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# Typicality of normal numbers with respect to the Cantor series expansion

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Abstract. Fix a sequence of integers $Q = \{q_n\}_{n=1}^{\infty}$ such that $q_n$ is greater than or equal to 2 for all $n$. In this paper, we improve upon results by J. Galambos and F. Schweiger showing that almost every (in the sense of Lebesgue measure) real number in $[0,1)$ is $Q$-normal with respect to the $Q$-Cantor series expansion for sequences $Q$ that satisfy a certain condition. We also provide asymptotics describing the number of occurrences of blocks of digits in the $Q$-Cantor series expansion of a typical number. The notion of strong $Q$-normality, that satisfies a similar typicality result, is introduced. Both of these notions are equivalent for the $b$-ary expansion, but strong normality is stronger than normality for the Cantor series expansion. In order to show this, we provide an explicit construction of a sequence $Q$ and a real number that is $Q$-normal, but not strongly $Q$-normal. We use the results in this paper to show that under a mild condition on the sequence $Q$, a set satisfying a weaker notion of normality, studied by A. Rényi, 1956, will be dense in $[0,1)$.

Contents



## 1. Introduction

**Definition 1.1.** Let $b$ and $k$ be positive integers. A *block of length $k$ in base $b$* is an ordered $k$-tuple of integers in $\{0,1,\ldots,b-1\}$. A *block of length*

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$k$ is a block of length $k$ in some base $b$. A *block* is a block of length $k$ in base $b$ for some integers $k$ and $b$.

**Definition 1.2.** Given an integer $b \geq 2$, the *$b$-ary expansion* of a real $x$ in $[0, 1)$ is the (unique) expansion of the form

$$x = \sum_{n=1}^{\infty} \frac{E_n}{b^n} = 0.E_1E_2E_3 \ldots$$

such that $E_n$ is in $\{0, 1, \ldots, b-1\}$ for all $n$ with $E_n \neq b-1$ infinitely often.

Denote by $N_n^b(B, x)$ the number of times a block $B$ occurs with its starting position no greater than $n$ in the $b$-ary expansion of $x$.

**Definition 1.3.** A real number $x$ in $[0, 1)$ is *normal in base $b$* if for all $k$ and blocks $B$ in base $b$ of length $k$, one has

$$\lim_{n \to \infty} \frac{N_n^b(B, x)}{n} = b^{-k}. \tag{1}$$

A number $x$ is *simply normal in base $b$* if (1) holds for $k = 1$.

Borel introduced normal numbers in 1909 and proved that almost all (in the sense of Lebesgue measure) real numbers in $[0, 1)$ are normal in all bases. The best known example of a number that is normal in base 10 is due to Champernowne [3]. The number

$$H_{10} = 0.1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12 \ldots,$$

formed by concatenating the digits of every natural number written in increasing order in base 10, is normal in base 10. Any $H_b$, formed similarly to $H_{10}$ but in base $b$, is known to be normal in base $b$. Since then, many examples have been given of numbers that are normal in at least one base. One can find a more thorough literature review in [4] and [5].

The $Q$-Cantor series expansion, first studied by Georg Cantor in [9], is a natural generalization of the $b$-ary expansion.

**Definition 1.4.** $Q = \{q_n\}_{n=1}^{\infty}$ is a *basic sequence* if each $q_n$ is an integer greater than or equal to 2.

**Definition 1.5.** Given a basic sequence $Q$, the *$Q$-Cantor series expansion* of a real $x$ in $[0, 1)$ is the (unique) expansion of the form

$$x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n} \tag{2}$$

such that $E_n$ is in $\{0, 1, \ldots, q_n - 1\}$ for all $n$ with $E_n \neq q_n - 1$ infinitely often. We abbreviate (2) with the notation $x = 0.E_1E_2E_3 \ldots$ with respect to $Q$.

Clearly, the $b$-ary expansion is a special case of (2) where $q_n = b$ for all $n$. If one thinks of a $b$-ary expansion as representing an outcome of repeatedly rolling a fair $b$-sided die, then a $Q$-Cantor series expansion may be thought of as representing an outcome of rolling a fair $q_1$ sided die, followed by a fair $q_2$ sided die and so on. For example, if $q_n = n+1$ for all $n$, then the $Q$-Cantor series expansion of $e-2$ is

$$e - 2 = \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots .$$

If $q_n = 10$ for all $n$, then the $Q$-Cantor series expansion for $1/4$ is

$$\frac{1}{4} = \frac{2}{10} + \frac{5}{10^2} + \frac{0}{10^3} + \frac{0}{10^4} + \cdots .$$

For a given basic sequence $Q$, let $N_n^Q(B,x)$ denote the number of times a block $B$ occurs starting at a position no greater than $n$ in the $Q$-Cantor series expansion of $x$. Additionally, define

$$Q_n^{(k)} = \sum_{j=1}^{n} \frac{1}{q_j q_{j+1} \cdots q_{j+k-1}}.$$

A. Rényi [7] defined a real number $x$ to be normal with respect to $Q$ if for all blocks $B$ of length 1,

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n^{(1)}} = 1. \tag{3}$$

If $q_n = b$ for all $n$, then (3) is equivalent to *simple normality in base* $b$, but not equivalent to *normality in base* $b$. Thus, we want to generalize normality in a way that is equivalent to normality in base $b$ when all $q_n = b$.

**Definition 1.6.** A real number $x$ is *$Q$-normal of order $k$* if for all blocks $B$ of length $k$,

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n^{(k)}} = 1.$$

We say that $x$ is *$Q$-normal* if it is $Q$-normal of order $k$ for all $k$. A real number $x$ is *$Q$-ratio normal of order $k$* if for all blocks $B$ and $B'$ of length $k$, we have

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{N_n^Q(B',x)} = 1.$$

$x$ is *$Q$-ratio normal* if it is $Q$-ratio normal of order $k$ for all positive integers $k$.

We make the following definitions:

**Definition 1.7.** A basic sequence $Q$ is *$k$-divergent* if $\lim_{n\to\infty} Q_n^{(k)} = \infty$. $Q$ is *fully divergent* if $Q$ is $k$-divergent for all $k$. $Q$ is *$k$-convergent* if it is not $k$-divergent.

