

The λ_u -function in JB^* -algebras

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ABSTRACT. Inspired by work of R. M. Aron and R. H. Lohman, Gert K. Pedersen introduced a geometric function, which is defined on the unit ball of a C^* -algebra and called the λ_u -function. Our goal here is to extend the notion of a λ_u -function to the context of JB^* -algebras. We study convex combinations of elements in a JB^* -algebra using the λ_u -function and earlier results we have obtained on the geometry of JB^* -algebras. A formula to compute λ_u -function is obtained for invertible elements in a JB^* -algebra. In the course of our analysis, some C^* -algebra results due to G. K. Pedersen, C. L. Olsen and M. Rørdam are extended to general JB^* -algebras.

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1. Introduction and preliminaries

R. M. Aron and R. H. Lohman [2] studied a geometric function, called the λ -function for normed spaces. Subsequently, G. K. Pedersen [11] introduced a related function, namely, the λ_u -function, defined on the unit ball of a C^* -algebra, where u is a unitary element of the algebra. In this article, we study the λ_u -function in the general setting of JB^* -algebras. It may be noted here that the study of the λ_u -function can not be further extended to more general JB^* -triple systems (cf. [19] or [16]) which have no invertible (hence, no unitary) elements.

The absence of associativity in JB^* -algebras causes great difficulties in calculations. In [15], the author proved some fundamental results on unitary isotopes of JB^* -algebras and by applying these we obtained various results on representing elements as convex combinations of unitaries in a JB^* -algebra (see [14, 16, 17, 18]); our approach also provides alternative proofs to certain related results for C^* -algebras.

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We introduce below two special sets of real numbers $\mathcal{S}(x)$ and $\mathcal{V}(x)$ that turn out to be intervals and are used to study certain convex combinations of elements in JB^* -algebras. Using some of our earlier results from [15, 18], we shall investigate the relationship between the intervals $\mathcal{S}(x)$, $\mathcal{V}(x)$ and the λ_u -function for JB^* -algebras. In the process, we obtain extensions to JB^* -algebras of some C^* -algebra results that appeared in [10, 11, 12]. In the end, a formula to compute λ_u -function is obtained for invertible elements of a JB^* -algebra.

Preliminaries. We begin by recalling that if x is an element in a Jordan algebra \mathcal{J} , then the x -homotope of \mathcal{J} , denoted by $\mathcal{J}_{[x]}$, is the Jordan algebra consisting of the same elements and linear space structure as \mathcal{J} but with a different product, denoted by “ \cdot_x ”, defined by $a \cdot_x b = \{axb\}$ for all a, b in $\mathcal{J}_{[x]}$. Here, $\{pqr\}$ denotes the Jordan triple product and is defined in any Jordan algebra by $\{pqr\} = (p \circ q) \circ r - (p \circ r) \circ q + (q \circ r) \circ p$ where “ \circ ” stands for the Jordan product in the original algebra. (See [5], for instance.)

The homotopes of interest here will be obtained when \mathcal{J} has a unit e and x is invertible: this means that there exists $x^{-1} \in \mathcal{J}$, called the inverse of x , such that $x \circ x^{-1} = e$ and $x^2 \circ x^{-1} = x$. The set of all invertible elements of \mathcal{J} will be denoted by \mathcal{J}_{inv} . In this case, as $x \cdot_{x^{-1}} y = \{xx^{-1}y\} = y + (x^{-1} \circ y) \circ x - (x \circ y) \circ x^{-1} = y$ (see [8]), x acts as the unit for the homotope $\mathcal{J}_{[x^{-1}]}$ of \mathcal{J} .

If $x \in \mathcal{J}_{\text{inv}}$ then x -isotope of \mathcal{J} , denoted by $\mathcal{J}^{[x]}$, is defined to be the x^{-1} -homotope $\mathcal{J}_{[x^{-1}]}$ of \mathcal{J} . We denote the multiplication “ $\cdot_{x^{-1}}$ ” of $\mathcal{J}^{[x]}$ by “ \circ_x ”. $\{, , \}_x, y^{-1x}$ will stand for the Jordan triple product and the multiplicative inverse (if it exists) of y in the isotope $\mathcal{J}^{[x]}$, respectively.

