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Convex combinations of unitaries in JB^*-algebras

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Abstract. We continue our recent efforts to exploit the notion of a unitary isotope to study convex combinations of unitaries in an arbitrary JB^*-algebra. Exact analogues of C^*-algebraic results, due to R. V. Kadison, C. L. Olsen and G. K. Pedersen, are proved for general JB^*-algebras. We show that if a contraction in a JB^*-algebra is a convex combination of n unitaries, then it is also a mean of n unitaries. This generalizes a well known theorem of Kadison and Pedersen. Our methods also provide alternative proofs of other results for C^*-algebras.

Contents

1. Introduction

In [17], B. Russo and H. A. Dye proved that the closed unit ball of any C^*-algebra is the closed convex hull of the set of its unitary elements. They also raised the question: Which operators lie in the purely algebraic convex hull of the unitaries of a C^*-algebra? Subsequently, in 1984, L. T. Gardner [5] obtained an elementary proof of the Russo–Dye theorem by proving that every element of norm less than a half in a C^*-algebra is a convex combination of unitaries. This result stimulated a number of mathematicians, including R. V. Kadison, G. K. Pedersen, C. L. Olsen and M. Rørdam and others, to study convex combinations of unitaries in C^*-algebras and related unitary approximation theorems (see [7, 13, 14, 16]).

It is well known that JB^*-algebras are an important generalization of C^*-algebras (see [24]). Hence, it is important to understand which results from C^*-algebra theory extend to JB^*-algebras. There already is a substantial

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literature in this area. See, in particular, [25, 20, 19, 21, 22] for a sample and further references.

In this paper, we build upon our earlier results from [20, 19, 22] to investigate convex combinations of unitaries in an arbitrary JB^*-algebra. We present some interesting generalizations of certain C^*-algebra results due to Kadison, Olsen and Pedersen [7, 14]. Our approach also provides alternative proofs to certain known results for C^*-algebras. We prove, in particular, that the distance from any positive element to the set $\mathcal{U}(\mathcal{J})$ of unitaries in a JB^*-algebra $\mathcal{J}$ is attained at the unit element of $\mathcal{J}$. We also prove that if $x \in (\mathcal{J})_1$ (closed unit ball of $\mathcal{J}$) with $\text{dist}(x, \mathcal{U}(\mathcal{J})) \leq 2\alpha$ and $\alpha < \frac{1}{2}$ then $x \in \alpha\mathcal{U}(\mathcal{J}) + (1-\alpha)\mathcal{U}(\mathcal{J})$. The proof requires a nontrivial application of some of our results involving the Stone–Weierstrass Theorem and the continuous functional calculus. In the last section, we obtain the inclusion

$$\alpha\mathcal{U}(\mathcal{J}) + (1-\alpha)\mathcal{U}(\mathcal{J}) \subseteq \beta\mathcal{U}(\mathcal{J}) + (1-\beta)\mathcal{U}(\mathcal{J})$$

for any $0 \leq \alpha \leq \beta \leq \frac{1}{2}$; this would lead us to JB^*-algebra analogues of additional C^*-algebra results. In the sequel, we shall show that if a contraction in a JB^*-algebra is a convex combination of n unitaries, then it is also a mean of n unitaries. This generalizes known results for C^*-algebras due to Kadison and Pedersen [7].

The concepts and notation that we shall use throughout the sequel are consistent with our previous papers [19, 20, 21, 22].

We begin by recalling (from [6]) the concept of a homotope of a Jordan algebra. Let $\mathcal{J}$ be a Jordan algebra and $x \in \mathcal{J}$. The *x-homotope*, $\mathcal{J}_{[x]}$ of $\mathcal{J}$ is the Jordan algebra that consists of the same elements and linear structure as $\mathcal{J}$, but with the new product " $\cdot_x$" defined by the equation:

$$a \cdot_x b = \{axb\}, \qquad a, b \in \mathcal{J}_{[x]}.$$

Here, and throughout, $\{pqr\}$ denotes the Jordan triple product of p, q, r defined in the Jordan algebra $\mathcal{J}$ by the formula:

$$\{pqr\} = (p \circ q) \circ r - (p \circ r) \circ q + (q \circ r) \circ p,$$

where $\circ$ stands for the original Jordan product in $\mathcal{J}$.

