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# Counterexamples on non-$\alpha$-normal functions with good integrability

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Abstract. Blaschke products are used to construct concrete examples of analytic functions with good integrability and bad behavior of spherical derivative. These examples are used to show that none of the classes $M_p^{\#}$, $0 < p < \infty$, is contained in the $\alpha$-normal class $\mathcal{N}^\alpha$ when $0 < \alpha < 2$. This implies that $M_p^{\#}$ is in a sense a much larger class than $Q_p^{\#}$.

Contents



## 1. Introduction and results

Let $\mathcal{M}(\mathbb{D})$ denote the class of all meromorphic functions in the unit disc $\mathbb{D} = \{z : |z| < 1\}$. A function $f \in \mathcal{M}(\mathbb{D})$ is called normal if

$$\|f\|_{\mathcal{N}} = \sup_{z\in\mathbb{D}} f^{\#}(z)(1-|z|^2) < \infty,$$

where $f^{\#}(z) = |f'(z)|/(1+|f(z)|^2)$ is the spherical derivative of $f$ at $z$. The class of normal functions is denoted by $\mathcal{N}$. For a given sequence $\{z_n\}_{n=1}^{\infty}$ of points in $\mathbb{D}$ for which $\sum_{n=1}^{\infty}(1-|z_n|^2)$ converges (with the convention

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$z_n/|z_n| = 1$ for $z_n = 0$), the Blaschke product associated with the sequence $\{z_n\}_{n=1}^{\infty}$ is defined as

$$B(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z}_n z}.$$

Allen and Belna [1] showed that the analytic function

$$f_s(z) = B(z)/(1-z)^s,$$

where $B(z)$ is the Blaschke product associated with $\{1 - e^{-n}\}_{n=1}^{\infty}$, is not a normal function if $0 < s < \frac{1}{2}$, but satisfies the integrability condition

$$\int_{\mathbb{D}} |f_s'(z)|\, dA(z) < \infty.$$

It is well-known that if $f$ is analytic in $\mathbb{D}$ and satisfies

$$\int_{\mathbb{D}} |f'(z)|^2\, dA(z) < \infty,$$

that is, $f$ belongs to the Dirichlet space (analytic functions in $\mathbb{D}$ with bounded area of image counting multiplicities), then $f \in \mathcal{N}$. Concerning the normality, the question arose if

$$\int_{\mathbb{D}} |f'(z)|^p\, dA(z) < \infty, \quad 1 < p < 2, \tag{1}$$

implies $f \in \mathcal{N}$. Yamashita [9] showed that this is not the case since the function

$$f(z) = B(z) \log \frac{1}{1-z}, \tag{2}$$

where $B$ is a Blaschke product associated with an exponential sequence $\{z_n\}_{n=1}^{\infty}$ whose limit is 1, is not normal but satisfies (1) for all $1 < p < 2$. Recall that a sequence $\{z_n\}_{n=1}^{\infty}$ is exponential if

$$1 - |z_{n+1}| \le \beta(1 - |z_n|), \quad n \in \mathbb{N}, \tag{3}$$

for some $\beta \in (0, 1)$. It is well known that every such sequence $\{z_n\}_{n=1}^{\infty}$ satisfies

$$\prod_{k \ne n} \left| \frac{z_n - z_k}{1 - \overline{z}_n z_k} \right| \ge \delta, \quad n \in \mathbb{N}, \tag{4}$$

for some $\delta = \delta(\beta) > 0$, and is therefore an interpolating sequence (uniformly separated sequence).

The basic idea in this note is to find a function $f$ that satisfies (1) (or another integrability condition) but the behavior of $f^{\#}$ is worse than the behavior of the spherical derivative of a nonnormal function necessarily is. To make this precise, for $0 < \alpha < \infty$, a function $f \in \mathcal{M}(\mathbb{D})$ is called $\alpha$-normal if

$$\sup_{z \in \mathbb{D}} f^{\#}(z)(1 - |z|^2)^{\alpha} < \infty.$$

The class of all $\alpha$-normal functions is denoted by $\mathcal{N}^\alpha$.

