Isospectral towers of Riemannian manifolds

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ABSTRACT. In this paper we construct, for $n \geq 2$, arbitrarily large families of infinite towers of compact, orientable Riemannian n-manifolds which are isospectral but not isometric at each stage. In dimensions two and three, the towers produced consist of hyperbolic 2-manifolds and hyperbolic 3-manifolds, and in these cases we show that the isospectral towers do not arise from Sunada's method.

CONTENTS

1. Introduction

Let M be a closed Riemannian n-manifold. The eigenvalues of the Laplace–Beltrami operator acting on the space $L^2(M)$ form a discrete sequence of nonnegative real numbers in which each value occurs with a finite multiplicity. This collection of eigenvalues is called the spectrum of M, and two Riemannian n-manifolds are said to be *isospectral* if their spectra coincide. Inverse spectral geometry asks the extent to which the geometry and topology of M is determined by its spectrum. Whereas volume and scalar curvature are spectral invariants, the isometry class is not. Although there is a long history of constructing Riemannian manifolds which are isospectral but not isometric, we restrict our discussion to those constructions most

relevant to the present paper and refer the reader to [3] for an excellent survey.

In 1980 Vignéras [15] constructed, for every $n \geq 2$, pairs of isospectral but not isometric Riemannian n-manifolds. These manifolds were obtained as quotients of products $\mathbf{H}_2^a \times \mathbf{H}_3^b$ of hyperbolic upper-half planes and upper-half spaces by discrete groups of isometries obtained via orders in quaternion algebras. A few years later Sunada [12] described an algebraic method of producing isospectral manifolds, which he used to construct further examples of isospectral but not isometric Riemann surfaces. Sunada's method is extremely versatile and is responsible for the majority of the known examples of isospectral but not isometric Riemannian manifolds.

Before stating our main result we require some additional terminology. We will say that a collection $\{M_i\}_{i=0}^{\infty}$ of Riemannian manifolds is a tower if for every $i \geq 0$ there is a finite cover $M_{i+1} \to M_i$. We shall say that two infinite towers $\{M_i\}$ and $\{N_i\}$ of Riemannian n-manifolds are *isospectral towers* if for every $i \geq 0$, M_i and N_i are isospectral but not isometric.

Theorem 1.1. *For all $m, n \geq 2$ there exist infinitely many families of isospectral towers of compact Riemannian n-manifolds, each family being of cardinality m.*

Like Vignéras' examples, our towers arise as quotients of products of hyperbolic upper-half planes and spaces so that in dimension two (respectively dimension three) the manifolds produced are hyperbolic 2-manifolds (respectively hyperbolic 3-manifolds). It follows from the Mostow Rigidity Theorem (see [11], [8, page 69]) that in dimension three, the fundamental groups of the manifolds at each stage of the isospectral towers are nonisomorphic.

The proof of Theorem 1.1 is entirely number-theoretic and relies on the construction of suitable chains of quaternion orders. To our knowledge the only constructions of isospectral towers are given by McReynolds [9], who used a variant of Sunada's method to produce manifolds as quotients of symmetric spaces associated to noncompact Lie groups. Although McReynolds' methods apply in our setting, our results are of independent interest. Not only do we produce arbitrarily large families of isospectral towers, but we additionally show that in dimensions two and three our isospectral towers cannot arise from Sunada's method.

Theorem 1.2. *In dimensions two and three the isospectral towers constructed in Theorem 1.1 do not arise from Sunada's method.*

Chen [1] has shown that Vignéras' two and three dimensional examples of isospectral but not isometric hyperbolic manifolds cannot arise from Sunada's method. Chen's proof relies on the fact that Vignéras' manifolds are obtained from maximal orders in quaternion algebras. We are able to extend Chen's argument to the nonmaximal orders contained in our chains of quaternion orders by employing Jørgensen and Thurston's proof [13, 10]

that the set of volumes of hyperbolic 3-orbifolds comprise a well-ordered subset of the nonnegative real numbers.

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2. Genera of quaternion orders

We begin with some facts concerning genera of quaternion orders. Most of these facts are well-known and can be found in [15, 8]; for proofs of the others, see Section 3 of [7].

Let K be a number field and B/K a quaternion algebra. We denote by $\mathrm{Ram}(B)$ the set of primes at which B ramifies, by $\mathrm{Ram}_f(B)$ the set of finite primes of $\mathrm{Ram}(B)$ and by $\mathrm{Ram}_\infty(B)$ the set of archimedean primes of $\mathrm{Ram}(B)$. If R is a subring of B then we denote by R^1 the multiplicative group consisting of the elements of R having reduced norm one. For a prime ν of K (finite or infinite) we denote by B_ν the quaternion algebra $B \otimes_K K_\nu$. Let $\mathcal{R}$ be an $\mathcal{O}_K$-order of B. If ν is a finite prime of K, denote by $\mathcal{R}_\nu$ the $\mathcal{O}_{K_\nu}$-order $\mathcal{R} \otimes_{\mathcal{O}_K} \mathcal{O}_{K_\nu}$ of B_ν.

