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Enhancements of rack counting invariants via dynamical cocycles

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Abstract. We introduce the notion of *N-reduced dynamical cocycles* and use these objects to define enhancements of the rack counting invariant for classical and virtual knots and links. We provide examples to show that the new invariants are not determined by the rack counting invariant, the Jones polynomial or the generalized Alexander polynomial.

Contents

1. Introduction

Racks were introduced in 1992 in [6] as an algebraic structure for defining representational and functorial invariants of framed oriented knots and links. A rack generalizes the notion of a *quandle*, an algebraic structure defined in 1982 in [8] and independently in [9] which defines invariants of unframed knots and links. More precisely, the number of quandle homomorphisms from the fundamental quandle of a knot or link to a finite quandle X defines a computable integer-valued invariant of unframed oriented knots and links known as the *quandle counting invariant*.

In [10], a property of finite racks known as *rack rank* or *rack characteristic* was used to define an integer-valued invariant of unframed oriented knots and links using nonquandle racks, known as the *integral rack counting invariant*; for quandles, this invariant coincides with the quandle counting

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invariant. An *enhancement* of a counting invariant uses a Reidemeister-invariant signature for each homomorphism rather than merely counting homomorphisms. In [3], the first enhancement of the quandle counting invariant was defined using *Boltzmann weights* determined by elements of the second cohomology of a finite quandle. The resulting *quandle* 2*-cocycle invariants* of knots and links have been the subject of much study ever since.

In [7] an enhancement of the integral rack counting invariant was defined using a modification of the *rack module* structure from [1], associating a vector space or module to each homomorphism. In this paper we further generalize the enhancement from [7] using a modified version of an algebraic structure first defined in [1] known as a *dynamical cocycle*. In particular, dynamical cocycles satisfying a condition we call N*-reduced* yield an enhancement of the rack counting invariant.

The paper is organized as follows. In Section 2 we review the basics of racks, the rack counting invariant, and the rack module enhancement. In Section 3 we define N-reduced dynamical cocycles and the N-reduced dynamical cocycle invariant. In Section 4 we provide some computations and examples, and we conclude in Section 5 with some questions for future study.

2. Racks, the counting invariant and the rack module enhancement

We start by reviewing some basic definitions from [6, 8].

Definition 1. A *rack* is a set X equipped with a binary operation

$$\triangleright : X \times X \to X$$

satisfying the following two conditions:

(i) For each $x \in X$, the map $f_x : X \to X$ defined by $f_x(y) = y \triangleright x$ is invertible, with inverse $f_x^{-1}(y)$ denoted by $y \triangleright^{-1} x$.

(ii) For each $x, y, z \in X$, we have $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

A *quandle* is a rack with the added condition:

(iii) For all $x \in X$, we have $x \triangleright x = x$.

Note that (ii) is equivalent to the requirement that each map $f_x : X \to X$ be a rack homomorphism, i.e.,

$$f_z(x \triangleright y) = (x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z) = f_z(x) \triangleright f_z(y),$$

so we can alternatively define a rack as a set X with a bijection $f_x : X \to X$ for each $x \in X$ such that every f_x is an automorphism of the structure on X defined by $x \triangleright y = f_y(x)$.

Standard examples of racks include:

- (t, s)*-racks.* Any module over $\ddot{\Lambda} = \mathbb{Z}[t^{\pm 1}, s]/(s^2 - (1-t)s)$ is a rack under

$$x \triangleright y = tx + sy.$$

If s is invertible, then $s^2-(1-t)s=0$ implies $s=1-t$ and we have a quandle known as an *Alexander quandle*.

- *Conjugation racks.* Every group G is a rack (indeed, a quandle) under n-fold conjugation for any $n \in \mathbb{Z}$:

$$x \triangleright y = y^{-n}xy^n.$$

- *The fundamental rack of a framed oriented link.* Let $L \subset S^3$ be a link of c components, $n(L)$ a regular neighborhood of L with set of framing curves $F=\{F_1,\dots,F_c\}$ giving the framing of L, $x_0 \in S^3 \setminus n(L)$ a base point and $FR(L)$ the set of isotopy classes of paths from x_0 to F_i where the terminal point of the path can wander along F_i during the isotopy. For each point $x_1 \in F_i$ there is a meridian $m(x_1)$ in $n(L)$, unique up to isotopy, linking the ith component of L once. Then for each path $y:[0,1]\to S^3\setminus n(L)$ representing an isotopy class in $FR(L)$, let $p(y)=y^{-1}*m(y(1))*y \in \pi_1(S^3\setminus n(L))$ where $*$ is path concatenation reading right-to-left. Then $FR(L)$ is a rack under the operation

$$[x]\triangleright[y]=[x*p(y)].$$

Combinatorially, $FR(L)$ can be understood as equivalence classes of rack words in a set of generators corresponding one-to-one with the set of arcs in a diagram of L under the equivalence r elation generated by the rack axioms and crossing relations in L. See [6] for more details.

