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On the secondary Steenrod algebra

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Abstract. We introduce a new model for the secondary Steenrod algebra at the prime 2 which is both smaller and more accessible than the original construction of H.-J. Baues.

We also explain how BP can be used to define a variant of the secondary Steenrod algebra at odd primes.

Contents

1. Introduction

Let A be the Steenrod algebra. In [Bau06], H.-J. Baues has constructed an exact sequence $B_\bullet$

$$A \rightarrowtail B_1 \xrightarrow{\partial} B_0 \twoheadrightarrow A \tag{1.1}$$

which captures the algebraic structure of secondary cohomology operations in ordinary mod p cohomology. This sequence is called the *secondary Steenrod algebra* and its knowledge allows, among other things, to give a purely

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algebraic description of the d_2-differential in the classical Adams spectral sequence (see [BJ06], [BJ04b] and Remark 4.13).

Unfortunately, the construction of $B_\bullet$ is not very explicit and apparently not many topologists have become familiar with it. The aim of the present note is to show that there is a smaller and much more accessible model which captures the same information. In fact our model is so simple that we can describe it in this introduction:

Fix $p = 2$ and let D_0 be the Hopf algebra that represents power series

$$f(x) = \sum_{k\geq 0} \xi_k x^{2^k} + \sum_{0\leq k<l} 2\xi_{k,l} x^{2^k+2^l}$$

under composition modulo 4. There is a natural map $\pi : D_0 \twoheadrightarrow A$ and a decomposition

$$D_0 = \mathbb{Z}/4\{\mathrm{Sq}(R)\} \oplus \sum_{-1\leq k<l} Y_{k,l} A \tag{1.2}$$

where $\mathrm{Sq}(R), Y_{k,l} \in D_0$ are dual to ξ^R resp. $\xi_{k+1,l+1}$ with respect to the natural basis $\{\, \xi^R,\ 2\xi^R\xi_{k,l}\,\}$ of $D_{0*} = \mathbb{Z}/4[\xi_n, 2\xi_{k,l}]$.

Here are some computations that can help to become familiar with D_0: $\mathrm{Sq}^1\mathrm{Sq}^1 = 2\mathrm{Sq}^2 + Y_{-1,0}$, $\mathrm{Sq}^1 Y_{-1,0} = Y_{-1,0}\mathrm{Sq}^1 + 2\mathrm{Sq}(0,1)$. Let $Q_k = \mathrm{Sq}(\Delta_{k+1})$ for the exponent sequence Δ_k with $\xi^{\Delta_k} = \xi_k$ and $P^s_t = \mathrm{Sq}(2^s\Delta_t)$. Then $Q_0Q_k = \mathrm{Sq}(\Delta_1 + \Delta_{k+1}) + Y_{-1,k}$ and $[Q_0, Q_k] = Y_{-1,k}$ if $k > 0$. One also finds

$$P^s_t P^s_t = \begin{cases} 2P^{s+1}_t & (s+1<t), \\ 2P^{s+1}_t + Y_{t-2,2s}\mathrm{Sq}\left((2^s-1)\Delta_t\right) & (s+1=t). \end{cases}$$

So for example $\mathrm{Sq}(0,2)\cdot\mathrm{Sq}(0,2) = 2\mathrm{Sq}(0,4) + Y_{0,2}\mathrm{Sq}(0,1)$. More computations can be found in Figure 1.

For products involving $Y_{k,l}$ there is the simple formula

$$aY_{k,l} = \sum_{i,j\geq 0} Y_{k+i,l+j} \urcorner(\xi_i^{2^{k+1}}\xi_j^{2^{l+1}}, a) \tag{1.3}$$

if we interpret the $Y_{k,l}$ with $k \geq l$ as

$$Y_{k,l} = \begin{cases} Y_{l,k} & (l<k), \\ 2\mathrm{Sq}(\Delta_{k+2}) & (l=k). \end{cases} \tag{1.4}$$

Here we have written $\urcorner(p, a)$ for the contraction of $a \in A$ by $p \in A_*$ defined via $\langle \urcorner(p,a), q\rangle = \langle a, pq\rangle$ for $q \in A_*$. Let $\kappa(a) = \urcorner(\xi_1, a)$.

Our model $D_\bullet$ for the secondary Steenrod algebra is the sequence

$$A \rightarrowtail D_1 \xrightarrow{\partial} D_0 \xrightarrow{\pi} A.$$

where

$$D_1 = \Bigg(A + \mu_0 A + \sum_{-1 \le k,\, 0 \le l} U_{k,l} A \Bigg) \Big/ \sim,$$

μ_0 and $U_{k,l}$ are symbols of degree $|\mu_0| = -1$ and $|U_{k,l}| = |Y_{k,l}| - 1 = 2^{k+1} + 2^{l+1} - 2$. We turn D_1 into an A-bimodule via $a\mu_0 = \mu_0 a + \kappa(a)$ and

$$aU_{k,l} = \sum_{i,j \ge 0} U_{k+i,l+j} \urcorner(\xi_i^{2^{k+1}} \xi_j^{2^{l+1}}, a). \tag{1.5}$$

The relations defining D_1 are

$$U_{k,l} = \begin{cases} U_{l,k} + \mathrm{Sq}(\Delta_{k+1} + \Delta_{l+1}) & (l < k), \\ \mu_0 \mathrm{Sq}(\Delta_{k+2}) + \mathrm{Sq}(2\Delta_{k+1}) & (l = k). \end{cases} \tag{1.6}$$

The boundary ∂ is zero on $A \subset D_1$ and otherwise given by $\partial \mu_0 a = 2a$ and $\partial U_{k,l} a = Y_{k,l} a$.

(Note that our grading convention differs slightly from the one that is used by Baues: our ∂ raises degrees by one whereas the inclusion $A \subset D_1$ is degree-preserving; in [Bau06] the inclusion $\Sigma A \subset B_1$ raises degrees but $\partial : B_1 \to B_0$ doesn't.)

The following is our main result:

Theorem 1.1. *There is a weak equivalence $B_\bullet \to D_\bullet$ of crossed algebras that is the identity on π_0 and π_1.*

Recall that a crossed algebra [Bau06, 5.1.6] is an exact sequence of the form $B_\bullet$ with B_0 an algebra, B_1 a B_0-bimodule and a bilinear differential $\partial : B_1 \to B_0$ with $(\partial b)b' = b(\partial b')$ for $b, b' \in B_1$. The homotopy groups $\pi_0(B_\bullet) := \operatorname{coker} \partial$ and $\pi_1(B_\bullet) := \ker \partial$ will mostly be A in our examples.

This theorem makes it easy to compute threefold Massey products in the Steenrod algebra. Think of $D_\bullet$ as the splice of the two short exact sequences

$$A \rightarrowtail D_1 \underset{\partial}{\overset{u}{\twoheadrightarrow}} R_D, \qquad R_D \rightarrowtail D_0 \overset{\sigma}{\twoheadrightarrow} A$$

and pick sections σ and u as indicated. For σ, for example, we can take the (nonadditive) map $\sigma(\sum c_i \mathrm{Sq}(R_i)) = \sum \widehat{c_i} \mathrm{Sq}(R_i)$ with $\widehat{(-)} : \mathbb{Z}/2 \to \mathbb{Z}/4$ given by $\widehat{0} = 0$ and $\widehat{1} = 1$. For u we can let

$$2\mathrm{Sq}(R) \mapsto \mu_0 \mathrm{Sq}(R), \qquad Y_{k,l}\mathrm{Sq}(R) \mapsto U_{k,l}\mathrm{Sq}(R) \qquad (\text{for } k < l) \tag{1.7}$$

which gives a right-linear section. For $a, b \in A$ one then has $\sigma(ab) = \sigma(a)\sigma(b) + \partial\tau(a,b)$ with $\tau(a,b) = u\,(\sigma(ab) - \sigma(a)\sigma(b)) \in D_1$. Associativity of the multiplication in A dictates that

$$\langle a, b, c \rangle := \tau(ab, c) - \tau(a,b)\sigma(c) - \tau(a, bc) + \sigma(a)\tau(b,c)$$

is a ∂-cycle, hence in A. $\langle a,b,c\rangle$ is the Massey product in question. It is only defined up to an indeterminacy coming from the choices of σ and u.

As an example, consider the case $a=b=c=\mathrm{Sq}(0,2)$. With σ and u chosen as above one has $\sigma(a)\sigma(b)=2\mathrm{Sq}(0,4)+Y_{0,2}\mathrm{Sq}(0,1)$, so $\tau(a,b)=\mu_0\mathrm{Sq}(0,4)+U_{0,2}\mathrm{Sq}(0,1)$. One finds

$$\begin{aligned}\langle a,b,c\rangle &= \mathrm{Sq}(0,2)\tau(b,c)-\tau(a,b)\mathrm{Sq}(0,2)\\ &= \mu_0\underbrace{[\mathrm{Sq}(0,2),\mathrm{Sq}(0,4)]}_{=\mathrm{Sq}(0,1,0,1)}+U_{0,2}\underbrace{[\mathrm{Sq}(0,2),\mathrm{Sq}(0,1)]}_{=0}+U_{2,2}\mathrm{Sq}(0,1)\\ &= \mu_0\mathrm{Sq}(0,1,0,1)+\left(\mu_0\mathrm{Sq}(0,0,0,1)+\mathrm{Sq}(0,0,2)\right)\mathrm{Sq}(0,1)\\ &= \mathrm{Sq}(0,1,2)\end{aligned}$$

which agrees with the calculation of Baues [Bau06, 16.6.7]. A straightforward computation, whose details we leave to the interested reader, now generalizes this to:

Corollary 1.2. *Let $t\geq 1$. Then $\langle P^s_t,P^s_t,P^s_t\rangle$ is zero for $s<t-1$ and $\langle P^{t-1}_t,P^{t-1}_t,P^{t-1}_t\rangle \ni \mathrm{Sq}\left((2^{t-1}-1)\Delta_t+2^t\Delta_{t+1}\right)$.*

The plan of the paper is as follows. In Section 2 we will review the definition and structure of $D_\bullet$ and sketch proofs for the claims in this introduction. In Section 3 we will construct an intermediate sequence $E_\bullet$ with a weak equivalence $E_\bullet\to D_\bullet$. We then construct a comparison map $B_\bullet\to E_\bullet$ in Section 4, thereby proving the main theorem. Finally, the appendix sketches the relation of the odd-primary secondary Steenrod algebra with the algebra of BP operations.