Two orders $\mathcal{R}, \mathcal{S}$ of B are said to lie in the same genus if $\mathcal{R}_\nu \cong \mathcal{S}_\nu$ for all finite primes ν of K. The genus of $\mathcal{R}$ is the disjoint union of isomorphism classes and it is known that the number of isomorphism classes in the genus of $\mathcal{R}$ is finite (and is in fact a power of 2 whenever there exists an archimedean prime of K not lying in $\mathrm{Ram}_\infty(B)$). When $\mathcal{R}$ is a maximal order of B, the number of isomorphism classes in the genus of $\mathcal{R}$ is called the type number of B.

The genus of an order $\mathcal{R}$ is profitably characterized adelically. Let J_K be the idele group of K and J_B the idele group of B. The genus of $\mathcal{R}$ is characterized by the coset space $J_B/\mathfrak{N}(\mathcal{R})$ where $\mathfrak{N}(\mathcal{R}) = J_B \cap \prod_\nu N(\mathcal{R}_\nu)$ and $N(\mathcal{R}_\nu)$ is the normalizer of $\mathcal{R}_\nu$ in $B_\nu^\times$. The isomorphism classes in the genus of $\mathcal{R}$ then correspond to double cosets of $B^\times \backslash J_B/\mathfrak{N}(\mathcal{R})$. This correspondence is given as follows. Suppose that $\mathcal{S}$ is in the genus of $\mathcal{R}$ and let $\tilde{x} = (x_\nu) \in J_B$ be such that $x_\nu \mathcal{S}_\nu x_\nu^{-1} = \mathcal{R}_\nu$. Then the isomorphism class of $\mathcal{S}$ corresponds to the double coset $B^\times \tilde{x} \mathfrak{N}(\mathcal{R})$.

The reduced norm induces a bijection

$$n : B^\times \backslash J_B/\mathfrak{N}(\mathcal{R}) \to K^\times \backslash J_K/n(\mathfrak{N}(\mathcal{R})).$$

Define $H_\mathcal{R} = K^\times n(\mathfrak{N}(\mathcal{R}))$ and $G_\mathcal{R} = J_K/H_\mathcal{R}$. As J_K is abelian there is an isomorphism $G_\mathcal{R} \cong K^\times \backslash J_K/n(\mathfrak{N}(\mathcal{R}))$. The group $H_\mathcal{R}$ is an open subgroup of J_K having finite index, so that by class field theory there exists a class field $K(\mathcal{R})$ associated to it. The extension $K(\mathcal{R})/K$ is an abelian extension of exponent 2 whose conductor is divisible only by the primes dividing the level ideal of $\mathcal{R}$ (that is, the primes ν such that $\mathcal{R}_\nu$ is not a maximal order of B_ν) and whose Galois group is isomorphic to $G_\mathcal{R}$. In the case that $\mathcal{R}$ is maximal we will often write $K(B)$ in place of $K(\mathcal{R})$. Note that $K(B)$ can

be characterized as the maximal abelian extension of K of exponent 2 which is unramified outside the real places of $\mathrm{Ram}(B)$ and in which all primes of $\mathrm{Ram}_f(B)$ split completely [2, page 39].

Given orders $\mathcal{S}, \mathcal{T}$ lying in the genus of $\mathcal{R}$, define the $G_{\mathcal{R}}$-*valued distance idele* $\rho_{\mathcal{R}}(\mathcal{S}, \mathcal{T})$ as follows. Let $\tilde{x}_{\mathcal{S}}, \tilde{x}_{\mathcal{T}} \in J_B$ be such that $x_{\mathcal{S}_\nu} \mathcal{S}_\nu x_{\mathcal{S}_\nu}^{-1} = \mathcal{R}_\nu$ and $x_{\mathcal{T}_\nu} \mathcal{T}_\nu x_{\mathcal{T}_\nu}^{-1} = \mathcal{R}_\nu$ for all ν. We define $\rho_{\mathcal{R}}(\mathcal{S}, \mathcal{T})$ to be the coset $n(\tilde{x}_{\mathcal{S}}^{-1} \tilde{x}_{\mathcal{T}}) H_{\mathcal{R}}$ in $G_{\mathcal{R}}$. The function $\rho_{\mathcal{R}}(-, -)$ is well-defined and has the property that $\rho_{\mathcal{R}}(\mathcal{S}, \mathcal{T})$ is trivial if and only if $\mathcal{S} \cong \mathcal{T}$ [7, Prop. 3.6].

3. Arithmetic groups derived from quaternion algebras

In this section we recall some facts about arithmetic groups derived from orders in quaternion algebras; see [8] for proofs.