**Definition 1.8.** A basic sequence $Q$ is *infinite in limit* if $q_n \to \infty$.

For $Q$ that are infinite in limit, it has been shown that the set of all $x$ in $[0,1)$ that are $Q$-normal of order $k$ has full Lebesgue measure if and only if $Q$ is $k$-divergent [7]. Therefore, if $Q$ is infinite in limit, then the set of all $x$ in $[0,1)$ that are $Q$-normal has full Lebesgue measure if and only if $Q$ is fully divergent. Suppose that $Q$ is 1-divergent. Given an arbitrary nonnegative integer $a$, F. Schweiger [8] proved that for almost every $x$ with $\epsilon > 0$, one has

$$N_n((a),x) = Q_n^{(1)} + O\left(\sqrt{Q_n^{(1)}} \cdot \log^{3/2+\epsilon} Q_n^{(1)}\right).$$

J. Galambos proved an even stronger result in [10]. He showed that for almost every $x$ in $[0,1)$ and for all nonnegative integers $a$,

$$N_n^Q((a),x) = Q_n^{(1)} + O\left(\sqrt{Q_n^{(1)}} \left(\log\log Q_n^{(1)}\right)^{1/2}\right).$$

We provide the following main results:

(1) A notion of strong $Q$-normality is provided and we construct an explicit example of a basic sequence $Q$ and a real number that is $Q$-normal, but not strongly $Q$-normal (Theorem 2.15).

(2) (Theorem 4.9) If $Q$ is a basic sequence that is infinite in limit and $B$ is a block of length $k$, then for almost every real number $x$ in $[0,1)$, we have

$$N_n^Q(B,x) = Q_n^{(k)} + O\left(\sqrt{Q_n^{(k)}} \left(\log\log Q_n^{(k)}\right)^{1/2}\right).$$

(3) If $Q$ is infinite in limit, then almost every real number is $Q$-normal of order $k$ if and only if $Q$ is $k$-divergent (Theorem 4.11).

(4) If $Q$ is $k$-convergent for some $k$, then the set of numbers that are $Q$-normal is empty (Proposition 5.1). If $Q$ is infinite in limit, then the set of $Q$-ratio normal numbers is dense in $[0,1)$ (Corollary 5.3).

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## 2. Strongly normal numbers

**2.1. Basic definitions and results.** In this section, we will introduce a notion of normality that is stronger than $Q$-normality. This notion of normality will arise naturally later in this paper and will be useful for studying the typicality of $Q$-normal numbers. We will first need to make definitions similar to those of $N_n^Q(B,x)$ and $Q_n^{(k)}$.

Given a real number $x \in [0,1)$, a basic sequence $Q$, a block $B$ of length $k$, a positive integer $p \in [1,k]$, and a positive integer $n$, we will denote by $N_{n,p}^Q(B,x)$ the number of times that the block $B$ occurs in the $Q$-Cantor series expansion of $x$ with starting position of the form $j \cdot k + p$ for $0 \leq j < \frac{n}{k}$.

If $n$ and $k$ are positive integers, define

$$\rho(n,k) = \lceil n/k \rceil - 1 = \max\left\{ i \in \mathbb{Z} : i < \frac{n}{k} \right\}.$$

Suppose that $Q$ is a basic sequence and that $n, p$, and $k$ are positive integers with $p \in [1,k]$. We will write

$$Q_{n,p}^{(k)} = \sum_{j=0}^{\rho(n,k)} \frac{1}{q_{jk+p} q_{jk+p+1} \cdots q_{jk+p+k-1}}.$$

**Definition 2.1.** Let $k$ be a positive integer. Then a basic sequence $Q$ is *strongly $k$-divergent*[1] if for all positive integers $p$ with $p \in [1,k]$, we have $\lim_{n \to \infty} Q_{n,p}^{(k)} = \infty$. A basic sequence $Q$ is *strongly fully divergent* if it is strongly $k$-divergent for all $k$.

Given a real number $x \in [0,1)$, a basic sequence $Q$, a block $B$ of length $k$, a positive integer $p \in [1,k]$, and a positive integer $n$, we will denote by $N_{n,p}^Q(B,x)$ the number of times the block $B$ occurs in the $Q$-Cantor series expansion of $x$ with positions of the form $j \cdot k + p$ for $0 \le j < \frac{n}{k}$.

**Definition 2.2.** Suppose that $Q$ is a basic sequence. A real number $x$ in $[0,1)$ is *strongly $Q$-normal of order $k$* if for all blocks $B$ of length $m \le k$ and all $p \in [1,m]$, we have

$$\lim_{n \to \infty} \frac{N_{n,p}^Q(B,x)}{Q_{n,p}^{(m)}} = 1.$$

A real number $x$ is *strongly $Q$-normal* if it is strongly $Q$ normal of order $k$ for all $k$.

We will use the following lemmas frequently and without mention:

**Lemma 2.3.** *Given a real number $x \in [0,1)$, a basic sequence $Q$, a block $B$ of length $k$, a positive integer $p \in [1,k]$, and a positive integer $n$, we have*

$$N_{n,1}^Q(B,x) + N_{n,2}^Q(B,x) + \cdots + N_{n,k}^Q(B,x) = N_n^Q(B,x) + O(1) \text{ and}$$

$$Q_{n,1}^{(k)} + Q_{n,2}^{(k)} + \cdots + Q_{n,k}^{(k)} = Q_n^{(k)} + O(1).$$

**Proof.** This follows directly from the definitions of $N_n^Q(B,x)$ and $Q_n^{(k)}$. □

[1] It is not true that $k$-divergent basic sequences must be strongly $k$-divergent. The following example of a 2-divergent basic sequence that is not strongly 2-divergent was suggested by C. Altomare (verbal communication): let the basic sequence $Q = \{q_n\}$ be given by

$$q_n = \begin{cases} \max(2, \lfloor n^{1/4} \rfloor) & \text{if } n \equiv 0 \pmod 4 \\ \max(2, \lfloor n^{1/4} \cdot \log^2 n \rfloor) & \text{if } n \equiv 1 \pmod 4 \\ \max(2, \lfloor n^{3/4} \rfloor) & \text{if } n \equiv 2 \pmod 4 \\ \max(2, \lfloor n^{3/4} \cdot \log^2 n \rfloor) & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