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2. The construction of $D_\bullet$

2.1. Definition. As in the introduction, we let

$$D_{0*}=\mathbb{Z}/4[\xi_k,2\xi_{k,l}\,|\,0\leq k<l,\,\xi_0=1].$$

This is turned into a Hopf algebra with coproduct

$$\begin{aligned}\Delta\left(\xi_n\right)&=\sum_{i+j=n}\xi_i^{2^j}\otimes\xi_j+2\sum_{0\leq k<l}\xi_{n-1-k}^{2^k}\xi_{n-1-l}^{2^l}\otimes\xi_{k,l}\\ \Delta\left(\xi_{n,m}\right)&=\xi_{n,m}\otimes 1+\sum_{k\geq 0}\xi_{n-k}^{2^k}\xi_{m-k}^{2^k}\otimes\xi_{k+1}\\ &\quad+\sum_{0\leq k<l}\left(\xi_{n-k}^{2^k}\xi_{m-l}^{2^l}+\xi_{m-k}^{2^k}\xi_{n-l}^{2^l}\right)\otimes\xi_{k,l}.\end{aligned}$$

We list some basic properties of its dual in the following

Lemma 2.1. *Let* $D_0 = \mathrm{Hom}(D_{0*}, \mathbb{Z}/4)$ *be the dual algebra and let* $\mathrm{Sq}(R)$, $Y_{k,l}(R) \in D_0$ *be defined by*

$$\langle \mathrm{Sq}(R), \xi^S \rangle = \delta_{R,S}, \qquad \langle \mathrm{Sq}(R), 2\xi_{m,n}\xi^S \rangle = 0,$$
$$\langle Y_{k,l}(R), \xi^S \rangle = 0, \qquad \langle Y_{k,l}(R), 2\xi_{m,n}\xi^S \rangle = 2\delta_{k+1,m}\delta_{l+1,n}\delta_{R,S}.$$

Write $Y_{k,l}$ *for* $Y_{k,l}(0)$. *The following is true:*

1. *There is a multiplicative map* $\pi : D_0 \to A$ *with* $\mathrm{Sq}(R) \mapsto \mathrm{Sq}(R)$.
2. *One has* $Y_{k,l}(R) = Y_{k,l}\mathrm{Sq}(R)$.
3. *The kernel* $R_D = \ker \pi$ *is* $2D_0 + \sum_{-1 \le k < l} Y_{k,l}A$ *and satisfies* $R_D^2 = 0$.
4. *The commutation rule* (1.3) *holds with* $Y_{k,l}$ *as in* (1.4) *for* $k \ge l$.

Proof. The verification is straightforward. □

We will encounter the following A-bimodules more than once.

Lemma 2.2. *There are* A*-bimodules* U, V *with*

$$V = \sum_{-1 \le k} V_k A, \quad U = \sum_{-1 \le k,l} U_{k,l} A$$

and relations

$$aV_k = \sum_{i \ge 0} V_{k+i} \urcorner(\xi_i^{2^{k+1}}, a), \quad aU_{k,l} = \sum_{i,j \ge 0} U_{k+i,l+j} \urcorner(\xi_i^{2^{k+1}}\xi_j^{2^{l+1}}, a).$$

Furthermore, let $R_{k,l} = U_{k,l} + U_{l,k}$ *and* $R_{k,k} = U_{k,k}$ *for* $-1 \le k < l$ *and*

$$K = \sum_{-1 \le k < l} R_{k,l}A + \sum_{-1 \le k} R_{k,k}A.$$

Then

$$aR_{k,l} = \sum_{-1 \le n < m} R_{n,m} \urcorner(\xi_{n-k}^{2^{k+1}}\xi_{m-l}^{2^{l+1}} + \xi_{m-k}^{2^{k+1}}\xi_{n-l}^{2^{l+1}}, a), \tag{2.1}$$

$$aR_{k,k} = \sum_{0 \le i} R_{k+i,k+i} \urcorner(\xi_i^{2^{k+2}}, a) + \sum_{0 \le i < j} R_{k+i,l+j} \urcorner(\xi_i^{2^{k+1}}\xi_j^{2^{l+1}}, a) \tag{2.2}$$

and K *is a bimodule, too. All of* U, V *and* K *are free* A*-modules from both left and right with basis the* $U_{k,l}$, V_k, *resp.* $R_{k,l}$ *and* $R_{k,k}$. *The same is true for the sub-bimodules*

$$V' = \sum_{0 \le k} V_k A, \qquad U' = \sum_{-1 \le k,\, 0 \le l} U_{k,l} A, \qquad K' = \sum_{0 \le k < l} R_{k,l}A + \sum_{0 \le k} R_{k,k}A$$

where the generators V_{-1}, $U_{*,-1}$ *and* $R_{-1,*}$ *have been left out.*

Proof. This is also straightforward. □

We will need the following computation in A.

Lemma 2.3. *Let $a \in A$ and $k \geq 0$, $l \geq 1$. Then*

$$aQ_k = \sum_{i\geq 0} Q_{k+i} \urcorner \left(\xi_i^{2^{k+1}}, a\right), \tag{2.3}$$

$$aP_l^1 = \sum_{i\geq 0} P_{l+i}^1 \urcorner \left(\xi_i^{2^{l+1}}, a\right) + \kappa(a)Q_{l+1} + \sum_{l\leq i<j} Q_iQ_j \urcorner \left(\xi_{l-i}^{2^l}\xi_{l-j}^{2^l}, a\right). \tag{2.4}$$

Proof. Recall that A_* is canonically an A-bimodule with

$$\Delta(p) = \sum_R \mathrm{Sq}(R)p \otimes \xi^R = \sum_R \xi^R \otimes p\mathrm{Sq}(R).$$

One has $\langle a\mathrm{Sq}(R), p\rangle = \langle a, \mathrm{Sq}(R)p\rangle$ and $\langle \mathrm{Sq}(R)a, p\rangle = \langle a, p\mathrm{Sq}(R)\rangle$. Upon dualization (2.3) therefore becomes the identity

$$Q_kp = \sum_{i\geq 0}(pQ_{k+i}) \cdot \xi_i^{2^{k+1}}.$$

Here both sides are derivations in p, so it only remains to check equality on the ξ_n which is easily done.

The second claim can be proved similarly, but with messier details. We leave this to the skeptical reader. □

The following lemma is the key to the definition of D_1. Recall that $A+\mu_0A$ carries the bimodule structure $a\mu_0 = \mu_0a + \kappa(a)$.

Lemma 2.4. *There is a bilinear map $\lambda : K' \to A + \mu_0A$ with*

$$R_{k,l} \mapsto \mathrm{Sq}(\Delta_{k+1} + \Delta_{l+1}),$$
$$R_{k,k} \mapsto \mathrm{Sq}(2\Delta_{k+1}) + \mu_0\mathrm{Sq}(\Delta_{k+2}).$$

Proof. We need to show that λ respects the relations (2.1) and (2.2).

By (2.3) one has

$$aQ_kQ_l = \sum_{i,j\geq 0} Q_{k+i}Q_{l+j} \urcorner(\xi_i^{2^{k+1}}\xi_j^{2^{l+1}}, a).$$

Using $Q_kQ_l = Q_lQ_k$ and $Q_k^2 = 0$ this immediately implies compatibility with (2.1).

For (2.2) note $a\lambda(R_{k,k}) = aP_{k+1}^1 + \kappa(a)Q_{k+1} + \mu_0aQ_{k+1}$. The claim is therefore equivalent to

$$aP_{k+1}^1 + \kappa(a)Q_{k+1} = \sum_{0\leq i} P_{k+i+1}^1\urcorner(\xi_i^{2^{k+2}}, a) + \sum_{0\leq i<j} Q_{k+i}Q_{l+j}\urcorner(\xi_i^{2^{k+1}}\xi_j^{2^{l+1}}, a),$$

$$aQ_{k+1} = \sum_{0\leq i} Q_{k+i+1} \urcorner(\xi_i^{2^{k+2}}, a).$$

These are again just variants of (2.3) and (2.4). □

Now let $D_1 = (A + \mu_0 A + U')/L$ where $L = \{\, (\lambda(x), x) \mid x \in K' \,\}$ is the graph of λ. This is easily seen to agree with the definition in the introduction.

Lemma 2.5. *Let $\partial U_{k,l} = Y_{k,l}$ and $\partial \mu_0 = 2$. This defines an exact sequence*

$$A \rightarrowtail D_1 \xrightarrow{\partial} D_0 \xrightarrow{\pi} A.$$

Proof. By Lemma 2.4 L is a sub-bimodule of $(A + \mu_0 A) \times K'$, so D_1 is indeed a bimodule. That ∂ is well-defined and bilinear follows from the relations (1.3). Finally, D_1 can be written as the direct sum

$$D_1 = A + \mu_0 A + \sum_{-1 \le k < l} U_{k,l} A.$$

From this the exactness of the sequence is obvious. □

2.2. Represented functors. Some of the previous constructions can be given meaningful descriptions when we look at their associated functors. Unfortunately, we have not been able to find a good explication for the map λ, so we eventually have to resort to pure algebra in our construction of $D_\bullet$.

Let $\mathrm{Alg}^{\mathrm{c}}_{\mathbb{Z}/4}$ be the category of commutative algebras over $\mathbb{Z}/4$.

Lemma 2.6. *There is a natural isomorphism $\mathrm{Hom}_{\mathrm{Alg}^{\mathrm{c}}_{\mathbb{Z}/4}}(D_{0*}, -) \xrightarrow{\cong} G(-)$ where $G(R) \subset R[[x]]$ is the group*

$$\left\{ f(x) = \sum_{k \ge 0} t_k x^{2^k} + \sum_{0 \le k < l} t_{k,l} x^{2^k + 2^l} \;\middle|\; t_0 = 1,\ J^2 = 0 \text{ for } J = (2, t_{k,l}) \subset R \right\}.$$

Proof. A $\phi : D_{0*} \to R$ maps to the f with $t_k = \phi(\xi_k)$ and $t_{k,l} = \phi(2\xi_{k,l})$. □

The bimodules U and V can be understood by looking at the functors

$$V_!(R) = G(R) \times \left\{ v(x) = \sum_{k \ge 0} v_k x^{2^k} \;\middle|\; v(x)^2 = 2v(x) = 0 \right\},$$

$$U_!(R) = G(R) \times \left\{ f_2(x,y) = \sum_{k,l \ge 0} u_{k,l} x^{2^k} y^{2^l} \;\middle|\; f_2(x,y)^2 = 2 f_2(x,y) = 0 \right\}.$$

The group operation is given by $(f_1, v) \circ (g_1, w) = (f_1 g_1, v g_1 + w)$ resp. $(f_1, f_2) \circ (g_1, g_2) = (f_1 g_1, f_2(g_1 \times g_1) + g_2)$.

$V_!$ and $U_!$ are represented by algebras $D_{0*}[v_k]/J^2$ and $D_{0*}[u_{k,l}]/J^2$ where J is the ideal $(2, v_k)$ resp. $(2, u_{k,l})$. V and U can then be recovered as the duals of the degree 1 part of these algebras.

We can use this to at least partially explain the map from U to D_0.

Lemma 2.7. *The map $\phi : U \to D_0$ with $U_{k,l} \mapsto Y_{k,l}$ and $U_{k,k} \mapsto 2Q_{k+1}$ is associated to the natural transformation*

$$U(R) \ni f = (f_1, f_2) \mapsto f^{\mathrm{eff}} \in G(R)$$

with $f^{\mathrm{eff}}(x) = f_1(x) + f_2(x, x)$.

