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A generalization of Jørgensen's inequality to infinite dimension

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Abstract. In this paper, we give a generalization of Jørgensen's inequality to hyperbolic Möbius transformations in infinite dimension by using Clifford algebras. We also give an application.

Contents

1. Introduction

In the theory of discrete groups, the following important and useful inequality is well known as Jørgensen's inequality, see [5].

Theorem J. *Suppose that $f, g \in M(\overline{\mathbb{R}}^2)$ generate a discrete and nonelementary group $\langle f, g\rangle$. Then*

$$|\mathrm{tr}^2(f) - 4| + |\mathrm{tr}([f, g]) - 2| \geq 1.$$

In [4], Hersonsky gave a partial generalization of Theorem J to Möbius transformations in $\overline{\mathbb{R}}^n$ by using Clifford algebra, which is stated in the following form.

Theorem H. *Let $f, g \in M(\overline{\mathbb{R}}^n)$ such that f and $[f, g]$ are hyperbolic, and suppose that $\langle f, g\rangle$ is a discrete and nonelementary group. Then*

$$|\mathrm{tr}^2(f) - 4| + |\mathrm{tr}([f, g]) - 2| \geq 1.$$

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In [12], Waterman generalized Jørgensen's inequality to high dimensional groups and obtained

Theorem WA. *Let f, $g \in M(\overline{\mathbb{R}}^n)$. If $\langle f, g\rangle$ is discrete and nonelementary, then*

$$\|f - I\| \cdot \|g - I\| \geq \frac{1}{32}.$$

In [11], Wang also studied the generalization of Jørgensen's inequality to hyperbolic Möbius transformations in high dimension, giving the following generalization of Theorem H.

Theorem W. *Let $f, g \in M(\overline{\mathbb{R}}^n)$ such that f is hyperbolic and $[f, g]$ is vectorial, and suppose that $\langle f, g\rangle$ is a discrete and nonelementary group. Then*

$$|\mathrm{tr}^2(f) - 4| + |\mathrm{tr}([f, g]) - 2| \geq 1.$$

We refer to [6, 9, 10, 11, 12, 13] for related investigations in this direction.

The main aim of this paper is to establish Jørgensen's inequality in the infinite dimensional case. Our main result is Theorem 3.1, which is a generalization of Theorems H and W and a partial generalization of Theorem J to infinite dimension. We will state and prove it in Section 3. In Section 4 we will give an application of Theorem 3.1.

2. Preliminaries

The Clifford algebra ℓ is the associative algebra over the real field $\mathbb{R}$, generated by a countable family $\{i_k\}_{k=1}^{\infty}$ subject to the following relations:

$$i_h i_k = -i_k i_h \ (h \neq k), \quad i_k^2 = -1, \quad \forall h, k \geq 1$$

and no others. Every element of ℓ can be expressed of the following type

$$a = \sum a_I I,$$

where $I = i_{v_1} i_{v_2} \ldots i_{v_p}, 1 \leq v_1 < v_2 < \cdots < v_p$, $p \leq n$, n is a fixed natural number depending on a, $a_I \in \mathbb{R}$ are the coefficients and $\sum_I a_I^2 < \infty$. If $I = \emptyset$, then a_I is called the real part of a and denoted by $\mathrm{Re}(a)$; the remaining part is called the imaginary part of a and denoted by $\mathrm{Im}(a)$.

In ℓ, the Euclidean norm is expressed by

$$|a| = \sqrt{\sum_I a_I^2} = \sqrt{|\mathrm{Re}(a)|^2 + |\mathrm{Im}(a)|^2}.$$

The algebra ℓ has three important involutions:

(1) "$'$": replacing each i_k ($k \geq 1$) of a by $-i_k$, we get a new number a'. $a \mapsto a'$ is an isomorphism of ℓ:

$$(ab)' = a'b', \quad (a + b)' = a' + b',$$

for $a, b \in \ell$.