Definition 2. Let $X=\{x_1,\dots,x_n\}$ be a finite set. We can specify a rack structure on X by a *rack matrix* M_X in which the (i,j)th entry is k when $x_k = x_i \triangleright x_j$. Rack axiom (i) is equivalent to the condition that every column of M_X is a permutation; rack axiom (ii) requires checking each triple for the condition $M_{M_{i,j},k}=M_{M_{i,k},M_{j,k}}$.

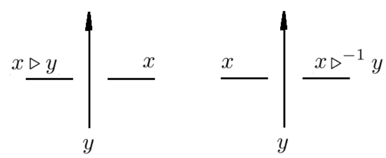
Example 1. The (t,s)-rack structure on

$$\mathbb{Z}_4=\{x_1=1,x_2=2,x_3=3,x_4=4\}$$

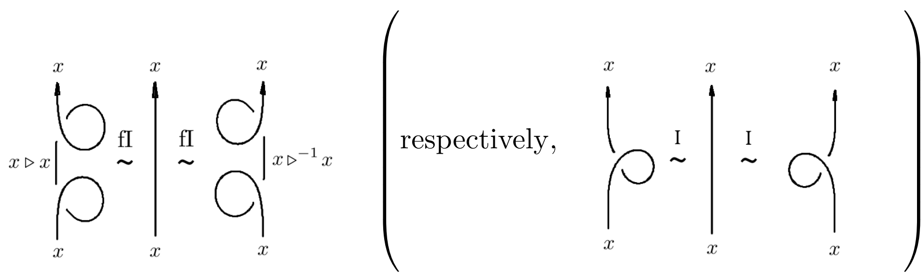
with $t=1$ and $s=2$ has rack matrix

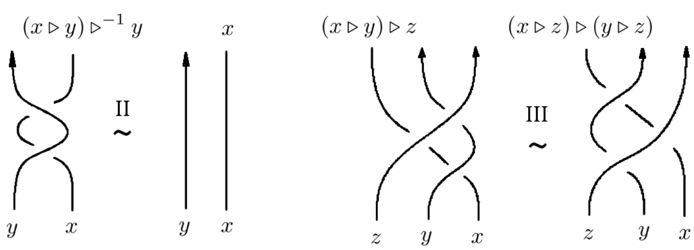
$$M_X=\begin{bmatrix} 3 & 1 & 3 & 1\\ 4 & 2 & 4 & 2\\ 1 & 3 & 1 & 3\\ 2 & 4 & 2 & 4 \end{bmatrix}.$$

Definition 3. Let X be a rack and L an oriented link diagram. An X-*labeling* or *rack labeling of L by X* is an assignment of an element of X to each arc in L such that the condition below is satisfied:



Indeed, the rack axioms are algebraic distillations of Reidemeister moves II and III under this labeling scheme; the quandle condition corresponds to the unframed Reidemeister move I, and the framed Reidemeister I moves do not impose any additional conditions. Accordingly, labelings of arcs of oriented framed knot or link diagrams by rack elements (respectively, quandle elements) as shown above are preserved by oriented framed Reidemeister moves (respectively, oriented unframed Reidemeister moves) as illustrated in the figures below.



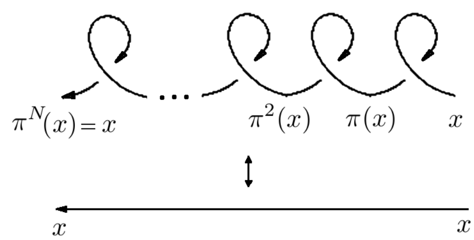


Definition 4. Let X be a rack. We call the map $\pi : X \to X$ defined by $\pi(x) = x \triangleright x$ the *kink map*. The *rack rank* or *rack characteristic* of X, denoted by $N(X)$, is the order of the permutation π considered as an element of the symmetric group $S_{|X|}$. Equivalently, for every element $x \in X$, the *rank* of x, denoted by $N(x)$, is the smallest positive integer N such that $\pi^N(x) = x$. Thus, $N(X)$ is the least common multiple of the ranks $N(x)$ for all $x \in X$. In particular, the kink map of a rack structure on a finite set $X = \{x_1, \dots, x_n\}$ given by a rack matrix M_X is the permutation in $S_{|X|}$ which sends k to the (k,k) entry of M_X. That is, the image of π is given by the entries along the diagonal of M_X.