Proof. We have an isomorphism $D_{0*}[u_{k,l}]/J^2 = D_{0*}[2w_{k,l}]$ and will use the $w_{k,l}$ in our computation for the sake of clarity. Recall that $\langle Q_k a, p\rangle = \langle a, (\partial p)/(\partial\xi_{k+1})\rangle$ for $a \in A$, $p \in A_*$. Therefore the dual $\phi_* : D_{0*} \to U_*$ is given by

$$p \mapsto 2\sum_{k\geq 0}(\partial p)/(\partial\xi_{k+1})w_{k,k} + \sum_{0\leq k<l} 2(\partial p)/(\partial\xi_{k,l})\,(w_{k,l}+w_{l,k})\,.$$

The map $\widehat{\phi_*} : D_{0*} \to D_{0*}[2w_{k,l}]$ with $p \mapsto p + \phi_*(p)$ is multiplicative since ϕ_* is a derivation. It therefore does correspond to a natural transformation $U_!(R) \to G(R)$. To see that this transformation is $f \mapsto f^{\mathrm{eff}}$ one just has to check that $\widehat{\phi_*}(\xi_{n+1}) = \xi_{n+1} + 2w_{n,n}$ and $\widehat{\phi_*}(2\xi_{k,l}) = 2\xi_{k,l} + 2w_{k,l} + 2w_{l,k}$. □

The bilinearity of ϕ expresses the fact, that $f \mapsto f^{\mathrm{eff}}$ is multiplicative. This is also easy to see computationally.

Lemma 2.8. *One has* $(fg)^{\mathrm{eff}} = f^{\mathrm{eff}} \circ g^{\mathrm{eff}}$.

Proof. We have

$$(fg)^{\mathrm{eff}}(x) = f_1(g_1(x)) + f_2(g_1(x), g_1(x)) + g_2(x,x),$$
$$f^{\mathrm{eff}}(g^{\mathrm{eff}}(x)) = f_1(g_1(x) + g_2(x,x)) + f_2(g_1(x) + g_2(x,x), g_1(x) + g_2(x,x)).$$

Since $g_2^k = 0$ for $k \geq 2$ we have

$$f_1(g_1(x) + g_2(x,x)) = f_1(g_1(x)) + g_2(x,x),$$
$$f_2(g_1(x) + g_2(x,x), g_1(x) + g_2(x,x)) = f_2(g_1(x), g_1(x))$$

which implies $(fg)^{\mathrm{eff}}(x) = f^{\mathrm{eff}}(g^{\mathrm{eff}}(x))$. □

3. The construction of $E_\bullet$

We now prepare ourselves for the comparison between our $D_\bullet$ and the $B_\bullet$ of Baues. It turns out that an intermediate $E_\bullet$ is required. The reason is that $D_\bullet$, although sufficient for the computational applications of the theory, does not capture all of the structure of $B_\bullet$. The latter carries a comultiplication which turns it into a *secondary Hopf algebra* and the associated invariants L and S are crucial for the comparison. We will therefore now pass to a slightly larger $E_\bullet$ where this extra structure can be expressed.

3.1. Definition. Let $X = \sum_{-1\leq k,l} X_{k,l}A$ be a copy of U with $U_{k,l}$ renamed $X_{k,l}$ and let $X' \subset X$ be the subspace without $X_{-1,-1}A$. Let $\widehat{E_k} = D_k + X' + \mu_0 X'$ for $k = 0, 1$. We will write $e = e_D + e_X$ for the decomposition of $e \in \widehat{E_k}$ into the D_k and $X + \mu_0 X$ components. Let $\rho : E_\bullet \to D_\bullet$ denote the projection $e \mapsto e_D$. We extend ∂ to $\widehat{E_\bullet}$ via $\partial e = \partial e_D + e_X$. This defines an exact sequence

$$A \rightarrowtail \widehat{E_1} \xrightarrow{\partial} \widehat{E_0} \xrightarrow{\pi} A. \tag{3.1}$$

Here the grading is given by $|\mu_0 X_{k,l}| = |X_{k,l}| - 1$ and $|X_{k,l}| = |Y_{k,l}|$ in $\widehat{E_0}$, $|X_{k,l}| = |Y_{k,l}| - 1$ in $\widehat{E_1}$.

We need to define a multiplication on $\widehat{E_0}$. Note that there is an isomorphism $U \cong V \otimes_A V$ where $U_{k,l} \leftrightarrow V_k \otimes V_l$. We can therefore write $X_{k,l} = X_k X_l$ where the X_k are generators of a copy V_X of V. Let $\psi : A \to V'_X$ be given by $\psi(a) = \sum_{k\geq 0} X_k \urcorner(\xi_{k+1}, a)$. ψ is a derivation because one has $\psi(a) = X_{-1}a - aX_{-1}$. Recall that $\kappa : A \to A$ is also a derivation.

Lemma 3.1. *Let* $* : D_0 \otimes D_0 \to D_0 + X + \mu_0 X$ *be given by*

$$a * b = ab + \psi(a)\psi(b)\mu_0 + X_{-1}\psi(a)\kappa(b) \tag{3.2}$$

and extend this to all of $\widehat{E_0}$ *via* $d * m = \pi(d)m$, $m * d = m\pi(d)$ *and* $mm' = 0$ *for* $d \in D_0$ *and* $m, m' \in X + \mu_0 X$. *Then* $*$ *is associative.*

Proof. The only questionable case is when all three factors are in D_0. But this is a straightforward computation:

$$\begin{aligned}
&(a * b) * c = \\
&= abc + \psi(ab)\psi(c)\mu_0 + X_{-1}\psi(ab)\kappa(c) + \psi(a)\psi(b)\mu_0 c + X_{-1}\psi(a)\kappa(b)c \\
&= abc + \psi(a)b\psi(c)\mu_0 + a\psi(b)\psi(c)\mu_0 + X_{-1}\psi(a)b\kappa(c) + X_{-1}a\psi(b)\kappa(c) \\
&\qquad + \psi(a)\psi(b)c\mu_0 + \psi(a)\psi(b)\kappa(c) + X_{-1}\psi(a)\kappa(b)c, \\
&a * (b * c) = \\
&= abc + \psi(a)\psi(bc)\mu_0 + X_{-1}\psi(a)\kappa(bc) + a\psi(b)\psi(c)\mu_0 + aX_{-1}\psi(b)\kappa(c) \\
&= abc + \psi(a)b\psi(c)\mu_0 + \psi(a)\psi(b)c\mu_0 + X_{-1}\psi(a)\kappa(b)c + X_{-1}\psi(a)b\kappa(c) \\
&\qquad + a\psi(b)\psi(c)\mu_0 + X_{-1}a\psi(b)\kappa(c) + \psi(a)\psi(b)\kappa(c).
\end{aligned}$$

□

Figure 1 illustrates the multiplication in E_0 with the computation of the first few Adem relations.

We will define $E_0 \subset \widehat{E_0}$ by a condition on the coefficients of $Y_{-1,*}$, $X_{-1,*}$ and $X_{*,-1}$. To formulate that condition we need to define two more maps.

Lemma 3.2. *Let* $\theta_D : D_0 \to V$ *be the map that extracts the* $Y_{-1,k}$. *In other words, let*

$$\theta_D(\mathrm{Sq}(R)) = 0, \quad \theta_D(Y_{-1,n}a) = V_n a, \quad \theta_D(Y_{k,l}a) = 0 \quad \textit{for } k \neq -1.$$

Then $\widehat{\theta_D} : D_0 \to V + \mu_0 V$ *with* $\widehat{\theta_D}(d) = \theta_D(d) + \psi(d)\mu_0$ *is a derivation.*

Proof. We sketch a quick computational proof here. A better argument will be given later from the functorial point of view.

We already know that ψ is a derivation, so we just need to show $\theta_D(de) = d\theta_D(e) + \theta_D(d)e + \psi(d)\kappa(e)$. Since θ_D sees only the $\xi_{0,n}$ we can compute $\theta_D(de)$ from the coproduct formula

$$\Delta\xi_{0,n} = \xi_{0,n} \otimes 1 + \sum_{k\geq 0} \xi_{n-k}^{2^k} \otimes \xi_{0,k} + \xi_{n-1} \otimes \xi_1$$

and these summands translate to $\theta_D(d)e$, $d\theta_D(e)$ and $\psi(d)\kappa(e)$. □

Similarly, let $\theta_E : \widehat{E_0} \to V$ extract the $X_{-1,k}$:

$$\theta_E(X_{-1,k}a) = V_k a, \quad \theta_E(X_{l,-1}a) = 0,$$
$$\theta_E\Big(D_0 + \mu_0 X + \sum\nolimits_{k,l\geq 0} X_{k,l}A\Big) = 0.$$

Lemma 3.3. *One has* $\theta_E(d * e) = \theta_E(d)e + d\theta_E(e) + \psi(d_D)\kappa(e_D)$ *for* $d, e \in \widehat{E_0}$.

Proof. This is a straightforward computation. See also the discussion in Remark 3.9 below. □

Lemma 3.4. *Define*

$$\widetilde{E_0} = D_0 + \sum_{k,l\geq 0} X_{k,l}A + \sum_{k,l\geq 0} \mu_0 X_{k,l}A + \sum_{k\geq 0} X_{-1,k}A \subset \widehat{E_0}$$

and let $E_0 \subset \widetilde{E_0}$ *be the subset where* $\theta_D \circ \rho$ *and* θ_E *coincide. Then* E_0 *is closed under the multiplication* $*$.

Proof. It's clear that $\widetilde{E_0}$ is multiplicatively closed since $*$ cannot generate any $X_{k,-1}$ if this is not already part of one factor.

That E_0 is also multiplicatively closed follows from the identical formulas for $\theta_D(de)$ and $\theta_E(de)$. □

Corollary 3.5. *Let* $E_1 = \partial^{-1}(E_0) \subset \widehat{E_1}$. *Then*

$$A \rightarrowtail E_1 \xrightarrow{\partial} E_0 \xrightarrow{\pi} A. \tag{3.3}$$

is a crossed algebra $E_\bullet$ *with a canonical projection* $\rho : E_\bullet \twoheadrightarrow D_\bullet$.

Proof. Clear. □

3.2. Represented functors.

Lemma 3.6. *For* $f(x) \in G(R)$ *let* $\tau_f(x)$ *and* $\theta_f(x)$ *be defined by the decomposition*

$$f(x) = x + \tau_f(x^2) + x\theta_f(x^2) \tag{3.4}$$

and write $\overline{f}(x) = f(x) - x$. *Then*

$$\overline{fg}(x) = \overline{f}(g(x)) + \overline{g}(x), \tag{3.5}$$

$$\theta_{fg}(x) = \theta_f(g(x)) + \theta_g(x) + \xi_1^f \overline{g}(x), \tag{3.6}$$

where $\xi_1^f = \tau_f'(0)$ *is the coefficient of* x^2 *in* $f(x)$.

