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Power bounded weighted composition operators

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Abstract. We study when weighted composition operators $C_{\phi,\psi}$ acting between weighted Bergman spaces of infinite order are power bounded resp. uniformly mean ergodic.

Contents

1. Introduction

Let $H(\mathbb{D})$ denote the set of all holomorphic functions on the open unit disk $\mathbb{D}$ and ϕ an analytic self-map of $\mathbb{D}$. We obtain the linear *composition operator* C_ϕ by composing an element of $H(\mathbb{D})$ with the map ϕ, that is,

$$C_\phi : H(\mathbb{D}) \to H(\mathbb{D}), \quad f \mapsto f \circ \phi.$$

Such operators occur naturally in various problems such as the study of multiplication operators and the theory of dynamical systems. Since composition operators link operator theoretical questions with classical results in complex analysis, many properties of such operatos have been investigated by several authors, see, e.g., [22], [14], [18], [10], [12], [11], [17], [21]. Since the literature on this subject is growing steadily, this can only be a sample of articles.

In this article we combine *multiplication operators*

$$M_\psi : H(\mathbb{D}) \to H(\mathbb{D}), \quad f \mapsto \psi f$$

with composition operators to get the *weighted composition operator*

$$C_{\phi,\psi} : H(\mathbb{D}) \to H(\mathbb{D}), \quad f \mapsto \psi(f \circ \phi).$$

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We are interested in such operators acting in the following setting: Let $v : \mathbb{D} \to (0, \infty)$ be a bounded and continuous function (*weight*) on $\mathbb{D}$. We consider the weighted Bergman spaces of infinite order

$$H_v^\infty := \left\{ f \in H(\mathbb{D});\ \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty \right\}$$

endowed with norm $\|.\|_v$. Such spaces arise naturally in functional analysis, complex analysis, partial differential equations and convolution equations as well as in distribution theory. They have been studied intensively in several articles. For further information see, e.g., [3], [6], [4] and [5].

For a Banach space X, we denote the space of all continuous linear operators from X into itself by $\mathcal{L}(X)$ and assume that $\mathcal{L}(X)$ is equipped with the operator norm topology. Given $T \in \mathcal{L}(X)$, its Cesàro means are defined by

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m, \quad n \in \mathbb{N}.$$

The following equality is well-known and can be checked easily

$$\frac{1}{n} T^n = T_{[n]} - \frac{n-1}{n} T_{[n-1]}, \quad n \in \mathbb{N},$$

where $T_{[0]} := I$ is the identity operator on X. An operator $T \in \mathcal{L}(X)$ is *uniformly mean ergodic* if $(T_{[n]})_n$ is a convergent sequence in $\mathcal{L}(X)$. Moreover, it is *power bounded* if and only if there is $C > 0$ such that

$$\sup_{n\in\mathbb{N}} \|T^n\| \le C.$$

We say that an operator T on X is *similar to a contraction* if we can find an invertible operator S on X and a contraction C on X such that

$$T = S^{-1} \circ C \circ S.$$

We call a contraction C on X *strict* if $\|C\| < 1$.

A good reference for information on ergodic theory is the monograph [16]. Additionally, interesting articles related to this topic are [1], [2] and [8]. In [9] Bonet and Ricker studied when multiplication operators acting on weighted Bergman spaces of infinite order are power bounded resp. uniformly mean ergodic. More precisely, they showed that $M_\psi : H_v^\infty \to H_v^\infty$ is power bounded if and only if $\|\psi\|_\infty = \sup_{z\in\mathbb{D}} |\psi(z)| \le 1$. Moreover, M_ψ is uniformly mean ergodic if and only if one of the following holds:

(1) There is $\xi \in \mathbb{C}$ with $|\xi| = 1$ such that $\psi(z) = \xi$, for $z \in \mathbb{D}$.
(2) $(1 - \psi)^{-1} \in H^\infty$.

Here H^∞ denotes the collection of all bounded analytic functions on $\mathbb{D}$.

Motivated by this, in [24] we analyzed when composition operators C_ϕ on spaces H_v^∞ are power bounded resp. uniformly mean ergodic. We proved that C_ϕ is power bounded if and only if it is similar to a contraction. Dealing with the property "uniformly mean ergodic" was much more difficult.

