \geq (c_1(K_{\overline{M}}^{-1} - \Sigma)^+)^2.$$

By the adjunction formula we have

$$(c_1(K_{\overline{M}}^{-1} - \Sigma)^+)^2 \geq (c_1(K_{\overline{M}}^{-1} - \Sigma))^2 = 2(\chi(\overline{M}) + 2(g-1)l)$$

where l is the number of cusp ends. By Propositions 8.1 and 8.2

$$\chi(M) = \chi(\overline{M}) + 2l(g-1),$$

and we conclude that

$$\frac{1}{32\pi^2}\int_M s_g^2 d\mu_g \geq 2\chi(\Sigma)\cdot\chi(\Sigma_g)$$

with equality if and only if g is Kähler with constant negative scalar curvature and the harmonic representative of $c_1(\mathcal{L})$ is self-dual. The latter condition implies that g is Kähler–Einstein. We can now apply Theorem A for a Spin^c structure of complex type compatible the reversed oriented $\overline{M}$. This implies that g must be Kähler–Einstein with respect to the commuting complex structures J and $\overline{J}$ on M. This implies that g is the product of two hyperbolic -1 metrics on Σ and Σ_g. □

Finally, we present an obstruction for Einstein metrics on blow-ups.

Theorem C. *Let (M, g) as above. Let M' be obtained from M by blowing up k points. If $k \geq \frac{4}{3}\chi(\Sigma)\chi(\Sigma_g)$, then M' does not admit a cuspidal Einstein metric.*

Proof. By a result of Morgan–Friedman [7], we know that the manifold $\overline{M}\sharp k\overline{\mathbb{C}P^2}$ admits at least 2^k different Spin^c structures with determinant line bundles

$$L = K_{\overline{M}}^{-1} \pm E_1 \pm \cdots \pm E_k$$

for which the SW equations have irreducible solutions for each metric. Since

$$\begin{aligned}(c_1(L)^+)^2 &= (c_1(\overline{M})^+ \pm E_1^+ \pm \cdots \pm E_k^+)^2 \\ &= (c_1(\overline{M})^+)^2 + 2\sum_i c_i(\overline{M})^+ \cdot \pm E_i^+ + \left(\sum_i \pm E_i^+\right)^2\end{aligned}$$

we can chose a Spin^c structure whose determinant line bundle satisfies

$$(c_1(L)^+)^2 \geq (c_1(\overline{M})^+)^2 \geq c_1(\overline{M})^2 = c_1^2(\overline{M}).$$

We can now apply Theorem A for any of the Spin^c structure above and with respect to the metric g on $M^{'}$. We then construct 2^k irreducible solutions $(A, \psi) \in L_1^2(M^{'}, g)$, where $A = C + a$ with C a fixed smooth connection on $L \otimes \mathcal{O}(-\Sigma)$ and $a \in L_1^2(\Omega_g^1(M^{'}))$. By appropriately choosing the Spin^c structure and using Theorem 3.2 we compute

$$\begin{aligned}\frac{1}{32\pi^2}\int_{M'} s^2 d\mu_g &\geq (c_1(L \otimes \mathcal{O}(-\Sigma))^+)^2 \\ &\geq (c_1(L)^+)^2 + \Sigma^2 + 2K_{\overline{M}} \cdot \Sigma \\ &\geq c_1^2(\overline{M}) + 2K_{\overline{M}} \cdot \Sigma\end{aligned}$$

where in the last inequality we used the fact that Σ has trivial self intersection. By the adjunction formula we have

$$\begin{aligned}\int_{M'} s^2 d\mu_g &\geq c_1^2(\overline{M}) + 4l(g-1) \\ &= 2\chi(\overline{M}) + 4l(g-1),\end{aligned}$$

where k is the number of distinct components of the divisor Σ. By an obvious modification of Proposition 8.1 one has

$$\begin{aligned}\chi(M^{'}, g) &= \chi(\overline{M}) + k + 2l(g-1) \\ \sigma(M^{'}, g) &= -k\end{aligned}$$

Thus, if we assume g to be Einstein

$$\begin{aligned}c_1^2(\overline{M}) + 4l(g-1) - k &= 2\chi(M^{'}) + 3\sigma(M^{'}) \\ &= \frac{1}{4\pi^2}\int_{M'} 2|W_+|^2 + \frac{s^2}{24} d\mu_g \\ &\geq \frac{1}{96\pi^2}\int_{M'} s^2 d\mu_g \\ &\geq \frac{1}{3}(c_1^2(\overline{M}) + 4l(g-1))\end{aligned}$$

so that

$$\frac{2}{3}(c_1^2(\overline{M}) + 4l(g-1)) \geq k.$$

In other words if

$$k > \frac{4}{3}(\chi(\Sigma) \cdot \chi(\Sigma_g))$$

we cannot have a cuspidal Einstein metric on $M\sharp k\overline{\mathbb{C}P^2}$. The equality case can also be included and the proof goes as in the compact case. For more details, see [16]. The proof is then complete. □

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