**Theorem 1.** *Let $B$ be the Blaschke product associated with an exponential sequence $\{z_n\}_{n=1}^\infty$ whose limit is 1. Let $1 \le \alpha < \infty$, $0 < p < 2$ and*

$$f_s(z) = \frac{B(z)}{(1-z)^s}, \quad 0 < s < \infty. \tag{5}$$

*Then $f_s \notin \mathcal{N}^\alpha$ for all $s > \alpha - 1$, but*

$$\int_{\mathbb{D}} |f_s'(z)|^p \, dA(z) < \infty$$

*for all $s \in (0, 2/p - 1)$.*

It is easy to see that $f_{\alpha-1} \in \mathcal{N}^\alpha$. Moreover, the following result proves the sharpness of Theorem 1.

**Theorem 2.** *Let $B$ be the Blaschke product associated with an exponential sequence $\{z_n\}_{n=1}^\infty$ such that $|z_n - \frac{1}{2}| = \frac{1}{2}$, $\Im z_n > 0$ and $\lim_{n\to\infty} z_n = 1$. Let $0 < p < 1$. Then*

$$f_{\frac{2}{p}-1}(z) = \frac{B(z)}{(1-z)^{\frac{2}{p}-1}}$$

*satisfies*

$$\int_{\mathbb{D}} |f'_{\frac{2}{p}-1}(z)|^p \, dA(z) = \infty.$$

The following result is of the same nature as Theorem 1.

**Theorem 3.** *Let $B$ be the Blaschke product associated with an exponential sequence $\{z_n\}_{n=1}^\infty$ whose limit is 1. Let $1 \le \alpha < \infty$ and*

$$f(z) = \log\frac{1}{1-z}\frac{B(z)}{(1-z)^{\alpha-1}}.$$

*Then $f \notin \mathcal{N}^\alpha$, but*

$$\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{2\alpha-2+\varepsilon} \, dA(z) < \infty \tag{6}$$

*for all $\varepsilon > 0$.*

Wulan [8] showed that the function $f$, defined in (2), satisfies

$$f \notin \bigcup_{0<p<\infty} Q_p^\# \quad \text{but} \quad f \in \bigcap_{0<p<\infty} M_p^\#, \tag{7}$$

where

$$Q_p^\# = \left\{ f \in \mathcal{M}(\mathbb{D}) : \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f^\#(z))^2 g^p(z,a) \, dA(z) < \infty \right\}$$

and

$$M_p^\# = \left\{ f \in \mathcal{M}(\mathbb{D}) : \|f\|^2_{M_p^\#} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f^\#(z))^2 (1-|\varphi_a(z)|^2)^p \, dA(z) < \infty \right\}.$$

Here $\varphi_a(z) = (a-z)/(1-\overline{a}z)$ is a Möbius transformation and $g(z,a) = -\log|\varphi_a(z)|$ is a Green's function of $\mathbb{D}$.

Keeping (7) in mind, we will show that the function $f_s$, defined in (5), belongs to $M_p^\#$ for certain values of $s$.

**Theorem 4.** *Let $B$ be the Blaschke product associated with an exponential sequence $\{z_n\}_{n=1}^\infty$ whose limit is 1. Then the function $f_s$, defined in (5), satisfies*

$$f_s \in \bigcap_{0<p<\infty} M_p^\#$$

*for all $0 < s \le 1$.*

Theorems 1 and 4 have the following immediate consequence.

**Corollary 5.**

$$\bigcap_{0<p<\infty} M_p^\# \not\subset \bigcup_{0<\alpha<2} \mathcal{N}^\alpha.$$

Using [6, Theorem 3.3.3], with $\alpha = 2 - 2/q$, we see that if $f \in \mathcal{M}(\mathbb{D})$ satisfies

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f^\#(z))^q (1-|z|^2)^{2q-4} (1-|\varphi_a(z)|^2)^p \, dA(z) < \infty,$$

for some $2 < q < \infty$ and $0 \le p < \infty$, then $f \in \mathcal{N}^2$. In view of this fact and Corollary 5 it is natural to ask the following questions.