Isotopy may produce essentially different Jordan algebras. The x -isotope $\mathcal{J}^{[x]}$ of a Jordan algebra \mathcal{J} need not be isomorphic to \mathcal{J} . For such examples see [9, 7]. However, some important features of Jordan algebras are unaffected by isotopy (see [15, Lemma 4.2 and Theorem 4.6], for examples).

JB^* -algebras. A Jordan algebra \mathcal{J} with product \circ is called a Banach Jordan algebra if there is a norm $\|\cdot\|$ on \mathcal{J} such that $(\mathcal{J}, \|\cdot\|)$ is a Banach space and $\|a \circ b\| \leq \|a\| \|b\|$. If, in addition, \mathcal{J} has unit e with $\|e\| = 1$ then \mathcal{J} is called a unital Banach Jordan algebra.

An important tool in Banach algebra theory is the spectrum of an element. Let \mathcal{J} be a complex unital Banach Jordan algebra with unit e and let $x \in \mathcal{J}$. The spectrum of x in \mathcal{J} , denoted by $\sigma_{\mathcal{J}}(x)$, is defined by

$$\sigma_{\mathcal{J}}(x) = \{\lambda \in \mathbb{C} : x - \lambda e \text{ is not invertible in } \mathcal{J}\}.$$

Here, \mathbb{C} denotes the field of complex numbers. When no confusion can arise, we shall write $\sigma(x)$ in place of $\sigma_{\mathcal{J}}(x)$.

We are interested in a special class of Banach Jordan algebras, called JB^* -algebras. A complex Banach Jordan algebra \mathcal{J} with involution $*$ is called a JB^* -algebra if $\|\{xx^*x\}\| = \|x\|^3$ for all $x \in \mathcal{J}$. It follows that

$\|x^*\| = \|x\|$ for all elements x of a JB^* -algebra (see [22], for instance). The class of JB^* -algebras was introduced by Kaplansky in 1976 and it includes all C^* -algebras as a proper subclass (cf. [20]).

As usual, an element x of a JB^* -algebra \mathcal{J} is said to be self-adjoint if $x^* = x$. A self-adjoint element x of \mathcal{J} is said to be positive in \mathcal{J} if its spectrum $\sigma_{\mathcal{J}}(x)$ is contained in the set of nonnegative real numbers. For basic theory of Banach Jordan algebras and JB^* -algebras, we refer to the sources [1, 4, 13, 19, 20, 21, 22].

Unitary Isotopes of a JB^* -algebra. Let \mathcal{J} be a unital JB^* -algebra. An element $u \in \mathcal{J}$ is called unitary if $u^* = u^{-1}$. We shall denote the set of all unitary elements of \mathcal{J} by $\mathcal{U}(\mathcal{J})$ and its convex hull by $co\mathcal{U}(\mathcal{J})$. If $u \in \mathcal{U}(\mathcal{J})$ then the isotope $\mathcal{J}^{[u]}$ is called a unitary isotope of \mathcal{J} .

It is well known (see [7, 3, 15]) that for any unitary element u of JB^* -algebra \mathcal{J} , the unitary isotope $\mathcal{J}^{[u]}$ is a JB^* -algebra with u as its unit with respect to the original norm and the involution $*_u$ defined as below:

$$x^{*_u} = \{ux^*u\}.$$

Like invertible elements [15, Theorem 4.2 (ii)], the set of unitary elements in the (unital) JB^* -algebra \mathcal{J} is invariant on passage to isotopes of \mathcal{J} [15, Theorem 4.6]. Moreover, every invertible element x of JB^* -algebra \mathcal{J} is positive in certain unitary isotope of \mathcal{J} [15, Theorem 4.12]; this is a tricky result and its proof (appeared in [15]) involves Stone–Weierstrass Theorem and standard functional calculus. In the sequel, we shall use these results as our main tools.