Proof. This is a straightforward computation. □

$[n,m]$	Definition	D_0	$X+\mu_0 X$
$[1,1]$	$1\cdot 1$	$2\,\mathrm{Sq}(2) + Y_{-1,0}$	$X_{-1,0} + \mu_0 X_{0,0}$
$[1,2]$	$1\cdot 2+3$	$Y_{-1,0}\mathrm{Sq}(1)$	$X_{-1,0}\mathrm{Sq}(1) + \mu_0 X_{0,0}\mathrm{Sq}(1) + X_{0,0}$
$[2,2]$	$2\cdot 2+3\cdot 1$	$2\,\mathrm{Sq}(1,1) + 2\,\mathrm{Sq}(4) + Y_{-1,0}\mathrm{Sq}(2)$	$X_{-1,0}\mathrm{Sq}(2) + X_{0,0}\mathrm{Sq}(1) + \mu_0 X_{0,0}\mathrm{Sq}(2) + \mu_0 X_{0,1}$
$[1,3]$	$1\cdot 3$	$Y_{-1,0}\mathrm{Sq}(2)$	$X_{-1,0}\mathrm{Sq}(2) + \mu_0 X_{0,0}\mathrm{Sq}(2) + X_{0,0}\mathrm{Sq}(1)$
$[3,2]$	$3\cdot 2$	$2\,\mathrm{Sq}(2,1) + 2\,\mathrm{Sq}(5) + Y_{-1,0}(\mathrm{Sq}(0,1)+\mathrm{Sq}(3))$	$X_{-1,0}(\mathrm{Sq}(0,1) + \mathrm{Sq}(3)) + X_{0,0}\mathrm{Sq}(2) + X_{0,1} + \mu_0 X_{0,0}(\mathrm{Sq}(0,1) + \mathrm{Sq}(3)) + \mu_0 X_{0,1}\mathrm{Sq}(1)$
$[2,3]$	$2\cdot 3+4\cdot 1+5$	$2\,\mathrm{Sq}(2,1)$	$X_{0,1} + \mu_0 X_{0,1}\mathrm{Sq}(1)$
$[1,4]$	$1\cdot 4+5$	$2\,\mathrm{Sq}(5) + Y_{-1,0}\mathrm{Sq}(3)$	$X_{-1,0}\mathrm{Sq}(3) + X_{0,0}\mathrm{Sq}(2) + \mu_0 X_{0,0}\mathrm{Sq}(3)$
$[3,3]$	$3\cdot 3+5\cdot 1$	$2\,\mathrm{Sq}(6) + Y_{-1,0}(\mathrm{Sq}(1,1)+\mathrm{Sq}(4))$	$X_{-1,0}(\mathrm{Sq}(1,1) + \mathrm{Sq}(4)) + X_{0,0}(\mathrm{Sq}(0,1) + \mathrm{Sq}(3)) + \mu_0 X_{0,0}(\mathrm{Sq}(1,1) + \mathrm{Sq}(4))$
$[2,4]$	$2\cdot 4+5\cdot 1+6$	$2\,\mathrm{Sq}(3,1) + 2\,\mathrm{Sq}(6) + Y_{-1,0}\mathrm{Sq}(4)$	$X_{-1,0}\mathrm{Sq}(4) + X_{0,0}\mathrm{Sq}(3) + X_{0,1}\mathrm{Sq}(1) + \mu_0 X_{0,0}\mathrm{Sq}(4) + \mu_0 X_{0,1}\mathrm{Sq}(2)$
$[1,5]$	$1\cdot 5$	$2\,\mathrm{Sq}(6) + Y_{-1,0}\mathrm{Sq}(4)$	$X_{-1,0}\mathrm{Sq}(4) + X_{0,0}\mathrm{Sq}(3) + \mu_0 X_{0,0}\mathrm{Sq}(4)$
$[4,3]$	$4\cdot 3+5\cdot 2$	$2\,\mathrm{Sq}(1,2) + 2\,\mathrm{Sq}(4,1) + Y_{-1,0}(\mathrm{Sq}(2,1) + \mathrm{Sq}(5))$	$X_{-1,0}(\mathrm{Sq}(2,1) + \mathrm{Sq}(5)) + X_{0,0}(\mathrm{Sq}(1,1) + \mathrm{Sq}(4)) + \mu_0 X_{0,0}(\mathrm{Sq}(2,1) + \mathrm{Sq}(5)) + \mu_0 X_{0,1}\mathrm{Sq}(0,1)$
$[3,4]$	$3\cdot 4+7$	$Y_{-1,0}\mathrm{Sq}(2,1)$	$X_{-1,0}\mathrm{Sq}(2,1) + X_{0,1}\mathrm{Sq}(2) + \mu_0 X_{0,0}\mathrm{Sq}(2,1) + \mu_0 X_{0,1}\mathrm{Sq}(3) + X_{0,0}\mathrm{Sq}(1,1)$
$[2,5]$	$2\cdot 5+6\cdot 1$	$2\,\mathrm{Sq}(4,1)$	$X_{0,1}\mathrm{Sq}(2) + \mu_0 X_{0,1}\mathrm{Sq}(3)$
$[1,6]$	$1\cdot 6+7$	$Y_{-1,0}\mathrm{Sq}(5)$	$X_{-1,0}\mathrm{Sq}(5) + \mu_0 X_{0,0}\mathrm{Sq}(5) + X_{0,0}\mathrm{Sq}(4)$

FIGURE 1. List of Adem relations in E_0.

Recall that V represents the functor

$$V_!(R) \cong G(R) \times \left\{ v(x) = \sum_{k\geq 1} v_k x^{2^k} \,\middle|\, v(x)^2 = 0,\ 2v(x) = 0 \right\}.$$

This extends to $M = V + \mu_0 V$ as

$$M_!(R) \cong G(R) \times \left\{ v(x) = v_0(x) + \mu_0 v_1(x) \,\middle|\, v_0,\ v_1 \text{ as in } V_!(R) \right\}$$

where

$$(f, v_0 + \mu_0 v_1) \circ (g, w_0 + \mu_0 w_1) = (fg, v_0 g + w_0 + \xi_1^f w_1 + \mu_0(v_1 g + w_1)).$$

We can use this to give an explanation of ψ and θ_D.

Lemma 3.7. *Let $\widehat{\theta_D}$ be the derivation $D_0 \to V + \mu_0 V = M$ from Lemma 3.2 and let $\widetilde{\theta_D} : \mathrm{Sym}_{D_{0*}}(M_*) \to D_{0*}$ be the multiplicative extension with $\widetilde{\theta_D}|_{M_*} = \widehat{\theta_{D*}}$. Then $\widetilde{\theta_D}$ represents the transformation $G(R) \to M_!(R)$ with $f \mapsto (f, \theta_f(x) + \mu_0 \overline{f}(x))$.*

Proof. For an $f(x)$ of the form $\sum_{k\geq 0} x^{2^k} + \sum_{0\leq k<l} 2\xi_{k,l} x^{2^k+2^l}$ one has

$$\tau_f(x) = \sum_{k\geq 1} \xi_k x^{2^{k-1}} + \sum_{1\leq k<l} 2\xi_{k,l} x^{2^{k-1}+2^{l-1}},$$
$$\theta_f(x) = \sum_{k\geq 0} 2\xi_{0,k} x^{2^k}.$$

The map $f \mapsto (f, \theta_f(x) + \mu_0 \overline{f}(x))$ therefore corresponds to the $M_* \to D_{0*}$ with $v_k \mapsto 2\xi_{0,k}$ and $\mu_0^* v_k \mapsto \xi_k$. But this is just $\widehat{\theta_{D*}}$. ∎

The multiplicative properties of ψ and θ_D that we established in Lemma 3.2 are therefore just a reformulation of (3.5) and (3.6).

We can now translate the definition of E_0 into the functorial context.

Lemma 3.8. *The ring $\widehat{E_0}$ represents pairs $(f_1(x), f_2(x,y))$ with $f_1(x) \in G(R)$ and $f_2(x,y) = f_2^{(0)}(x,y) + \mu_0 f_2^{(1)}(x,y)$ with $(f_1, f_2^{(j)}) \in U_!(R)$. The multiplication $*$ corresponds to the composition*

$$\begin{aligned}(f\circ g)_2(x,y) = f_2(g_1(x), g_1(y)) + \xi_1^f \cdot g_2^{(1)}(x,y) + g_2(x,y) \\ + \mu_0^f \overline{g}(x) \cdot \overline{f}(g(y)) + \xi_1^f x \cdot \overline{g}(y).\end{aligned}$$

The subset of those (f_1, f_2) with

$$f_2(x,y) = x \cdot \theta_{f_1}(y^2) + f_2^{(0)}(x^2, y^2) + \mu_0 f_2^{(1)}(x^2, y^2)$$

is closed under $*$ and represented by E_0.

Proof. Again this is straightforward. ∎

Remark 3.9. Rephrasing the previous discussion one could say that in E_0 we are studying certain pairs $f = (f_1, f_2)$ under the transformation rule

$$(fg)_1 = f_1 g_1, \quad (fg)_2(x, y) = (fg)_2^{\text{basic}}(x, y) + \text{correction terms}$$

where

$$(fg)_2^{\text{basic}}(x, y) = f_2\left(g_1(x), g_1(y)\right) + \xi_1^f \cdot g_2^{(1)}(x, y) + g_2(x, y).$$

Here the correction terms are specifically crafted to preserve the conditions

$$f_2(x, y) \equiv 0 \mod y^2,$$
$$f_2(x, y) \equiv x\theta_{f_1}(y^2) \mod x^2$$

that define E_0. To us this suggests that the basic object of study should be the composition $(fg)_2^{\text{basic}}$ and the subspace E_0, both of which have a reasonably elementary definition. The precise structure of the correction terms might then count as an artifact of the retraction from $\widehat{E_0}$ to E_0.

4. The Hopf structure on $E_\bullet$

The secondary Steenrod algebra comes equipped with a diagonal $B_\bullet \to B_\bullet \hat{\otimes} B_\bullet$ that extends the usual coproducts on A and B_0. This extra structure is essential for the characterization of $B_\bullet$ in the Uniqueness Theorem [Bau06, 15.3.13]. In this section we are going to exhibit a similar structure on $E_\bullet$, which is a key step in our proof that $B_\bullet \sim E_\bullet$.

4.1. E_0 as Hopf algebra.

Lemma 4.1. *There is a unique multiplicative $\Delta_0 : E_0 \to E_0 \otimes E_0$ with*

$$\Delta_0 \left(\mathrm{Sq}(R)\right) = \sum_{E+F=R} \mathrm{Sq}(E) \otimes \mathrm{Sq}(F)$$

and $\Delta_0(Z) = Z \otimes 1 + 1 \otimes Z$ for $Z \in \{Y_{k,l}, X_{k,l}, \mu_0 X_{k,l}\}$.

Proof. The uniqueness is clear. To show existence, we begin with the dual of the multiplication map $D_{0*} \otimes D_{0*} \to D_{0*}$. This defines a $\Delta_0 : D_0 \to D_0 \otimes D_0$ with $\Delta_0(Y_{k,l}) = Y_{k,l} \otimes 1 + 1 \otimes Y_{k,l}$. We extend this to all of E_0 via $\Delta_0(Z \cdot \mathrm{Sq}(R)) = (Z \otimes 1 + 1 \otimes Z) \cdot \Delta(\mathrm{Sq}(R))$ for $Z \in \{X_{k,l}, \mu_0 X_{k,l}\}$. We have to show that this map is multiplicative.