**Question 6.** For which values of $p$ the class $M_p^\#$ is contained in $\mathcal{N}^2$?

**Question 7.** Is the class

$$\mathcal{B}^\# = \left\{ f \in \mathcal{M}(\mathbb{D}) : \sup_{a\in\mathbb{D}} \int_{D(a,r)} (f^\#(z))^2 \, dA(z) < \infty \right\}$$

contained in $\mathcal{N}^2$?

Recall that $M_p^\# = \mathcal{B}^\#$ for all $1 < p < \infty$, see [7, 8].

Before embarking on the proofs of the results presented in this section, a word about the notation. We will write $A \lesssim B$ if there is a positive constant $C$ such that $A \le CB$. The notation $A \gtrsim B$ is understood in an analogous manner. In particular, if $A \lesssim B \lesssim A$, we will write $A \simeq B$.

## 2. Proof of Theorem 1

The fact $f_s \notin \mathcal{N}^\alpha$ for all $s > \alpha - 1$ follows at once by the following lemma, see also [7, Theorem 4.4.2].

**Lemma 8.** *Let $1 \leq \alpha < \infty$ and let $B$ be an interpolating Blaschke product associated with the sequence $\{z_n\}_{n=1}^{\infty}$. If $g$ is analytic in $\mathbb{D}$ and satisfies*

$$|g(z_{n_k})|(1-|z_{n_k}|)^{\alpha-1} \to \infty, \quad k \to \infty,$$

*for some subsequence $\{z_{n_k}\}_{k=1}^{\infty}$, then $Bg \notin \mathcal{N}^{\alpha}$.*

**Proof.** Since $B$ is interpolating, there exists $\delta > 0$ such that

$$\begin{aligned}(Bg)^{\#}(z_{n_k})(1-|z_{n_k}|^2)^{\alpha} &= |B'(z_{n_k})||g(z_{n_k})|(1-|z_{n_k}|^2)^{\alpha} \\ &\geq \delta |g(z_{n_k})|(1-|z_{n_k}|)^{\alpha-1} \to \infty, \quad k \to \infty,\end{aligned}$$

and hence $Bg \notin \mathcal{N}^{\alpha}$. □

To prove the assertion on the integrability, let first $0 < p \leq 1$ and $s < \frac{2}{p} - 1$. Then

$$\int_{\mathbb{D}} |f_s'(z)|^p \, dA(z) \leq \int_{\mathbb{D}} \frac{|B'(z)|^p}{|1-z|^{ps}} \, dA(z) + \int_{\mathbb{D}} \frac{s^p |B(z)|^p}{|1-z|^{p(s+1)}} \, dA(z) = I_1 + I_2,$$

where $I_2 < \infty$ since $s < \frac{2}{p} - 1$. To estimate $I_1$, write

$$\frac{B'(z)}{B(z)} = \sum_{n=1}^{\infty} \frac{|z_n|^2 - 1}{(1-\overline{z}_n z)(z_n - z)}$$

so that

$$\begin{aligned}|B'(z)| &= \left| \sum_{n=1}^{\infty} \frac{1-|z_n|^2}{(1-\overline{z}_n z)(z_n - z)} \right| \left| \prod_{k=1}^{\infty} \frac{|z_k|}{z_k} \frac{z_k - z}{1-\overline{z}_k z} \right| \\ &\leq \sum_{n=1}^{\infty} \frac{1-|z_n|^2}{|z_n - z||1-\overline{z}_n z|} \frac{|z_n - z|}{|1-\overline{z}_n z|} |B_n(z)| \\ &\leq \sum_{n=1}^{\infty} |\varphi_{z_n}'(z)|,\end{aligned} \tag{8}$$

where $B_n(z) = \prod_{k \neq n} \frac{|z_k|}{z_k} \frac{z_k - z}{1-\overline{z}_k z}$. Therefore