An element x of a Jordan algebra $\mathcal{J}$, with unit e, is said to be *invertible* if there exists $x^{-1} \in \mathcal{J}$, called the *inverse* of x, such that $x \circ x^{-1} = e$ and $x^2 \circ x^{-1} = x$. The set of all invertible elements of unital Jordan $\mathcal{J}$ will be denoted by $\mathcal{J}_{\text{inv}}$. It is easy to see that any invertible element x acts as the unit of the x^{-1}-homotope $\mathcal{J}_{[x^{-1}]}$ of $\mathcal{J}$. For an invertible element x of a unital Jordan algebra $\mathcal{J}$, the x^{-1}-homotope $\mathcal{J}_{[x^{-1}]}$ of $\mathcal{J}$ is called the *x-isotope* of $\mathcal{J}$, and is denoted by $\mathcal{J}^{[x]}$. Any two isotopes of an associative algebra are *isomorphic* to each other (see [6, p. 56]). Thus in the assiciative case, *isotopy* basically just changes the unit element and does not produce new structures. However, it may be convenient to change isotopes when performing certain calculations; such an example is given in [12, p. 617]. But for a general Jordan algebra, the process of forming isotopes may produce essentially different Jordan algebras (for examples, see [12, 10].) Fortunately, the set of invertible elements in a unital Jordan algebra remains invariant on passage to isotopes [20, Lemma 4.2].

A Jordan algebra $\mathcal{J}$ with product $\circ$ is called a *Banach Jordan algebra* if there is a norm $\|.\|$ on $\mathcal{J}$ such that $(\mathcal{J}, \|.\|)$ is a Banach space and

$$\|a \circ b\| \leq \|a\|\|b\|.$$

If, in addition, $\mathcal{J}$ has unit e with $\|e\| = 1$ then $\mathcal{J}$ is called a *unital Banach Jordan algebra*. Many elementary properties of Banach Jordan algebras are similar to those of Banach algebras and their proofs are fairly routine [2, 4, 18, 23]. Throughout the sequel, we will only be considering unital Banach Jordan algebras. The norm closure of the Jordan subalgebra $J(x_1, \ldots, x_r)$, generated by $x_1, \ldots, x_r$ of Banach Jordan algebra $\mathcal{J}$, will be denoted by $\mathcal{J}(x_1, \ldots, x_r)$. Let $\mathcal{J}$ be a complex unital Banach Jordan algebra and let $x \in \mathcal{J}$. As usual, the spectrum $\sigma_{\mathcal{J}}(x)$ of x in $\mathcal{J}$ is defined by

$$\sigma_{\mathcal{J}}(x) = \{\lambda \in \mathbb{C} : x - \lambda e \text{ is not invertible in } \mathcal{J}\}.$$

In this note, we are interested in a special class of Banach Jordan algebras, called JB^*-algebras ; these include all C^*-algebras as a proper subclass (see [24, 26]). A complex Banach Jordan algebra $\mathcal{J}$ with involution $*$ (cf. [8, 9]) is called a *JB^*-algebra* if $\|\{xx^*x\}\| = \|x\|^3$ for all $x \in \mathcal{J}$. It is easily seen that $\|x^*\| = \|x\|$ for all elements x of a JB^*-algebra (see [26], for instance).

There is a more tractable subclass of these algebras: Let $\mathcal{H}$ be any complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the full algebra of bounded linear operators on $\mathcal{H}$. Then, any closed self-adjoint complex Jordan subalgebra of $\mathcal{B}(\mathcal{H})$ is called a JC^*-algebra. A JB^*-algebra is also called a JC^*-algebra if it is isometrically $*$-isomorphic to a JC^*-algebra. It is easily seen that every JC^*-algebra is a JB^*-algebra; the converse generally is not true (cf. [2]).

An element x of a JB^*-algebra $\mathcal{J}$ is said to be *self-adjoint* if $x^* = x$. A self-adjoint element x of $\mathcal{J}$ is said to be *positive* (in $\mathcal{J}$) if its spectrum $\sigma_{\mathcal{J}}(x)$ is contained in the set of nonnegative real numbers. An element $u \in \mathcal{J}$ is called *unitary* if $u^* = u^{-1}$.