Let K be a number field with r_1 real primes and r_2 complex primes so that $[K : \mathbb{Q}] = r_1 + 2r_2$, and denote by V_∞ the set of archimedean primes of K. Let B/K a quaternion algebra and consider the isomorphism

$$B \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H}^r \times M_2(\mathbb{R})^s \times M_2(\mathbb{C})^{r_2} \qquad (r + s = r_1).$$

There exists an embedding

$$\rho : B^\times \hookrightarrow \prod_{\nu \in V_\infty \backslash \mathrm{Ram}_\infty(B)} B_\nu^\times \longrightarrow G = GL_2(\mathbb{R})^s \times GL_2(\mathbb{C})^{r_2}.$$

Restricting ρ to B^1 gives an embedding $\rho : B^1 \hookrightarrow G^1 = SL_2(\mathbb{R})^s \times SL_2(\mathbb{C})^{r_2}$. Let N be a maximal compact subgroup of G^1 so that $X := G^1/N = \mathbf{H}_2^s \times \mathbf{H}_3^{r_2}$ is a product of two and three dimensional hyperbolic spaces.

If $\mathcal{R}$ is an order of B then $\rho(\mathcal{R}^1)$ is a discrete subgroup of G^1 of finite covolume. The orbifold $\rho(\mathcal{R}^1) \backslash X$ is compact if and only if B is a division algebra and will be a manifold if $\rho(\mathcal{R}^1)$ contains no elliptic elements.

4. Isospectral towers and chains of quaternion orders

As was the case with Vignéras' [15] examples of Riemannian manifolds which are isospectral but not isometric, our construction will make use of the theory of orders in quaternion algebras. In particular, our isospectral towers will arise from chains of quaternion orders.

Theorem 4.1. *Let K be a number field and B/K a quaternion algebra of type number $t \geq 2$. Then there exist infinitely many families of chains of quaternion orders $\{\{\mathcal{R}_i^j\} : 1 \leq j \leq t\}$ of B satisfying:*

(1) *For all $i \geq 0$ and $1 \leq j \leq t$, $\mathcal{R}_0^j$ is a maximal order and $\mathcal{R}_{i+1}^j \subsetneq \mathcal{R}_i^j$.*
(2) *For all $i \geq 0$ and $1 \leq j_1 < j_2 \leq t$, $\mathcal{R}_i^{j_1}$ and $\mathcal{R}_i^{j_2}$ are in the same genus but are not conjugate.*
(3) *For all $i \geq 0$ and $1 \leq j \leq t$, there is an equality of class fields $K(\mathcal{R}_i^j) = K(B)$.*

Before proving Theorem 4.1, we show that it implies our main result. We will make use of the following proposition.

Proposition 4.2. *Suppose that* $\mathrm{Ram}_f(B)$ *is not empty and that there is an archimedean prime of* K *not lying in* $\mathrm{Ram}_\infty(B)$. *Let* L *be a maximal subfield of* B *and* $\Omega \subset L$ *a quadratic* $\mathcal{O}_K$-*order. If* $\{\{\mathcal{R}_i^j\} : 1 \leq j \leq t\}$ *is a family of chains of quaternion orders of* B *satisfying conditions* (1)–(3) *of Theorem* 4.1, *then for every* $i \geq 0$ *and* $1 \leq j_1 < j_2 \leq t$, $\mathcal{R}_i^{j_1}$ *admits an embedding of* Ω *if and only if* $\mathcal{R}_i^{j_2}$ *admits an embedding of* Ω.

Proof. Our hypothesis that $\mathrm{Ram}_f(B)$ be nonempty means that there exists a finite prime of K which ramifies in B. Such a prime splits completely in $K(B)/K$ and thus in every subfield of $K(B)$ as well. It follows that $L \not\subset K(B)$, for otherwise would contradict the Albert–Brauer–Hasse–Noether theorem. Suppose now that there exists an $i \geq 0$ and $1 \leq j_1 \leq t$ such that $\mathcal{R}_i^{j_1}$ admits an embedding of Ω. By condition (3), $L \not\subset K(\mathcal{R}_i)$, hence every order in the genus of $\mathcal{R}_i^{j_1}$ admits an embedding of Ω [7, Theorem 5.7]. In particular, $\mathcal{R}_i^{j_2}$ admits an embedding of Ω for all $1 \leq j_2 \leq t$. $\qquad\square$

Theorem 4.3. *For all* $m, n \geq 2$ *there exist infinitely many families of isospectral towers of compact Riemannian* n-*manifolds, each family being of cardinality* m.

Proof. Fix integers $m, n \geq 2$. Vignéras [14, Theorem 8] proved the existence of a number field K and a quaternion algebra B/K with the following properties:

- $\mathrm{Ram}_f(B)$ is nonempty.
- The type number of B is at least 2.
- The only $\mathbb{Q}$-automorphisms of B are inner automorphisms.
- If $\mathcal{O}$ is an order in B then $\rho(\mathcal{O}^1) \backslash (\mathbf{H}_2^s \times \mathbf{H}_3^{r_2})$ is a Riemannian n-manifold.

The proof of Vignéras' theorem is easily modified to construct quaternion algebras B having arbitrarily large type number. In particular we will assume that B satisfies the above properties and has type number greater than or equal to m. Because the type number of B is a power of 2, we may assume without loss of generality that the type number t of B is equal to m and write $t = m = 2^\ell$.