$$|B'(z)|^p \leq \sum_{n=1}^{\infty} |\varphi_{z_n}'(z)|^p, \tag{9}$$

and the Hölder inequality yields

$$\begin{aligned}I_1 &\leq \int_{\mathbb{D}} \left( \sum_{n=1}^{\infty} |\varphi_{z_n}'(z)|^p \right) \frac{dA(z)}{|1-z|^{ps}} \\ &= \sum_{n=1}^{\infty} (1-|z_n|^2)^p \int_{\mathbb{D}} \frac{dA(z)}{|1-\overline{z}_n z|^{2p}|1-z|^{ps}} \\ &\leq \sum_{n=1}^{\infty} (1-|z_n|^2)^p \left( \int_{\mathbb{D}} \frac{dA(z)}{|1-\overline{z}_n z|^{2p\mu}} \right)^{\frac{1}{\mu}} \left( \int_{\mathbb{D}} \frac{dA(z)}{|1-z|^{ps\lambda}} \right)^{\frac{1}{\lambda}},\end{aligned}$$

where $\mu \in (\frac{2}{2-sp}, \frac{2}{p})$ and $\lambda \in (\frac{2}{2-p}, \frac{2}{sp})$ such that $\lambda^{-1} + \mu^{-1} = 1$. The last integral above is finite because $ps\lambda < 2$. Moreover, we may choose $\mu$ such that $p\mu > 1$. Then

$$\left(\int_{\mathbb{D}} \frac{dA(z)}{|1-\overline{z}_n z|^{2p\mu}}\right)^{\frac{1}{\mu}} \lesssim \frac{1}{(1-|z_n|^2)^{2p-2/\mu}},$$

and it follows that

$$I_1 \lesssim \sum_{n=1}^{\infty} (1-|z_n|^2)^{\frac{2}{\mu}-p} \leq (1-|z_1|^2)^{\frac{2}{\mu}-p} \sum_{n=1}^{\infty} \beta^{(n-1)(\frac{2}{\mu}-p)} < \infty$$

since $\frac{2}{\mu} - p > 0$.

Let now $1 < p < 2$ and $s < \frac{2}{p} - 1$. Then

$$\begin{aligned}
&\int_{\mathbb{D}} |f_s'(z)|^p \, dA(z) \\
&\qquad \leq 2^{p-1} \int_{\mathbb{D}} \frac{|B'(z)|^p}{|1-z|^{ps}} \, dA(z) + 2^{p-1} \int_{\mathbb{D}} \frac{s^p |B(z)|^p}{|1-z|^{p(s+1)}} \, dA(z) = I_1 + I_2,
\end{aligned}$$

where $I_2 < \infty$ since $s < \frac{2}{p} - 1$. By using the inequality (9) with $p = 1$ and the Schwarz–Pick lemma, we obtain

$$\begin{aligned}
I_1 &\leq 2^{p-1} \int_{\mathbb{D}} \frac{|B'(z)|}{(1-|z|)^{p-1}|1-z|^{ps}} \, dA(z) \\
&\leq 2^{p-1} \sum_{n=1}^{\infty} (1-|z_n|^2) \int_0^1 \frac{1}{(1-r)^{ps+p-1}} \int_0^{2\pi} \frac{d\theta}{|1-\overline{z}_n r e^{i\theta}|^2} dr \\
&\simeq \sum_{n=1}^{\infty} (1-|z_n|^2) \int_0^1 \frac{dr}{(1-r)^{ps+p-1}(1-|z_n|r)}.
\end{aligned}$$

Choose the conjugate indexes $\mu$ and $\lambda$ such that $\mu < 1/(ps+p-1)$. Then the Hölder inequality implies

$$\begin{aligned}
I_1 &\lesssim \sum_{n=1}^{\infty} (1-|z_n|) \left(\int_0^1 \frac{dr}{(1-r)^{\mu(ps+p-1)}}\right)^{\frac{1}{\mu}} \left(\int_0^1 \frac{dr}{(1-|z_n|r)^{\lambda}}\right)^{\frac{1}{\lambda}} \\
&\lesssim \sum_{n=1}^{\infty} (1-|z_n|) \frac{1}{(1-|z_n|)^{\frac{\lambda-1}{\lambda}}} = \sum_{n=1}^{\infty} (1-|z_n|)^{\frac{1}{\lambda}}.
\end{aligned}$$

Since $\{z_n\}_{n=1}^{\infty}$ is exponential, it follows that $I_1 < \infty$, and we are done.