However, we could show that in case that ϕ has an attracting fixed point inside the disk the induced composition operator C_ϕ must be uniformly mean ergodic. The same holds true for symbols ϕ that have a super-attracting fixed point in $\mathbb{D}$ and the weight $v(z) = 1 - |z|$. While we could show that each composition operator C_ϕ induced by one of the following symbols:

(1) $\varphi_p(z) := \frac{p-z}{1-\overline{p}z}$ for fixed $p \in \mathbb{D}$ and every $z \in \mathbb{D}$,
(2) $\phi_\Theta(z) = e^{i\Theta\pi}$ for fixed rational Θ and every $z \in \mathbb{D}$,

must be uniformly mean ergodic, it is still an open question what happens in case that Θ is not rational.

However, the articles [9] and [24] gave rise to the question we address in the sequel, namely, under which conditions is the combination of a multiplication and a composition operator, that is a weighted composition operator, power bounded resp. uniformly mean ergodic?

2. Basics

For an introduction to as well as for a deep study of composition operators we refer the reader to the monographs [13] and [23]. In the setting of weighted spaces of holomorphic functions the so called *associated weights* play an important role. For a weight v we can define the associated weight as follows:

$$\tilde{v}(z) := \frac{1}{\sup\{|f(z)|;\ f \in H_v^\infty, \|f\|_v \leq 1\}} = \frac{1}{\|\delta_z\|_{H_v^{\infty\prime}}},$$

where δ_z denotes the point evaluation of z. By [5] the associated weight $\tilde{v}$ is continuous, $\tilde{v} \geq v > 0$ and for every $z \in \mathbb{D}$ we can find $f_z \in H_v^\infty$ with $\|f_z\|_v \leq 1$ such that $|f_z(z)| = \frac{1}{\tilde{v}(z)}$. Furthermore, it is well-known that if a weight v is radial and satisfies the Lusky condition

$$\text{(L1)} \qquad \inf_n \frac{v(1-2^{-n-1})}{v(1-2^{-n})} > 0,$$

then v and $\tilde{v}$ are equivalent, which means that we can find a constant $k > 0$ with

$$kv(z) \geq \tilde{v}(z) \geq v(z) \text{ for every } z \in \mathbb{D}.$$

Weights with this property are called *essential*. Since often it is quite difficult to compute the associated weight, it is very useful to know under which conditions weights are essential.

Moreover, by [12] the norm of a weighted composition operator $C_{\phi,\psi}$ acting on H_v^∞ is given by

$$\|C_{\phi,\psi}\| = \sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)|}{\tilde{v}(\phi(z))}.$$

Furthermore, Contreras and Hernández–Díaz also showed that such an operator is compact if and only if

$$\limsup_{|\phi(z)|\to 1} \frac{v(z)|\psi(z)|}{\tilde{v}(\phi(z))} = 0.$$

Obviously, $[C^n_{\phi,\psi} f](z) = \psi(\phi_{n-1}(z)) \cdots \psi(\phi(z))\psi(z) f(\phi_n(z))$ for every $z \in \mathbb{D}$ and every $f \in H(\mathbb{D})$, where $\phi_n := \underbrace{\phi \circ \cdots \circ \phi}_{n-\text{times}}$ for every $n \in \mathbb{N}$ and $\phi_0(z) = z$. Thus,

$$\|C^n_{\phi,\psi}\| = \sup_{z\in\mathbb{D}} \prod_{k=0}^{n-1} |\psi(\phi_k(z))| \frac{v(z)}{\tilde{v}(\phi_n(z))}.$$

In our investigations analytic self-maps ϕ of $\mathbb{D}$ which have a fixed point a inside the open unit disk $\mathbb{D}$ will play a great role. We will discuss the following cases.

(1) a is an *attracting* fixed point of ϕ, i.e., $\phi'(a) \neq 0$. Model maps are functions $g(z) = \lambda z$ for $z \in \mathbb{D}$ with $\lambda \in \mathbb{C}$, $|\lambda| < 1$.