Example 2. The rack in Example 1 has kink map satisfying $\pi(1) = 3$, $\pi(2) = 4$, $\pi(3) = 1$ and $\pi(4) = 2$ (or, in cycle notation, $\pi = (13)(24)$) and hence has rack rank $N = 2$.

Remark 3. The quandle condition implies that the rank of every quandle element is 1, and thus the rack rank of a quandle is always 1. Indeed, quandles are simply racks with rack rank $N = 1$.

Rack rank can be understood geometrically in terms of the Reidemeister type I move: if an arc in a knot diagram is labeled with a rack element x, going through a positive kink changes the label to $\pi(x)$. A natural question is then: how many kinks must we go though to end up again with x? This notion of order is the rank of x. We can illustrate the concept of rack rank with the *N-phone cord move* pictured below:

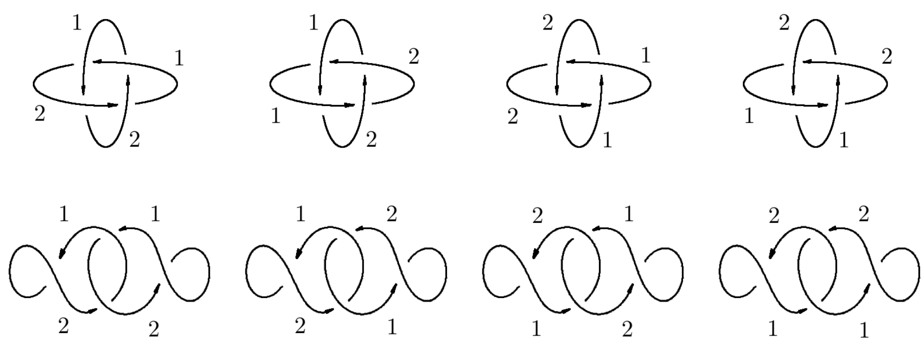


If N is the rank of X, then labelings of a link diagram L by X are preserved by N-phone cord moves. In particular, if X is a rack of rack rank N, and L and L' are framed oriented links related by framed Reidemeister moves with framings congruent modulo N, then the sets of X-labelings of L and L' are in bijective correspondence and we have $|\mathrm{Hom}(L, X)| = |\mathrm{Hom}(L', X)|$. It follows that the number of homomorphisms is periodic in the framing number with period N. Since each component of a link L has its own independent framing number, a link of c components has a $\mathbb{Z}^c$-lattice of framings, and the numbers of X-labelings of these framed links form a tiling of the lattice by blocks of side length N. In particular, while the number of X-labelings of a diagram is an invariant only of framed isotopy, the number of labelings over a complete tile is an invariant of unframed ambient isotopy.

Definition 5. Let X be a rack with rank N and let L be an oriented link of c components. Let $\mathbf{w} \in (\mathbb{Z}_N)^c$ be a framing vector specifying a framing modulo N for each component of L, and let us denote a diagram of L with framing vector $\mathbf{w}$ by $(L, \mathbf{w})$. We thus obtain a set of N^c diagrams of framings of L mod N. For each such diagram $(L, \mathbf{w})$, we have a set of X-labelings corresponding to homomorphisms $f : FR(L, \mathbf{w}) \to X$. Summing the numbers of X-labelings over the set $\{(L, \mathbf{w}) \mid \mathbf{w} \in (\mathbb{Z}_N)^c\}$, we obtain an invariant of unframed links known as the *integral rack counting invariant,* which is denoted by:

$$\Phi_X^{\mathbb{Z}}(L) = \sum_{\mathbf{w}\in(\mathbb{Z}_N)^c} |\mathrm{Hom}(FR(L,\mathbf{w}),X)|.$$

Example 4. Let X be the rack with rack matrix $M_X = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$. As a labeling rule, the rack structure of X says that at a crossing, the understrand switches from 1 to 2 or from 2 to 1 since $1 \triangleright x = 2$ and $2 \triangleright x = 1$ for $x = 1, 2$. The kink map is the transposition (12), so $N = 2$. Thus, to compute $\Phi_X^{\mathbb{Z}}$ on a link of $c = 2$ components, we must count X-labelings on the set of $N^c = 2^2 = 4$ diagrams with writhe vectors in $(\mathbb{Z}_N)^c$. The $(4,2)$-torus link $L4a1$ and the Hopf link $L2a1$ both have four X-labelings as depicted below, so we have $\Phi_X^{\mathbb{Z}}(L4a1) = \Phi_X^{\mathbb{Z}}(L2a1) = 4$.