Let $\{\{\mathcal{R}_i^j\} : 1 \leq j \leq t\}$ be a family of chains of quaternion orders as in Theorem 4.1. For $1 \leq j \leq t$, denote by $\{M_i^j\}$ the infinite tower of finite covers of Riemannian n-manifolds associated to the $\{\mathcal{R}_i^j\}$. By Theorem 4.3.5 of [15] and the discussion immediately afterwords, that condition (2) of Theorem 4.1 is satisfied implies that $M_i^{j_1}$ and $M_i^{j_2}$ are not isometric for any $i \geq 0$ and $1 \leq j_1 < j_2 \leq t$.

It remains only to show that for all $i \geq 0$ and $1 \leq j_1 < j_2 \leq t$, the manifolds $M_i^{j_1}$ and $M_i^{j_2}$ are isospectral. To ease notation set $\mathcal{O}_1 = \mathcal{R}_i^{j_1}$ and $\mathcal{O}_2 = \mathcal{R}_i^{j_2}$. By [14, Theorem 6], it suffices to show that $\mathcal{O}_1^1$ and $\mathcal{O}_2^1$ have the same number of conjugacy classes of elements of a given reduced trace.

If $h \in \mathcal{O}_1^1$, then the number of conjugacy classes of elements of $\mathcal{O}_1$ with reduced trace $\mathrm{tr}(h)$ is equal to the number of embeddings of $\mathcal{O}_K[h]$ into $\mathcal{O}_1$ modulo the action of $\mathcal{O}_1^1$. It is well known that this quantity, when nonzero, is independent of the choice of order in the genus of $\mathcal{O}_1$. This can be proven as follows. Let Ω be a quadratic $\mathcal{O}_K$-order in a maximal subfield L of B and note that every embedding of Ω into $\mathcal{O}_1$ extends to an optimal embedding of some overorder Ω^* of Ω into $\mathcal{O}_1$. Conversely, if Ω^* is an overorder of Ω, then every optimal embedding of Ω^* into $\mathcal{O}_1$ restricts to an embedding of Ω into $\mathcal{O}_1$. It therefore suffices to show that if $\mathcal{O}_2$ admits an embedding of Ω, then the number of optimal embeddings of Ω into $\mathcal{O}_1$ modulo $\mathcal{O}_1^1$ is equal to the number of optimal embeddings of Ω into $\mathcal{O}_2$ modulo $\mathcal{O}_2^1$. This can be proven adelically using the same proof that Vignéras employed in the case of Eichler orders [15, Theorem 3.5.15].

By the previous paragraph, in order to prove that $M_i^{j_1}$ and $M_i^{j_2}$ are isospectral, it suffices to prove that if Ω is a quadratic order in a maximal subfield L of B, then Ω embeds into $\mathcal{R}_i^{j_1}$ if and only if Ω embeds into $\mathcal{R}_i^{j_2}$. This follows from Proposition 4.2. $\qquad\square$

5. Proof of Theorem 4.1

5.1. Orders in split quaternion algebras over local fields.
We begin by defining a family of orders in the quaternion algebra $M_2(k)$, k a nondyadic local field.

Let k be a nondyadic local field with ring of integers $\mathcal{O}_k$ and maximal ideal $P_k = \pi_k \mathcal{O}_k$. Let F/k be the unramified quadratic field extension of k with ring of integers $\mathcal{O}_F$ and maximal ideal $P_F = \pi_F \mathcal{O}_F$. We may choose π_F so that $\pi_F = \pi_k$.

Let $\mathfrak{B} = M_2(k)$ and $\xi \in \mathfrak{B}$ be such that $\mathfrak{B} = F + \xi F$ and $x\xi = \xi \overline{x}$ for all $x \in F$. Let $m \geq 0$. Following Jun [6], we define a family of orders in $\mathfrak{B}$:

$$(5.1) \qquad\qquad R_{2m}(F) = \mathcal{O}_F + \xi \pi_F^m \mathcal{O}_F.$$

For convenience we will denote $R_{2m}(F)$ by R_{2m}. It is easy to see that R_0 is a maximal order of $\mathfrak{B}$ and that we have inclusions:

$$\cdots \subsetneq R_{2m+2} \subsetneq R_{2m} \subsetneq \cdots \subsetneq R_2 \subsetneq R_0.$$

Proposition 5.1. *If $m \geq 1$, then $n(N(R_{2m})) = \mathcal{O}_k^\times k^{\times 2}$, where $N(R_{2m})$ is the normalizer of R_{2m} in $\mathfrak{B}^\times$.*

Proof. Suppose that $x \in N(R_{2m})$. Then $x\mathcal{O}_F x^{-1} \subseteq R_{2m} \subseteq R_0$ because $\mathcal{O}_F$ is contained in R_{2m}. Similarly, $x\pi_F^m \xi \mathcal{O}_F x^{-1} \subseteq R_{2m}$. Because $\pi_F = \pi_k$ is in the center of $\mathfrak{B}$, it follows that $x\xi \mathcal{O}_F x^{-1} \subseteq \pi_F^{-m} R_{2m} = \pi_F^{-m} \mathcal{O}_F + \xi \mathcal{O}_F$.