This is a straightforward computation, and we will work out only one representative case. Let $a \in A$ and $\Delta a = \sum a' \otimes a''$. Then

$$\begin{aligned}\Delta_0\,(aX_{k,l}) &= \Delta_0\Big(\sum\nolimits_{i,j\geq 0} X_{k+i,l+j} \urcorner \left(\xi_i^{2^{k+1}}\xi_j^{2^{l+1}}, a\right)\Big)\\ &= \sum_{i,j\geq 0} \left(X_{k+i,l+j}\otimes 1 + 1\otimes X_{k+i,l+j}\right)\Delta_0\left(\urcorner\left(\xi_i^{2^{k+1}}\xi_j^{2^{l+1}}, a\right)\right)\\ &= \sum_{a',a''}\sum_{i,j\geq 0}\Big\{\left(X_{k+i,l+j}\urcorner\left(\xi_i^{2^{k+1}}\xi_j^{2^{l+1}}, a'\right)\right)\otimes a''\\ &\qquad\qquad + a'\otimes\left(X_{k+i,l+j}\urcorner\left(\xi_i^{2^{k+1}}\xi_j^{2^{l+1}}, a''\right)\right)\Big\}\\ &= \sum_{a',a''}\left(a'X_{k,l}\otimes a'' + a'\otimes a''X_{k,l}\right)\end{aligned}$$

where we have used $\Delta \urcorner(p,a) = \sum \urcorner(p,a')\otimes a'' = \sum a'\otimes \urcorner(p,a'')$. This shows $\Delta_0(aX_{k,l}) = \Delta_0(a)\Delta_0(X_{k,l})$. We leave the remaining cases to the reader. ☐

There is also a canonical augmentation $\epsilon : E_0 \to \mathbb{Z}/4$ which is dual to the inclusion $\mathbb{Z}/4 \subset D_{0*} \subset E_{0*}$. The following corollary is then obvious.

Corollary 4.2. *E_0 is a Hopf algebra over $\mathbb{Z}/4$ with augmentation ϵ and coproduct Δ_0. The projection $E_0 \to A$ is a map of Hopf algebras.*

4.2. The folding product. We next want to define a secondary diagonal $\Delta_1 : E_1 \to (E\,\hat{\otimes}\,E)_1$. This requires a short discussion of the folding product $(E\,\hat{\otimes}\,E)_\bullet$ that figures on the right hand side. The necessary algebraic background is developped in [Bau06, Ch. 12] and [Bau06, Introduction (B5-B6)].

Let p for the moment be an arbitrary prime and $\mathbb{G} = \mathbb{Z}/p^2$. We consider exact sequences of $\mathbb{G}$-modules of the form

$$M_\bullet = \left(A^{\otimes m} \overset{\iota}{\rightarrowtail} M_1 \xrightarrow{\partial} M_0 \overset{\pi}{\twoheadrightarrow} A^{\otimes m}\right).$$

Under certain assumptions (e.g., if both factors are $[p]$-algebras in the sense of [Bau06, 12.1.2]) one can define the folding product

$$(M\,\hat{\otimes}\,N)_\bullet = \left(A^{\otimes(m+n)} \overset{\iota_\sharp}{\rightarrowtail} (M\,\hat{\otimes}\,N)_1 \xrightarrow{\partial_\sharp} \underbrace{(M\,\hat{\otimes}\,N)_0}_{=M_0\otimes N_0} \overset{\pi\otimes\pi}{\twoheadrightarrow} A^{\otimes(m+n)}\right)$$

of two such sequences. Here $(M\,\hat{\otimes}\,N)_1$ is a quotient of $M_1\otimes N_0 \oplus N_0\otimes M_1$, so we can represent its elements as tensors $m\,\hat{\otimes}\,n$ where either $m \in M_1$, $n \in N_0$ or $m \in M_0$, $n \in N_1$. Let $R_M = \ker(M_0 \to A)$ and $R_N = \ker(N_0 \to A)$ be the relation modules. Then $(M\,\hat{\otimes}\,N)_1$ fits into the short exact sequence

$$A^{\otimes(m+n)} \overset{\iota_\sharp}{\rightarrowtail} (M\,\hat{\otimes}\,N)_1 \overset{\partial}{\twoheadrightarrow} R_M\otimes N_0 + M_0\otimes R_N = R_{M\,\hat{\otimes}\,N}$$

with $\partial(m \hat{\otimes} n) = (\partial m) \otimes n + (-1)^{|m|} m \otimes (\partial n)$.

Unfortunately, $D_\bullet$ and $E_\bullet$ are not $[p]$-algebras in the sense of [Bau06, 12.1.2], because D_0 and E_0 fail to be $\mathbb{G}$-free. It is easy to see, however, that in both cases ∂ restricts to an isomorphism $\mu_0 M_0 \to p M_0$, so the reduction $\tilde{M}_\bullet$ with $\tilde{M}_1 = M_1/\mu_0 M_0$ and $\tilde{M}_0 = M_0/pM_0$ is again an exact sequence. A careful reading of Baues's theory shows that this suffices for the construction of the folding product.

Assume now that we have a right-linear splitting $u : R_M \hookrightarrow M_1$ of ∂. For $B_\bullet$ such a splitting has been established in [Bau06, 16.1.3-16.1.5]. For $D_\bullet$ we take the map $R_D \to D_1$

$$2\mathrm{Sq}(R) \mapsto \mu_0 \mathrm{Sq}(R), \quad Y_{k,l} a \mapsto U_{k,l} a \quad (\text{for } k < l,\ a \in A)$$

from (1.7) in the introduction. We extend this to $R_E = R_D \oplus W \to E_1 = D_1 \oplus W$ via $u_E = u_D \oplus \mathrm{id}_W$ where $W = X + \mu_0 X$. We then get an induced splitting $u_\sharp$ for $(M \hat{\otimes} M)_\bullet$ with $u_\sharp(r \otimes m) = u(r) \hat{\otimes} m$ and $u_\sharp(m \otimes r) = m \hat{\otimes} u(r)$ for $r \in R_M$, $m \in M_0$.

The splitting u allows us to decompose M_1 as the direct sum $M_1 = \iota(A) \oplus u(R_M)$. However, this decomposition is only valid for the *right* action of M_0 on $M_\bullet$. We also have an action from the left and this is described by the associated *multiplication map*[1] $\mathrm{op} : M_0 \otimes R_M \to A^{\otimes m}$ with

$$m \cdot u(r) = u(m \cdot r) + \iota(\mathrm{op}(m, r)).$$

In our examples, op actually factors through $M_0 \otimes R_M \twoheadrightarrow A \otimes R_M$. For $B_\bullet$ this is proved in [Bau06, 16.3.3]. For $D_\bullet$ and $E_\bullet$ it is obvious as both D_1 and E_1 are A-bimodules to begin with.

We will now compute op and $\mathrm{op}_\sharp$ explicitly for $D_\bullet$ and $E_\bullet$.

Lemma 4.3. *For $d \in D_0$ and $-1 \le k < l$ one has $\mathrm{op}(a, 2d) = \kappa(a)\pi(d)$ and*

$$\mathrm{op}(a, Y_{k,l}) = \sum_{\substack{i,j \ge 0,\\ k+i \ge l+j}} \mathrm{Sq}(\Delta_{k+i+1} + \Delta_{l+j+1}) \urcorner(\xi_i^{2^{k+1}} \xi_j^{2^{l+1}}, a).$$

Furthermore, $\mathrm{op}(a, x) = 0$ *for all* $x \in X + \mu_0 X$*.*

Proof. Since $u(2d) = \mu_0 \pi(d)$ one finds $au(2d) = \kappa(a)\pi(d) + u(a \cdot 2d)$ which proves $\mathrm{op}(a, 2d) = \kappa(a)\pi(d)$.

We have $a \cdot u(Y_{k,l}) = \sum_{i,j \ge 0} U_{k+i,l+j} \urcorner(\xi_i^{2^{k+1}} \xi_j^{2^{l+1}}, a)$. Using the relations (1.6) we can write

$$U_{k+i,l+j} = \begin{cases} u(Y_{k+i,l+j}) & (k+i < l+j), \\ u(2\mathrm{Sq}(\Delta_{k+i+2})) + \mathrm{Sq}(2\Delta_{k+i+1}) & (k+i = l+j), \\ u(Y_{l+j,k+i}) + \mathrm{Sq}(\Delta_{k+i+1} + \Delta_{l+j+1}) & (k+i > l+j). \end{cases}$$

[1] This map is denoted A in Baues's theory.

Therefore

$$a \cdot u(Y_{k,l}) = u(aY_{k,l}) + \sum_{\substack{i,j\geq 0,\\ k+i\geq l+j}} \mathrm{Sq}(\Delta_{k+i+1} + \Delta_{l+j+1}) \urcorner(\xi_i^{2^{k+1}} \xi_j^{2^{l+1}}, a)$$

as claimed.

Finally, $\mathrm{op}(a, -)$ vanishes on $M = X + \mu_0 X$ because $u|_M = \mathrm{id}$ is left-linear. □

For $\mathrm{op}_\sharp$ there is a similar result.

Lemma 4.4. *Write* $B_{k,l,i,j} = \mathrm{Sq}(\Delta_{k+i+1} + \Delta_{l+j+1})$. *Then*

$$\mathrm{op}_\sharp(a, \Delta(2d)) = \Delta\,\mathrm{op}(a, 2d), \quad (\textit{for } d \in D_0),$$
$$\mathrm{op}_\sharp(a, \Delta(Y_{k,l})) = \sum_{\substack{i,j\geq 0,\\ k+i\geq l+j}} (B_{k,l,i,j} \otimes 1 + 1 \otimes B_{k,l,i,j}) \urcorner(\xi_i^{2^{k+1}} \xi_j^{2^{l+1}}, a).$$

One has $\mathrm{op}_\sharp(a, \Delta(x)) = 0$ *for* $x \in X + \mu_0 X$.

Proof. The first claim follows from

$$\mathrm{op}_\sharp(a, \Delta(2d)) = \kappa(a)\Delta(2d) = \Delta\left(\kappa(a) \cdot 2d\right) = \Delta\,\mathrm{op}(a, 2d).$$

For the second we use $\mathrm{op}_\sharp(a, \Delta(Y_{k,l})) = \mathrm{op}_\sharp(a, Y_{k,l} \otimes 1 + 1 \otimes Y_{k,l})$. From Lemma 4.3 we find

$$\begin{aligned}\mathrm{op}_\sharp(a, Y_{k,l} \otimes 1) &= \sum \mathrm{op}(a', Y_{k,l}) \otimes a'' \\ &= \sum B_{k,l,i,j} \urcorner(\cdots, a') \otimes a'' \\ &= \sum (B_{k,l,i,j} \otimes 1) \urcorner(\cdots, a)\end{aligned}$$

where we have temporarily suppressed some details. There is a similar formula for $\mathrm{op}_\sharp(a, 1 \otimes Y_{k,l})$ and together they make up the second claim.