2. λ_u -function

The λ_u -function, defined on the unit ball of a C^* -algebra, was introduced by Pedersen [11] where u is some unitary element of the algebra. In this section, we study the λ_u -function for general JB^* -algebras; of course, this study can not be further extended to more general JB^* -triple systems (cf. [19] or [16]) that do not have unitary elements. In the sequel, we shall introduce two special classes of sets $\mathcal{V}(x)$ and $\mathcal{S}(x)$ of real numbers. Indeed, these sets are intervals and allow us to study some special convex combinations. We investigate relationships between intervals $\mathcal{S}(x)$, $\mathcal{V}(x)$ and the λ_u -function. In doing this, some C^* -algebra results due to G. K. Pedersen, C. L. Olsen and M. Rørdam [10, 11, 12] will be extended to JB^* -algebras.

We define the set $\mathcal{V}(x)$, for each element x of the closed unit ball of a JB^* -algebra as follows:

Definition 2.1. Let \mathcal{J} be a unital JB^* -algebra.

(1) For each $\delta \geq 1$, we define $co_{\delta}\mathcal{U}(\mathcal{J}) \subseteq co\mathcal{U}(\mathcal{J})$ by

$$co_{\delta}\mathcal{U}(\mathcal{J}) = \left\{ \delta^{-1} \sum_{i=1}^{n-1} u_i + \delta^{-1}(1 + \delta - n)u_n : u_j \in \mathcal{U}(\mathcal{J}), j = 1, \dots, n \right\}$$

where n is the integer given by $n - 1 < \delta \leq n$.

(2) For each $x \in (\mathcal{J})_1$, we define the set $\mathcal{V}(x)$ by

$$\mathcal{V}(x) = \{\beta \geq 1 : x \in \text{co}_\beta \mathcal{U}(\mathcal{J})\}.$$

Part (a) of the next theorem extends a C^* -algebra result due to Rørdam (see [12, Proposition 3.1]); the proof follows his argument with suitable changes necessitated by the nonassociativity of Jordan algebras.

Theorem 2.2. *Let \mathcal{J} be a unital JB^* -algebra and let $x \in (\mathcal{J})_1$.*

(a) *Let $\|\gamma x - u_o\| \leq \gamma - 1$ for some $\gamma \geq 1$ and some $u_o \in \mathcal{U}(\mathcal{J})$. Let $(\alpha_2, \dots, \alpha_m) \in \mathfrak{R}^{m-1}$ such that $0 \leq \alpha_j < \gamma^{-1}$ and $\gamma^{-1} + \sum_{j=2}^m \alpha_j = 1$. Then there exist unitaries u_1, \dots, u_m in \mathcal{J} such that*

$$(i) \quad x = \gamma^{-1}u_1 + \sum_{j=2}^m \alpha_j u_j.$$

Moreover,

$$(ii) \quad (\gamma, \infty) \subseteq \mathcal{V}(x).$$

(b) *On the other hand, if (ii) holds then for all $r > \gamma$ there is $u_1 \in \mathcal{U}(\mathcal{J})$ such that $\|rx - u_1\| \leq r - 1$.*

Proof. (a) If $\gamma = 1$, then $\|\gamma x - u_o\| \leq \gamma - 1$ means that $\gamma x = u_o$ and so there is nothing left to prove (i) in this case.

Now, assume $\gamma > 1$ and put $y = (\gamma - 1)^{-1}(\gamma x - u_o)$. Since $\|\gamma x - u_o\| \leq \gamma - 1$, $\|y\| = (\gamma - 1)^{-1}\|\gamma x - u_o\| \leq (\gamma - 1)^{-1}(\gamma - 1) = 1$ so that $y \in (\mathcal{J})_1$. Further, since $\gamma^{-1} + \sum_{j=2}^m \alpha_j = 1$ and since $\gamma x - u_o - (\gamma - 1)y = 0$, we have

$$(iii) \quad \gamma x = u_o + (\gamma - 1)y = u_o + \gamma \sum_{j=2}^m \alpha_j y.$$