**Lemma 2.4.** *If $g_1, g_2, \dots, g_n$ are nonnegative functions on the natural numbers, then*

$$o(g_1) + o(g_2) + \cdots + o(g_n) = o(g_1 + g_2 + \cdots + g_n).$$

**Theorem 2.5.** *If $Q$ is a basic sequence and $x$ is strongly $Q$-normal of order $k$, then $x$ is $Q$-normal of order $k$.*

**Proof.** Let $m \le k$ be a positive integer and let $B$ be a block of length $k$. Since $x$ is strongly $Q$-normal of $k$, we know that for all $p \in [1, m]$, $N_{n,p}^Q(B, x) = Q_{n,p}^{(k)} + o\left(Q_{n,p}^{(k)}\right)$. Thus, we see that

$$\begin{aligned} N_n^Q(B,x) &= \sum_{p=1}^{m} N_{n,p}^Q(B,x) = \sum_{p=1}^{m} \left(Q_{n,p}^{(k)} + o\left(Q_{n,p}^{(k)}\right)\right) \\ &= \sum_{p=1}^{m} Q_{n,p}^{(k)} + o\left(\sum_{p=1}^{m} Q_{n,p}^{(k)}\right) = Q_n^{(k)} + o\left(Q_n^{(k)}\right), \end{aligned}$$

so $\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n^{(k)}} = 1$. Therefore, $x$ is $Q$-normal of order $k$. $\square$

**Corollary 2.6.** *Suppose that $Q$ is a basic sequence. If $x$ is strongly $Q$-normal, then $x$ is $Q$-normal.*

**2.2. Construction of a number that is $Q$-normal, but not strongly $Q$-normal of order 2.** In this subsection, we will work towards giving an example of a basic sequence $Q$ and a real number $x$ that is $Q$-normal, but not strongly $Q$-normal of order 2. We will use the conventions found in [6].

Given a block $B$, $|B|$ will represent the length of $B$. Given nonnegative integers $l_1, l_2, \dots, l_n$, at least one of which is positive, and blocks $B_1, B_2, \dots, B_n$, the block $B = l_1 B_1 l_2 B_2 \dots l_n B_n$ will be the block of length $l_1|B_1| + \cdots + l_n|B_n|$ formed by concatenating $l_1$ copies of $B_1$, $l_2$ copies of $B_2$, through $l_n$ copies of $B_n$. For example, if $B_1 = (2, 3, 5)$ and $B_2 = (0, 8)$, then $2B_1 1B_2 0B_2 = (2, 3, 5, 2, 3, 5, 0, 8)$. We will need the following definitions:

**Definition 2.7.** A *weighting* $\mu$ is a collection of functions $\mu^{(1)}, \mu^{(2)}, \mu^{(3)}, \dots$ with $\sum_{j=0}^{\infty} \mu^{(1)}(j) = 1$ such that for all $k$, $\mu^{(k)} : \{0, 1, 2, \dots\}^k \to [0, 1]$ and $\mu^{(k)}(b_1, b_2, \dots, b_k) = \sum_{j=0}^{\infty} \mu^{(k+1)}(b_1, b_2, \dots, b_k, j)$.

**Definition 2.8.** The *uniform weighting in base $b$* is the collection $\lambda_b$ of functions $\lambda_b^{(1)}, \lambda_b^{(2)}, \lambda_b^{(3)}, \dots$ such that for all $k$ and blocks $B$ of length $k$ in base $b$

$$\lambda_b^{(k)}(B) = b^{-k}. \tag{4}$$

**Definition 2.9.** Let $p$ and $b$ be positive integers such that $1 \le p \le b$. A weighting $\mu$ is $(p, b)$-*uniform* if for all $k$ and blocks $B$ of length $k$ in base $p$, we have

$$\mu^{(k)}(B) = \lambda_b^{(k)}(B) = b^{-k}. \tag{5}$$

Given blocks $B$ and $y$, let $N(B, y)$ be the number of occurrences of the block $B$ in the block $y$.

**Definition 2.10.** Let $\epsilon$ be a real number such that $0 < \epsilon < 1$ and let $k$ be a positive integer. Assume that $\mu$ is a weighting. A block of digits $y$ is $(\epsilon, k, \mu)$-*normal*[2] if for all blocks $B$ of length $m \leq k$, we have

$$\mu^{(m)}(B)|y|(1-\epsilon) \leq N(B, y) \leq \mu^{(m)}(B)|y|(1+\epsilon). \tag{6}$$

For the rest of this subsection, we use the following conventions. Given sequences of nonnegative integers $\{l_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ with each $b_i \geq 2$ and a sequence of blocks $\{x_i\}_{i=1}^{\infty}$, we set

$$L_i = |l_1 x_1 \dots l_i x_i| = \sum_{j=1}^{i} l_j |x_j|, \tag{7}$$

$$q_n = b_i \text{ for } L_{i-1} < n \leq L_i, \tag{8}$$

and

$$Q = \{q_n\}_{n=1}^{\infty}. \tag{9}$$

Moreover, if $(E_1, E_2, \dots) = l_1 x_1 l_2 x_2 \dots$, we set

$$x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n}. \tag{10}$$

Given $\{q_n\}_{n=1}^{\infty}$ and $\{l_i\}_{i=1}^{\infty}$, it is assumed that $x$ and $Q$ are given by the formulas above.

**Definition 2.11.** A *block friendly family* is a 6-tuple

$$W = \{(l_i, b_i, p_i, \epsilon_i, k_i, \mu_i)\}_{i=1}^{\infty}$$

with nondecreasing sequences $\{l_i\}_{i=1}^{\infty}$, $\{b_i\}_{i=1}^{\infty}$, $\{p_i\}_{i=1}^{\infty}$ and $\{k_i\}_{i=1}^{\infty}$ of nonnegative integers for which $b_i \geq 2$, $b_i \to \infty$ and $p_i \to \infty$, such that $\{\mu_i\}_{i=1}^{\infty}$ is a sequence of $(p_i, b_i)$-uniform weightings and $\{\epsilon_i\}_{i=1}^{\infty}$ strictly decreases to 0.