One can change variables analytically in a neighbourhood of a and conjugate ϕ to the map $g(z) = \lambda z$ for $\lambda = \phi'(a)$. More precisely, there is an analytic map σ which sends a small neighbourhood of a conformally onto a small neighbourhood of 0 such that

$$\sigma \circ \phi \circ \sigma^{-1}(z) = \phi'(a) z$$

for all z near 0. For more details see article [20]. Originally the existence of such a map σ was shown by Koenigs in [15].

(2) a is a *super-attracting* fixed point of ϕ, i.e., $\phi'(a) = 0$. In this case, in 1905 Böttcher showed that one can change variables analytically to conjugate ϕ to a map $g(z) = z^n$, $n \geq 2$, in a neighbourhood of a, see [7]. For more details we again refer the reader to the article [20].

We close this section by stating the famous Denjoy–Wolff theorem which will play a great role in this article.

Theorem 1 (Denjoy–Wolff Theorem)**.** *Let ϕ be an analytic self-map of $\mathbb{D}$. If ϕ is not the identity and not an automorphism with exactly one fixed point in the open unit disk $\mathbb{D}$, then there is a unique point $p \in \overline{\mathbb{D}}$ such that $(\phi_n)_n$ converges to p uniformly on the compact subsets of $\mathbb{D}$.*

3. Power boundedness

Proposition 2. *Let ϕ have an attracting fixed point in $\mathbb{D}$, $\psi \in H^\infty$, and v be a radial weight satisfying (L1). Then $C_{\phi,\psi} : H^\infty_v \to H^\infty_v$ is power bounded if and only if $C_{\phi,\psi}$ is a contraction.*

Proof. Obviously, if $C_{\phi,\psi} : H^\infty_v \to H^\infty_v$ is a contraction, then it must be power bounded. Conversely, let us assume that $C_{\phi,\psi}$ is power bounded. We have to distinguish the following cases. We first assume that we can find a point $z_0 \in \mathbb{D}$ such that:

(1) $\sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)|}{v(\phi(z))} = \frac{v(z_0)|\psi(z_0)|}{v(\phi(z_0))}$.
(2) $z_0 = \phi(z_0)$.

In this case, since $C_{\phi,\psi}$ is power bounded, we can find a constant $C > 0$ such that

$$\sup_{z\in\mathbb{D}} |\psi(z)| \cdots |\psi(\phi_{n-1}(z))| \frac{v(z)}{v(\phi_n(z))} = \sup_{z\in\mathbb{D}} \prod_{k=1}^{n} |\psi(\phi_{k-1}(z))| \frac{v(\phi_{k-1}(z))}{v(\phi_k(z))}$$
$$= \left(|\psi(z_0)| \frac{v(z_0)}{v(\phi(z_0))} \right)^n \leq C$$

for every $n \in \mathbb{N}$. This immediately implies that $\sup_{z\in\mathbb{D}} \frac{|\psi(z)|v(z)}{v(\phi(z))} \leq 1$, i.e., $C_{\phi,\psi}$ is a contraction.

Next, we assume that there is $z_0 \in \mathbb{D}$ with the following properties:

(1) $\sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)|}{v(\phi(z))} = \frac{v(z_0)|\psi(z_0)|}{v(\phi(z_0))}$.
(2) $z_0 \neq \phi(z_0)$.

The strategy now is to reduce this to the situation given in the first case. To do this let z_1 be the attracting fixed point of ϕ, that is $\phi(z_1) = z_1$. We consider the weighted composition operator

$$C_{\phi_1,\psi_1} : H_v^\infty \to H_v^\infty, \ f \mapsto \psi_1(f \circ \phi_1),$$

defined by $\phi_1 = (\varphi_{z_0} \circ \varphi_{z_1}) \circ \phi \circ (\varphi_{z_1} \circ \varphi_{z_0})$ and $\psi_1 = \psi \circ (\varphi_{z_1} \circ \varphi_{z_0})$. Hence, C_{ϕ_1,ψ_1} is similar to $C_{\phi,\psi}$ and obviously has the same norm. Moreover, $\phi_1(z_0) = z_0$. Thus, we have the same situation as in the first case and obtain the claim.