We note that $0 \leq \gamma \alpha_j < 1$ since $0 \leq \alpha_j < \gamma^{-1}$ for all $j = 2, \dots, m$. Therefore, by [18, Theorem 3.6] there exist unitaries $v_1, \dots, v_m, u_2, \dots, u_m$ in \mathcal{J} such that $v_1 = u_o$ and $v_{k-1} + \gamma \alpha_k y = v_k + \gamma \alpha_k u_k$ for $k = 2, \dots, m$. We put $v_m = u_1$. Then (iii) becomes that $\gamma x = v_2 + \gamma \alpha_2 u_2 + \sum_{j=3}^m \gamma \alpha_j y = \gamma \alpha_2 u_2 + v_3 + \gamma \alpha_3 u_3 + \sum_{j=4}^m \gamma \alpha_j y = \dots = u_1 + \sum_{j=2}^m \gamma \alpha_j u_j$. This gives the representation (i) of x .

Now, let $\delta > \gamma$ and m, n be integers given by $m - 2 < (1 - \gamma^{-1})\delta \leq m - 1$ and $n - 1 < \delta \leq n$. Then $(1 - \gamma^{-1})\delta + 1 = \delta + (1 - \gamma^{-1})\delta < \delta$ so that $m \leq n$. Moreover, by setting $\alpha_m = 1 - \gamma^{-1} - (m - 2)\delta^{-1}$, we have $0 < \alpha_m \leq \delta^{-1} < \gamma^{-1}$. Since $\alpha_m + \gamma^{-1} + (m - 2)\delta^{-1} = 1$, by the first part of this proof we get the existence of unitaries w_1, \dots, w_n in \mathcal{J} such that

$$(iv) \quad \sum_{j=1}^n \alpha_j u_j = x = \gamma^{-1}w_1 + \delta^{-1} \sum_{k=2}^{m-1} w_k + \alpha_m w_m + 0w_{m+1} + \dots + 0w_n.$$

For (ii), we only have to show that $\delta \in \mathcal{V}(x)$. For this, we must find unitaries w'_1, \dots, w'_n in \mathcal{J} such that

$$(v) \quad x = \delta^{-1} \sum_{j=1}^{n-1} w'_j + \delta^{-1}(\delta + 1 - n)w'_n = \sum_{j=1}^n \gamma_j w'_j$$

with $\gamma_1 = \dots = \gamma_{n-1} = \delta^{-1}$ and $\gamma_n = \delta^{-1}(\delta + 1 - n)$; recall that $n-1 < \delta \leq n$. The existence of such unitaries w'_1, \dots, w'_n in \mathcal{J} follows from [18, Theorem 5.5] if the n -tuple $(\gamma_1, \dots, \gamma_n) \in \text{co}\{(\alpha_{\pi(1)}, \dots, \alpha_{\pi(n)}) : \pi \in S_n\}$, where S_n denotes the group of all permutations on the set $\{1, \dots, n\}$; or equivalently, if $\sum_{j=1}^k \gamma_j \leq \sum_{j=1}^k \alpha_j$ for all $k = 1, \dots, n-1$ because $0 \leq \gamma_n \leq \gamma_{n-1} = \dots = \gamma_1$ since $\delta \leq n$ by (see [18, Lemma 5.4]). However, by (iv) and (v), for $k < m$, $\sum_{j=1}^k \gamma_j = k\delta^{-1} \leq \gamma^{-1} + (k-1)\delta^{-1} = \sum_{j=1}^k \alpha_j$ since $\delta^{-1} < \gamma^{-1}$. Also, for $k \geq m$, $\sum_{j=1}^k \gamma_j \leq 1 = \sum_{j=1}^k \alpha_j$.

(b) We suppose $(\gamma, \infty) \subseteq \mathcal{V}(x)$. If $r > \gamma$ then $x \in \text{co}_r \mathcal{U}(\mathcal{J})$ so that $x = r^{-1}(u_1 + \dots + u_{n-1} + (1+r-n)u_n)$ for some unitaries $u_1, \dots, u_n \in \mathcal{U}(\mathcal{J})$ and integer n with $n-1 < r \leq n$; hence, $\|rx - u_1\| \leq n-2 + (1+r-n) = r-1$. \square