**Definition 2.12.** Let $W = \{(l_i, b_i, p_i, \epsilon_i, k_i, \mu_i)\}_{i=1}^{\infty}$ be a block friendly family. If $\lim k_i = K < \infty$, then let $R(W) = \{0, 1, 2, \dots, K\}$. Otherwise, let $R(W) = \{0, 1, 2, \dots\}$. A sequence $\{x_i\}_{i=1}^{\infty}$ of $(\epsilon_i, k_i, \mu_i)$-normal blocks of nondecreasing length is said to be $W$-*good* if for all $k$ in $R$, the following three conditions hold:

$$\frac{b_i^k}{\epsilon_{i-1} - \epsilon_i} = o(|x_i|); \tag{11}$$

$$\frac{l_{i-1}}{l_i} \cdot \frac{|x_{i-1}|}{|x_i|} = o(i^{-1} b_i^{-k}); \tag{12}$$

[2] Definition 2.10 is a generalization of the concept of $(\epsilon, k)$-normality, originally due to Besicovitch [2].

$$\frac{1}{l_i} \cdot \frac{|x_{i+1}|}{|x_i|} = o(b_i^{-k}). \tag{13}$$

We now state a key theorem of [6].

**Theorem 2.13.** *Let $W$ be a block friendly family and $\{x_i\}_{i=1}^{\infty}$ a $W$-good sequence. If $k \in R(W)$, then $x$ is $Q$-normal of order $k$. If $k_i \to \infty$, then $x$ is $Q$-normal.*

If $b$ and $w$ are positive integers where $b$ is greater than or equal to 2 and $w \geq 3$ is odd, then we let $C_{b,w}$ be one of the blocks formed by concatenating all the blocks of length $w$ in base $b$ in such a way that there are at least twice as many copies of the block $(0)$ at odd positions as the block $(1)$. For example, we could pick

$$\begin{aligned} C_{2,3} &= 1(0,0,0)1(1,0,1)1(0,1,0)1(0,0,1)1(0,1,1)1(1,0,0)1(1,1,0)1(1,1,1) \\ &= (0,0,0,1,0,1,0,1,0,0,0,1,0,1,1,1,0,0,1,1,0,1,1,1), \end{aligned}$$

which has 9 copies of $(0)$ at the odd positions and 3 copies of $(1)$ at the odd positions. Note that $|C_{b,w}| = wb^w$. The next lemma is proven identically to Lemma 4.2 in [6]:

**Lemma 2.14.** *If $K < w$ and $\epsilon = \frac{K}{w}$, then $C_{b,w}$ is $(\epsilon, K, \lambda_b)$-normal.*

**Theorem 2.15.** [3] *There exists a basic sequence $Q$ and a real number $x$ such that $x$ is $Q$-normal, but not strongly $Q$-normal of order 2.*

**Proof.** Let $x_1 = (0,1)$, $b_1 = 2$, and $l_1 = 0$. For $i \geq 2$, let $x_i = C_{2i,(2i+1)^2}$, $b_i = 2i$, and $l_i = (2i)^{9i+8}$. Set $\epsilon_1 = 1/2$, $k_1 = 1$, $p_1 = 2$ and $\mu_1 = \lambda_2$. For $i \geq 2$, put $\epsilon_i = 1/(2i+1)$, $k_i = 2i+1$, $p_i = b_i$, $\mu_i = \lambda_{2i}$, and

$$W = \{(l_i, b_i, p_i, \epsilon_i, k_i, \mu_i)\}_{i=1}^{\infty}.$$

Thus, since $x_i = C_{b,w}$ where $b = 2i$ and $w = (2i+1)^2$, $x_i$ is $(\epsilon_i, k_i, \lambda_{b_i})$-normal by Lemma 2.14.

In order to show that $\{x_i\}$ is a $W$-good sequence we need to verify (11), (12), and (13). Since $k_i \to \infty$, we let $k$ be an arbitrary positive integer. We will make repeated use of the fact that $|x_i| = (2i+1)^2 \cdot (2i)^{(2i+1)^2}$. We first verify (11):

$$\lim_{i\to\infty} |x_i| \Bigg/ \left( \frac{(2i)^k}{\frac{1}{2(i-1)+1} - \frac{1}{2i+1}} \right) = \lim_{i\to\infty} \frac{2(2i+1)^2 \cdot (2i)^{(2i+1)^2}}{(2i)^k \cdot (4i^2-1)} = \infty.$$

[3]Theorem 2.13 may be used to construct other explicit examples of $Q$-normal numbers that satisfy some unusual conditions. Given a basic sequence $Q$, we say that $x$ is $Q$-*distribution normal* if the sequence $\{q_1 q_2 \cdots q_n x\}_n$ is uniformly distributed mod 1. [1] uses Theorem 2.13 to give an example of a basic sequence $Q$ and a real number $x$ such that $x$ is $Q$-normal, but $q_1 q_2 \cdots q_n x \pmod 1 \to 0$, so $x$ is not $Q$-distribution normal.