It remains to assume that there is a sequence $(z_m)_m \subset \mathbb{D}$ with $|z_m| \to 1$ such that

$$\sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)|}{v(\phi(z))} = \lim_{m\to\infty} \frac{v(z_m)|\psi(z_m)|}{v(\phi(z_m))}.$$

Since w.l.o.g. ϕ is of type $\phi(z) = \lambda z$ with $\lambda \in \mathbb{C}$, $|\lambda| < 1$, this means that $\lim_{m\to\infty} \frac{v(z_m)}{\tilde{v}(\phi(z_m))} = 0$ and hence $\psi \notin H^\infty$, since otherwise $\sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)|}{\tilde{v}(\phi(z))} = 0$ which cannot be the case. Finally, the claim follows. □

Proposition 3. *Let ϕ have a fixed point in $\mathbb{D}$ and $C_{\phi,\psi} : H_v^\infty \to H_v^\infty$ be compact. Then $C_{\phi,\psi} : H_v^\infty \to H_v^\infty$ is power bounded if and only if it is a contraction.*

Proof. If $C_{\phi,\psi} : H_v^\infty \to H_v^\infty$ is a contraction, the power boundedness is obvious. Thus, let us assume that the operator is power bounded. Again, we have to distinguish three cases. The first two are analogous to the first two cases in the proof of the previous proposition. In order to treat the third case we assume that there is a sequence $(z_m)_m \subset \mathbb{D}$ with $|z_m| \to 1$ such that

$$\sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)|}{\tilde{v}(\phi(z))} = \lim_{m\to\infty} \frac{v(z_m)|\psi(z_m)|}{\tilde{v}(\phi(z_m))}.$$

Hence it follows, that $\limsup_{|\phi(z)|\to 1} \frac{v(z)|\psi(z)|}{\tilde{v}(\phi(z))} \neq 0$, but as we mentioned above, by [12] this means that $C_{\phi,\psi} : H_v^\infty \to H_v^\infty$ cannot be compact, which is a contradiction. □

Proposition 4. *Let ϕ be a conformal automorphism. Then $C_{\phi,\psi} : H_v^\infty \to H_v^\infty$ is power bounded if and only if it is a contraction.*

Proof. Let $C_{\phi,\psi}$ be power bounded. Then, we can find a constant $C > 0$ such that for every $n \in \mathbb{N}$:

$$\sup_{z\in\mathbb{D}} |\psi(\phi_{n-1}(z))| \cdots |\psi(z)| \frac{v(z)}{\tilde{v}(\phi_n(z))} \leq C.$$

Let us rewrite the supremum on the left-hand side of the previous equation:

$$\begin{aligned} \sup_{z\in\mathbb{D}} |\psi(\phi_{n-1}(z))| \cdots |\psi(z)| \frac{v(z)}{\tilde{v}(\phi_n(z))} &= \prod_{k=1}^{n} \sup_{z\in\mathbb{D}} |\psi(\phi_{k-1}(z))| \frac{v(\phi_{k-1}(z))}{v(\phi_k(z))} \\ &= \left(\sup_{z\in\mathbb{D}} |\psi(z)| \frac{v(z)}{\tilde{v}(\phi(z))} \right)^n \end{aligned}$$

since ϕ is an automorphism and v and ψ are continuous. Hence

$$\left(\sup_{z\in\mathbb{D}} |\psi(z)| \frac{v(z)}{\tilde{v}(\phi(z))} \right)^n \leq C$$

for every $n \in \mathbb{N}$. Finally, the operator must be a contraction. □

Proposition 5. *Let ϕ have no fixed point inside the disk and let us assume that there is a sequence $(z_m)_m \subset \mathbb{D}$ with $|z_m| \to 1$ such that*

$$\sup_{z\in\mathbb{D}} \frac{|\psi(z)|v(z)}{\tilde{v}(\phi(z))} = \lim_{m\to\infty} \frac{|\psi(z_m)|v(z_m)}{\tilde{v}(\phi(z_m))}.$$

Then the composition operator $C_{\phi,\psi} : H_v^\infty \to H_v^\infty$ is power bounded if and only if it is a contraction.