Corollary 2.3. *For any unital JB^* -algebra \mathcal{J} , $\text{co}_\gamma \mathcal{U}(\mathcal{J}) \subseteq \text{co}_\delta \mathcal{U}(\mathcal{J})$ whenever $1 \leq \gamma \leq \delta$. In particular, for each $x \in (\mathcal{J})_1$, $\mathcal{V}(x)$ is either empty or equal to $[\gamma, \infty)$ or (γ, ∞) for some $\gamma \geq 1$.*

Proof. This follows directly from Theorem 2.2 and constructions of sets $\text{co}_\delta \mathcal{U}(\mathcal{J})$, $\text{co}_\gamma \mathcal{U}(\mathcal{J})$ and $\mathcal{V}(x)$. \square

Next, we define another set $\mathcal{S}(x)$ and the λ_u -function:

Definition 2.4. Let \mathcal{J} be a unital JB^* -algebra. For each $x \in (\mathcal{J})_1$, we define

$$\mathcal{S}(x) = \{0 \leq \lambda \leq 1 : x = \lambda v + (1 - \lambda)y \text{ with } v \in \mathcal{U}(\mathcal{J}), y \in (\mathcal{J})_1\}.$$

and the λ_u -function by

$$\lambda_u(x) = \sup \mathcal{S}(x)$$

We observe some interesting relationships between the sets $\mathcal{S}(x)$ and $\mathcal{V}(x)$:

Theorem 2.5. *Let \mathcal{J} be a unital JB^* -algebra and let $x \in (\mathcal{J})_1$.*

- (i) *If $\lambda \in \mathcal{S}(x)$ and $\lambda > 0$ then $(\lambda^{-1}, \infty) \subseteq \mathcal{V}(x)$.*
- (ii) *If $\delta \in \mathcal{V}(x)$ then $\delta^{-1} \in \mathcal{S}(x)$.*
- (iii) *$\lambda_u(x) = 0$ if and only if $\mathcal{V}(x) = \emptyset$.*
- (iv) *If $\lambda_u(x) > 0$ then $\mathcal{S}(x) = [0, \lambda_u(x))$ or $[0, \lambda_u(x)]$.*
- (v) *If $\lambda_u(x) > 0$ and if $0 < \lambda < \lambda_u(x)$ then $\lambda^{-1} \in \mathcal{V}(x)$.*
- (vi) *If $\lambda_u(x) > 0$ then $(\inf \mathcal{V}(x))^{-1} = \lambda_u(x)$.*
- (vii) *If $\inf(\mathcal{V}(x)) \in \mathcal{V}(x)$ then $\lambda_u(x) \in \mathcal{S}(x)$.*

Proof. (i) This follows from Theorem 2.2.

(ii) Immediate from the definitions.

(iii) Suppose $\lambda_u(x) = 0$ and $\mathcal{V}(x) \neq \emptyset$. Then there is at least one (in fact, infinitely many by Corollary 2.3) $\delta \in \mathcal{V}(x)$, so $\delta^{-1} \in \mathcal{S}(x)$ by part (ii). Hence, $\lambda_u(x) \geq \delta^{-1} > 0$; a contradiction. Conversely, suppose $\mathcal{V}(x) = \emptyset$. If $\lambda_u(x) \neq 0$, there exists at least one $\lambda \in \mathcal{S}(x)$ with $\lambda > 0$. Hence, by part (i), $(\lambda^{-1}, \infty) \subseteq \mathcal{V}(x)$. In particular, $\mathcal{V}(x) \neq \emptyset$; a contradiction.

(iv) By definition, $0 \in \mathcal{S}(x)$. Let $\lambda \in (0, \lambda_u(x))$. By definition of $\lambda_u(x)$, there exists an increasing sequence (λ_n) in $\mathcal{S}(x)$ such that $\lim_{n \rightarrow \infty} \lambda_n = \lambda_u(x)$. As $\lambda < \lambda_u(x)$, there exists integer N such that when $n \geq N$, $\lambda_n > \lambda$. Hence, by part (i), $\lambda^{-1} \in (\lambda_n^{-1}, \infty) \subseteq \mathcal{V}(x)$ so $\lambda \in \mathcal{S}(x)$ by part (ii). So $[0, \lambda_u(x)] \subseteq \mathcal{S}(x) \subseteq [0, \lambda_u(x)]$ as $\lambda_u(x) = \sup \mathcal{S}(x)$. Thus $\mathcal{S}(x) = [0, \lambda_u(x)]$ or $[0, \lambda_u(x)]$.