If $u \in \mathcal{U}(\mathcal{J})$ (the set of unitary elements in $\mathcal{J}$), then the isotope $\mathcal{J}^{[u]}$ is called a *unitary isotope* of $\mathcal{J}$. It is well known (see [10, 3, 20]) that *any unitary isotope $\mathcal{J}^{[u]}$ is a JB^*-algebra with u as its unit with respect to the original norm and the involution $*_u$ defined by*

$$x^{*_u} = \{ux^*u\}.$$

Notice that for nonunitary $x \in \mathcal{J}_{\text{inv}}$, the isotope $\mathcal{J}^{[x]}$ of the JB^*-algebra $\mathcal{J}$ may not be a JB^*-algebra with the "$*_u$" as involution.

Like invertible elements, the set of unitary elements in the (unital) JB^*-algebra $\mathcal{J}$ is invariant on passage to unitary isotopes of $\mathcal{J}$ [20, Theorem 4.6] and every invertible element x of a JB^*-algebra $\mathcal{J}$ is positive in the unitary isotope $\mathcal{J}^{[u]}$ of $\mathcal{J}$, where $u \in \mathcal{U}(\mathcal{J})$ is given by the usual polar decomposition $x = u|x|$ of x considered as an operator in the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a suitable Hilbert space $\mathcal{H}$ [20, Theorem 4.12]. This is one of the principal results we proved in [20]. The somewhat involved proof uses the Stone–Weierstrass theorem and the standard functional calculus. In this note, we shall make free use of this fact as one of our main tools.

2. Convex combinations of unitaries

We continue using our earlier results on unitary isotopes to study convex combinations of unitaries in an arbitrary JB^*-algebra. The following definition is inspired by [7]:

Definition 2.1. Let $\mathcal{J}$ be a unital JB^*-algebra and let $x \in \mathcal{J}$. We define two numbers $u_c(x)$ and $u_m(x)$ by

$$u_c(x) = \min\left\{ n : x = \sum_{j=1}^{n} \alpha_j u_j \text{ with } u_j \in \mathcal{U}(\mathcal{J}),\ \alpha_j \geq 0,\ \sum_{j=1}^{n} \alpha_j = 1 \right\},$$

$$u_m(x) = \min\left\{ n : x = \frac{1}{n}\sum_{j=1}^{n} u_j,\ u_j \in \mathcal{U}(\mathcal{J}) \right\}.$$

If x has no decomposition as a convex combination of elements of $\mathcal{U}(\mathcal{J})$, we define $u_c(x)$ to be ∞.

Remark 2.2. From this definition, it is clear that $u_c(x) \leq u_m(x)$ and $u_c(x) = u_m(x) = \infty$ whenever $\|x\| > 1$. In [22, Theorem 2.3], the author proved that for general JB^*-algebra $\mathcal{J}$ there exist $u_i \in \mathcal{U}(\mathcal{J})$, $i = 1, 2, \ldots, n$ satisfying $x = \frac{1}{n}\sum_{i=1}^{n} u_i$ whenever $\|x\| < 1 - 2n^{-1}$ with $n \geq 3$ (for the special case of C^*-algebra, see [7, Theorem 2.1]). Thus, $u_m(x) < \infty$ whenever $\|x\| < 1$.

The following result is clear from [15, 4.3.10] and also from the facts given in [8, exercises 4.6.16 and 4.6.31]. The same facts are used in [7, Lemma 6 and Corollary 11].