Proof. Obviously, if $C_{\phi,\psi} : H_v^\infty \to H_v^\infty$ is a contraction, then it must be power bounded. Thus, let us assume that $C_{\phi,\psi}$ is power bounded. In order to show that it is a contraction we have to distinguish the following cases. First let us assume that $(z_m)_m$ tends to a fixed point z_0 of ϕ. In that case we can find $C > 0$ such that

$$\begin{aligned} C \geq \sup_{z\in\mathbb{D}} \frac{|\psi(z)| \cdots |\psi(\phi_{n-1}(z))| v(z)}{\tilde{v}(\phi(z))} &= \prod_{k=1}^{n} \frac{|\psi(\phi_{k-1}(z))| v(\phi_{k-1}(z))}{\tilde{v}(\phi_k(z))} \\ &\geq \left(\sup_{z\in\mathbb{D}} |\psi(z)| \frac{v(z)}{\tilde{v}(\phi(z))} \right)^n \end{aligned}$$

for every $n \in \mathbb{N}$. If we assume that $(z_m)_m$ tends to a boundary point $z_1 \in \partial\mathbb{D}$ that is not a fixed point of ϕ, we consider the rotation $\varphi_\Theta(z) = e^{i\Theta\pi} z$ that takes the point z_1 to the fixed point z_0. then we consider the operator

$$C_{\phi_1,\psi_1} : H_v^\infty \to H_v^\infty$$

given by $\phi_1 = \varphi_{-\Theta} \circ \phi \circ \varphi_\Theta$ and $\psi_1 = \psi \circ \varphi_\Theta$. Then the operator is similar to the operator $C_{\phi,\psi}$. Moreover,

$$\phi_1(z_1) = \varphi_{-\Theta}(\phi(\varphi_\Theta(z_1))) = \varphi_{-\Theta}(\phi(z_0)) = \varphi_{-\Theta}(z_0) = z_1$$

and we have reduced the problem to the situation described above. □

4. Uniformly mean ergodicity

Proposition 6. *Let v be a weight and $T : H_v^\infty \to H_v^\infty$ a linear operator. If $T : H_v^\infty \to H_v^\infty$ is similar to a strict contraction, then T is uniformly mean ergodic.*

Proof. We will show that $\|T_{[n]}\| \to 0$ if $n \to \infty$. By hypothesis, we can find an invertible operator S on H_v^∞ and an operator C on H_v^∞ with $\|C\| < 1$ such that $T = S^{-1} \circ C \circ S$. Thus, we arrive at the following estimate

$$\|T_{[n]}\| \leq \frac{1}{n}\sum_{m=1}^{n} \|T^m\| \leq \frac{1}{n}\sum_{m=1}^{n} \|S^{-1}\|\|C\|^m\|S\| \leq \frac{1}{n}\|S^{-1}\|\|S\|M \to 0$$

if $n \to \infty$, where $M := \sum_{m=1}^{\infty} \|C\|^m = \frac{1}{1-\|C\|} < \infty$. □

The converse is not true, as the following trivial example shows.

Example 7. If we take $v(z) = 1 - |z|$ and $\phi(z) = \mathrm{id}(z) = z$ for every $z \in \mathbb{D}$ we obtain $\phi_n(z) = z$ for every $n \in \mathbb{N}$. Obviously we have that $\|C_\phi\| = 1$ and $(C_\phi)_{[n]} = \frac{1}{n}\sum_{m=1}^{n} C_\phi^m = C_\phi$ for every $n \in \mathbb{N}$. Hence C_ϕ is uniformly mean ergodic.

Next, we need some auxiliary results regarding differences of weighted composition operators. Dealing with such differences requires the so-called *pseudohyperbolic distance* given by

$$\rho(z,p) = |\varphi_p(z)| \text{ for every } z, p \in \mathbb{D},$$

where φ_p denotes the Möbius transformation which interchanges 0 and p, that is

$$\varphi_p(z) := \frac{p - z}{1 - \overline{p}z} \text{ for every } z \in \mathbb{D}.$$

Lemma 8 (Bonet–Lindström–Wolf [11])**.** *Let v be a radial weight satisfying condition* (L1)*. Then, there exists a constant $C_v > 0$ (depending only on the weight v) such that, for all $f \in H_v^\infty$,*

$$|f(z) - f(p)| \leq C_v\|f\|_v \max\left\{\frac{1}{v(z)}, \frac{1}{v(p)}\right\}\rho(z,p)$$

for all $z, p \in \mathbb{D}$.