An *enhancement* of $\Phi_X^{\mathbb{Z}}(L)$ is a link invariant defined by associating to each X-labeling of L a quantity which is unchanged by X-labeled framed Reidemeister moves and N-phone cord moves. Examples include:

- *Image Enhanced Invariant.* The image subrack of a rack homomorphism is closed under $\triangleright$ and thus is unchanged by N-phone cord moves. Hence we have an enhancement:
$$\Phi_X^{\mathrm{Im}}(L) = \sum_{\mathbf{w}\in(\mathbb{Z}_N)^c} \left(\sum_{f\in\mathrm{Hom}(FR(L,\mathbf{w}),X)} u^{|\mathrm{Im}(f)|} \right)$$
where u is a formal variable.
- *Writhe Enhanced Invariant.* Keeping track of which labelings are contributed by which writhes yields another enhancement:
$$\Phi_X^{\mathrm{W}}(L) = \sum_{\mathbf{w}\in(\mathbb{Z}_N)^c} |\mathrm{Hom}(FR(L,\mathbf{w}),X)| q^{\mathbf{w}}$$
where $q^{(w_1,\dots,w_c)} = q_1^{w_1}\dots q_c^{w_c}$ is a product of formal variables.
- *Cocycle Invariants.* A finite rack X has a cohomology theory analogous to group cohomology. For any $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[X^n],\mathbb{Z})$, define

$\delta^n : \mathbb{Z}[X^n] \to \mathbb{Z}[X^{n+1}]$ by

$$(\delta^n f)(x_1,\dots,x_{n+1}) = \sum_{k=2}^{n+1}(-1)^k\big(f(x_1,\dots,x_{k-1},x_{k+1},\dots,x_{n+1}) \\ - f(x_1\triangleright x_k,\dots,x_{k-1}\triangleright x_k,x_{k+1},\dots,x_{n+1})\big)$$

and extend linearly. Let D^n be the subgroup of $\mathbb{Z}[X^n]$ generated by elements of the form

$$\sum_{k=1}^{N}(x_1,\dots,\pi^k(x_j),\pi^{k+1}(x_j),\dots,x_n), \quad j=1,\dots,n-1,$$

where N is the rack rank of X. Then (D^n,δ^n) is a subcomplex of $(\mathbb{Z}[X^n],\delta^n)$; the quotient complex $(\mathbb{Z}[X^n]/D^n,\delta^n)$ is the N*-reduced rack cochain complex* (or the *quandle cochain complex* if $N=1$), with cohomology groups denoted by $H^n_{R/ND}(X)$. For every element $\phi\in H^2_{R/ND}(X)$ (such a ϕ is called an N*-reduced* 2*-cocycle*) we have an enhancement

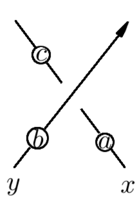
$$\Phi_X^{\phi}(L) = \sum_{\mathbf{w}\in(\mathbb{Z}_N)^c}\left(\sum_{f\in\mathrm{Hom}(FR(L,\mathbf{w}),X)} u^{BW(f)}\right)$$

where $BW(f)$, the *Boltzmann weight* of f, is the sum over all crossings in f of ϕ evaluated at the arc labelings of each crossing.

See [3, 5, 10] for further details.

Example 5. In Example 4, the links $L2a1$ and $L4a1$ have the same number of X-labelings over a complete period of framings mod N, but these labelings occur at different framing vectors. In particular, all four labelings of $L4a1$ occur with writhe vector $\mathbf{w}=(0,0)$ while all four labelings of $L2a1$ occur with writhe vector $\mathbf{x}=(1,1)$. Thus the writhe enhanced invariant Φ_X^W distinguishes the links, with $\Phi_X^W(L4a1)=4\neq 4q_1q_2=\Phi_X^W(L2a1)$.

In [1] an algebra known as the *rack algebra* $\mathbb{Z}[X]$ was associated to each finite rack X; in [7] a modified form of the rack algebra was used to define an enhancement of $\Phi_X^{\mathbb{Z}}$. The idea is to add a secondary labeling to an X-labeled link diagram by putting *beads* on each arc and defining a (t,s)-rack style operation on the beads at a crossing with t and s values indexed by the arc labels in X as depicted below:



$$c = t_{x,y}a + s_{x,y}b$$

Definition 6. Let X be a finite rack with rack rank N. The *rack algebra* of X, denoted by $\mathbb{Z}[X]$, is the quotient of the polynomial algebra $\mathbb{Z}[t^{\pm 1}_{x,y}, s_{x,y}]$ generated by noncommuting variables $t^{\pm 1}_{x,y}$ and $s_{x,y}$ for each $x, y \in X$ modulo the ideal I generated by the relators