That $\mathrm{op}_\sharp(-, \Delta(X + \mu_0 X))$ vanishes is clear from the vanishing of op on $A \otimes (X + \mu_0 X)$. □

4.3. The secondary coproduct. We can now define the secondary diagonal $\Delta_\bullet : E_\bullet \to (E \hat{\otimes} E)_\bullet$. We still need a few preparations.

Lemma 4.5. *Let* $U'' \subset U$ *be the sub-bimodule on the* $U_{k,l}$ *with* $k, l \geq 0$. *There is a bilinear* $\nabla : U'' \to A \otimes A$ *with* $U_{k,l} \mapsto Q_l \otimes Q_k$.

Proof. One has

$$\begin{aligned}a\,(Q_k \otimes 1) = \sum (a' Q_k \otimes a'') &= \sum_{i\geq 0} Q_{k+i} \urcorner(\xi_i^{2^{k+1}}, a') \otimes a'' \\ &= \sum_{i\geq 0} (Q_{k+i} \otimes 1) \urcorner(\xi_i^{2^{k+1}}, a).\end{aligned}$$

Therefore

$$a\,(Q_k \otimes Q_l) = a\,(Q_k \otimes 1)\,(1 \otimes Q_l) = \sum_{i,j \geq 0} (Q_{k+i} \otimes Q_{l+j}) \urcorner (\xi_i^{2^{k+1}} \xi_j^{2^{l+1}}, a)$$

which is the same commutation relation as for the $U_{k,l}$. □

Lemma 4.6. *There is a right-linear* $\nabla : R_E \to A \otimes A \oplus \mu_0 A \otimes A$ *with*

$$\begin{aligned} \nabla X_{k,l} &= Q_l \otimes Q_k, \quad \nabla \mu_0 X_{k,l} = \mu_0 Q_l \otimes Q_k && (0 \leq k, l) \\ \nabla Y_{k,l} &= Q_l \otimes Q_k && (0 \leq k < l) \end{aligned}$$

and $\nabla|_{2D_0} = \nabla|_{Z_*} = 0$ *where* $Z_k = X_{-1,k} + Y_{-1,k}$. *Let* $\Phi(a,r) = \nabla(ar) - a(\nabla r)$ *be the left linearity defect of* ∇. *Then*

$$\Phi(a,r) = \Delta\,\mathrm{op}(a,r) + \mathrm{op}_\sharp(a, \Delta r) \tag{4.1}$$

for $a \in A$ *and* $r \in R_E$.

Proof. R_E is free as a right A-module with basis 2, Z_k (for $0 \leq k$), $Y_{k,l}$ (for $0 \leq k < l$) and $X_{k,l}$, $\mu_0 X_{k,l}$ (for $0 \leq k, l$). Therefore ∇ is well-defined and right-linear.

We have $\Phi(a, X_{k,l}) = 0$ and $\Phi(a, \mu_0 X_{k,l}) = 0$ by Lemma 4.5, $\Phi(a, 2) = 0$ and $\Delta\,\mathrm{op}(a, 2) + \mathrm{op}_\sharp(a, \Delta 2) = 0$ by Lemma 4.4, so it just remains to prove the formula for $r = Y_{k,l}$ and $r = Z_k$.

Combining Lemmas 4.3 and 4.4 we find

$$\begin{aligned} &\Delta\,\mathrm{op}(a, Y_{k,l}) + \mathrm{op}_\sharp(a, \Delta Y_{k,l}) \\ &\quad = \sum_{\substack{i,j \geq 0, \\ k+i \geq l+j}} \underbrace{(\Delta B_{k,l,i,j} - B_{k,l,i,j} \otimes 1 + 1 \otimes B_{k,l,i,j})}_{=: C_{k,l,i,j}} \urcorner (\xi_i^{2^{k+1}} \xi_j^{2^{l+1}}, a) \end{aligned}$$

where

$$C_{k,l,i,j} = \begin{cases} Q_{k+i+1} \otimes Q_{l+j+1} + Q_{l+j+1} \otimes Q_{k+i+1} & (k+i+1 \neq l+j+1), \\ Q_{k+i+1} \otimes Q_{l+j+1} & (k+i+1 = l+j+1). \end{cases}$$

To see that this is $\Phi(a, Y_{k,l})$ note first that $\nabla(a U_{k,l}) - a \nabla(U_{k,l}) = 0$ by Lemma 4.5. We can compute $\Phi(a, Y_{k,l}) = \nabla(a Y_{k,l}) - a \nabla(Y_{k,l})$ from this by changing every $\nabla U_{n,m}$ to $\nabla Y_{n,m}$. Since $\nabla U_{k,l} = \nabla Y_{k,l}$ for $k < l$ and

$$\nabla U_{k+i,l+j} = \begin{cases} \nabla Y_{k+i,l+j} + C_{k,l,i,j} & (k+i \geq l+j) \\ \nabla Y_{k+i,l+j} & (k+i < l+j) \end{cases}$$

this introduces exactly the error terms from the $C_{k,l,i,j}$.

The case of Z_k is similar and left to the reader. □

Now define $\mathfrak{X}$, $L : R_E \to A \otimes A$ by $\nabla(r) = \mathfrak{X}(r) + \mu_0 L(r)$. Recall that $E_1 = \iota(A) \oplus u(R_E)$ and let $\Delta_1 : E_1 \to (E \hat{\otimes} E)_1$ be given by

$$\Delta_1 (\iota(a)) = \iota_\sharp (\Delta(a)), \quad \Delta_1 (u(r)) = u_\sharp (\Delta_0(r)) + \iota_\sharp (\mathfrak{X}(r)). \tag{4.2}$$

Lemma 4.7. *With this coproduct $E_\bullet$ becomes a secondary Hopf algebra.*

Proof. First note that Δ_1 is right-linear and fits into a commutative diagram

$$\begin{array}{ccccccc}
A & \overset{\iota}{\rightarrowtail} & E_1 & \overset{\partial}{\longrightarrow} & E_0 & \twoheadrightarrow & A \\
\downarrow \Delta & & \downarrow \Delta_1 & & \downarrow \Delta_0 & & \downarrow \Delta \\
A \otimes A & \overset{\iota_\sharp}{\rightarrowtail} & (E \hat{\otimes} E)_1 & \overset{\partial}{\longrightarrow} & E_0 \otimes E_0 & \twoheadrightarrow & A \otimes A.
\end{array}$$

$\Delta_\bullet : E_\bullet \to (E \hat{\otimes} E)_\bullet$ is therefore a map of $[p]$-algebras in the sense of [Bau06, 12.1.2 (4)]. There is also a natural augmentation $\epsilon_\bullet : E_\bullet \to G_\bullet$ where $G_\bullet = (\mathbb{F} \hookrightarrow \mathbb{F} + \mu_0 \mathbb{F} \to \mathbb{G} \twoheadrightarrow \mathbb{F})$ is the unit object for the folding product.

It remains to verify the usual identities

$$(\epsilon_\bullet \hat{\otimes} \mathrm{id})\Delta_\bullet = \mathrm{id} = (\mathrm{id} \hat{\otimes} \epsilon_\bullet)\Delta_\bullet, \quad (\Delta_\bullet \hat{\otimes} \mathrm{id})\Delta_\bullet = (\mathrm{id} \hat{\otimes} \Delta_\bullet)\Delta_\bullet.$$

This can be done on the A generators $\mu_0, U_{k,l}, X_{k,l}, \mu_0 X_{k,l} \in E_1$. We have $\Delta_1(\mu_0) = \mu_0 \hat{\otimes} 1 = 1 \hat{\otimes} \mu_0$ and

$$\begin{aligned}
\Delta_1 (U_{k,l}) &= U_{k,l} \hat{\otimes} 1 + 1 \hat{\otimes} U_{k,l} + Q_l \hat{\otimes} Q_k, \qquad \text{(for } k < l) \\
\Delta_1 (X_{k,l}) &= X_{k,l} \hat{\otimes} 1 + 1 \hat{\otimes} X_{k,l} + Q_l \hat{\otimes} Q_k, \\
\Delta_1 (\mu_0 X_{k,l}) &= \mu_0 X_{k,l} \hat{\otimes} 1 + 1 \hat{\otimes} \mu_0 X_{k,l}.
\end{aligned}$$

Then, for example,

$$\begin{aligned}
(\mathrm{id} \hat{\otimes} \Delta_1)\Delta_1 (U_{k,l}) &= (\mathrm{id} \otimes \Delta_1) \left(U_{k,l} \hat{\otimes} 1 + 1 \hat{\otimes} U_{k,l} + Q_l \hat{\otimes} Q_k\right) \\
&= U_{k,l} \hat{\otimes} 1 \hat{\otimes} 1 + 1 \hat{\otimes} U_{k,l} \hat{\otimes} 1 + 1 \hat{\otimes} 1 \hat{\otimes} U_{k,l} \\
&\quad + 1 \hat{\otimes} Q_l \hat{\otimes} Q_k + Q_l \hat{\otimes} 1 \hat{\otimes} Q_k + Q_l \hat{\otimes} Q_k \hat{\otimes} 1 \\
&= (\Delta_1 \hat{\otimes} \mathrm{id})\Delta_1 (U_{k,l}).
\end{aligned}$$

We leave the remaining cases to the reader. □

Our Δ_1 fails to be left-linear or symmetric; as in [Bau06, 14.1] that failure is captured by the *left action operator* L and the *symmetry operator* S as defined in the following lemma.

Lemma 4.8. *For $e \in E_1$ and $a \in A$ one has*

$$\Delta_1(ae) = a\Delta_1(e) + \iota_\sharp (\kappa(a)L(\partial e)), \quad T\Delta_1(e) = \Delta_1(e) + \iota_\sharp (S(\partial e))$$

with $S(r) = (1 + T)\mathfrak{X}(r)$ where $T : A \otimes A \to A \otimes A$ is the twist map.

Proof. That $S(r) = (1+T)\mathfrak{X}(r)$ is obvious from the definition. For the left-linearity defect one computes

$$\begin{aligned}
\Delta_1\left(a \cdot u(r)\right) &= \Delta_1\left(u(ar) + \iota\left(\operatorname{op}(a,r)\right)\right) \\
&= u_\sharp\left(\Delta_0(ar)\right) + \iota_\sharp\left(\mathfrak{X}(ar) + \Delta \operatorname{op}(a,r)\right), \\
a \cdot \Delta_1\left(u(r)\right) &= a \cdot \left(u_\sharp\left(\Delta_0(r)\right) + \iota_\sharp\left(\mathfrak{X}(r)\right)\right) \\
&= u_\sharp\left(a \cdot \Delta_0(r)\right) + \iota_\sharp\left(\operatorname{op}_\sharp\left(a, \Delta_0(r)\right) + a \cdot \mathfrak{X}(r)\right).
\end{aligned}$$

Therefore $\Delta_1\left(au(r)\right) - a\Delta_1\left(u(r)\right)$ is

$$\iota_\sharp\left(\mathfrak{X}(ar) - a\mathfrak{X}(r) + \Delta \operatorname{op}(a,r) - \operatorname{op}_\sharp\left(a, \Delta_0(r)\right)\right)$$

which by Lemma 4.6 is

$$\iota_\sharp\left(\mathfrak{X}(ar) - a\mathfrak{X}(r) + \nabla(ar) - a\nabla(r)\right) = \iota_\sharp\left(\kappa(a)L(r)\right). \qquad \square$$

Note that in Baues's book L was originally defined as a certain map $L : A \otimes R \to A \otimes A$. However, it was shown in [BJ04a, 12.7] that $L(a \otimes r) = \kappa(a)L(\mathrm{Sq}^1 \otimes r)$, so our $L(r)$ corresponds to $L(\mathrm{Sq}^1 \otimes a)$ in [Bau06].