(2) "$*$": replacing each $i_{v_1}i_{v_2}\dots i_{v_p}$ of a by $i_{v_p}i_{v_{p-1}}\dots i_{v_1}$. We know that $a \mapsto a^*$ is an anti-isomorphism of ℓ:

$$(ab)^* = b^*a^*, \quad (a+b)^* = b^* + a^*.$$

(3) "$^-$": $\bar{a} = (a^*)' = (a')^*$. It is obvious that $a \mapsto \bar{a}$ is also an anti-isomorphism of ℓ.

We refer to elements of the following type as vectors:

$$x = x_0 + x_1 i_1 + \cdots + x_n i_n + \cdots \in \ell.$$

The set of all such vectors is denoted by ℓ_2 and we let $\overline{\ell_2} = \ell_2 \bigcup \{\infty\}$. For any $x \in \ell_2$, we have $x^* = x$ and $\bar{x} = x'$. For $x, y \in \ell_2$, the inner product $(x \cdot y)$ of x and y is given by

$$(x \cdot y) = x_0y_0 + x_1y_1 + \cdots + x_ny_n + \dots,$$

where $x = x_0 + x_1i_1 + \cdots + x_ni_n + \dots,\ y = y_0 + y_1i_1 + \cdots + y_ni_n + \dots.$

Obviously, any nonzero vector x is invertible in ℓ with $x^{-1} = \frac{\bar{x}}{|x|^2}$. The inverse of a vector is invertible too. Since any product of nonzero vectors is invertible, we conclude that any product of nonzero vectors is invertible in ℓ. The set of products of finitely many nonzero vectors is a multiplicative group, called Clifford group and denoted by Γ.

Definition 2.1. If a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies:

(1) $a, b, c, d \in \Gamma \bigcup \{0\}$,
(2) $\triangle(g) = ad^* - bc^* = 1$,
(3) $ab^*,\ d^*b,\ cd^*,\ c^*a \in \ell_2$,

then we call g a Clifford matrix in infinite dimension; the set of all such matrices is denoted by $\mathrm{SL}(\Gamma)$.

Let

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ g^{-1} = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}.$$

Obviously, $gg^{-1} = g^{-1}g = I$, that is, g^{-1} is the inverse of g. By a simple computation, we know that $\mathrm{SL}(\Gamma)$ is a multiplicative group of matrices.

For any $g = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma)$, the corresponding mapping

$$g(x) = (ax + b)(cx + d)^{-1}$$

is a bijection of $\overline{\ell_2}$ onto itself, which we call a Möbius transformation in infinite dimension. Correspondingly, the set of all such mappings is also a group, which is still denoted by $\mathrm{SL}(\Gamma)$.

Now, we give a classification of the nontrivial elements of $\mathrm{SL}(\Gamma)$ as follows:

- f is *loxodromic* if it is conjugate in $\mathrm{SL}(\Gamma)$ to $\begin{pmatrix} r\lambda & 0 \\ 0 & r^{-1}\lambda' \end{pmatrix}$, where $r \in \mathbb{R}\backslash\{\pm 1,\ 0\}$, $\lambda \in \Gamma$ and $|\lambda| = 1$; if $\lambda = \pm 1$, then f is called *hyperbolic*.
- f is *parabolic* if it is conjugate in $\mathrm{SL}(\Gamma)$ to $\begin{pmatrix} a & b \\ 0 & a' \end{pmatrix}$, where $a,\ b \in \Gamma$, $|a| = 1$, $b \neq 0$ and $ab = ba'$; if $a = \pm 1$, then f is called *strictly parabolic*.
- Otherwise we say f is *elliptic*.

Definition 2.2. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma)$, we define the trace of g as

$$\mathrm{tr}(g) = a + d^*.$$

For a nontrivial element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma)$, if $b^* = b$, $c^* = c$ and $\mathrm{tr}(g) \in \mathbb{R}$, then we call g vectorial.

For the trace, we have the following result (see [8]).

Lemma 2.3. *Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma)$. *Then* $\mathrm{Re}(\mathrm{tr}(g))$ *is invariant under conjugation.*

The following two lemmas come from [8].