(v) Let $\lambda \in (0, \lambda_u(x))$. Let $\lambda_1 = \frac{1}{2}(\lambda + \lambda_u(x))$. Then $\lambda < \lambda_1 < \lambda_u(x)$, and so by part (vi), $\lambda_1 \in \mathcal{S}(x)$. Hence, $\lambda^{-1} \in (\lambda_1^{-1}, \infty) \subseteq \mathcal{V}(x)$.

(vi) Let $\lambda \in (0, \lambda_u(x))$. Then by part (v), $\lambda^{-1} \in \mathcal{V}(x)$ so $\lambda^{-1} \geq \inf \mathcal{V}(x)$. Therefore, $\lambda \leq (\inf \mathcal{V}(x))^{-1}$ and so $(\inf \mathcal{V}(x))^{-1}$ is an upper bound for $(0, \lambda_u(x))$. Hence, $\lambda_u(x) \leq (\inf \mathcal{V}(x))^{-1}$. Next, suppose $\lambda_u(x) < \delta < (\inf \mathcal{V}(x))^{-1}$. Then $\delta^{-1} > \inf \mathcal{V}(x)$ so $\delta^{-1} \in \mathcal{V}(x)$ by Corollary 2.3. Hence by part (ii), $\delta \in \mathcal{S}(x)$; a contradiction. Thus $\lambda_u(x) = (\inf \mathcal{V}(x))^{-1}$.

(vii) Follows immediately from the parts (iii), (vi) and (ii). \square

Corollary 2.6. *For any $x \in (\mathcal{J})_1 \setminus \mathcal{J}_{\text{inv}}$, the following statements are equivalent:*

- (i) $\text{dist}(x, \mathcal{J}_{\text{inv}}) < 1 \Rightarrow \mathcal{V}(x) \neq \emptyset$.
- (ii) $\lambda_u(x) = 0 \Rightarrow \text{dist}(x, \mathcal{J}_{\text{inv}}) = 1$.
- (iii) $\text{dist}(x, \mathcal{J}_{\text{inv}}) < 1 \Rightarrow \lambda_u(x) > 0$.

Proof. Immediate from Theorem 2.5(iii). \square

We now look for the elements with $\mathcal{V}(x) \cap [1, 2) \neq \emptyset$.

Theorem 2.7. *Let $0 \leq \alpha < \frac{1}{2}$. Let \mathcal{J} be a unital JB^* -algebra and let $x \in (\mathcal{J})_1$. Then the following statements are equivalent:*

- (i) $\text{dist}(x, \mathcal{U}(\mathcal{J})) \leq 2\alpha$.
- (ii) $x \in \alpha\mathcal{U}(\mathcal{J}) + (1 - \alpha)\mathcal{U}(\mathcal{J})$.
- (iii) $(1 - \alpha)^{-1} \in \mathcal{V}(x)$.
- (iv) $(1 - \alpha) \in \mathcal{S}(x)$.

Proof. (i) \Rightarrow (ii): See the proof of [18, Corollary 4.4].

(ii) \Rightarrow (iii): If $x = \alpha u_1 + (1 - \alpha)u_2$ where $u_1, u_2 \in \mathcal{U}(\mathcal{J})$ then as $\alpha < \frac{1}{2}$ we have $1 \leq (1 - \alpha)^{-1} < 2$. Therefore, $x = (1 - \alpha)u_2 + (1 - \alpha)(1 + \frac{1}{1 - \alpha} - 2)u_1$ so that $(1 - \alpha)^{-1} \in \mathcal{V}(x)$.

(iii) \Rightarrow (iv): Clear from part (ii) of Theorem 2.5.