$$t_{x\triangleright y,z}t_{x,y} - t_{x\triangleright z,y\triangleright z}t_{x,z}, \quad t_{x\triangleright y,z}s_{x,y} - s_{x\triangleright z,y\triangleright z}t_{x,z},$$
$$s_{x\triangleright y,z} - s_{x\triangleright z,y\triangleright z}s_{y,z} - t_{x\triangleright z,y\triangleright z}s_{x,z} \quad \text{and} \quad 1 - \prod_{k=0}^{N-1}\left(t_{\pi^k(x),\pi^k(x)} + s_{\pi^k(x),\pi^k(x)}\right)$$

for all $x, y, z \in X$. An X-*module* is a representation of $\mathbb{Z}[X]$, that is, an abelian group G with automorphisms $t_{x,y} : G \to G$ and endomorphisms $s_{x,y} : G \to G$ such that the maps defined by the relators of I are zero.

Example 6. Let R be a commutative ring. Then any R-module becomes an X-module with a choice of automorphisms and endomorphisms given by multiplication by invertible elements $t_{x,y} \in R$ and generic elements $s_{x,y} \in R$ such that the ideal I is zero. We can express such a structure conveniently with a block matrix $M_R = [\ T \mid S\ ]$ where the (i, j) entries of T and S are t_{x_i,y_j} and s_{x_i,y_j} respectively.

Example 7. Let X be a rack and let $f \in \mathrm{Hom}(FR(L), X)$ be an X-labeled link diagram. The *fundamental* $\mathbb{Z}[X]$-*module* of f, denoted by $\mathbb{Z}[f]$, is the quotient of the free $\mathbb{Z}[X]$-module generated by the set of arcs in f modulo the ideal generated by the crossing relations.

In [7] an enhancement of $\Phi_X^{\mathbb{Z}}$ was defined using the number of bead labelings of an X-labeled diagram of a framed oriented link L as a signature as follows:

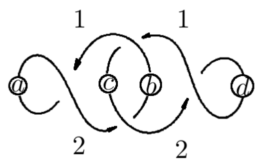
Definition 7. Let X be a finite rack and R a commutative ring with an X-module structure. The *rack module enhanced invariant* is given by:

$$\Phi_{X,R}(L) = \sum_{\mathbf{w}\in(\mathbb{Z}_N)^c}\left(\sum_{f\in\mathrm{Hom}(FR(L,\mathbf{w}),X)} u^{|\mathrm{Hom}(\mathbb{Z}[f],R)|}\right).$$

Example 8. Let X be the rack from Example 4 and let $R = \mathbb{Z}_3$. The matrix

$$M_R = [T|S] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{array}\right]$$

defines an X-module structure on R. To compute $\Phi_{X,R}$ for the Hopf link $L2a1$, we must compute $|\mathrm{Hom}(\mathbb{Z}[f], R)|$ for each valid X-labeling of $L2a1$. For instance, the following X-labeled diagram has fundamental $\mathbb{Z}[X]$-module with listed presentation matrix:



$$M_{\mathbb{Z}[f]} = \left[\begin{array}{cccc} t_{2,2}+s_{2,2} & -1 & 0 & 0 \\ 0 & s_{1,1} & -1 & t_{1,1} \\ t_{2,2} & -1 & s_{2,2} & 0 \\ 0 & 0 & -1 & t_{1,1}+s_{1,1} \end{array}\right]$$

Replacing each $t_{x,y}$ and $s_{x,y}$ with its value from M_R and row-reducing over $\mathbb{Z}_3$, we have

$$\left[\begin{array}{cccc} 2 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 2 \end{array}\right] \to \left[\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so the solution space (i.e., the set of bead labelings) is the set

$$\{(0,0,0,0),(2,1,2,1),(1,2,1,2)\}$$

and this X-labeling contributes u^3 to $\Phi_{X,R}(L2a1)$. Repeating for the other labelings, we have $\Phi_{X,R}(L2a1) = 4u^3$.

3. Dynamical cocycles and enhancements of the counting invariant

In this section we generalize the rack module idea to remove the restrictions of the abelian group structure, keeping only those conditions required by the Reidemeister moves. The result is a rack structure on the product $X \times S$ defined via a map $\alpha : X \times X \to \mathrm{Maps}(S \times S, S)$ known as a *dynamical cocycle*. Dynamical cocycles were defined in [1] and used to construct extension racks; we will use dynamical cocycles satisfying an extra condition, which we call *N-reduced dynamical cocycles*, to define an enhancement of the rack counting invariant $\Phi_X^{\mathbb{Z}}$.