## 3. Proof of Theorem 2

Denote $\mathbb{D}^- = \{z \in \mathbb{D} : \Im z \leq 0\}$. By [3, Theorem 1] there exists $\delta > 0$ such that $|B(z)| \geq \delta$ for all $z \in \mathbb{D}^-$. Since $0 < p < 1$, we have

$$\begin{aligned}
(10) \qquad \int_{\mathbb{D}} |f'_{\frac{2}{p}-1}(z)|^p \, dA(z) + |f_{\frac{2}{p}-1}(0)|^p &\simeq \int_{\mathbb{D}} |f_{\frac{2}{p}-1}(z)|^p (1-|z|^2)^{-p} \, dA(z) \\
&\geq \int_{\mathbb{D}^-} |B(z)|^p \frac{(1-|z|^2)^{-p}}{|1-z|^{2-p}} \, dA(z) \\
&\gtrsim \delta^p \int_0^1 \frac{ds}{1-s} = \infty,
\end{aligned}$$

and the assertion follows.

Note that an application of the asymptotic equality in (10) gives an alternative way to prove the case $0 < p < 1$ in Theorem 1.

## 4. Proof of Theorem 3

Denote

$$g_\alpha(z) = (1-z)^{1-\alpha} \log \frac{1}{1-z}, \quad 1 \leq \alpha < \infty.$$

Then Lemma 8 implies that $f = B g_\alpha \notin \mathcal{N}^\alpha$.

To show that (6) is satisfied for all $\varepsilon > 0$, observe first that $|f'|^2 \leq 4(|g'_\alpha|^2 + |g_\alpha|^2 |B'|^2)$. Now

$$\begin{aligned}
I_1 &= \int_{\mathbb{D}} |g'_\alpha(z)|^2 (1-|z|^2)^{2\alpha-2+\varepsilon} \, dA(z) \\
&\lesssim \int_0^1 \left( \log \frac{e}{1-r} \right)^2 (1-r)^{2\alpha-2+\varepsilon} \left( \int_0^{2\pi} \frac{d\theta}{|1-re^{i\theta}|^{2\alpha}} \right) dr \\
&\lesssim \int_0^1 \left( \log \frac{e}{1-r} \right)^3 (1-r)^{\varepsilon-1} \, dr < \infty
\end{aligned}$$

for all $\varepsilon > 0$, so it suffices to show that

$$I_2 = \int_{\mathbb{D}} |g_\alpha(z) B'(z)|^2 (1-|z|^2)^{2\alpha-2+\varepsilon} \, dA(z) < \infty.$$

By using the inequality (9) with $p = 1$ and the Schwarz–Pick lemma, we obtain

$$\begin{aligned}
I_2 &\leq \int_{\mathbb{D}} |g_\alpha(z)|^2 |B'(z)| (1-|z|^2)^{2\alpha-3+\varepsilon} \, dA(z) \\
&\lesssim \sum_{n=1}^\infty (1-|z_n|^2) \int_0^1 \left( \log \frac{e}{1-r} \right)^2 (1-r)^{\varepsilon-1} \int_0^{2\pi} \frac{d\theta}{|1-\overline{z}_n r e^{i\theta}|^2} dr \\
&\simeq \sum_{n=1}^\infty (1-|z_n|^2) \int_0^1 \left( \log \frac{e}{1-r} \right)^2 (1-r)^{\varepsilon-1} (1-|z_n|r)^{-1} dr.
\end{aligned}$$

Choose the conjugate indexes $p$ and $q$ such that $p(1-\varepsilon)<1$. Here we assume, without loss of generality, that $\varepsilon\in(0,1)$. Then the Hölder inequality implies

$$I_2 \lesssim \sum_{n=1}^{\infty}(1-|z_n|)\left(\int_0^1\left(\log\frac{e}{1-r}\right)^{2p}(1-r)^{p(\varepsilon-1)}dr\right)^{\frac{1}{p}}\left(\int_0^1\frac{dr}{(1-|z_n|r)^q}\right)^{\frac{1}{q}}$$
$$\lesssim \sum_{n=1}^{\infty}(1-|z_n|)\frac{1}{(1-|z_n|)^{\frac{q-1}{q}}}=\sum_{n=1}^{\infty}(1-|z_n|)^{\frac{1}{q}}.$$

Since $\{z_n\}_{n=1}^{\infty}$ is exponential, there exists $\beta\in(0,1)$ such that $(1-|z_{n+1}|)\le\beta(1-|z_n|)$ for all $n\in\mathbb{N}$. Therefore $I_2\lesssim(1-|z_1|)^{\frac{1}{q}}\sum_{n=1}^{\infty}\beta^{\frac{n-1}{q}}<\infty$, and we are done.