Lemma 9. *Let v and w be weights such that v is typical with* (L1)*. Moreover, let ϕ^1, ϕ^2 be analytic self-maps of $\mathbb{D}$ and $\psi^1, \psi^2 \in H(\mathbb{D})$. Then there is* *a constant $C > 0$ such that, we obtain the following estimate for the norm of the operator $C_{\phi^1,\psi^1} - C_{\phi^2,\psi^2} : H_v^\infty \to H_w^\infty$:*

$$\begin{aligned}&\|C_{\phi^1,\psi^1} - C_{\phi^2,\psi^2}\| \\ &\leq C \max\left\{ \sup_{z\in\mathbb{D}} |\psi^1(z)| \max\left\{ \frac{w(z)}{v(\phi^1(z))}, \frac{w(z)}{v(\phi^2(z))} \right\} \rho(\phi^1(z),\phi^2(z)), \right. \\ &\qquad\qquad \left. \sup_{z\in\mathbb{D}} \frac{w(z)}{v(\phi^2(z))} |\psi^1(z) - \psi^2(z)| \right\}.\end{aligned}$$

Proof. First recall that the norm of $C_{\phi^1,\psi^1} - C_{\phi^2,\psi^2} : H_v^\infty \to H_w^\infty$ is given by

$$\begin{aligned}&\|C_{\phi^1,\psi^1} - C_{\phi^2,\psi^2}\| \\ &= \sup\left\{ \sup_{z\in\mathbb{D}} w(z)|\psi^1(z)f(\phi^1(z)) - \psi^2(z)f(\phi^2(z)),\ f \in H_v^\infty,\ \|f\|_v \leq 1 \right\}.\end{aligned}$$

Now, for every $f \in H_v^\infty$ with $\|f\|_v \leq 1$ we obtain by using Lemma 8

$$\begin{aligned}&\sup_{z\in\mathbb{D}} w(z)|\psi^1(z)f(\phi^1(z)) - \psi^2(z)f(\phi^2(z))| \\ &\leq \sup_{z\in\mathbb{D}} |\psi^1(z)|w(z)|f(\phi^1(z)) - f(\phi^2(z))| \\ &\quad + w(z)|f(\phi^2(z))||\psi^1(z) - \psi^2(z)| \\ &\leq \sup_{z\in\mathbb{D}} |\psi^1(z)| \max\left\{ \frac{w(z)}{v(\phi^1(z))}, \frac{w(z)}{v(\phi^2(z))} \right\} \rho(\phi^1(z),\phi^2(z)) \\ &\quad + \sup_{z\in\mathbb{D}} |\psi^1(z) - \psi^2(z)| \frac{w(z)}{v(\phi^2(z))}.\end{aligned}$$

Hence the claim follows. □

Theorem 10. *Let v be a typical weight with (L1) and ϕ be an analytic self-map but not a conformal automorphism of $\mathbb{D}$. Let us assume that ϕ has an* attracting *fixed point a in $\mathbb{D}$, i.e., $\phi'(a) \neq 0$. Furthermore, let $\psi \in H^\infty$ with $\sup_{z\in\mathbb{D}} |\psi(z)| \leq M < \infty$ such that we can find $\Theta \in \mathbb{C}$ $\sup_{z\in\mathbb{D}} |\prod_{k=0}^{n-1} \psi(\phi_k(z)) - \Theta| < |\mu|^n$ for some $|\mu| < 1$ and some $n \geq n_0$. Then $C_{\phi,\psi} : H_v^\infty \to H_v^\infty$ is uniformly mean ergodic.*

Proof. W.l.o.g. we may assume that $\phi(z) = \lambda z$ for every $z \in \mathbb{D}$ with $|\lambda| < 1$ and that $n_0 = 1$ in the hypothesis. Obviously we have that $\phi_n(z) = \lambda^n z$ for every $z \in \mathbb{D}$ and every $n \in \mathbb{N}$ as well as $\|C_{\phi_n}\| = \|C_\phi\| = 1$ for every $n \in \mathbb{N}$. If $C_{0,\Theta}$ is the weighted composition operator defined by $C_{0,\Theta} : H_v^\infty \to H_v^\infty$, $(C_{0,\Theta}f)(z) = \Theta f(0)$ for every $z \in \mathbb{D}$ we obtain by using Lemma 9