Definition 8. Let X be a finite rack of rack rank N and S be a finite set. The elements of S will be called *beads*. A map $\alpha : X \times X \to \mathrm{Maps}(S \times S, S)$ may be understood as a collection of binary operations $\cdot_{x,y} : S \times S \to S$ indexed by pairs of elements of X where where we write $a \cdot_{x,y} b = \alpha(x,y)(a,b)$. Such a map α is a *dynamical cocycle* on S if the maps satisfy:

(i) For all $x, y \in X$ and $b \in S$, the map $f_b^{x,y} : S \to S$ defined by $f_b^{x,y}(a) = a \cdot_{x,y} b$ is a bijection.

(ii) For all $x, y, z \in X$ and $a, b, c \in S$, we have

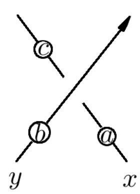
$$(a \cdot_{x,y} b) \cdot_{x \triangleright y, z} c = (a \cdot_{x,z} c) \cdot_{x \triangleright z, y \triangleright z} (b \cdot_{y,z} c).$$

Definition 9. Let X be a rack of rack rank N and $\alpha : X \times X \to \mathrm{Maps}(S \times S, S)$ a dynamical cocycle. Define $\rho_x : S \to S$ by $\rho_x(a) = a \cdot_{x,x} a$. Then if the diagram

$$\begin{tikzcd} M \arrow[r, "\rho_x"] \arrow[rrrr, bend right, "\mathrm{Id_M}"'] & M \arrow[r, "\rho_{\pi(x)}"] & M \arrow[r, dash] & \cdots \arrow[r, "\rho_{\pi^{N-1}(x)}"] & M \end{tikzcd}$$

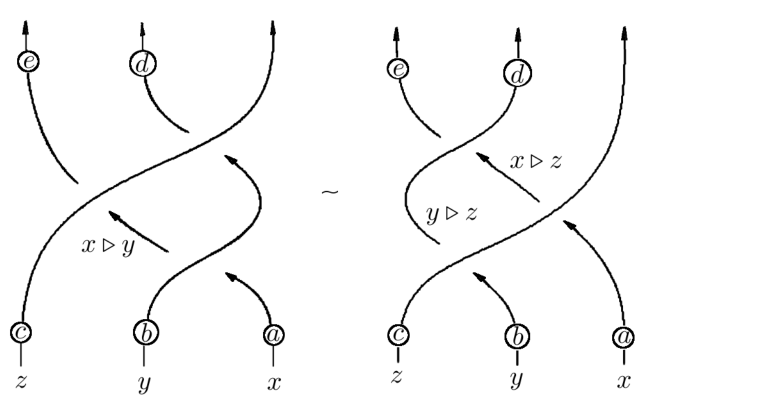
commutes for every $x \in X$ and $a \in S$, we say the cocycle α is N-*reduced.*

The definition of a dynamical cocycle is chosen so that bead labelings of an X-labeled diagram according to the rule



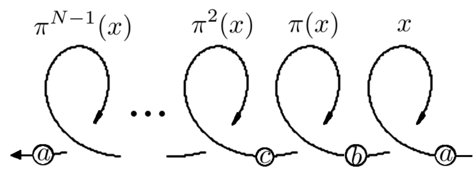
$$c = a \cdot_{x,y} b$$

are preserved under X-labeled framed oriented Reidemeister moves as shown below:



$$\begin{aligned} d &= b \cdot_{y,z} c \\ e &= (a \cdot_{x,y} b) \cdot_{x \triangleright y, z} c \end{aligned} \qquad \begin{aligned} d &= b \cdot_{y,z} c \\ e &= (a \cdot_{x,z} c) \cdot_{x \triangleright z, y \triangleright z} (b \cdot_{y,z} c) \end{aligned}$$

The Reidemeister II and framed type I moves require the operations $\cdot_{x,y} : S \times S \to S$ to be right-invertible; the N-reduced condition is required by the N-phone cord move:



$$\begin{aligned} b &= \rho_x(a) \\ c &= \rho_{\pi(x)}(b) = \rho_{\pi(x)}(\rho_x(a)) \\ &\vdots \\ a &= \rho_{\pi^N(x)}(\rho_{\pi^{N-1}(x)}(\dots(\rho_x(a))\dots)) \end{aligned}$$

Example 9. Let X be a finite rack and M an X-module as defined in Section 2. Then the operations

$$a \cdot_{x,y} b = t_{x,y}a + s_{x,y}b$$

define an N-reduced dynamical cocycle on M.