Lemma 2.4. $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma)$ $(c \neq 0)$ *is hyperbolic if and only if* $\mathrm{tr}(g) \in \mathbb{R}$, $\mathrm{tr}^2(g) > 4$ *and* $c \in \ell_2$. *If* g *is hyperbolic, then the two fixed points of* g *are*

$$u, v = -\frac{1}{2}(c^{-1}d - ac^{-1}) \pm \frac{1}{2}c^{-1}((a + d^*)^2 - 4)^{\frac{1}{2}}.$$

Lemma 2.5. $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{SL}(\Gamma)$ $(b \neq 0)$ *is hyperbolic if and only if* $\mathrm{tr}(g) \in \mathbb{R}$, $\mathrm{tr}^2(g) > 4$ *and* $b \in \ell_2$. *If* g *is hyperbolic, then the two fixed points of* g *are* ∞ *and* $-b(a-d)^{-1}$.

Definition 2.6. For a subgroup $G \subset \mathrm{SL}(\Gamma)$, we call G elementary if G has a finite G-orbit, that is, there exists a point $x \in \overline{\ell_2}$ such that

$$G(x) = \{g(x) | g \in G\}$$

is finite; otherwise, we call G nonelementary.

We say that G is discrete if $g,\ f_1, f_2, \cdots \in G$ and $f_i \to g$ imply $f_i = g$ for all sufficiently large i. Otherwise, G is not discrete.

Lemma 2.7. *Let* $f \in \mathrm{SL}(\Gamma)$ *be not elliptic, and let* $\theta : \mathrm{SL}(\Gamma) \to \mathrm{SL}(\Gamma)$ *be defined by*

$$\theta(g) = gfg^{-1}.$$

Suppose that there exists n such that $\theta^n(g) = f$, then the group $\langle f, g\rangle$ generated by f and g is elementary.

Proof. Define $g_0 = g$ and $g_n = \theta^n(g)$. So for some $m \geq 0$,

$$g_{m+1} = g_m f g_m^{-1}.$$

Suppose first that f is parabolic. Since f has exactly one fixed point, we may assume that $f(\infty) = \infty$. As $g_1, \dots, g_n$ are conjugate to f, they are each parabolic and so have a unique fixed point. Thus if g_{r+1} fixes ∞, then so does g_r, where $r \geq 0$. As $g_n(= f)$ fixes ∞, we deduce that each g_j $(j = 0, 1, \dots, n)$ fixes ∞. This shows that $\langle f, g\rangle$ is elementary.

Suppose now that f is loxodromic and the two fixed points of f are x and y. Clearly, $g_1, \dots, g_n$ each have exactly two fixed points. Now suppose that g_{r+1} fixes x and y (as does g_n): then

$$\{x, y\} = \{g_r(x), g_r(y)\}.$$

Since g_r cannot interchange x and y for $r \geq 1$, we know that if g_{r+1} fixes x and y, then so does g_r for $r \geq 1$. It follows that $g_1, \dots, g_n$ each fix x and y. This shows that f and g leave the set $\{x, y\}$ invariant and so $\langle f, g\rangle$ is elementary. □

3. The main result and its proof

Now we come to state and prove our main result.

Theorem 3.1. *Let $f, g \in \mathrm{SL}(\Gamma)$ such that f is hyperbolic and $[f, g]$ is vectorial, and suppose that $\langle f, g\rangle$ is discrete and nonelementary, then*

$$|\mathrm{tr}^2(f) - 4| + |\mathrm{tr}([f, g]) - 2| \geq 1. \tag{3.1}$$

Proof. By Lemmas 2.4, 2.5 and 2.3, we know that $\mathrm{tr}(f) \in \mathbb{R}$, and $\mathrm{tr}(f)$ and $\mathrm{tr}([f, g])$ are invariant under conjugation. Without loss of generality, we may assume that

$$f = \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $\tau > 0$ and $\tau \neq 1$. Let κ denote the left side of relation (3.1) and suppose that (3.1) fails. Then

$$\kappa = (\tau - \tau^{-1})^2(1 + |bc|) < 1. \tag{3.2}$$

We let

$$g_0 = g, \quad g_{m+1} = g_m f {g_m}^{-1}, \quad g_m = \begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix}, \quad m = 0, 1, \dots .$$