We may write an arbitrary element of $x\xi \mathcal{O}_F x^{-1}$ as $\pi_F^{-m} a + \xi b$, where $a, b \in \mathcal{O}_F$. It is straightforward to show that $n(\pi_F^{-m} a + \xi b) = n(\pi_F^{-m} a) - n(b)$. As b is an element of $\mathcal{O}_F$ we see that $n(b)$ is integral. Similarly, $n(\pi_F^{-m} a + \xi b)$ must be integral. This follows from observing that $\pi_F^{-m} a + \xi b \in x\xi \mathcal{O}_F x^{-1}$

is integral since every element of $\xi\mathcal{O}_F$ is integral and conjugation preserves integrality. Therefore $n(\pi_F^{-m}a)$ is integral, hence $a \in P_F^m$. We have shown that every element of $x\xi\mathcal{O}_F x^{-1}$ lies in $\mathcal{O}_F + \xi\mathcal{O}_F = R_0$.

If $x \in N(R_{2m})$ then $x\mathcal{O}_F x^{-1}, x\xi\mathcal{O}_F x^{-1} \subseteq R_0$. Therefore

$$xR_0 x^{-1} = x\left(\mathcal{O}_F + \xi\mathcal{O}_F\right)x^{-1} \subseteq R_0.$$

Equivalently, $x \in N(R_0)$ hence $N(R_{2m}) \subseteq N(R_0)$. Recalling that R_0 is conjugate to $M_2(\mathcal{O}_k)$, whose normalizer is $GL_2(\mathcal{O}_k)k^\times$, we deduce that $n(N(R_{2m})) \subseteq \mathcal{O}_k^\times k^{\times^2}$. To show the reverse inclusion we note that both $\mathcal{O}_F^\times$ and $k^\times$ lie in $N(R_{2m})$ and that $n(\mathcal{O}_F^\times k^\times) = \mathcal{O}_k^\times k^{\times^2}$ (since F/k is unramified). $\qquad\square$

Proposition 5.2. *If $\mathcal{O} = \mathcal{O}_k[R_{2m}^1]$ is the ring generated over $\mathcal{O}_k$ by R_{2m}^1, then $\mathcal{O} = R_{2m}$.*

Proof. Consider the map $f : R_{2m}^\times \to \mathcal{O}_k^\times/\mathcal{O}_k^{\times^2}$ induced by composing the reduced norm on $R_{2m}^\times$ with the projection $\mathcal{O}_k^\times \to \mathcal{O}_k^\times/\mathcal{O}_k^{\times^2}$. This map is surjective as $n(\mathcal{O}_F^\times) = \mathcal{O}_k^\times$ and $\mathcal{O}_F \subset R_{2m}$. The kernel of f is the disjoint union of the cosets gR_{2m}^1 as g varies over $\mathcal{O}_k^\times$. Because k is a nondyadic local field, $\mathcal{O}_k^\times/\mathcal{O}_k^{\times^2}$ has order 2. As $\mathcal{O}^\times$ contains the kernel of f as well as elements of nonsquare reduced norm (this follows from the fact that $\mathcal{O}^\times$ contains a k-basis for $\mathfrak{B}$. Fix an element of $\mathfrak{B}$ of nonsquare reduced norm and write it as a k-linear combination of this basis, then clear denominators by multiplying through by an element of $\mathcal{O}_k^2$. The resulting element will lie in $\mathcal{O}^\times$ and have a nonsquare reduced norm), $\mathcal{O}^\times = R_{2m}^\times$. Using equation (5.1) and the observation that one may take as ξ the matrix with zeros on the diagonal and ones on the anti-diagonal, it is easy to see that R_{2m} has an $\mathcal{O}_k$-basis consisting entirely of elements of $R_{2m}^\times$, hence $R_{2m} \subset \mathcal{O}$. As the opposite inclusion is immediate, we conclude that $\mathcal{O} = R_{2m}$. $\qquad\square$

5.2. Proof of Theorem 4.1. Let notation be as above and assume that B has type number $t = 2^\ell$ with $\ell \geq 1$. Let $\mathcal{R}$ be a fixed maximal order of B. We will prove Theorem 4.1 in the case that $\ell = 2$ as the other cases are similar though in general more tedious in terms of notation.

Because the type number of B is 4 there is an isomorphism

$$\mathrm{Gal}(K(B)/K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Let μ_1, μ_2 be primes of K such that the Frobenius elements $(\mu_1, K(B)/K)$ and $(\mu_2, K(B)/K)$ generate $\mathrm{Gal}(K(B)/K)$. By the Chebotarev density theorem we may choose μ_1 and μ_2 from sets of primes of K having positive density.

Let $\delta_i = \mathrm{diag}(\pi_{K_{\mu_i}}, 1)$ and F_i be the unramified quadratic field extension of K_{μ_i} (for $i = 1, 2$).

