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Wandering subspaces and quasi-wandering subspaces in the Bergman space

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Abstract. An elementary proof of the Aleman–Richter–Sundberg theorem concerning the invariant subspaces of the Bergman space is given. Also we study the quasi-wandering subspaces of the Bergman space.

Contents

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and let dA denote the normalized Lebesgue measure on $\mathbb{D}$. The Bergman space L_a^2 is a Hilbert space consisting of square integrable analytic functions on $\mathbb{D}$. We denote by $\mathcal{B}$ the multiplication operator by the coordinate function z on L_a^2, $\mathcal{B}f = zf$, which is called the Bergman shift. On the Hardy space H^2 on $\mathbb{D}$, we denote by T_z the multiplication operator by z. For an operator T on a Hilbert space H and an invariant subspace M of T, the subspace $M \ominus TM$ is called a wandering subspace of M. In this paper, the space $P_M T(H \ominus M)$ is called the quasi-wandering subspace for M, where P_M is the orthogonal projection from H onto M. By the definitions, a quasi-wandering subspace is a counterpart of a wandering subspace. For a subset E of H, we denote by $[E]$ the smallest invariant subspace of H for T containing E.

In 1949, Beurling showed that for an invariant subspace M in H^2, $M = \theta H^2$ for a nonconstant inner function θ, and its wandering subspace has dimension 1 [Beu49]. It holds that $[M \ominus T_z M] = M$ for all invariant subspaces M of T_z. So we say that the Beurling type theorem holds for T if

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$[M \ominus TM] = M$ for all invariant subspaces M of T, where $[M \ominus TM]$ is the smallest invariant subspace of T containing $M \ominus TM$. We also have

$$M \ominus T_z M = \mathbb{C} \cdot \theta \quad \text{and} \quad [M \ominus T_z M] = M,$$
$$T_z^*(H^2 \ominus M) \subset (H^2 \ominus M)$$

and

$$P_M T_z(H^2 \ominus M) = \mathbb{C} \cdot \theta.$$

Hence $[P_M T_z(H^2 \ominus M)] = M$ and $[H^2 \ominus M] = H^2$. In the one variable Hardy space case, a wandering subspace coincides with a quasi-wandering subspace for M.

On the Bergman space, the situation is different. There are studies of the dimensions of wandering subspaces of invariant subspaces of Bergman shift, and it is known that the dimension ranges from 1 to ∞ (see [ABFP85, Hed93, HRS96]). In 1996, Aleman, Richter, and Sundberg [ARS96] made significant progress in the study of invariant subspaces of $\mathcal{B}$. They proved the Beurling type theorem for the Bergman shift. This result reveals the internal structure of invariant subspaces of the Bergman space and becomes a fundamental theorem in the function theory on L^2_a [DS04, HKZ00]. Later, different proofs of the the Beurling type theorem are given in [MR02, Olo05, Shi01, SZ09]. In [Shi01], Shimorin proved the following theorem.

Shimorin's Theorem. *Let T be a bounded linear operator on a Hilbert space H. If T satisfies the following conditions:*

(i) $\|Tx + y\|^2 \le 2(\|x\|^2 + \|Ty\|^2), \quad x, y \in H,$
(ii) $\bigcap\{T^n H : n \ge 0\} = \{0\},$

then $H = [H \ominus TH]$.

If T satisfies conditions (i) and (ii), then $T|_M : M \to M$ also satisfies conditions (i) and (ii). Hence by Shimorin's theorem, the Beurling type theorem holds for T. As an application of this theorem, Shimorin gave a simpler proof of the Aleman, Richter, and Sundberg theorem. In [SZ09], Sun and Zheng gave another proof of this theorem. Their idea was to lift up the Bergman shift as the compression of a commuting pair of isometries on the subspace of the Hardy space over the bidisk. Sun and Zheng's idea has two aspects. One is to show some identities in the Bergman space. Another one is a technique how to prove the Beurling type theorem.