$$\begin{aligned}&\|(C_{\phi,\psi})_{[n]} - C_{0,\Theta}\| \\ &= \left\| \frac{1}{n} \sum_{m=1}^{n} C_{\phi,\psi}^m - C_{0,\Theta} \right\|\end{aligned}$$

$$\leq \frac{1}{n}\sum_{m=1}^{n}\left\|C_{\phi,\psi}^{m}-C_{0,\Theta}\right\|$$
$$\leq C\frac{1}{n}\sum_{m=1}^{n}\max\Bigg\{\sup_{z\in\mathbb{D}}|\psi(z)|\max\left\{\frac{v(z)}{v(\phi_m(z))},\frac{v(z)}{v(0)}\right\}\rho(\phi_m(z),0),$$
$$\sup_{z\in\mathbb{D}}\frac{v(z)}{v(0)}\left|\prod_{k=0}^{m-1}\psi(\phi_k(z))-\Theta\right|\Bigg\}$$
$$\leq C\frac{1}{n}\sum_{m=1}^{n}\max\{M|\lambda|^m,|\mu|^m\}\to 0$$

since $|\lambda|<1$. Hence, in this case, $((C_{\phi,\psi})_{[n]})_{n\in\mathbb{N}}$ tends to $C_{0,\Theta}$ in $\mathcal{L}(H_v^\infty)$. Thus, $C_{\phi,\psi}$ is uniformly mean ergodic, and the claim follows. □

Theorem 11. *Let $v(z)=1-|z|$ for every $z\in\mathbb{D}$. Moreover, let ϕ be an analytic self-map but not a conformal automorphism of $\mathbb{D}$ such that ϕ has a* super-attracting *fixed point $a\in\mathbb{D}$, i.e., $\phi'(a)=0$. Furthermore let $\psi\in H^\infty$ with $\sup_{z\in\mathbb{D}}|\psi(z)|\leq M<\infty$ such that we can find $\Theta\in\mathbb{C}$ with $\sup_{z\in\mathbb{D}}|\prod_{k=0}^{n-1}\psi(\phi_k(z))-\Theta|<|\mu|^n$ for some $|\mu|<1$ and some $n\geq n_0$. Then $C_{\phi,\psi}:H_v^\infty\to H_v^\infty$ is uniformly mean ergodic.*

Proof. W.l.o.g. we may assume that ϕ is given by $\phi(z)=z^n$ for every $z\in\mathbb{D}$, $n\geq 2$. Hence, the iterates are given by $\phi_k(z)=z^{n^k}$ for every $z\in\mathbb{D}$, $k\in\mathbb{N}$. We will show that the sequence $((C_{\phi,\psi})_{[k]})_k$ tends to $C_{0,\Theta}$ with respect to the operator norm $\|.\|$, where $C_{0,\Theta}$ is given by $(C_{0,\Theta}f)(z)=\Theta f(0)$ for every $z\in\mathbb{D}$.

The function $f:[0,1)\to\mathbb{R},\ f(r)=\frac{1-r}{1-r^{n^k}}$ is monotone decreasing since

$$f'(r)=\frac{-1+(1-n^k)r^{n^k}+n^kr^{n^k-1}}{(1-r^{n^k})^2}\leq 0 \text{ for every } r\in[0,1).$$

Moreover, we have that $\lim_{r\to1}\frac{1-r}{1-r^{n^k}}=\lim_{r\to1}\frac{1}{n^kr^{n^k-1}}=\frac{1}{n^k}$ and $\sum_{k=1}^\infty\frac{1}{n^k}=\frac{n}{n-1}$. Hence there has to be $0<r_0<1$ such that $\sum_{k=1}^\infty\frac{1-r_0}{1-r_0^{n^k}}=L<\infty$. Now, we choose such an $0<r_0<1$ and obtain with an application of Lemma 9

$$\|(C_{\phi,\psi})_{[k]}-C_{0,\Theta}\|$$
$$\leq\frac{1}{k}\sum_{m=1}^{k}\|C_{\phi,\psi}^m-C_{0,\Theta}\|$$
$$\leq\frac{C}{k}\sum_{m=1}^{k}\max\Bigg\{\sup_{|z|\leq r_0}|\psi(z)|\max\left\{\frac{1-|z|}{1-|z|^{n^m}},1-|z|\right\}\rho(\phi_m(z),0),$$