Then, we have

$$
\begin{aligned}
a_{m+1} &= \tau a_m {d_m}^* - \tau^{-1} b_m {c_m}^*, \\
b_{m+1} &= (\tau^{-1} - \tau) a_m {b_m}^*, \\
c_{m+1} &= -(\tau^{-1} - \tau) c_m {d_m}^*, \\
d_{m+1} &= \tau^{-1} d_m {a_m}^* - \tau c_m {b_m}^*, \\
b_{m+1} {c_{m+1}}^* &= -(\tau^{-1} - \tau)^2 (1 + b_m {c_m}^*) b_m {c_m}^*.
\end{aligned} \tag{3.3}
$$

Let $f : [0, +\infty) \longrightarrow [0, +\infty)$ be defined by

$$f(x) = x(1 + x)(\tau^{-1} - \tau)^2.$$

Let $r = (\tau^{-1} - \tau)^{-2} - 1$. It is obvious that $f(x)$ is an increasing function on $[0, +\infty)$ such that $f(x) \le x$ on $[0, r]$. It follows from (3.2) that $|bc| < r$. The above facts and relations (3.3) show that

$$
\begin{aligned}
|b_{m+1} {c_{m+1}}^*| &\le f(|b_m {c_m}^*|) \le \cdots \le f^{m+1}(|bc^*|) \le |bc^*|, \\
|b_{m+1} {c_{m+1}}^*| &\le (\tau^{-1} - \tau)^2 (1 + |b_m {c_m}^*|) |b_m {c_m}^*| \\
&\le (\tau^{-1} - \tau)^2 (1 + |bc^*|) |b_m {c_m}^*| = \kappa |b_m {c_m}^*|, \\
|b_{m+1} {c_{m+1}}^*| &\le \kappa^{m+1} |bc|.
\end{aligned}
$$

So

$$\lim_{m\to\infty} b_m {c_m}^* = 0, \quad \lim_{m\to\infty} a_m {d_m}^* = 1.$$

The above relation and (3.3) imply that

$$\lim_{m\to\infty} a_m = \tau, \quad \lim_{m\to\infty} d_m = \tau^{-1}.$$

Now

$$|b_m^{-1} b_{m+1}| = |(\tau^{-1} - \tau) {a_m}^*| \to |\tau(\tau^{-1} - \tau)| < \sqrt{\kappa}\tau.$$

So for sufficiently large m, we have

$$\left| \frac{b_{m+1}}{\tau^{m+1}} \right| \le \sqrt{\kappa} \left| \frac{b_m}{\tau^m} \right|.$$

It follows that

$$\left| \frac{b_m}{\tau^m} \right| \to 0.$$

In a very similar way, we get that

$$\lim_{m\to\infty} c_m \tau^m = 0.$$

It follows that

$$\lim_{m\to\infty} f^{-m} g_{2m} f^m = f.$$

Since $\langle f, g \rangle$ is discrete, we must have $g_{2m} = f$ for some m. By Lemma 2.7, $\langle f, g \rangle$ must be elementary, which violates the assumption. The contradiction shows that κ cannot be less than 1. □

Remark 3.2. Theorem 3.1 is a generalization of Theorem B in [4] and the corresponding result in [11].

4. An application

For $f_r = \begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix}$, where $a_r, b_r, c_r, d_r \in \Gamma \cup \{0\}$ and $r = 1,\ 2$, define

$$\|f_r\| = \sqrt{|a_r|^2 + |b_r|^2 + |c_r|^2 + |d_r|^2},$$

$$\|f_1 - f_2\| = \sqrt{|a_1 - a_2|^2 + |b_1 - b_2|^2 + |c_1 - c_2|^2 + |d_1 - d_2|^2}.$$

Then

Lemma 4.1 ([7])**.** *For any* $U = \begin{pmatrix} a & b \\ -b' & a' \end{pmatrix} \in \mathrm{SL}(\Gamma)$ *(U is called unitary),* $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, *we have* $\|g\| = \|gU\| = \|Ug\|$, *where* $\alpha,\ \beta,\ \gamma,\ \delta \in \Gamma \cup \{0\}$.