Lemma 2.3. *Let x be a self-adjoint element of a C^*-algebra $\mathcal{A}$. If $\lambda \in \sigma_{\mathcal{A}}(x)$, then there exists a pure state ρ of $\mathcal{A}$ such that $\rho(x) = \lambda$ and $\rho(yx) = \rho(y)\rho(x)$ for all $y \in \mathcal{A}$.* □

The next lemma extends [19, Lemma 4.1] and [7, Lemma 6]. Notice that the authors used [7, Lemma 6] as a principal tool in their paper:

Lemma 2.4. *Let x be any self-adjoint element of a unital JB^*-algebra $\mathcal{J}$ and $\alpha \in [0, \frac{1}{2}]$. Define I_α to be the set $[-1, 1] \setminus (-(1-2\alpha), (1-2\alpha))$. Then $\sigma_{\mathcal{J}}(x) \subseteq I_\alpha$ if and only if $x = \alpha u_1 + (1-\alpha)u_2$ for some $u_1, u_2 \in \mathcal{U}(\mathcal{J})$.*

Proof. Assume $x = \alpha u_1 + (1-\alpha)u_2$ with $u_1, u_2 \in \mathcal{U}(\mathcal{J})$. Then $u_i^* = u_i^{-1}$ for $i = 1, 2$ and hence, by [20, Corollary 2.5], the JB^*-subalgebra $\mathcal{J}(e, u_1, u_2, u_1^{-1}, u_2^{-1})$ of $\mathcal{J}$ generated by the identity element e, u_1, u_2 and their inverses is a JC^*-algebra. Let $\mathcal{J}(e, u_1, u_2, u_1^{-1}, u_2^{-1})$ be embedded into $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Clearly, $x \in \mathcal{J}(e, u_1, u_2, u_1^{-1}, u_2^{-1})$.

Suppose that $\lambda \in \sigma_{\mathcal{J}}(x)$. Since x is self-adjoint, [20, Lemma 2.1(v)] gives

$$\sigma_{\mathcal{J}(e,u_1,u_2,u_1^{-1},u_2^{-1})}(x) \;=\; \sigma_{\mathcal{B}(\mathcal{H})}(x),$$

so that $\lambda \in \sigma_{\mathcal{B}(\mathcal{H})}(x)$. Hence, by Lemma 2.3, there exists a pure state ρ of $\mathcal{B}(\mathcal{H})$ such that $\rho(x) = \lambda$ and $\rho(yx) = \rho(y)\rho(x)$ for every $y \in \mathcal{B}(\mathcal{H})$. In particular, $\rho(u_2^* x) = \rho(u_2^*)\rho(x)$. Since ρ is a pure state, ρ has norm 1,

$$\begin{aligned}
1 - 2\alpha &= (1-\alpha) - \alpha \\
&\le |(1-\alpha)\rho(e)| - |\alpha\rho(u_2^* u_1)| \\
&\le |\rho(\alpha u_2^* u_1 + (1-\alpha)e)| \\
&= |\rho(u_2^*(\alpha u_1 + (1-\alpha)u_2))| \\
&= |\rho(u_2^* x)| \text{(by our hypothesis)} \\
&= |\rho(u_2^*)\rho(x)| = |\rho(u_2^*)|\,|\rho(x)| \le |\lambda|.
\end{aligned}$$

But $|\lambda| \le \|x\| = \|\alpha u_1 - (1-\alpha)u_2\| \le 1$. Therefore, $\lambda \in I_\alpha$ by its construction. Thus, $\sigma_{\mathcal{J}}(x) \subseteq I_\alpha$.

Conversely, suppose $\sigma_{\mathcal{J}}(x) \subseteq I_\alpha$. As x is self-adjoint, the JB^*-subalgebra $\mathcal{J}(e, x)$ of $\mathcal{J}$ generated by x and the unit e is a C^*-algebra (see [20, Remark 2.6]). So, by [7, Lemma 6], $x = \alpha u_1 + (1-\alpha)u_2$ for some unitaries u_1 and u_2 in $\mathcal{J}(e, x)$ and hence in $\mathcal{J}$. □

Remark 2.5. Explicit formulae for the unitaries u_1, u_2 appearing in the converse part of the above proof can be given as follows:

Case (i). If $\alpha \ = 0$, then x is a symmetry (a self-adjoint unitary) and so $u_1 = ie$ and $u_2 = x$ work. (Of course, any unitary can be taken for u_1, in this case).

Case (ii). If $\alpha = \frac{1}{2}$ then $u_1 = x + i(e - x^2)^{\frac{1}{2}}$ and $u_2 = x - i(e - x^2)^{\frac{1}{2}}$ (seen the proof of [19, Lemma 2.11]).