More generally, if X is a finite rack of cardinality n, we can describe a dynamical cocycle on a finite set $S = \{b_1, \dots, b_k\}$ with an $(nk) \times (nk)$ block matrix, $M_{x,y}$, encoding the operations tables for $\cdot_{x,y}$

$$M_{x,y} = \left[\begin{array}{c|c|c|c} M_{1,1} & M_{1,2} & \dots & M_{1,n} \\ \hline M_{2,1} & M_{2,2} & \dots & M_{2,n} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline M_{n,1} & M_{n,2} & \dots & M_{n,n} \end{array}\right]$$

where the (i,j)th entry of $M_{x,y}$ is l when $b_i \cdot_{x,y} b_j = b_l$.

Definition 10. Let X be a finite rack and α an N-reduced dynamical cocycle on a set S. For an X-labeled link diagram f, let $\mathcal{L}(f)$ be the set of S-labelings of f. Then we define the *N-reduced dynamical cocycle enhanced invariant* or *α-enhanced invariant* $\Phi_{X,\alpha}(L)$ by:

$$\Phi_{X,\alpha}(L) = \sum_{\mathbf{w}\in W} \left(\sum_{f\in \mathrm{Hom}(FR(L,\mathbf{w}))} u^{|\mathcal{L}(f)|} \right).$$

By construction, we have:

Theorem 1. *Let X be a finite rack and α an N-reduced dynamical cocycle on a set S. If L and L' are ambient isotopic links, then $\Phi_{X,\alpha}(L) = \Phi_{X,\alpha}(L')$.*

Remark 10. The α-enhanced invariant is well-defined for virtual knots by the usual convention of ignoring virtual crossings.

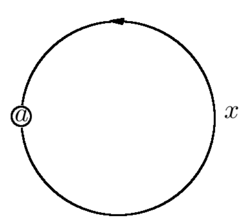
4. Computations and examples

In this section we present example computations of the N-reduced dynamical cocycle enhanced invariant.

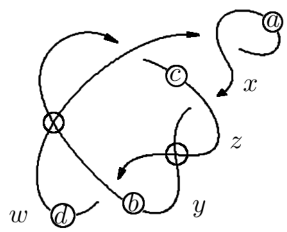
Example 11. Let X be the rack with rack matrix $M_X = \left[\begin{array}{cc} 2 & 2 \\ 1 & 1 \end{array}\right]$ and let α be the dynamical cocycle on $S = \{1, 2, 3\}$ given by the block matrix

$$M_\alpha = \left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 2 & 1 & 3 \\ 1 & 2 & 3 & 3 & 2 & 1 \\ 2 & 3 & 1 & 1 & 3 & 2 \\ \hline 2 & 1 & 3 & 3 & 1 & 2 \\ 3 & 2 & 1 & 1 & 2 & 3 \\ 1 & 3 & 2 & 2 & 3 & 1 \end{array}\right].$$

The virtual knots 3.7 and the unknot both have Jones polynomial 1 and integral rack counting invariant $\Phi_X^{\mathbb{Z}} = 2$. Let us compare $\Phi_{X,\alpha}(3.7)$ with $\Phi_{X,\alpha}(\text{Unknot})$. Since X has rank $N = 2$, we need to consider diagrams with writhes mod 2. The odd writhe diagrams have no valid X-labelings, and there are two valid X-labelings of the even writhe diagrams. We collect the valid bead labelings in the tables below.



x	a	x	a
1	1	2	1
1	2	2	2
1	3	2	3



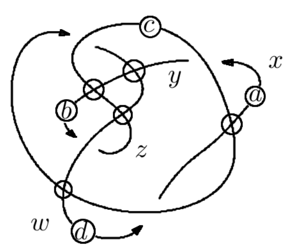
x	y	z	w	a	b	c	d	x	y	z	w	a	b	c	d
1	2	1	2	1	1	2	3	2	1	2	1	1	1	2	3
1	2	1	2	1	2	3	3	2	1	2	1	1	2	3	3
1	2	1	2	1	3	1	3	2	1	2	1	1	3	1	3
1	2	1	2	2	1	1	2	2	1	2	1	2	1	1	2
1	2	1	2	2	2	2	2	2	1	2	1	2	2	2	2
1	2	1	2	2	3	3	2	2	1	2	1	2	3	3	2
1	2	1	2	3	1	3	1	2	1	2	1	3	1	3	1
1	2	1	2	3	2	1	1	2	1	2	1	3	2	1	1
1	2	1	2	3	3	2	1	2	1	2	1	3	3	2	1

Hence, we have $\Phi_{X,\alpha}(3.7) = 2u^9 \neq 2u^3 = \Phi_{X,\alpha}(\text{Unknot})$ and $\Phi_{X,\alpha}$ is not determined by the Jones polynomial or the integral rack counting invariant $\Phi_X^{\mathbb{Z}}$.