## 5. Proof of Theorem 4

This proof uses ideas from [2]. Let $0<p<\infty$ and $0<s\le 1$. Let $\delta^*=\delta/4$, and consider the pseudohyperbolic discs

$$D_n=D(z_n,\delta^*)=\{z:|\varphi_{z_n}(z)|<\delta^*\}.$$

Then $D_n\cap D_k=\emptyset$ if $n\ne k$. Denote $g_s(z)=(1-z)^{-s}$, so that

$$|f_s'(z)|\le|B'(z)||g_s(z)|+|B(z)||g_s'(z)|.$$

Further, denote $E_1=\cup_{n=1}^{\infty}D_n$ and $E_2=\mathbb{D}\setminus E_1$. Then

$$\|f_s\|_{M_p^\#}^2=\int_{\mathbb{D}}|f_s^\#(z)|^2(1-|\varphi_a(z)|^2)^p\,dA(z)$$
$$\le 2\left(\sup_{a\in\mathbb{D}}I_1(E_1)+\sup_{a\in\mathbb{D}}I_1(E_2)+\sup_{a\in\mathbb{D}}I_2(E_1)+\sup_{a\in\mathbb{D}}I_2(E_2)\right),$$

where

$$I_1(F)=\int_F\frac{|B'(z)g_s(z)|^2}{(1+|B(z)g_s(z)|^2)^2}(1-|\varphi_a(z)|^2)^p\,dA(z)$$

and

$$I_2(F)=\int_F\frac{|B(z)g_s'(z)|^2}{(1+|B(z)g_s(z)|^2)^2}(1-|\varphi_a(z)|^2)^p\,dA(z)$$

for $F\subset\mathbb{D}$.

Recall that $\{z_n\}_{n=1}^{\infty}$ satisfies (4) for some $\delta=\delta(\beta)>0$, and therefore $|B(z)|\ge\gamma=\gamma(\delta)>0$ for all $z\in E_2$ by [3, Theorem 1]. It follows that

$$I_1(E_2)=\int_{E_2}\frac{|B'(z)g_s(z)|^2}{(1+|B(z)g_s(z)|^2)^2}(1-|\varphi_a(z)|^2)^p\,dA(z)$$
$$\le\frac{1}{\gamma^2}\int_{E_2}|B'(z)|^2\frac{\gamma^2|g_s(z)|^2}{(1+\gamma^2|g_s(z)|^2)^2}(1-|\varphi_a(z)|^2)^p\,dA(z)$$
$$\le\frac{1}{4\gamma^2}\int_{\mathbb{D}}|B'(z)|^2(1-|\varphi_a(z)|^2)^p\,dA(z)\le M_1<\infty$$

for all $a \in \mathbb{D}$ by [4, p. 208 (2)]. Moreover,

$$\begin{aligned}
I_2(E_2) &= \int_{E_2} \frac{|B(z)g_s'(z)|^2}{(1+|B(z)g_s(z)|^2)^2}(1-|\varphi_a(z)|^2)^p\,dA(z)\\
&\le \frac{1}{\gamma^4}\int_{E_2} \frac{|g_s'(z)|^2}{|g_s(z)|^4}(1-|\varphi_a(z)|^2)^p\,dA(z)\\
&\le \frac{s^2}{\gamma^4}\int_{\mathbb{D}} \frac{1}{|1-z|^{1-3s}}(1-|\varphi_a(z)|^2)^p\,dA(z) \le M_2 < \infty
\end{aligned}$$

for all $p > 0$ and $0 < s \le 1$.