Case (iii). If $0 < \alpha < \frac{1}{2}$ then x is invertible (as $0 \notin \sigma_{\mathcal{J}}(x)$ in this case) and so with $a = \frac{1}{2}\alpha^{-1}(x - (1-2\alpha)x^{-1})$, $b = \frac{1}{2}(1-\alpha)^{-1}(x + (1-2\alpha)x^{-1})$ and $c = (1-\alpha)^{-1}(e - a \circ a)^{\frac{1}{2}} = \alpha^{-1}(e - b \circ b)^{\frac{1}{2}}$, we can take

$$u_1 \;=\; a + i(1-\alpha)c \;\; \textit{and} \;\; u_2 \;=\; b - i\alpha c\,.$$

For this, we observe that

$$\begin{aligned}
\alpha u_1 + (1-\alpha)u_2 &= \alpha a + (1-\alpha)b \\
&= \frac{1}{2}(x - (1-2\alpha)x^{-1}) + \frac{1}{2}(x + (1-2\alpha)x^{-1}) = x.
\end{aligned}$$

Further,

$$u_1^* u_1 = a^2 + (1-\alpha)^2 c^2 = a^2 + (e - a^2) = e,$$
$$u_2^* u_2 = b^2 + \alpha^2 c^2 = b^2 + (e - b^2) = e.$$

Similarly, $u_1 u_1^* = e$ and $u_2 u_2^* = e$.

Using Lemma 2.4, we generalize [14, Lemma 2.1] to JB^*-algebras:

Theorem 2.6. *Let $\mathcal{J}$ be a JB^*-algebra and $x \in \mathcal{J}$ with $\|x\| \le \epsilon < 1$. Then for each $u \in \mathcal{U}(\mathcal{J})$ there exist $u_1, u_2 \in \mathcal{U}(\mathcal{J})$ such that $u + x = u_1 + \epsilon u_2$.*

Proof. Let $\mathcal{P} = \mathcal{J}(u, x, x^{*u})$ be the JC^*-subalgebra of the JB^*-algebra $\mathcal{J}^{[u]}$, generated by its identity u, x and x^{*u}. As $\|x\| < 1$, $u + x$ is invertible by [20, Lemma 2.1(iii)]. So, by [20, Theorem 4.12], there exists a unitary $v \in \mathcal{P}$ such that $u + x$ is positive (and invertible) in the isotope $\mathcal{P}^{[v]}$. Hence by the *functional calculus of positive elements* $\inf \sigma_{\mathcal{J}^{[v]}}(u+x) = \|(u+x)^{-1_v}\|^{-1}$. Moreover, by using the geometric series representation $(u+x)^{-1_v} = \sum_{n=0}^{\infty}(-x)^n$ (see [20, Lemma 2.1(iii)]), we get

$$\begin{aligned}
\|(u+x)^{-1_v}\|^{-1} &= \left\| \sum_{n=0}^{\infty} (-x)^n \right\|^{-1} \\
&\ge \left(\sum_{n=0}^{\infty} \|x^n\| \right)^{-1} \\
&= \left(\frac{1}{1 - \|x\|} \right)^{-1} \\
&= 1 - \|x\| \ge 1 - \epsilon, \quad \text{as } \|x\| \le \epsilon.
\end{aligned}$$

Of course, $\sup \sigma_{\mathcal{J}^{[v]}}(u+x) \le 1 + \epsilon$. So, $\sigma_{\mathcal{J}^{[v]}}(u+x) \subseteq [1-\epsilon, 1+\epsilon]$. Hence, $\sigma_{\mathcal{J}^{[v]}}(y) \subseteq [\frac{1-\epsilon}{1+\epsilon}, 1]$ with $y = (1+\epsilon)^{-1}(u+x)$.