Lemma 4.2. *Let* $f \in \mathrm{SL}(\Gamma)$ *be hyperbolic. Then*

$$\|f - I\|^2 \geq \frac{1}{2}|\mathrm{tr}^2(f) - 4|.$$

Proof. Since $\|f - I\|$ and $\mathrm{tr}^2(f)$ are invariant under conjugation by unitary transformations by Lemmas 2.3, 2.4, 2.5 and 4.1, without loss of generality, we may assume that

$$f = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix},$$

where $u > 1$. By a simple computation, the conclusion follows. □

Lemma 4.3. *Let* $f, g \in \mathrm{SL}(\Gamma)$ *be hyperbolic such that* $[f, g]$ *is vectorial. Then*

$$\|f - I\|^2 \cdot \|g - I\|^2 \geq |\mathrm{tr}([f, g]) - 2|.$$

Proof. Since the two sides of the above inequality are invariant under conjugation by unitary transformations, we may assume that

$$f = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix},\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $u > 1$. By computation, we see that

$$[f, g] = \begin{pmatrix} ad^* - u^2 bc^* & (u^2 - 1)ab^* \\ (u^{-2} - 1)cd^* & da^* - u^{-2} cb* \end{pmatrix},\ \ |\mathrm{tr}([f, g]) - 2| = (u - u^{-1})^2 |bc^*|,$$

$$\|f - I\|^2 \cdot \|g - I\|^2 = [(u - 1)^2 + (u^{-1} - 1)^2][|a - 1|^2 + |b|^2 + |c|^2 + |d - 1|^2].$$

Therefore, we have

$$\|f - I\|^2 \cdot \|g - I\|^2 \geq (u - u^{-1})^2 |bc| = |\mathrm{tr}([f, g]) - 2|.$$ □

We will use Theorem 3.1 to prove

Theorem 4.4. *Let $f, g \in \mathrm{SL}(\Gamma)$ be hyperbolic such that $[f,g]$ and $[g,f]$ are vectorial. If $\langle f, g\rangle$ is discrete and nonelementary, then*

$$\|f-I\| \cdot \|g-I\| \geq \sqrt{2}-1.$$

Proof. Let $x=\min\{|\mathrm{tr}^2(f)-4|,\ |\mathrm{tr}^2(g)-4|\}$.

We first suppose that $x \leq 2\sqrt{2}-2$. By assumptions and Theorem 3.1,

$$|\mathrm{tr}^2(f)-4|+|\mathrm{tr}([f,g])-2| \geq 1, \quad |\mathrm{tr}^2(g)-4|+|\mathrm{tr}([g,f])-2| \geq 1.$$

Therefore, by Lemma 4.3, we have that

$$\|f-I\|^2 \cdot \|g-I\|^2 \geq |\mathrm{tr}([f,g])-2| \geq 1-|\mathrm{tr}^2(f)-4|,$$

and

$$\|g-I\|^2 \cdot \|f-I\|^2 \geq |\mathrm{tr}([g,f])-2| \geq 1-|\mathrm{tr}^2(g)-4|.$$

Thus,

$$\|f-I\|^2 \cdot \|g-I\|^2 \geq 1-(2\sqrt{2}-2)=(\sqrt{2}-1)^2.$$

Now we suppose that $x \geq 2\sqrt{2}-2$. By Lemma 4.2, we have

$$\|f-I\|^2 \geq \frac{1}{2}|\mathrm{tr}^2(f)-4|, \quad \|g-I\|^2 \geq \frac{1}{2}|\mathrm{tr}^2(g)-4|.$$

We hence know that

$$\|f-I\|^2 \cdot \|g-I\|^2 \geq \frac{1}{4}|\mathrm{tr}^2(f)-4||\mathrm{tr}^2(g)-4| \geq (\sqrt{2}-1)^2. \qquad \square$$

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