4.4. Proof of $B_\bullet \sim E_\bullet$. We are now very close to establishing the weak equivalence between $E_\bullet$ and the secondary Steenrod algebra $B_\bullet$. Recall that B_0 is the free associative algebra over $\mathbb{Z}/4$ on the Sq^k with $k > 0$. Let $\mathfrak{c}_0 : B_0 \to E_0$ be the multiplicative map with $B_0 \ni \mathrm{Sq}^n \mapsto \mathrm{Sq}^n \in D_0$. It's easily checked that $\mathfrak{c}_0$ is also comultiplicative.

Let $\mathfrak{c}_0^* E_1$ be defined as the pullback of $E_1 \to E_0$ along $\mathfrak{c}_0$. We then have a commutative diagram

$$\begin{array}{ccccccc}
A & \rightarrowtail & E_1 & \longrightarrow & E_0 & \twoheadrightarrow & A \\
\| & & \uparrow{\scriptstyle \mathfrak{c}_1} & & \uparrow{\scriptstyle \mathfrak{c}_0} & & \| \\
A & \rightarrowtail & \mathfrak{c}_0^* E_1 & \longrightarrow & B_0 & \twoheadrightarrow & A
\end{array}$$

that defines a new sequence $\mathfrak{c}^* E_\bullet$ together with a weak equivalence to $E_\bullet$. We will prove that $\mathfrak{c}^* E_\bullet \cong B_\bullet$.

Lemma 4.9. *$\mathfrak{c}^* E$ inherits a secondary Hopf algebra structure from $E_\bullet$ such that the map $\mathfrak{c}^* E_\bullet \to E_\bullet$ is a map of secondary Hopf algebras.*

Proof. Indeed, using the splitting $(\mathfrak{c}^* E \,\hat{\otimes}\, \mathfrak{c}^* E)_1 = \iota'_\sharp\left(A \otimes A\right) \oplus u'_\sharp\left(R_{B \otimes B}\right)$ we can transport the definition (4.2) to

$$\Delta_1\left(\iota'(a)\right) = \iota'_\sharp\left(\Delta(a)\right), \quad \Delta_1\left(u'(r)\right) = u'_\sharp\left(\Delta_0(r)\right) + \iota'_\sharp\left(\mathfrak{X}(\mathfrak{c}_0(r))\right).$$

We leave the details to the reader. $\square$

Note that the left action and symmetry operators of $\mathfrak{c}^* E_\bullet$ are given by $L' = L \circ \mathfrak{c}_0$ and $S' = S \circ \mathfrak{c}_0$. The following lemma therefore shows that these agree with the operators from the secondary Steenrod algebra.

Lemma 4.10. *Decompose* $\nabla\mathfrak{c}_0|_{R_B} : R_B \to A \otimes A \oplus \mu_0 A \otimes A$ *as*

$$\nabla\left(\mathfrak{c}_0(r)\right) = \mathfrak{X}(r) + \mu_0 L(r) \quad \textit{with} \quad \mathfrak{X},\, L : R_B \to A \otimes A.$$

Then $r \mapsto L(r)$ *resp.* $r \mapsto (1+T)\mathfrak{X}(r)$ *coincide with the left-action resp. symmetry operator of* $B_\bullet$.

Proof. For $0 < n < 2m$ let $[n,m] \in R_B$ denote the Adem relation

$$\underbrace{\mathrm{Sq}^n \otimes \mathrm{Sq}^m + \sum_{1 \le k \le \frac{n}{2}} \binom{m-k-1}{n-2k} \mathrm{Sq}^{m+n-k} \otimes \mathrm{Sq}^k}_{=\langle n,m\rangle} + \underbrace{\binom{m-1}{n} \mathrm{Sq}^{m+n}}_{=\Lambda_{n,m}}.$$

Together with $2 \in R_B$ the $[n,m]$ generate R_B as a B_0-bimodule. We let $F^1 = \mathbb{Z}/2\{\mathrm{Sq}^n | n \ge 1\}$, so $\langle n,m\rangle \in F^1 \otimes F^1$ and $\Lambda_{n,m} \in F^1$.

According to [BJ04a, 12.7] or [Bau06, 14.4.3] the left action map is the unique bilinear $L : R_B \to A \otimes A$ with $L([n,m]) = L_R\left(\langle n,m\rangle\right)$ where $L_R : F^1 \otimes F^1 \to A \otimes A$ is given by

$$L_R\left(\mathrm{Sq}^n \otimes \mathrm{Sq}^m\right) = \sum_{\substack{n_1+n_2=n \\ m_1+m_2=m \\ m_1,n_2 \text{ odd}}} \mathrm{Sq}^{n_1}\mathrm{Sq}^{m_1} \otimes \mathrm{Sq}^{n_2}\mathrm{Sq}^{m_2}.$$

Lemma 4.6 proves that the L that we extracted from ∇ is also bilinear, so we only have to verify that it gives the right value on the Adem relations. We now compute

$$\begin{aligned} \text{(4.3)} \quad \mathrm{Sq}^n * \mathrm{Sq}^m &= \mathrm{Sq}^n\mathrm{Sq}^m + \psi(\mathrm{Sq}^n)\psi(Sq^m)\mu_0 + X_{-1}\psi(\mathrm{Sq}^n)\kappa(\mathrm{Sq}^m) \\ &= \mathrm{Sq}^n\mathrm{Sq}^m + X_0\mathrm{Sq}^{n-1}X_0\mathrm{Sq}^{m-1}\mu_0 + X_{-1,0}\mathrm{Sq}^{n-1}\mathrm{Sq}^{m-1}. \end{aligned}$$

For the μ_0-component we then find

$$\begin{aligned} &\nabla(X_0\mathrm{Sq}^{n-1}X_0\mathrm{Sq}^{m-1}) \\ &= \big((1\otimes Q_0)\Delta\mathrm{Sq}^{n-1}\big) \cdot \big((Q_0 \otimes 1)\Delta\mathrm{Sq}^{m-1}\big) \\ &= \left(\sum_{\substack{n_1+n_2=n, \\ n_2 \text{ odd}}} \mathrm{Sq}^{n_1} \otimes \mathrm{Sq}^{n_2}\right) \cdot \left(\sum_{\substack{m_1+m_2=m, \\ m_1 \text{ odd}}} \mathrm{Sq}^{m_1} \otimes \mathrm{Sq}^{m_2}\right) \end{aligned}$$

as claimed.

The identification of $S = (1+T)\mathfrak{X}$ with the symmetry operator proceeds similarly. We first evaluate $S([n,m])$. Moving μ_0 to the right gives

$$\nabla(\mathfrak{c}_0(r)) = \mu_0 L(r) + \mathfrak{X}(r) = L(r)\mu_0 + \underbrace{\kappa(L(r)) + \mathfrak{X}(r)}_{=:\tilde{\mathfrak{X}}(r)}.$$

We claim that $\mathrm{Sq}^n\mathrm{Sq}^m \in D_0$ does not have any $Y_{k,l}$-component with $0 \le k, l$. Indeed, from the coproduct formula in D_0 we find

$$\Delta\xi_{n,m} \equiv \xi_n\xi_m \otimes \xi_1 \bmod \xi_{k,l} \otimes 1,\, 1 \otimes \xi_{k,l},\, 1 \otimes \xi_j \text{ with } j \ge 2.$$

From (4.3) we then find

$$\tilde{\mathfrak{X}}(\mathrm{Sq}^n\mathrm{Sq}^m) = \nabla\mathrm{Sq}^n\mathrm{Sq}^m + \nabla X_{-1,0}\mathrm{Sq}^{n-1}\mathrm{Sq}^{m-1} = 0.$$

It follows that $S([n,m]) = (1+T)\kappa\left(L([n,m])\right) = (1+T)L\left(\kappa([n,m])\right)$. We still need to show that this is the expected outcome. Let $\langle n,m\rangle = \sum_i \mathrm{Sq}^{n_i}\otimes \mathrm{Sq}^{m_i}$. Expanding slightly on the computation above, we see that

$$L([n,m]) = \sum_i \nabla\left(X_{0,0}\mathrm{Sq}^{n_i-1}\mathrm{Sq}^{m_i-1} + X_{0,1}\mathrm{Sq}^{n_i-3}\mathrm{Sq}^{m_i-1}\right).$$

Therefore

$$(1+T)L(\kappa([n,m])) = \sum_i (1+T)\nabla X_{0,1}\left(\mathrm{Sq}^{n_i-4}\mathrm{Sq}^{m_i-1} + \mathrm{Sq}^{n_i-3}\mathrm{Sq}^{m_i-2}\right)$$

where we have ignored the $X_{0,0}(\cdots)$ because $(1+T)\nabla X_{0,0} = 0$. Since $\Lambda_{n,m} = \sum_i \mathrm{Sq}^{n_i}\mathrm{Sq}^{m_i} \in F^1$ we have

$$0 = \urcorner(\xi_2, \sum_i \mathrm{Sq}^{n_i}\mathrm{Sq}^{m_i}) = \sum_i \mathrm{Sq}^{n_i-2}\mathrm{Sq}^{m_i-1},$$
$$0 = \urcorner(\xi_1^2, \urcorner(\xi_2, \sum_i \mathrm{Sq}^{n_i}\mathrm{Sq}^{m_i})) = \sum_i \left(\mathrm{Sq}^{n_i-4}\mathrm{Sq}^{m_i-1} + \mathrm{Sq}^{n_i-2}\mathrm{Sq}^{m_i-3}\right).$$

We finally arrive at

$$(1+T)L(\kappa([n,m])) = \sum_i (1+T)\nabla X_{0,1}\left(\mathrm{Sq}^{n_i-2}\mathrm{Sq}^{m_i-3} + \mathrm{Sq}^{n_i-3}\mathrm{Sq}^{m_i-2}\right).$$

In the notation of the remark following [Bau06, 16.2.3] this is just $(1+T)K[n,m]$ where it is also affirmed that this is the correct value for $S\left([n,m]\right)$.