Taking $\alpha = \epsilon(1+\epsilon)^{-1}$ we see that $\sigma_{\mathcal{J}^{[v]}}(y) \subseteq [1-2\alpha, 1]$ (indeed, $1 - 2\alpha = 1 - 2\epsilon(1+\epsilon)^{-1} = \frac{1+\epsilon-2\epsilon}{1+\epsilon}$). Hence, by Lemma 2.4, $y = \alpha v_1 + (1-\alpha) v_2$ for some $v_1, v_2 \in \mathcal{U}(\mathcal{J}^{[v]})$. Thus, by [20, Theorem 4.6], the required result follows with $u_1 = v_2$ and $u_2 = v_1$. □

3. Distance from a positive element to the unitaries

Here, we prove that the distance from any positive element to the set of unitaries is attained at the unit element of the JB^*-algebra. As consequence of this fact and Lemma 2.4, we will obtain a precise analogue of [7, Corollary 11] for general JB^*-algebras (see Corollary 3.4 below) that also provides an alternative proof of the corresponding result for C^*-algebras.

Theorem 3.1. *Let x be a positive noninvertible element of the unital JB^*-algebra $\mathcal{J}$ with $\|x\| \le 1$. Then* $\operatorname{dist}(x, \mathcal{U}(\mathcal{J})) = \|e - x\|$.

Proof. Clearly, $0 \in \sigma_{\mathcal{J}}(x)$ and so $1 \in \sigma_{\mathcal{J}}(e-x)$. Also, $e-x \geq 0$ because $x \geq 0$ and $\|x\| \leq 1$. Therefore, $1 \in \sigma_{\mathcal{J}}(e-x) \subseteq [0,1]$. Hence, $\gamma_{\mathcal{J}}(e-x) = 1$. But $\gamma_{\mathcal{J}}(e-x) = \|e-x\|$ since $e-x \geq 0$. Therefore, $\|e-x\| = 1$.

Now, let $\|u-x\| < \|e-x\|$ for some $u \in \mathcal{U}(\mathcal{J})$. Then, $\|u-x\| < 1$. Therefore, by [20, Theorem 4.4, Lemma 2.1(iii)], x is invertible in $\mathcal{J}^{[u]}$. But by [20, Lemma 4.2(ii)], $\mathcal{J}^{[u]}_{\text{inv}} = \mathcal{J}_{\text{inv}}$. Hence, x is invertible in $\mathcal{J}$; a contradiction. Thus, $\|e-x\| \leq \|u-x\|, \forall u \in \mathcal{U}(\mathcal{J})$ and so the required result follows. □

Obviously, the above proof does not work if x is invertible. However, the same conclusion can be obtained without assuming the noninvertibility of x:

Theorem 3.2. *Let x be a positive element of unital JB^*-algebra $\mathcal{J}$. Then*

$$\|x-e\| = \operatorname{dist}(x,\mathcal{U}(\mathcal{J})) \leq \|x+e\|.$$

Proof. Let $u \in \mathcal{U}(\mathcal{J})$. Then the JB^*-subalgebra $\mathcal{J}(e,x,u,u^*)$, generated by $x, u, u^* = u^{-1}$ and unit e of $\mathcal{J}$ is a unital JC^*-algebra by the Shirshov–Cohn theorem with inverses (cf. [11]). Considering $\mathcal{J}(e,x,u,u^*)$ a JC^*-subalgebra of the C^*-algebra $\mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, we see that x is positive and u is unitary in $\mathcal{B}(\mathcal{H})$. Therefore, by [1, Theorem 3.1], $\|x-e\| \leq \|x-u\| \leq \|x+e\|$. Thus, the required result follows. □

Corollary 3.3. *For all $x \in \mathcal{J}_{\text{inv}}$, $\|x-u\| = \operatorname{dist}(x,\mathcal{U}(\mathcal{J}))$ where $u \in \mathcal{U}(\mathcal{J})$ is given by the polar decomposition $x = u|x|$ of x considered in a suitable $\mathcal{B}(\mathcal{H})$.*

Proof. By [20, Theorem 4.12], x is positive in the isotope $\mathcal{J}^{[u]}$ with unit u. Hence, $\|x-u\| = \operatorname{dist}(x,\mathcal{U}(\mathcal{J}^{[u]})) = \operatorname{dist}(x,\mathcal{U}(\mathcal{J}))$ by above Theorem 3.2 and [20, Theorem 4.6]. □

Now, we are able to obtain the following extension of the above mentioned C^*-algebra result due to Kadison and Pedersen (namely, [7, Corollary 11]). The proof we give exploits some of our previous results and standard continuous functional calculus.