For any $i \geq 0$ and $0 \leq a, b \leq 1$ define the orders $\mathcal{R}_i^{a,b}$ via:

$$(\mathcal{R}_i^{a,b})_\nu = \begin{cases} \mathcal{R}_\nu & \text{if } \nu \notin \{\mu_1, \mu_2\}, \\ \delta_1^a R_{2i}(F_1)\delta_1^{-a} & \text{if } \nu = \mu_1, \\ \delta_2^b R_{2i}(F_2)\delta_2^{-b} & \text{if } \nu = \mu_2. \end{cases}$$

We now set $\mathcal{R}_i^1 = \mathcal{R}_i^{0,0}, \mathcal{R}_i^2 = \mathcal{R}_i^{1,0}, \mathcal{R}_i^3 = \mathcal{R}_i^{0,1}$ and $\mathcal{R}_i^4 = \mathcal{R}_i^{1,1}$.

Having constructed our chains $\{\mathcal{R}_i^1\}, \{\mathcal{R}_i^2\}, \{\mathcal{R}_i^3\}, \{\mathcal{R}_i^4\}$, we show that they satisfy the required properties. That $\mathcal{R}_0^j$ is a maximal order and $\mathcal{R}_{i+1}^j \subsetneq \mathcal{R}_i^j$ for all $i \geq 0$ and $1 \leq j \leq 4$ is immediate. We claim that $K(\mathcal{R}_i^j) = K(B)$ for all $i \geq 0$ and $1 \leq j \leq 4$. Indeed, as $n(N(\mathcal{R}_{i\,\nu}^j)) = n(N(\mathcal{R}_\nu))$ for all primes ν of K (by Proposition 5.1), it is immediate that $H_{\mathcal{R}_i^j} = K^\times n(\mathfrak{N}(\mathcal{R}_i^j))$ is equal to $H_\mathcal{R} = K^\times n(\mathfrak{N}(\mathcal{R}))$. It follows that $K(\mathcal{R}_i^j) = K(\mathcal{R})$ and because $K(\mathcal{R}) = K(B)$ by definition, we have proven our claim.

By construction $\mathcal{R}_i^1, \mathcal{R}_i^2, \ldots, \mathcal{R}_i^4$ all lie in the same genus. That they represent distinct isomorphism classes can be proven by considering the $G_{\mathcal{R}_i^1}$-valued distance idele $\rho_{\mathcal{R}_i^1}(-,-)$. For instance, to show that $\mathcal{R}_i^1 \not\cong \mathcal{R}_i^2$ note that

$$\rho_{\mathcal{R}_i^1}(\mathcal{R}_i^1, \mathcal{R}_i^2) = n(\tilde{x}_{\mathcal{R}_i^2}) H_{\mathcal{R}_i^1} = e_{\mu_1} H_{\mathcal{R}_i^1},$$

where $e_{\mu_1} = (1, \ldots, 1, \pi_{K_{\mu_1}}, 1, \ldots) \in J_K$. This corresponds, under the Artin reciprocity map, to the element $(\mu_1, K(B)/K) \in \mathrm{Gal}(K(B)/K)$, which is nontrivial by choice of μ_1. This shows that properties (1)–(3) are satisfied, concluding our proof.

6. The Sunada construction

In this section we prove that in dimensions two and three the isospectral towers constructed in Theorem 4.3 do not arise from Sunada's method [12]. Our proof follows Chen's proof [1] that the two and three dimensional examples of isospectral but not isometric Riemannian manifolds that Vignéras constructed in [14] do not arise from Sunada's construction, though several difficulties arise from our introduction of nonmaximal quaternion orders.

We begin by reviewing Sunada's construction. Let H be a finite group with subgroups H_1 and H_2. We say that H_1 and H_2 are *almost conjugate* if for every $h \in H$,

$$\#([h] \cap H_1) = \#([h] \cap H_2),$$

where $[h]$ denotes the H-conjugacy class of h. In this case we call (H, H_1, H_2) a *Sunada triple*.

Let M_1 and M_2 be Riemannian manifolds and (H, H_1, H_2) a Sunada triple. If M is a Riemannian manifold then as in Chen [1] we say that M_1 and M_2 are *sandwiched* between M and M/H with triple (H, H_1, H_2) if there is an embedding of H into the group of isometries of M such that the projections $M \to M/H_i$ are Riemannian coverings and M_i is isometric

to M/H_i (for $i = 1, 2$). The following theorem of Sunada [12] provides a versatile means of producing isospectral Riemannian manifolds.

Theorem 6.1 (Sunada). *If M_1 and M_2 are Riemannian manifolds which can be sandwiched with a triple (H, H_1, H_2) then M_1 and M_2 are isospectral.*

The majority of the known examples of isospectral but not isometric Riemannian manifolds have been constructed by means of Sunada's theorem and its variants (see [4] for a survey of work inspired by Sunada's theorem). We will show that in dimensions two and three, none of the manifolds produced in Theorem 4.3 arise from Sunada's construction.