To estimate the integrals over $E_1$, note first that

$$|\varphi_a(z)| = |\varphi_{\varphi_a(w)}(\varphi_w(z))|, \quad z, w, a \in \mathbb{D},$$

and hence

$$\begin{aligned}
|B(z)| = \left|\prod_{n=1}^{\infty} \varphi_{z_n}(z)\right| &= |\varphi_{z_k}(z)|\left|\prod_{n\ne k} \varphi_{z_n}(z)\right|\\
&= |\varphi_{z_k}(z)|\left|\prod_{n\ne k} \varphi_{\varphi_{z_n}(z_k)}(\varphi_{z_k}(z))\right|.
\end{aligned}$$

This yields

$$\frac{3\delta}{4}|\varphi_{z_k}(z)| \le |B(z)| < |\varphi_{z_k}(z)|, \quad z \in D(z_k, \delta^*). \tag{11}$$

Therefore

$$\begin{aligned}
&\int_{D_k} \frac{|B(z)|^2|g_s'(z)|^2}{(1+|B(z)g_s(z)|^2)^2}(1-|\varphi_a(z)|^2)^p\,dA(z)\\
&\le \int_{D_k} |\varphi_{z_k}(z)|^2|g_s'(z)|^2(1-|\varphi_a(z)|^2)^p\,dA(z)\\
&= \int_{D(0,\delta^*)} |w|^2|g_s'(\varphi_{z_k}(w))|^2(1-|\varphi_a(\varphi_{z_k}(w))|^2)^p|\varphi_{z_k}'(w)|^2\,dA(w)\\
&\lesssim (1-|\varphi_a(z_k)|^2)^p \int_{D(0,\delta^*)} |g_s'(\varphi_{z_k}(w))|^2|\varphi_{z_k}'(w)|^2\,dA(w),
\end{aligned}$$

where

$$\begin{aligned}
&\int_{D(0,\delta^*)} |g_s'(\varphi_{z_k}(w))|^2|\varphi_{z_k}'(w)|^2\,dA(w)\\
&\le \int_{D(0,\delta^*)} \frac{|\varphi_{z_k}'(w)|^2}{|1-\varphi_{z_k}(w))|^{2(s+1)}}\,dA(w)\\
&\lesssim (1-|z_k|)^{4-2(s+1)} \le 1.
\end{aligned}$$

Hence

$$I_2(E_1) = \int_{E_1} \frac{|B(z)g_s'(z)|^2}{(1+|B(z)g_s(z)|^2)^2}(1-|\varphi_a(z)|^2)^p\,dA(z)$$
$$\lesssim \sup_{a\in\mathbb{D}}\sum_{k=1}^{\infty}(1-|\varphi_a(z_k)|^2)^p < \infty$$

by [5].

A similar reasoning as above together with the Schwarz–Pick lemma yields

$$\int_{D_k} \frac{|B'(z)|^2|g_s(z)|^2}{(1+|B(z)g_s(z)|^2)^2}(1-|\varphi_a(z)|^2)^p\,dA(z)$$
$$\lesssim (1-|\varphi_a(z_k)|^2)^p \int_{D(0,\delta^*)} \frac{|g_s(\varphi_{z_k}(w))|^2}{(1-|\varphi_{z_k}(w)|^2)^2}|\varphi_{z_k}'(w)|^2\,dA(w),$$

where

$$\int_{D(0,\delta^*)} \frac{|g_s(\varphi_{z_k}(w))|^2}{(1-|\varphi_{z_k}(w)|^2)^2}|\varphi_{z_k}'(w)|^2\,dA(w)$$
$$\le 4\int_{D(0,\delta^*)} \frac{|\varphi_{z_k}'(w)|^2}{(1-|\varphi_{z_k}(w)|^2)^{2+2s}}\,dA(w) \lesssim (1-|z_k|)^{2-2s} \le 1.$$

It follows that

$$I_1(E_1) = \int_{E_1} \frac{|B'(z)|^2|g_s(z)|^2}{(1+|B(z)g_s(z)|^2)^2}(1-|\varphi_a(z)|^2)^p\,dA(z)$$
$$\lesssim \sup_{a\in\mathbb{D}}\sum_{k=1}^{\infty}(1-|\varphi_a(z_k)|^2)^p < \infty.$$

Putting everything together, we obtain $f \in M_p^{\#}$.

## References


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