(iv) \Rightarrow (i): Suppose $x = (1 - \alpha)u_1 + \alpha y_1$ where $u_1 \in \mathcal{U}(\mathcal{J})$ and $y_1 \in (\mathcal{J})_1$. Then $\|x - u_1\| = \alpha\|y_1 - u_1\| \leq 2\alpha$. Hence, $\text{dist}(x, \mathcal{U}(\mathcal{J})) \leq 2\alpha$. \square

Corollary 2.8. *Let \mathcal{J} be a unital JB^* -algebra and $x \in (\mathcal{J})_1$.*

- (a) *The following statements are equivalent:*

- (i) x is invertible.
 - (ii) $x \in \alpha\mathcal{U}(\mathcal{J}) + (1 - \alpha)\mathcal{U}(\mathcal{J})$ for some $0 \leq \alpha < \frac{1}{2}$.
 - (iii) $\text{dist}(x, \mathcal{U}(\mathcal{J})) \leq 2\alpha$ for some $0 \leq \alpha < \frac{1}{2}$.
 - (iv) $(1 - \alpha) \in \mathcal{S}(x)$ for some $0 \leq \alpha < \frac{1}{2}$.
 - (v) $(1 - \alpha)^{-1} \in \mathcal{V}(x)$ for some $0 \leq \alpha < \frac{1}{2}$.
 - (vi) $\lambda \in \mathcal{V}(x)$ for some $1 \leq \lambda < 2$.
- (b) Moreover, if x is invertible then $\inf \mathcal{V}(x) = 2(1 + \|x^{-1}\|^{-1})^{-1}$ and $\mathcal{V}(x) = [2(1 + \|x^{-1}\|^{-1})^{-1}, \infty)$.

Proof. (a) By Theorem 2.7, (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v) with the same α .

(i) \Rightarrow (ii): x being invertible is positive in some unitary isotope $\mathcal{J}^{[u]}$ of \mathcal{J} by [15, Theorem 4.12], and $\sigma(x) \subseteq [-1, 1] \setminus (2\alpha - 1, 1 - 2\alpha)$ for some $\alpha > 0$ as $0 \notin \sigma(x)$. Then by [18, Lemma 3.4], $x \in \alpha\mathcal{U}(\mathcal{J}^{[u]}) + (1 - \alpha)\mathcal{U}(\mathcal{J}^{[u]})$ and hence $x \in \alpha\mathcal{U}(\mathcal{J}) + (1 - \alpha)\mathcal{U}(\mathcal{J})$ by [15, Theorem 4.6].

(ii) \Rightarrow (i): Follows from [15, lemmas 2.2(iii) and 4.2(ii)].

(v) \Leftrightarrow (vi): Follows from the fact that $0 \leq \alpha < \frac{1}{2}$ iff $1 \leq (1 - \alpha)^{-1} < 2$.

(b) By [15, Theorem 14.2], x being invertible is positive invertible in certain unitary isotope $\mathcal{J}^{[u]}$. Hence by [18, Lemma 3.4] and [15, Theorem 4.2], for any $0 \leq \beta < \frac{1}{2}$,

$$x \in \beta\mathcal{U}(\mathcal{J}) + (1 - \beta)\mathcal{U}(\mathcal{J}) \quad \text{iff} \quad \inf \sigma_{\mathcal{J}^{[u]}}(x) \geq 1 - 2\beta.$$

By [15, Lemma 4.2(iii)], $x^{-1u} = \{ux^{-1}u\}$. Since $x^{-1} = \{u^*\{ux^{-1}u\}u^*\}$ and $\|x^{-1}\| = \|\{u^*\{ux^{-1}u\}u^*\}\| \leq \|\{ux^{-1}u\}\| \leq \|x^{-1}\|$, it follows that $\|x^{-1u}\| = \|x^{-1}\|$. Thus, by the functional calculus for positive elements,