We next verify (12). Since $l_{i-1}/l_i < 1$, $(2i-1)^2/(2i+1)^2 < 1$ and

$$\left(1-\frac{1}{i}\right)^{(2i+1)^2} < e^{-2(2i+1)},$$

we have

$$\begin{aligned}\lim_{i\to\infty}\frac{\frac{l_{i-1}}{l_i}\cdot\frac{x_{i-1}}{x_i}}{i^{-1}(2i)^{-k}} &\le \lim_{i\to\infty} i\cdot(2i)^k\cdot 1\cdot\frac{(2i-1)^2}{(2i+1)^2}\cdot\frac{(2i-2)^{(2i-1)^2}}{(2i)^{(2i+1)^2}}\\ &\le \lim_{i\to\infty} i(2i)^k\cdot 1\cdot\left(1-\frac{1}{i}\right)^{(2i+1)^2}\cdot(2i-2)^{-8i}\\ &\le \lim_{i\to\infty} i(2i+1)^k e^{-2(2i+1)}(2i-2)^{-8i} = 0.\end{aligned}$$

Lastly, we verify (13). Since $(2i+3)^2/(2i+1)^2 \le 2$, $(1+2/(2i+1))^{8i} < e^8$, and

$$\left(1+\frac{2}{2i+1}\right)^{(2i+1)^2} < 2e^{2(2i+1)},$$

we have

$$\begin{aligned}\lim_{i\to\infty}\frac{\frac{1}{l_i}\cdot\frac{|x_{i+1}|}{|x_i|}}{(2i)^{-k}} &= \lim_{i\to\infty}(2i)^{-9i-8+k}\cdot\frac{(2i+3)^2}{(2i+1)^2}\cdot\frac{(2i+2)^{(2i+3)^2}}{(2i)^{(2i+1)^2}}\\ &\le \lim_{i\to\infty}(2i)^{-9i-8+k}\cdot 2\cdot\left(1+\frac{1}{i}\right)^{(2i+1)^2}\cdot(2i+2)^{(8i+8)}\\ &\le \lim_{i\to\infty} 4e^{2(2i+1)}\left(1+\frac{1}{i}\right)^{8i+8}(2i)^{-i+k}\\ &\le \lim_{i\to\infty} 4e^{2(2i+1)+8}\cdot(2i)^{-i+k} = 0.\end{aligned}$$

Since $\lambda_{b_i}$ is $(p_i, b_i)$-uniform, $\{x_i\}$ is a $W$-good sequence and by Theorem 2.13, $x$ is $Q$-normal.

Since the length of each block $x_i$ is even, so there will always be at least twice as many copies of the block $(0)$ as the block $(1)$ in any initial segment of digits of $x$, so $x$ is not strongly $Q$-normal of order 2. □

## 3. Random variables associated with normality

For this section, we must recall a few basic notions from probability theory. Given a random variable $X$, we will denote the expected value of $X$ as $\mathrm{E}[X]$. We will denote the variance of $X$ as $\mathrm{Var}[X]$. Lastly, $\mathrm{P}(X=j)$ will represent the probability that $X=j$.

We consider $x$ as a random variable which has uniform distribution on the interval $[0,1)$. If $x = 0.E_1(x)E_2(x)E_3(x)\ldots$ with respect to $Q$, then we consider $E_1(x), E_2(x), E_3(x), \ldots$ to be random variables. So for all $n$, we have

$$\mathrm{P}\left(E_n(x)=j\right)=\begin{cases}\frac{1}{q_n} & \text{if } 0\le j\le q_n-1\\ 0 & \text{if } j\ge q_n.\end{cases}$$

**Lemma 3.1.** *If $Q$ is a basic sequence, then the random variables $E_1(x)$, $E_2(x)$, $E_3(x),\ldots$ are independent.*

**Proof.** Suppose $n_1$ and $n_2$ are distinct positive integers and $0\le F_j<q_j-1$ for all $j$. Then

$$\begin{aligned}\mathrm{P}&\left(E_{n_1}(x)=F_{n_1},E_{n_2}(x)=F_{n_2}\right)\\ &=\lambda\left(\{x\in[0,1):E_{n_1}(x)=F_{n_1}\text{ and }E_{n_2}(x)=F_{n_2}\}\right)\\ &=\frac{1}{q_{n_1}q_{n_2}}=\frac{1}{q_{n_1}}\cdot\frac{1}{q_{n_2}}=\mathrm{P}\left(E_{n_1}(x)=F_{n_1}\right)\cdot\mathrm{P}\left(E_{n_2}(x)=F_{n_2}\right). \quad\square\end{aligned}$$

Suppose that $Q$ is a basic sequence, $b$ is a natural number, $B$ is a block of length $k$, and $m=ik+p$ is an integer with $p\in[0,k-1]$. We set

$$\zeta^Q_{b,n}(x)=\begin{cases}1 & \text{if } E_n(x)=b\\ 0 & \text{if } E_n(x)\ne n,\end{cases}$$

$$\zeta^Q_{B,i,p}(x)=\begin{cases}1 & \text{if } E_{ik+p,k}(x)=B\\ 0 & \text{if } E_{ik+p,k}(x)\ne B,\end{cases}$$

$$F^{(k)}_m=\mathrm{E}\left[\zeta^Q_{B,i,p}(x)\right],\quad V^{(k)}_m=\mathrm{Var}\left[\zeta^Q_{B,i,p}(x)\right],\quad t^{(k)}_{n,p}=\sum_{i=0}^{\rho(n,k)}V^{(k)}_{ik+p}.$$

**Lemma 3.2.** *For all non-negative integers $b$, the random variables $\zeta^Q_{b,1}(x)$, $\zeta^Q_{b,2}(x)$, $\zeta^Q_{b,3}(x),\ldots$ are independent.*

**Proof.** This follows directly from Lemma 3.1 as the random variables $E_1(x)$, $E_2(x)$, $E_3(x),\ldots$ are independent. $\square$

**Lemma 3.3.** *If $B=(b_1,b_2,\ldots,b_k)$ is a block of length $k$, then*

$$\zeta^Q_{B,i,p}(x)=\zeta^Q_{b_1,ik+p}(x)\cdot\zeta^Q_{b_2,ik+p+1}(x)\cdots\zeta^Q_{b_k,ik+p+k-1}(x).$$

**Proof.** By definition,

$$\zeta^Q_{B,i,p}(x)=\begin{cases}1 & \text{if } E_{ik+p,k}=B\\ 0 & \text{if } E_{ik+p,k}\ne B.\end{cases}$$