In Section 2, we prove the following theorem:

Theorem 1. *Let T be a bounded linear operator on a Hilbert space H. Suppose T satisfies the following conditions:*

(i) $\|Tx\|^2 + \|T^{*2}Tx\|^2 \le 2\|T^*Tx\|^2, \quad x \in H.$
(ii) *T is bounded below, that is, there is $c > 0$ satisfying $\|Tx\| \ge c\|x\|$ for every $x \in H$.*
(iii) $\|T\| \le 1.$
(iv) *$\|T^{*k}x\| \to 0$ as $k \to \infty$ for every $x \in H$.*

Then $H = [H \ominus TH]$.

Our proof is just rewriting Sun and Zheng's proof [SZ09] in the most elementary way. We also give some identities in the Bergman space L_a^2, and as application of Theorem 1 we prove the Aleman, Richter, and Sundberg theorem.

In Section 3, we study quasi-wandering subspaces in the Bergman space.

This paper is the summary of [III1, III2].

2. The Beurling type theorem

We begin with the following:

Proof of Theorem 1. Let $N = H \ominus [H \ominus TH]$. It is sufficient to show that $N = \{0\}$. To do this, let $x \in N$. Since $x \perp T^k(H \ominus TH)$, $T^{*k}x \perp H \ominus TH$ for every $k \geq 0$. By condition (ii), TH is closed. Hence $T^{*k}x \in TH$ and there is $y_k \in H$ such that $Ty_k = T^{*k}x$. By condition (iii), we have $\|T^{*(k+1)}x\| \leq \|T^{*k}x\|$. Let

$$r_k = \|T^{*k}x\|^2 - \|T^{*(k+1)}x\|^2.$$

Then we have

$$\begin{aligned} r_k - r_{k+1} &= \|T^{*k}x\|^2 + \|T^{*(k+2)}x\|^2 - 2\|T^{*(k+1)}x\|^2 \\ &= \|Ty_k\|^2 + \|T^{*2}Ty_k\|^2 - 2\|T^*Ty_k\|^2 \\ &\leq 0 \qquad \text{by (i)}. \end{aligned}$$

Thus we get $0 \leq r_k \leq r_{k+1}$. By condition (iv), we have

$$r_k = \|T^{*k}x\|^2 - \|T^{*(k+1)}x\|^2 \to 0 \quad \text{as } k \to \infty.$$

Hence $r_k = 0$ for every $k \geq 0$. Thus we get $\|x\|^2 = \|T^{*k}x\|^2$ for every $k \geq 0$. By condition (iv) again, we get $x = 0$, so $N = \{0\}$. This completes the proof. □

As an application of Theorem 1 we give a simple and elementary proof of Aleman, Richter, and Sundberg theorem in the Bergman space L_a^2.

Lemma 2. *Let M be an invariant subspace of $\mathcal{B}$. Then for each $f \in M$,*

$$\|\mathcal{B}f\|^2 + \|\mathcal{B}^{*2}\mathcal{B}f\|^2 = 2\|(\mathcal{B}|_M)^*\mathcal{B}f\|^2 + \frac{1}{2}\|P_{M^\perp}\mathcal{B}\mathcal{B}^{*2}\mathcal{B}f\|^2.$$

By this lemma, we have the following theorem:

Theorem 3. *Let M be an invariant subspace of $\mathcal{B}$. Then for each $f \in M$,*

$$\begin{aligned} &\|\mathcal{B}f\|^2 + \|(\mathcal{B}|_M)^{*2}\mathcal{B}f\|^2 - 2\|(\mathcal{B}|_M)^*\mathcal{B}f\|^2 \\ &\qquad = \frac{1}{2}\|P_{M^\perp}\mathcal{B}P_{M^\perp}\mathcal{B}^{*2}\mathcal{B}f\|^2 - \|P_{M^\perp}\mathcal{B}^{*2}\mathcal{B}f\|^2. \end{aligned}$$

Since $\|P_{M^\perp}\mathcal{B}\| \leq 1$, we have the following corollary:

Corollary 4. *Let M be an invariant subspace of $\mathcal{B}$. Then for each $f \in M$,*

$$\|\mathcal{B}f\|^2 + \|(\mathcal{B}|_M)^{*2}\mathcal{B}f\|^2 + \frac{1}{2}\|P_{M^\perp}\mathcal{B}^{*2}\mathcal{B}f\|^2 \leq 2\|(\mathcal{B}|_M)^*\mathcal{B}f\|^2.$$

By Corollary 4, we get the following:

Corollary 5. *The Beurling type theorem holds for $\mathcal{B}$.*

3. Quasi-wandering subspaces

Let M be an invariant subspace of L_a^2 with $M \neq \{0\}$ and $M \neq L_a^2$. In this section, we consider the following:

Question 6. When is $P_M\mathcal{B}(L_a^2 \ominus M)$ dense in M?

Our question is equivalent to this one: does there exist $g \in M$ with $g \neq 0$ satisfying $\mathcal{B}^*g \in M$?

Let $\mathcal{D}$ be the Dirichlet space, that is, $\mathcal{D}$ is the space of analytic functions f on $\mathbb{D}$ satisfying $f' \in L_a^2$. It is well known that $f \in \mathcal{D}$ if and only if

$$\sum_{n=0}^{\infty}(n+1)|a_n|^2 < \infty,$$

where $f = \sum_{n=0}^{\infty} a_n z^n$. Note that $\mathcal{D} \subset H^2$. The following is the main theorem in this section.

Theorem 7 ([III1]). *Let M be an invariant subspace of L_a^2 for $\mathcal{B}$ with $M \neq \{0\}$ and $M \neq L_a^2$. Then the following conditions are equivalent.*

(i) *$P_M\mathcal{B}(L_a^2 \ominus M)$ is not dense in M.*
(ii) *There exists $f \in M$ with $f \neq 0$ satisfying $\mathcal{B}^*f \in M$.*
(iii) *$M \cap \mathcal{D} \neq \{0\}$.*
(iv) *There exists $f \in M$ with $f \neq 0$ satisfying $f' \in M$.*
(v) *There exists $f \in M$ with $f \neq 0$ satisfying $(M - \mathcal{B}\mathcal{B}^*)f \in M$.*

Corollary 8. *Let f be a nonzero function in $\mathcal{D}$ and M be the invariant subspace L_a^2 for $\mathcal{B}$ generated by f. Then $P_M\mathcal{B}(L_a^2 \ominus M)$ is not dense in M.*

Since $\mathcal{D} \subset H^2$, we have the following:

Corollary 9. *Let M be an invariant subspace of L_a^2 for $\mathcal{B}$ with $M \neq \{0\}$ and $M \neq L_a^2$. If $M \cap H^2 = \{0\}$, then $P_M\mathcal{B}(L_a^2 \ominus M)$ is dense in M.*

If $M \cap H^2 \neq \{0\}$, then by [Ri86] $\dim(M \ominus \mathcal{B}M) = 1$. So if $\dim(M \ominus \mathcal{B}M) \geq 2$, then $M \cap H^2 = \{0\}$.

Example 10. Let $\{\alpha_k\}_k \subset \mathbb{D}$ be a L_a^2-zero sequence. Suppose that

$$\sum_{k=1}^{\infty}(1 - |\alpha_k|) = \infty.$$

Horowitz showed the existence of such a sequence. Let M be the space of functions $f \in L_a^2$ satisfying $f(\alpha_k) = 0$ for every $k \geq 1$. Then M is an invariant subspace of L_a^2 for $\mathcal{B}$, $M \cap H^2 = \{0\}$ and $\dim(M \ominus \mathcal{B}M) = 1$.

The following is a counterpart of the Aleman, Richter, and Sundberg theorem.

Theorem 11 ([III1])**.** *Let I be an invariant subspace of L^2_a for $\mathcal{B}$ with $I \neq \{0\}$ and $I \neq L^2_a$. Then $\left[P_I \mathcal{B}(L^2_a \ominus I)\right]_{L^2_a} = I$.*

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