$$
\begin{aligned}
&\qquad \sup_{|z|\le r_0}\left|\prod_{k=0}^{m-1}\psi(\phi_k(z))-\Theta\right|(1-|z|)\Bigg\}\\
&\quad +\frac{C}{k}\sum_{m=1}^{k}\max\Bigg\{\sup_{|z|>r_0}|\psi(z)|\max\left\{\frac{1-|z|}{1-|z|^{n^m}},1-|z|\right\}\rho(\phi_m(z),0),\\
&\qquad \sup_{|z|>r_0}\left|\prod_{k=0}^{m-1}\psi(\phi_k(z))-\Theta\right|(1-|z|)\Bigg\}\\
&\le \frac{C}{k}\sum_{m=1}^{k}\max\{M|r_0|^{n^m},|\mu|^n\}+\frac{C}{k}\sum_{m=1}^{k}\max\left\{M\frac{1-r_0}{1-r_0^{n^m}},|\mu|^n\right\}\\
&\le \frac{C}{k}\max\left\{\frac{M}{1-r_0^n},\frac{1}{1-|\mu|}\right\}+\max\left\{\frac{1}{k}LM,\frac{1}{1-|\mu|}\right\}\to 0
\end{aligned}
$$

if $k\to\infty$. Thus, the claim follows. □

Next, let us give an example when such a Θ as in the hypothesis of Theorems 10 and 11 does exist.

Example 12.

(1) Let us consider $\phi(z)=\lambda z$ for every $z\in\mathbb{D}$ with $|\lambda|<1$ and $\psi(z)=z^k$ for some $k\in\mathbb{N}$. Then

$$\prod_{l=0}^{n-1}\psi(\phi_l(z))=\lambda^{k\cdot\sum_{l=0}^{n-1}l}z^{nk}.$$

Then, with $\Theta=0$, $n_0=3$ and $\mu=\lambda^k$ we obtain the desired inequality.

(2) Take $\phi(z)=\lambda z$ for every $z\in\mathbb{D}$ with $|\lambda|<1$ and $\psi(z)=1-z$ for every $z\in\mathbb{D}$. Then $\prod_{l=0}^{n-1}\psi(\phi_l(z))=\prod_{l=0}^{n-1}(1-\lambda^l z)$. Hence $\sup_{z\in\mathbb{D}}\prod_{l=0}^{n-1}|\psi(\phi_l(z))|=\prod_{l=0}^{n-1}(1+|\lambda|^l)$. Now, the product $\prod_{l=0}^{\infty}(1+|\lambda|^l)$ converges if and only if the series $\sum_{l=0}^{\infty}\mathrm{Log}(1+|\lambda|^l)$ converges. The quotient criterion shows that this is the case for every $\lambda\in\mathbb{C}$ with $|\lambda|<1$. Now, choose $\Theta=\prod_{l=0}^{\infty}(1+|\lambda|^l)$.

(3) Let $\phi(z)=z^k$, $k\ge 2$, and $\psi(z)=\lambda z$ with $|\lambda|<1$. Then

$$\prod_{l=0}^{n-1}\psi(\phi_l(z))=\lambda^n z^{k^{n-1}+k^{n-2}+\dots+k+1}.$$

Hence we obtain the desired inequality for $\Theta=0$, $n_0=1$ and $\mu=\lambda$.

Remark 13. If v is a typical weight with (L1) such that $\frac{v(r)}{v(r^n)}$ is monotone decreasing with respect to r and such that there is $C<1$ with $\lim_{r\to 1}\frac{v(r)}{v(r^n)}\le C^n$ for every $n\in\mathbb{N}$, then - with the same proof as above - we can show that an analytic self-map of $\mathbb{D}$ with a super-attracting fixed point $a\in\mathbb{D}$ induces a uniformly mean ergodic weighted composition operator $C_{\phi,\psi} : H_v^\infty \to H_v^\infty$. An example of this is, e.g., the weight $v(z) = \frac{1}{1-\ln(1-|z|)}$.

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