In other words, $\zeta^Q_{B,i,p}(x)=1$ if

$$\zeta^Q_{b_1,ik+p}(x)=\zeta^Q_{b_2,ik+p+1}(x)=\cdots=\zeta^Q_{b_k,ik+p+k-1}(x)=1$$

and $\zeta^Q_{B,i,p}(x)=0$ otherwise. $\square$

**Corollary 3.4.** *For all blocks $B = (b_1, b_2, \ldots, b_k)$ of length $k$ and nonnegative integers $p_1, p_2 \in [1, k]$, $i_1$, and $i_2$ with $(i_1, p_1) \neq (i_2, p_2)$, the random variables $\zeta^Q_{B,i_1,p_1}(x)$ and $\zeta^Q_{B,i_2,p_2}(x)$ are independent.*

**Proof.** Using Lemma 3.2 and Lemma 3.3, we see that

$$\begin{aligned}
&\mathrm{E}\left[\zeta^Q_{B,i_1,p_1}(x)\cdot\zeta^Q_{B,i_2,p_2}(x)\right]\\
&=\mathrm{E}\left[\left(\Pi_{j=0}^{k-1}\zeta^Q_{b_j,i_1k+p_1+j}(x)\right)\cdot\left(\Pi_{j=0}^{k-1}\zeta^Q_{b_j,i_2k+p_2+j}(x)\right)\right]\\
&=\left(\Pi_{j=0}^{k-1}\mathrm{E}\left[\zeta^Q_{b_j,i_1k+p_1+j}(x)\right]\right)\cdot\left(\Pi_{j=0}^{k-1}\mathrm{E}\left[\zeta^Q_{b_j,i_2k+p_2+j}(x)\right]\right)\\
&=\mathrm{E}\left[\Pi_{j=0}^{k-1}\zeta^Q_{b_j,i_1k+p_1+j}(x)\right]\cdot\mathrm{E}\left[\Pi_{j=0}^{k-1}\zeta^Q_{b_j,i_2k+p_2+j}(x)\right]\\
&=\mathrm{E}\left[\zeta^Q_{B,i_1,p_1}(x)\right]\cdot\mathrm{E}\left[\zeta^Q_{B,i_2,p_2}(x)\right].
\end{aligned}$$

□

**Lemma 3.5.** *If $B = (b_1, b_2, \ldots, b_k)$ is a block of length $k$, then*

$$F_m^{(k)} = \frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} \text{ and}$$

$$V_m^{(k)} = \frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} - \left(\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}}\right)^2.$$

**Proof.** We first compute the expected value of $\zeta^Q_{B,i,p}(x)$. By Lemma 3.2 and Lemma 3.3, we see that

$$\begin{aligned}
\mathrm{E}\left[\zeta^Q_{B,i,p}(x)\right] &= \mathrm{E}\left[\zeta^Q_{b_1,ik+p}(x)\cdot\zeta^Q_{b_2,ik+p+1}(x)\cdots\zeta^Q_{b_k,ik+p+k-1}(x)\right]\\
&=\mathrm{E}\left[\zeta^Q_{b_1,ik+p}(x)\right]\cdot\mathrm{E}\left[\zeta^Q_{b_2,ik+p+1}(x)\right]\cdots\mathrm{E}\left[\zeta^Q_{b_k,ik+p+k-1}(x)\right]\\
&=\frac{1}{q_{ik+p}}\cdot\frac{1}{q_{ik+p+1}}\cdots\frac{1}{q_{ik+p+k-1}}\\
&=\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}}.
\end{aligned}$$

Next, we recall that $\mathrm{Var}\left[\zeta^Q_{B,i,p}(x)\right] = \mathrm{E}\left[\zeta^Q_{B,i,p}(x)^2\right] - \mathrm{E}\left[\zeta^Q_{B,i,p}(x)\right]^2$. Since $\zeta^Q_{B,i,p}(x)$ may only be 0 or 1, we see that $\left(\zeta^Q_{B,i,p}(x)\right)^2 = \zeta^Q_{B,i,p}(x)$, so

$$\begin{aligned}
&\mathrm{Var}\left[\zeta^Q_{B,i,p}(x)\right]\\
&\quad=\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} - \left(\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}}\right)^2.
\end{aligned}$$

□

Lastly, we remark that $Q_{n,p}^{(k)} = \sum_{i=0}^{\rho(n,k)} F_{ik+p}^{(k)}$ by Lemma 3.5 and will use this fact frequently and without mention.

## 4. Typicality of normal numbers

We will need the following:

**Theorem 4.1.** [4] *Let $X_1, X_2, \ldots, X_n$ be independent random variables. Assume that there exists a constant $c > 0$ such that $|X_j| < c$ for all $j$. Let $G_j = E[X_j], U_j = Var[X_j]$, and $t_n = \sum_{j=1}^{n} U_j$. If $t_n \to \infty$, then, with probability one,*

$$\limsup_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n - G_1 - G_2 - \ldots G_n}{\sqrt{2t_n \log\log t_n}} = 1.$$

**Corollary 4.2.** *Under the same assumptions of Theorem* 4.1*, with probability one,*

$$X_1 + X_2 + \cdots + X_n = G_1 + G_2 + \cdots + G_n + O\left(t_n^{1/2}(\log\log t_n)^{1/2}\right).$$

We will also need the Borel–Cantelli Lemma:

**Theorem 4.3** (The Borel–Cantelli Lemma)**.** *If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A_n \text{ i.o.}) = 0$.*

Given a basic sequence $Q$, we will define $t_{n,p}^{(k)} = \sum_{i=0}^{\rho(n,k)} V_{jk+p}^{(k)}$.

**Lemma 4.4.** *If $Q$ is a basic sequence and $n, k$, and $p$ are positive integers with $p \in [1, k]$, then*

$$\frac{1}{2} Q_{n,p}^{(k)} \le t_{n,p}^{(k)} < Q_{n,p}^{(k)}.$$

**Proof.**

$$t_{n,p}^{(k)} = \sum_{i=0}^{\rho(n,k)} \left( \frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} - \left( \frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} \right)^2 \right)$$
$$< \sum_{i=0}^{\rho(n,k)} \frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} = \sum_{i=0}^{\rho(n,k)} F_{ik+p}^{(k)} = Q_{n,p}^{(k)}.$$

To show the other direction of the inequality, we recall that since $Q$ is a basic sequence, $q_m \ge 2$ for all $m$, so for all $i$

$$\sum_{i=0}^{\rho(n,k)} \left( \frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} - \left( \frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} \right)^2 \right)$$
$$\ge \sum_{i=0}^{\rho(n,k)} \left( \frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} - \frac{1}{2^k} \left( \frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} \right) \right)$$
$$\ge \sum_{i=0}^{\rho(n,k)} \frac{1}{2} \cdot \frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} = \frac{1}{2} Q_{n,p}^{(k)}.$$

□

[4] See, for example, [11].