Example 12. Similarly, the virtual knots 3.7 and 4.85 both have generalized Alexander polynomial

$$\Delta = (t^2 - 1)(s^2 - 1)(st - 1)$$

but are distinguished by $\Phi_{X,\alpha}$ with $\Phi_{X,\alpha}(3.7) = 2u^9 \neq 2u^3 = \Phi_{X,\alpha}(4.85)$ for the rack X and dynamical cocycle α from Example 11.



x	y	z	w	a	b	c	d	x	y	z	w	a	b	c	d
1	2	1	2	1	3	1	3	2	1	2	1	1	3	1	3
1	2	1	2	2	2	2	2	2	1	2	1	2	2	2	2
1	2	1	2	3	1	3	1	2	1	2	1	3	1	3	1

Hence, $\Phi_{X,\alpha}$ is not determined by the generalized Alexander polynomial.

Example 13. We randomly selected a small dynamical cocycle α on the set $S = \{1, 2, 3\}$ for the dihedral quandle X with matrices below:

$$M_X = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}, \qquad M_\alpha = \left[\begin{array}{ccc|ccc|ccc} 1 & 3 & 2 & 3 & 2 & 1 & 1 & 3 & 2 \\ 3 & 2 & 1 & 2 & 1 & 3 & 3 & 2 & 1 \\ 2 & 1 & 3 & 1 & 3 & 2 & 2 & 1 & 3 \\ \hline 3 & 2 & 1 & 1 & 3 & 2 & 2 & 1 & 3 \\ 2 & 1 & 3 & 3 & 2 & 1 & 1 & 3 & 2 \\ 1 & 3 & 2 & 2 & 1 & 3 & 3 & 2 & 1 \\ \hline 1 & 3 & 2 & 2 & 1 & 3 & 1 & 3 & 2 \\ 3 & 2 & 1 & 1 & 3 & 2 & 3 & 2 & 1 \\ 2 & 1 & 3 & 3 & 2 & 1 & 2 & 1 & 3 \end{array}\right].$$

We then computed $\Phi_{X,\alpha}$ for the list of prime classical knots with up to eight crossings and prime classical links with up to seven crossings as listed at the `knot atlas` [2]. The results are collected below. In particular, note that the invariant values $6+3u^9 \neq 9u^9$ both specailize to the same rack counting invariant value $\Phi_X^{\mathbb{Z}} = 9$, and we see that $\Phi_{X,\alpha}$ is not determined by $\Phi_X^{\mathbb{Z}}$.

$\Phi_{X,\alpha}(L)$	L
$3u^3$	Unknot, $4_1, 5_1, 5_2, 6_2, 6_3, 7_1, 7_2, 7_3, 7_5, 7_6, 8_1, 8_2, 8_3, 8_4, 8_6, 8_7, 8_8, 8_9, 8_{12}, 8_{13}, 8_{14}, 8_{16}, 8_{17}, L2a1, L4a1, L5a1, L6a2, L6a4, L6n1, L7a2, L7a3, L7a4, L7a6, L7a7, L7n1, L7n2$
$6+3u^9$	$3_1, 7_4, 7_7, 8_5, 8_{15}, 8_{19}, 8_{21}, L6a1, L6a3, L6a5, L7a1$
$9u^9$	$6_1, 8_{10}, 8_{11}, 8_{20}, L7a5$
$24+3u^{27}$	8_{18}

Our python results indicate that of the 116 prime virtual knots with up to 4 classical crossings listed at the knot atlas, $\Phi_{X,\alpha}$ for this α is $6+3u^9$ for the virtual knots 3.6, 3.7, 4.61, 4.61, 4.63, 4.64, 4.65, 4.66, 4.67, 4.68 and 4.98, $\Phi^{\alpha}_{X,S} = 9u^9$ for 4.99, and $\Phi_{X,\alpha} = 3u^3$ for the other virtual knots in the list.

Our `python` code for computing N-reduced dynamical cocycles and their link invariants is available at `www.esotericka.org`.

5. Questions for future research

In this section we collect a few questions for future research.

For a given pair of knots or links, how can we choose X and α to maximize the liklihood of $\Phi_{X,\alpha}$ distinguishing the knots or links in question? Is there an algorithm, perhaps starting with presentations of the fundamental racks of the knots, to construct a rack X and dynamical cocycle α such that $\Phi_{X,\alpha}$ always distinguishes inequivalent knots?

A natural direction of generalization is to look at knotted surfaces in $\mathbb{R}^4$, which have an integral quandle counting invariant which should be susceptible to enhancement by beads. What analog of the dynamical cocycle condition arises from the Roseman moves with beads on each sheet?

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