Let $\{M_i\}, \{N_i\}$ be isospectral towers of Riemannian n-manifolds as constructed in Theorem 4.3. Recall that these towers were constructed from chains $\{\mathcal{R}_i\}, \{\mathcal{S}_i\}$ of orders in a quaternion algebra B defined over a number field K. These towers will consist of hyperbolic 2-manifolds or hyperbolic 3-manifolds if and only if B is unramified at a unique archimedean prime of K.

Theorem 6.2. *If B is unramified at a unique archimedean prime of K then for every $i \geq 0$ the isospectral hyperbolic manifolds M_i and N_i do not arise from Sunada's construction.*

Our proof of Theorem 6.2 will require the following lemma.

Lemma 6.3. *Let $\mathcal{O} \in \{\mathcal{R}_i : i \geq 0\}$ and $J \supseteq \mathcal{O}^1$ be a subgroup of B^1. If $[J : \mathcal{O}^1] < \infty$ then $J \subseteq \mathcal{R}_0^1$.*

Proof. Denote by $\mathcal{J} = \mathcal{O}_K[J]$ the ring generated over $\mathcal{O}_K$ by J. Write $J = \bigcup_{j=1}^{\infty} \{g_j \mathcal{O}^1\}$ and $\mathcal{J} = \sum \mathcal{O}_K \{g_j \mathcal{O}^1\}$. As $[J : \mathcal{O}^1] < \infty$, $\mathcal{J}$ is a finitely generated $\mathcal{O}_K$-module containing $\mathcal{O}_k[\mathcal{O}^1]$ and hence an order of B. Let $\mathcal{M}$ be a maximal order of B containing $\mathcal{J}$. Then $\mathcal{O}^1 \subseteq \mathcal{M} \cap \mathcal{R}_0$, hence $\mathcal{O}_\nu \subseteq \mathcal{M}_\nu \cap \mathcal{R}_{0\nu}$ for all primes ν of K. Recall from Section 5.2 that the chain $\{\mathcal{R}_i\}$ was constructed using a finite set of primes $\{\mu_1, \ldots, \mu_t\}$ of K so that for every $i \geq 0$, $\mathcal{R}_{i\nu}$ is maximal if $\nu \notin \{\mu_1, \ldots, \mu_t\}$ and $\mathcal{R}_{i\mu_j}$ is conjugate to $R_{2i}(F_j)$ for $j = 1, \ldots, t$. We will prove the lemma in the case that $\mathcal{R}_{i\mu_j} = R_{2i}(F_j)$ for $j = 1, \ldots, t$; the other cases follow from virtually identical proofs upon appropriately conjugating every appearance of $R_{2i}(F_j)$.
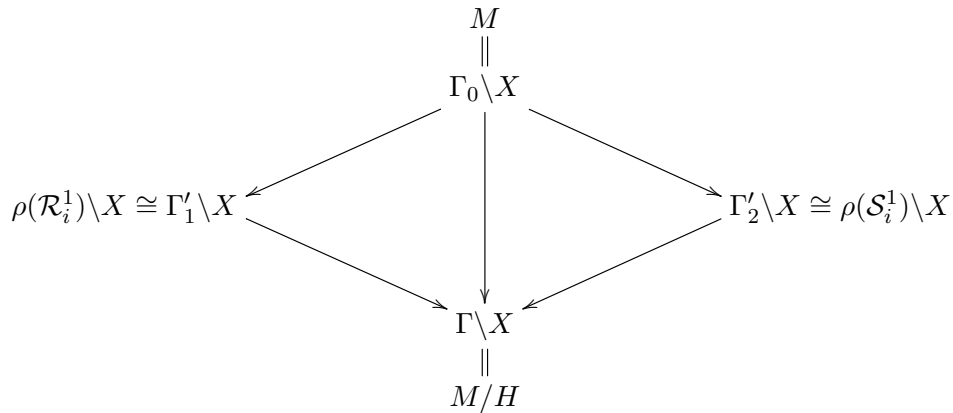
If $\nu \notin \{\mu_1, \ldots, \mu_t\}$ then $\mathcal{M}_\nu = \mathcal{R}_{0\nu} = \mathcal{O}_\nu$. At a prime $\mu \in \{\mu_1, \ldots, \mu_t\}$ we see that $\mathcal{O}_\mu^1 \subseteq \mathcal{M}_\mu \cap \mathcal{R}_{0\mu}$, hence $\mathcal{O}_\mu \subseteq \mathcal{M}_\mu \cap \mathcal{R}_{0\mu}$ by Proposition 5.2. Therefore $\mathcal{M}_\mu$ contains $\mathcal{O}_F$ (since $\mathcal{O}_F \subset R_{2i}(F) = \mathcal{O}_\mu$). By Proposition 2.2 of [6] there is a unique maximal order of B_μ containing $\mathcal{O}_F$, hence $\mathcal{M}_\mu = \mathcal{R}_{0\mu}$. We have shown that for all ν, $\mathcal{M}_\nu = \mathcal{R}_{0\nu}$, hence $\mathcal{M} = \mathcal{R}_0$. As $J \subset \mathcal{J}^1 \subseteq \mathcal{M}^1 = \mathcal{R}_0^1$, we have proven the lemma. $\square$

Remark 6.4. By conjugating every appearance of $\mathcal{R}_i$ and $\mathcal{R}_0$ in the proof of Lemma 6.3 one obtains the following: Let $x \in B^\times$, $\mathcal{O} \in \{x\mathcal{R}_i x^{-1} : i \geq 0\}$ and $J \supseteq \mathcal{O}^1$ be a subgroup of B^1. If $[J : \mathcal{O}^1] < \infty$ then $J \subseteq x\mathcal{R}_0^1 x^{-1}$.