**Lemma 4.5.** *If $Q$ is infinite in limit and $B$ is a block of length $k$, then for almost every real number $x$ in $[0,1)$, we have*

$$N_{n,p}^{Q}(B,x) = Q_{n,p}^{(k)} + O\left(\sqrt{Q_{n,p}^{(k)}}\left(\log\log Q_{n,p}^{(k)}\right)^{1/2}\right). \tag{14}$$

**Proof.** We consider two cases. The first case is when $\lim_{n\to\infty} Q_{n,p}^{(k)} < \infty$. We see that

$$\lim_{n\to\infty} Q_{n,p}^{(k)} = \lim_{n\to\infty} \sum_{i=0}^{\rho(n,k)} \mathrm{P}\left(\zeta_{B,i,p}^{Q} = 1\right) < \infty,$$

so by Theorem 4.3, we have $\mathrm{P}\left(\zeta_{B,i,p}^{Q} = 1 \text{ i.o.}\right) = 0$. Thus, for almost every $x \in [0,1)$, $\lim_{n\to\infty} N_{n,p}^{Q}(B,x) < \infty$ and (14) holds.

Second, we consider the case where $\lim_{n\to\infty} Q_{n,p}^{(k)} = \infty$. By Lemma 4.4, we have $\lim_{n\to\infty} t_{n,p}^{(k)} \geq \lim_{n\to\infty} Q_{n,p}^{(k)} = \infty$. Note that

$$N_{n,p}^{Q}(B,x) = \sum_{i=0}^{\rho(n,k)} \zeta_{B,i,p}(x).$$

By Corollary 4.2,

$$N_{n,p}^{Q}(B,x) = \sum_{i=0}^{\rho(n,k)} F_{ik+p}^{(k)} + O\left(\sqrt{t_{n,p}^{(k)}}\left(\log\log t_{n,p}^{(k)}\right)^{1/2}\right)$$

for almost every $x \in [0,1)$. By Lemma 4.4, $t_{n,p}^{(k)} < Q_{n,p}^{(k)}$, so the lemma follows. □

Lemma 4.5 allows us to prove the following results on strongly normal numbers:

**Theorem 4.6.** *Suppose that $Q$ is strongly $k$-divergent and infinite in limit. Then almost every $x \in [0,1)$ is strongly $Q$-normal of order $k$.*

**Proof.** Let $B$ be a block of length $m \leq k$ and $p \in [1,m]$. Then by Lemma 4.5, for almost every $x \in [0,1)$, we have that

$$N_{n,p}^{Q}(B,x) = Q_{n,p}^{(m)} + O\left(\sqrt{Q_{n,p}^{(m)}}\left(\log\log Q_{n,p}^{(m)}\right)^{1/2}\right),$$

so

$$\frac{N_{n,p}^{Q}(B,x)}{Q_{n,p}^{(m)}} = 1 + O\left(\frac{\sqrt{Q_{n,p}^{(m)}}\left(\log\log Q_{n,p}^{(m)}\right)^{1/2}}{Q_{n,p}^{(m)}}\right).$$

However, $Q$ is strongly $k$-divergent, so $Q_{n,p}^{(m)} \to \infty$ and

$$\lim_{n\to\infty} \frac{N_{n,p}^Q(B,x)}{Q_{n,p}^{(m)}} = \lim_{n\to\infty} \left(1 + O\left(\frac{\sqrt{Q_{n,p}^{(m)}}\left(\log\log Q_{n,p}^{(m)}\right)^{1/2}}{Q_{n,p}^{(m)}}\right)\right) = 1.$$

Since there are finitely many choices of $m$ and $p$ and only countably many choices of $B$, the result follows. □

**Corollary 4.7.** *If $Q$ is strongly fully divergent and infinite in limit, then almost every real $x \in [0,1)$ is strongly $Q$-normal.*

We now work towards proving a result much stronger than Corollary 4.7 on the typicality of $Q$-normal numbers. We will need the following lemma in addition to Lemma 4.5:

**Lemma 4.8.** *If $Q$ is a basic sequence and $k$ and $p$ are positive integers with $p \in [1,k]$, then*

$$\sum_{p=1}^{k} \left(Q_{n,p}^{(k)} + O\left(\sqrt{Q_{n,p}^{(k)}}\left(\log\log Q_{n,p}^{(k)}\right)^{1/2}\right)\right) = Q_n^{(k)} + O\left(\sqrt{Q_n^{(k)}}\left(\log\log Q_n^{(k)}\right)^{1/2}\right).$$

**Proof.** We first note that

$$\sum_{p=1}^{k} Q_{n,p}^{(k)} \le Q_n^{(k)} + \left(Q_n^{(k)} - Q_{n-k}^{(k)}\right).$$

Since $Q_n^{(k)} - Q_{n-k}^{(k)} \le (k+1)2^{-k} \to 0$, we see that

$$\sum_{p=1}^{k} Q_{n,p}^{(k)} = Q_n^{(k)} + o(1). \tag{15}$$

Next, note that

$$\sum_{p=1}^{k} \sqrt{Q_{n,p}^{(k)}}\left(\log\log Q_{n,p}^{(k)}\right)^{1/2} \le k\sqrt{\sum_{p=1}^{k} Q_{n,p}^{(k)}}\left(\log\log \sum_{p=1}^{k} Q_{n,p}^{(k)}\right)^{1/2}. \tag{16}$$