Corollary 3.4. *Let $\mathcal{J}$ be a JB^*-algebra with identity element e. Let $x \in (\mathcal{J})_1$ be such that $\operatorname{dist}(x,\mathcal{U}(\mathcal{J})) \leq 2\alpha$ with $\alpha < \frac{1}{2}$. Then*

$$x \in \alpha\mathcal{U}(\mathcal{J}) + (1-\alpha)\mathcal{U}(\mathcal{J}).$$

Proof. Let β be any number such that $\alpha < \beta < \frac{1}{2}$. Since $\operatorname{dist}(x,\mathcal{U}(\mathcal{J})) \leq 2\alpha$, there exists unitary $u \in \mathcal{U}(\mathcal{J})$ such that $\|x-u\| < 2\beta < 1$. So, by [20, Theorem 4.4, Lemma 2.1(iii)], x is invertible in the unitary isotope $\mathcal{J}^{[u]}$ and hence $x \in \mathcal{J}_{\text{inv}}$ by [20, Lemma 4.2(ii)]. Then, by Corollary 3.3,

$$\|x-v\| = \operatorname{dist}(x,\mathcal{U}(\mathcal{J})) \tag{1}$$

where $v \in \mathcal{U}(\mathcal{J})$ is given by the polar decomposition $x = v|x|$ in some $\mathcal{B}(\mathcal{H})$. So that $\|x-v\| \leq 2\alpha \leq 2\beta$. By [20, Theorem 4.12], x is positive in the isotope

$\mathcal{J}^{[v]}$ in which v is the unit. Therefore, by the continuous functional calculus, $\sigma_{\mathcal{J}^{[v]}}(x) \subseteq [1-2\beta, 1]$. Notice that the existence of $\mathcal{J}^{[v]}$ and the positivity of x in $\mathcal{J}^{[v]}$ with (1) depend only on the invertibility of x in $\mathcal{J}$. So that $\sigma_{\mathcal{J}^{[v]}}(x) \subseteq [1-2\alpha, 1]$. Hence, by Lemma 2.4, $x = \alpha w_1 + (1-\alpha)w_2$ for some $w_1, w_2 \in \mathcal{U}(\mathcal{J}^{[v]})$. Thus, by [20, Theorem 4.6], $x \in \alpha\mathcal{U}(\mathcal{J}) + (1-\alpha)\mathcal{U}(\mathcal{J})$. □

4. Means of unitaries

In this section, we extend [7, Corollary 10] to general JB^*-algebras, which in turn would lead us to the JB^*-algebra analogues of [7, Corollary 12, Theorem 14] and the conclusion that *every element in the convex hull of* n *unitaries in a* JB^**-algebra is the arithmetic mean of* n *unitaries* in the same algebra (an exact analogue of [7, Corollary 15]).

We need the following result:

Lemma 4.1. *Let* $\mathcal{J}$ *be a* JB^**-algebra with unit* e *and let* $u \in \mathcal{J}$. *Then* $\mathcal{J}(e, u, u^*)$ *is a unital* C^**-algebra.*

Proof. By Jacobson's Theorem [6], any Jordan algebra is integrally power associative, provided the inverses involved exist. It follows that $\mathcal{J}(e, u, u^*)$ is a C^*-algebra by [20, Lemma 2.1, Corollary 2.5]. □

Next, we prove a JB^*-algebra analogue of [7, Corollary 10]; observe that [7, Corollary 10] is used in our proof.

Theorem 4.2. *Let* $\mathcal{J}$ *be a unital* JB^**-algebra . Then for any* $0 \le \alpha \le \beta \le \frac{1}{2}$, $\alpha\mathcal{U}(\mathcal{J}) + (1-\alpha)\mathcal{U}(\mathcal{J}) \subseteq \beta\mathcal{U}(\mathcal{J}) + (1-\beta)\mathcal{U}(\mathcal{J})$.