$$\inf \sigma_{\mathcal{J}^{[u]}}(x) = \|x^{-1u}\|^{-1} = \|x^{-1}\|^{-1}.$$

Therefore, $x \in \beta\mathcal{U}(\mathcal{J}) + (1 - \beta)\mathcal{U}(\mathcal{J})$ if and only if $\|x^{-1}\|^{-1} \geq 1 - 2\beta$. However, by part (a) (as x is invertible) there exists $\lambda \in \mathcal{V}(x)$ with $1 \leq \lambda < 2$ so that $(1 - \lambda^{-1}) \in \mathcal{V}(x)$ with $0 < \lambda^{-1} < \frac{1}{2}$, hence by Theorem 2.7 we get $x \in \alpha\mathcal{U}(\mathcal{J}) + (1 - \alpha)\mathcal{U}(\mathcal{J})$ with $\alpha = 1 - \lambda^{-1}$. It follows that $\|x^{-1}\|^{-1} \geq 1 - 2\alpha = 2\lambda^{-1} - 1$. Hence $\inf \mathcal{V}(x) \geq 2(1 + \|x^{-1}\|^{-1})^{-1}$.

Since x is positive in $\mathcal{J}^{[u]}$, setting $\alpha = 1 - \frac{1}{2}(1 + \inf \sigma_{\mathcal{J}^{[u]}}(x))$ we have $0 \leq \alpha < \frac{1}{2}$ and $\inf \sigma_{\mathcal{J}^{[u]}}(x) = 1 - 2\alpha$ so that $\sigma_{\mathcal{J}^{[u]}}(x) \subseteq [1 - 2\alpha, 1]$. Hence by [18, Lemma 3.4] and [15, Theorem 4.6], $x \in \alpha\mathcal{U}(\mathcal{J}) + (1 - \alpha)\mathcal{U}(\mathcal{J})$. Thus, by Theorem 2.7, $2(1 + \|x^{-1}\|^{-1})^{-1} = (1 - \alpha)^{-1} \in \mathcal{V}(x)$. \square

Corollary 2.9. *Let \mathcal{J} be a unital JB^* -algebra and for $1 \leq \delta < 2$, the set $K = \{x \in (\mathcal{J})_1 \cap \mathcal{J}_{\text{inv}} : \|x^{-1}\| \leq (2\delta^{-1} - 1)^{-1}\}$. Then $\text{cos}_\delta\mathcal{U}(\mathcal{J}) = K$ and K is closed.*

Proof. The first part is immediate from Corollary 2.8. For the other part, let $\{y_n\}$ be a sequence in K which converges to $y \in \mathcal{J}$. Clearly, $y \in (\mathcal{J})_1$. By part (ai), $\delta \in \mathcal{V}(y_n)$ for all n . Let $\alpha = 1 - \delta^{-1}$. Then $0 \leq \alpha < \frac{1}{2}$ and $\text{dist}(y_n, \mathcal{U}(\mathcal{J})) \leq 2\alpha$ for all n by Corollary 2.8(a). Hence by the continuity of the distance, we get $\text{dist}(y, \mathcal{U}(\mathcal{J})) \leq 2\alpha$. Therefore, y is invertible and hence $\|y^{-1}\| = \lim_{n \rightarrow \infty} \|y_n^{-1}\| \leq (2\delta^{-1} - 1)^{-1}$. \square

Corollary 2.10. *Let \mathcal{J} be a unital JB^* -algebra and $x \in (\mathcal{J})_1$ be invertible. Then $\lambda_u(x) = \frac{1}{2}(1 + \|x^{-1}\|^{-1})$ and there exist unitaries u_1, u_2 in $\mathcal{U}(\mathcal{J})$ such that $x = \lambda_u(x)u_1 + (1 - \lambda_u(x))u_2$.*

Proof. By Corollary 2.8(b), $\inf \mathcal{V}(x) = 2(1 + \|x^{-1}\|^{-1})^{-1} \in \mathcal{V}(x)$. So, by Theorem 2.5, $\lambda_u(x) = \frac{1}{2}(1 + \|x^{-1}\|^{-1})$ and $\frac{1}{2} < \lambda_u(x) \leq 1$. Thus, $0 \leq 1 - \lambda_u(x) < \frac{1}{2}$, so there exist unitaries u_1, u_2 in $\mathcal{U}(\mathcal{J})$ such that $x = \lambda_u(x)u_1 + (1 - \lambda_u(x))u_2$, by Theorem 2.7. \square

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