By (15) and (16),

$$\sum_{p=1}^{k} O\left(\sqrt{Q_{n,p}^{(k)}}\left(\log\log Q_{n,p}^{(k)}\right)^{1/2}\right) = O\left(\sqrt{Q_n^{(k)}}\left(\log\log Q_n^{(k)}\right)^{1/2}\right). \tag{17}$$

Thus, the lemma follows by combining (15) and (17). □

**Theorem 4.9.** *If $Q$ is a basic sequence that is infinite in limit and $B$ is a block of length $k$, then for almost every real number $x$ in $[0,1)$, we have*

$$N_n^Q(B,x) = Q_n^{(k)} + O\left(\sqrt{Q_n^{(k)}}\left(\log\log Q_n^{(k)}\right)^{1/2}\right).$$

**Proof.** We first note that

$$N_n^Q(B,x) = \sum_{p=1}^{k} N_{n,p}(B,x) + O(1). \tag{18}$$

Thus, by (18) and Lemma 4.5, for almost every $x \in [0,1)$, we have

$$N_n^Q(B,x) = \sum_{p=1}^{k}\left(Q_{n,p}^{(k)} + O\left(\sqrt{Q_{n,p}^{(k)}}\left(\log\log Q_{n,p}^{(k)}\right)^{1/2}\right)\right) + O(1). \tag{19}$$

Thus, the theorem follows by applying Lemma 4.8 to (19). □

We recall the following standard result on infinite products:

**Lemma 4.10.** *If $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that $0 \le a_n < 1$ for all $n$, then the infinite product $\prod_{n=1}^{\infty}(1-a_n)$ converges if and only if the sum $\sum_{n=1}^{\infty} a_n$ is convergent.*

**Theorem 4.11.** *Suppose that $Q$ is a basic sequence that is infinite in limit. Then almost every real number in $[0,1)$ is $Q$-normal of order $k$ if and only if $Q$ is $k$-divergent.*

**Proof.** First, we suppose that $Q$ is $k$-divergent. Then by Theorem 4.9, for almost every $x \in [0,1)$, we have

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n^{(k)}} = \lim_{n\to\infty} \frac{Q_n^{(k)} + O\left(\sqrt{Q_n^{(k)}}\left(\log\log Q_n^{(k)}\right)^{1/2}\right)}{Q_n^{(k)}} = 1.$$

We now suppose that $Q$ is $k$-convergent.We will now use similar reasoning to that found in [7]. Set $B = (0,0,\dots,0)$ ($k$ zeros). We will show that the set of real numbers in $[0,1)$ whose $Q$-Cantor series expansion does not contain the block $B$ has positive measure. Call this set $V$. We see that

$$\lambda(V) = \prod_{n=1}^{\infty}\left(1 - \frac{1}{q_n q_{n+1}\cdots q_{n+k-1}}\right).$$

Set $a_n = q_n q_{n+1} \cdots q_{n+k-1}$. Since $Q$ is $k$-convergent, we have $\sum a_n < \infty$. Thus, $\lambda(V) > 0$ by Lemma 4.10. □

**Corollary 4.12.** *Suppose that $Q$ is a basic sequence that is infinite in limit. Then almost every real number in $[0,1)$ is $Q$-normal if and only if $Q$ is fully divergent.*

## 5. Ratio normal numbers

We are now in a position to compare the prevelance of $Q$-normal numbers to $Q$-ratio normal numbers, depending on properties of the basic sequence $Q$. In particular, we will show that if $Q$ is infinite in limit, then the set of $Q$-ratio normal numbers is dense in $[0,1)$ even though the set of $Q$-normal numbers may be empty. Suppose that $Q$ is a $k$-convergent basic sequence and define

$$Q_\infty^{(k)} = \lim_{n\to\infty} Q_n^{(k)} < \infty. \tag{20}$$

**Proposition 5.1.** *If $Q$ is a basic sequence that is $k$-convergent for some $k$, then the set of $Q$-normal numbers is empty.*

**Proof.** We make the observation that since $q_n \geq 2$ for all $n$, $Q_\infty^{(k)} \leq \frac{1}{2} Q_\infty^{(k-1)}$ for all $k$. Thus, there exists a $K > 0$ such that for all $k > K$, we have $Q_\infty^{(k)} < 1$. Thus, no blocks of length $k > K$ can occur in any $Q$-normal number and the set of $Q$-normal numbers is empty. □

If $B = (b_1, b_2, \cdots, b_k)$ is a block of length $k$, we write

$$\max(B) = \max(b_1, b_2, \cdots, b_k).$$

If $E = (E_1, E_2, \cdots)$, then set $E_{n,k} = (E_n, E_{n+1}, \cdots, E_{n+k-1})$.

**Proposition 5.2.** *If $Q = \{q_n\}_{n=1}^\infty$ is infinite in limit, then there exists a real number that is $Q$-ratio normal.*

**Proof.** Let $Q' = \{q'_n\}_{n=1}^\infty$ be any fully divergent basic sequence that is infinite in limit. Then we know that there exists a $Q'$-normal number by Corollary 4.12. Let $x = 0.E'_1 E'_2 E'_3 \ldots$ with respect to $Q'$ be $Q'$-normal and let $E' = (E'_1, E'_2, \ldots)$. Set

$$M_k = \min\{m : q_n > k \ \forall n \geq m\},$$

$E_n = \min(E'_n, q_n - 1)$, and $E = (E_1, E_2, \ldots)$. Suppose that $B$ and $B'$ are two blocks of length $k$ and let $l = \max(\max(B), \max(B')) + 2$.

Thus, if $n > M_l$, then $E'_{n,k} = B$ is equivalent to $E_{n,k} = B$ and $E'_{n,k} = B'$ is equivalent to $E_{n,k} = B'$. Since $x$ is $Q'$-normal, there are infinitely many occurences of every block. Additionally, $E_n \leq q_n - 1$ for all $n$, so $\sum_{n=1}^\infty \frac{E_n}{q_1 q_2 \ldots q_n}$ is $Q$-ratio normal. □

**Corollary 5.3.** *If $Q$ is infinite in limit, then the set of numbers that are $Q$-ratio normal is dense in $[0,1)$.*

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