Proof. Let u_1, u_2 be arbitrary but fixed elements of $\mathcal{U}(\mathcal{J})$. By [20, Theorem 4.4], the isotope $\mathcal{J}^{[u_1]}$ is a JB^*-algebra with identity element u_1. By [20, Theorem 4.6],

$$\mathcal{U}(\mathcal{J}) = \mathcal{U}(\mathcal{J}^{[u_1]}). \tag{2}$$

In particular, $u_2 \in \mathcal{U}(\mathcal{J}^{[u_1]})$. Let $\mathcal{J}(u_1, u_2, u_2^{-1_{u_1}})$ denote the norm closed Jordan subalgebra of $\mathcal{J}^{[u_1]}$, generated by the unitary u_2, its inverse $u_2^{-1_{u_1}}$ and the unit u_1. By Lemma 4.1, $\mathcal{J}(u_1, u_2, u_2^{-1_{u_1}})$ is a C^*-algebra. Moreover, we see that

$$\alpha u_1 + (1-\alpha)u_2 \in \alpha\mathcal{U}(\mathcal{J}(u_1, u_2, u_2^{-1_{u_1}})) + (1-\alpha)\mathcal{U}(\mathcal{J}(u_1, u_2, u_2^{-1_{u_1}})).$$

Hence, by [7, Corollary 10], there exist unitaries u_3, u_4 in the C^*-algebra $\mathcal{J}(u_1, u_2, u_2^{-1_{u_1}})$ such that

$$\alpha u_1 + (1-\alpha)u_2 = \beta u_3 + (1-\beta)u_4. \tag{3}$$

From Equation (2), we deduce $\mathcal{U}(\mathcal{J}(u_1, u_2, u_2^{-1_{u_1}})) \subseteq \mathcal{U}(\mathcal{J})$. In particular, $u_3, u_4 \in \mathcal{U}(\mathcal{J})$. This together with (3) gives the required result. □

Now, proceeding on the lines of [7], one can easily obtain the JB^*-algebra analogues of certain results due to Kadison and Pedersen (namely, [7, Corollary 12, Theorem 14 and its Corollary 15]): the proofs of these results as they appeared in [7] for C^*-algebras work well in the general case after appropriate translation of the terms for JB^*-algebras and using Theorem 4.2 in place of [7, Corollary 10]:

Corollary 4.3. *Let $\mathcal{J}$ be a unital JB^*-algebra . Then, for any nonnegative real numbers α_1, α_2, $\alpha_1\mathcal{U}(\mathcal{J}) + \alpha_2\mathcal{U}(\mathcal{J}) \subseteq \beta_1\mathcal{U}(\mathcal{J}) + \beta_2\mathcal{U}(\mathcal{J})$ provided the point (β_1, β_2) lies on the line segment joining (α_1, α_2) to (α_2, α_1) in the plane $\mathbb{R}^2$.*

Proof. The result follows from Theorem 4.2 (see the proof of [7, Corollary 12]). □

The next result extends Corollary 4.3 from two to any positive integer number of unitaries.

Theorem 4.4. *Let $\mathcal{J}$ be a unital JB^*-algebra and $(\alpha_1, \ldots, \alpha_n) \in \Re^n$ (Euclidean n-space) be such that each $\alpha_j \geq 0$. Let $(\beta_1, \ldots, \beta_n) \in coK$ (the convex hull of K), where*

$$K = \{(\alpha_{\pi(1)}, \ldots, \alpha_{\pi(n)}) : \pi \text{ is a permutation on } \{1, \ldots, n\}\}.$$

Then $\sum_{j=1}^n \alpha_j\mathcal{U}(\mathcal{J}) \subseteq \sum_{j=1}^n \beta_j\mathcal{U}(\mathcal{J})$.

Proof. The proof is immediate from Corollary 4.3 and [7, Lemma 13] (see the proof of [7, Theorem 14]). □

We conclude with the following strict analogue of [7, Corollary 15]):

Corollary 4.5. *Any convex combination of unitaries in a unital JB^*-algebra is the mean of same number of unitaries in the algebra. Hence, $u_m(x) = u_c(x)$.*

Proof. Immediate from Theorem 4.4 (see the proof of [7, Corollary 15]). □

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