We now prove Theorem 6.2. Write $M_i = \rho(\mathcal{R}_i^1)\backslash X$ and $N_i = \rho(\mathcal{S}_i^1)\backslash X$, where X is either two or three dimensional hyperbolic space (depending on whether the archimedean prime of K which is unramified in B is real or complex). Finally, suppose that M_i and N_i arose from Sunada's construction. That is, there exists a Riemannian manifold M and a Sunada triple (H, H_1, H_2) such that M_i and N_i are sandwiched between M and M/H. Let Γ be the discrete subgroup of $G = GL_2(\mathbb{R})^s \times GL_2(\mathbb{C})^{r_2}$ such that $M/H = \Gamma\backslash X$.

Recall that the set of volumes of hyperbolic 3-orbifolds comprise a well-ordered subset of the real numbers [13, 5, 10] and that an analogous statement holds for hyperbolic 2-orbifolds by the Riemann–Hurwitz formula. We may therefore choose the Sunada triple (H, H_1, H_2) and the Riemannian manifold M so that the following property is satisfied: If (H', H_1', H_2') is a Sunada triple and M' a Riemannian manifold such that M_i and N_i are sandwiched between M' and M'/H', then writing $M'/H' = \Gamma'\backslash X$ we have that $\mathrm{CoVol}(\Gamma' \cap \rho(B^1)) \geq \mathrm{CoVol}(\Gamma \cap \rho(B^1))$.

We have the following diagram of coverings:

$$
\begin{array}{ccc}
 & M & \\
 & \| & \\
 & \Gamma_0\backslash X & \\
\end{array}
$$

$$\rho(\mathcal{R}_i^1)\backslash X \cong \Gamma_1'\backslash X \qquad\qquad \Gamma_2'\backslash X \cong \rho(\mathcal{S}_i^1)\backslash X$$

$$
\begin{array}{c}
\Gamma\backslash X \\
\| \\
M/H
\end{array}
$$

where $\Gamma_0 \subseteq \Gamma_1', \Gamma_2' \subseteq \Gamma$ are discrete subgroups of G. We have $\Gamma_1' = \gamma_1 \rho(\mathcal{R}_i^1)\gamma_1^{-1}$ and $\Gamma_2' = \gamma_2 \rho(\mathcal{S}_i^1)\gamma_2^{-1}$ for some $\gamma_1, \gamma_2 \in G^\times$. By conjugating if necessary, we may assume that $\gamma_1 = \mathrm{id}$. Since Γ_0 is of finite index in both $\rho(\mathcal{R}_i^1)$ and $\gamma_2 \rho(\mathcal{S}_i^1)\gamma_2^{-1}$, by Lemma 4 of [1] there is an element $g \in \rho(B^\times)$ such that $\gamma_2 \rho(\mathcal{S}_i^1)\gamma_2^{-1} = g\rho(\mathcal{S}_i^1)g^{-1}$. If Γ is of finite index over $\rho(\mathcal{R}_i^1)$ then $\Gamma \cap \rho(B^1)$ is of finite index over $\rho(\mathcal{R}_i^1)$ and by Lemma 6.3, $\Gamma \cap \rho(B^1) \subseteq \rho(\mathcal{R}_0^1)$. Similarly, if Γ is of finite index over $g\rho(\mathcal{S}_i^1)g^{-1}$ then Remark 6.4 shows that $\Gamma \cap \rho(B^1) \subseteq g\rho(\mathcal{S}_0^1)g^{-1}$. This shows that $\rho(\mathcal{R}_i^1)$ and $g\rho(\mathcal{S}_i^1)g^{-1}$ are almost conjugate subgroups of both $\rho(\mathcal{R}_0^1)$ and $g\rho(\mathcal{S}_0^1)g^{-1}$ (they cannot be conjugate in either group because M_i and N_i are not isometric). Because we have chosen (H, H_1, H_2) so that the covolume of $\Gamma \cap \rho(B^1)$ is minimal, it must be the case that $\rho(\mathcal{R}_0^1) = \Gamma \cap \rho(B^1) = g\rho(\mathcal{S}_0^1)g^{-1}$. This is a contradiction as $M_0 = \rho(\mathcal{R}_0^1)\backslash X$ and $N_0 = \rho(\mathcal{S}_0^1)\backslash X$ are not isometric.

Thus Γ is not of finite index over $\rho(\mathcal{R}_i^1)$, hence the cover

$$M = \Gamma_0 \backslash X \to \Gamma \backslash X = M/H$$

is not finite, a contradiction. This finishes the proof of Theorem 6.2.

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