The proof of the lemma will be complete, once we have verified that S has the right linearity properties. From Lemma 4.6 we see that the linearity defect of ∇ is symmetrical; therefore $(1+T)\nabla = S + \mu_0(1+T)L$ is actually bilinear. For S this translates into

$$S(ra) = S(r)a, \quad S(ar) = aS(r) + (1+T)\kappa(a)L(r).$$

This agrees with the characterization in [Bau06, 14.5.2]. □

Corollary 4.11. *There is an isomorphism* $\mathfrak{c}^*E_\bullet \cong B_\bullet$.

Proof. Apply the Uniqueness Theorem [Bau06, 15.3.13]. □

This also proves Theorem 1.1 since we have by construction a chain of weak equivalences $\mathfrak{c}^*E_\bullet \xrightarrow{\sim} E_\bullet \xrightarrow{\sim} D_\bullet$.

Remark 4.12. The map $S : R_E \to A \otimes A$ does not factor through the projection $R_E \to R_D$. This can be seen from the computation

$$\begin{aligned}[3,2] &= 2\mathrm{Sq}(2,1) + 2\mathrm{Sq}(5) + (X_{-1,0} + Y_{-1,0})(\mathrm{Sq}(0,1) + \mathrm{Sq}(3)) \\ &\quad + X_{0,0}\mathrm{Sq}(2) + X_{0,1} + \mu_0 X_{0,0}(\mathrm{Sq}(0,1) + \mathrm{Sq}(3)) + \mu_0 X_{0,1}\mathrm{Sq}(1), \\ [2,2]\mathrm{Sq}^1 &= 2\mathrm{Sq}(2,1) + 2\mathrm{Sq}(5) + (X_{-1,0} + Y_{-1,0})(\mathrm{Sq}(0,1) + \mathrm{Sq}(3)) \\ &\quad + \mu_0 X_{0,0}(\mathrm{Sq}(0,1) + \mathrm{Sq}(3)) + \mu_0 X_{0,1}\mathrm{Sq}(1).\end{aligned}$$

One finds that $S([3,2]) = Q_1 \otimes Q_0 + Q_0 \otimes Q_1$ and $S([2,2]\mathrm{Sq}^1) = 0$ even though $[3,2]$ and $[2,2]\mathrm{Sq}^1$ have the same image in D_0. This shows that the secondary diagonal $\Delta_1 : B_1 \to (B \hat{\otimes} B)_1$ has no analogue over $D_\bullet$.

Remark 4.13. One can use $D_\bullet$ as a replacement for $B_\bullet$ in the computation of the d_2-differential in the Adams spectral sequence. To see this we first need to recall the description of this computation from [BJ04b].

Let $d : C_* \to C_{*-1}$ be an A-free resolution of $\mathbb{F}_2$ and let $G_* \subset C_*$ be an A-basis. Write $d(g) = \sum_h a_{g,h} \cdot h$ for $g, h \in G_*$, $a_{g,h} \in A$ and choose liftings $\hat{a}_{g,h} \in B_0$ of the $a_{g,h}$. Then $r_{g,l} = \sum_h \hat{a}_{g,h}\hat{a}_{h,l}$ lies in R_B since $d^2 = 0$. We then get an A-linear map $\rho : C_* \to R_B \otimes_A C_{*-2}$ with $\rho(ag) = \sum_l a r_{g,l} \otimes l$.

Now recall from [BJ04b, 8.6] that the d_2-differential on $\mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ is computed from a nonlinear chain map $\delta : C_* \to C_{*-2}$ with $\delta\partial = \partial\delta$ and

$$\delta(ax) = a\delta(x) + \mathrm{op}_B(a, \rho(x)). \tag{4.4}$$

Here $\mathrm{op}_B : A \otimes R_B \to A$ is the multiplication map for $B_\bullet$. But since $\mathrm{op}_B(a, r) = \mathrm{op}_D(a, \mathfrak{c}_0(r))$ we can express the condition (4.4) also through the $\mathfrak{c}_0$-images of the $r_{g,l}$. It follows that we could just as well have started with the D_0-liftings $\mathfrak{c}_0(\hat{a}_{g,h})$ in place of the $\hat{a}_{g,h}$, which would have avoided all references to $B_\bullet$.

Appendix A. EBP and a model at odd primes

Let p be a prime and let BP denote the Brown-Peterson spectrum at p. In this appendix we show how a model of the secondary Steenrod algebra can be extracted from BP if $p > 2$.

Recall that the homology $H_*\mathrm{BP}$ is the polynomial algebra over $\mathbb{Z}_{(p)}$ on generators $(m_k)_{k=1,2,\dots}$ and that $\mathrm{BP}_* \subset H_*\mathrm{BP}$ is the subalgebra generated by the Araki generators $(v_k)_{k=1,2,\dots}$. Let $\mathrm{EBP}_* = E(\mu_k \,|\, k \geq 0) \otimes \mathrm{BP}_*$ with exterior algebra generators μ_k of degree $|\mu_k| = |v_k| + 1$. EBP_* is a free BP_*-module and defines a Landweber exact homology theory EBP. Obviously, the representing spectrum is just a wedge of copies of BP. As usual, we let $I = (v_k) \subset \mathrm{BP}_*$ be the maximal invariant ideal.

The cooperation Hopf algebroid $\mathrm{EBP}_*\mathrm{EBP}$ is very easy to compute:

Lemma A.1. *One has* $\mathrm{EBP}_*\mathrm{EBP} = E(\mu_k) \otimes_{\mathbb{Z}_{(p)}} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathbb{Z}_{(p)}} E(\tau_k)$ *with*

$$\eta_R(\mu_n) = \sum_{k=0}^{n} \mu_k t_{n-k}^{p^k} + \tau_n \tag{A.1}$$

and

$$\Delta\tau_n = 1 \otimes \tau_n + \sum_{k=0}^{n} \tau_k \otimes t_{n-k}^{p^k} + \sum_{0 \le a \le n} \mu_a \left(-\Delta t_{n-a}^{p^a} + \sum_{b+c=n-a} t_b^{p^a} \otimes t_c^{p^{a+b}} \right).$$

The other structure maps are inherited from $\mathrm{BP}_*\mathrm{BP}$.

Proof. We use (A.1) to define the $\tau_k \in \mathrm{EBP}_*\mathrm{EBP} = E(\mu_k) \otimes \mathrm{BP}_*\mathrm{BP} \otimes E(\mu_k)$. $\Delta\tau_n$ can then be computed from $(\eta_R \otimes \mathrm{id})\eta_R(\mu_n) = \Delta\eta_R(\mu_n)$. $\square$

We can put a differential on EBP by setting $\partial\mu_k = v_k$ and this turns $\mathrm{EBP}_*\mathrm{EBP}$ into a differential Hopf algebroid.

Corollary A.2. *For* $p > 2$ *the homology Hopf algebroid of* $\mathrm{EBP}_*\mathrm{EBP}$ *with respect to* ∂ *is the dual Steenrod algebra* A_*.

Proof. We have $\partial\tau_n = \eta_R(v_n) - \sum_{k=0}^{n} v_k t_{n-k}^{p^k} \equiv 0 \bmod I^2$, so there are $\tau_n' \equiv \tau_n \bmod I$ with $\partial\tau_n' = 0$. Therefore $H^*(\mathrm{EBP}_*; \partial) = \mathbb{F}_p$ and

$$H^*(\mathrm{EBP}_*\mathrm{EBP}; \partial) = \mathbb{F}_p[t_k | k \ge 1] \otimes E(\tau_n' | n \ge 0) = A_*.$$

Lemma A.1 then shows that the induced coproduct on A_* coincides with the usual one. $\square$

We prefer to work with operations rather than cooperations. Write $E = \mathrm{EBP}_*$, $\Gamma_* = \mathrm{EBP}_*\mathrm{EBP}$ and let $\Gamma = \mathrm{Hom}_E(\Gamma_*, E)$ be the operation algebra $\mathrm{EBP}^*\mathrm{EBP}$ of EBP. Then Γ is a differential algebra and for odd p its homology $H(\Gamma; \partial)$ can be identified with the Steenrod algebra A. We therefore get an exact sequence $P_\bullet$

$$A \rightarrowtail \operatorname{coker}\partial \xrightarrow{\partial} \ker\partial \twoheadrightarrow A. \tag{A.2}$$

by splicing $H(\Gamma;\partial) \hookrightarrow \Gamma/\operatorname{im}\partial \twoheadrightarrow \operatorname{im}\partial$ and $\operatorname{im}\partial \hookrightarrow \ker\partial \twoheadrightarrow H(\Gamma;\partial)$. We claim that for odd p this sequence is a model for the secondary Steenrod algebra.

Theorem A.3. *Let* $p > 2$ *and let* $B_\bullet \to G_\bullet$ *be the secondary Steenrod algebra with its canonical augmentation to*

$$G_\bullet = (\mathbb{F}_p \hookrightarrow \mathbb{F}_p\{1, \mu_0\} \to \mathbb{Z}_{(p)} \twoheadrightarrow \mathbb{F}_p).$$

Then there is a diagram of crossed algebras

$$\begin{array}{ccccccc} P_\bullet & \twoheadrightarrow & (P/J^2)_\bullet & \leftarrowtail & T_\bullet & \longleftarrow & B_\bullet \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G^P{}_\bullet & \twoheadrightarrow & G^{P/J^2}{}_\bullet & \leftarrowtail & G^T{}_\bullet & \longleftarrow & G_\bullet \end{array} \tag{A.3}$$

where all horizontal maps are weak equivalences.

Note that $P_\bullet$ itself cannot be the target of a comparison map from $B_\bullet$ as p^2 is zero in B_0 but not in P_0. In the statement we have also singled out an intermediate sequence $T_\bullet$. This sequence is of independent interest because it is quite small and given by explicit formulas.

To construct (A.3) we first establish the diagram of augmentations. Let $J = I \cdot E \subset E$.

Lemma A.4. *Let $ZE = \ker E \xrightarrow{\partial} E$ and $w_k = v_k\mu_0 - p\mu_k = -\partial(\mu_0\mu_k) \in J$. Then there is a commutative diagram*

$$
\begin{array}{ccccccccc}
G^P{}_\bullet & & \mathbb{F}_p & \rightarrowtail & E/\partial E & \xrightarrow{\partial} & ZE & \twoheadrightarrow & \mathbb{F}_p \\
\downarrow & & \| & & \downarrow & & \downarrow & & \| \\
G^{P/J^2}{}_\bullet & & \mathbb{F}_p & \rightarrowtail & E/J & \xrightarrow{\partial} & \ker(E/J^2 \xrightarrow{\partial} E/J^3) & \twoheadrightarrow & \mathbb{F}_p \\
\uparrow & & \| & & \uparrow & & \uparrow & & \| \\
G^T{}_\bullet & & \mathbb{F}_p & \rightarrowtail & \mathbb{F}_p\{1, \mu_k, \mu_0\mu_k\} & \xrightarrow{\partial} & \mathbb{Z}_{(p)}[v_k, w_k]/J^2 & \twoheadrightarrow & \mathbb{F}_p \\
\uparrow & & \| & & \uparrow & & \uparrow & & \| \\
G_\bullet & & \mathbb{F}_p & \rightarrowtail & \mathbb{F}_p\{1, \mu_0\} & \xrightarrow{\partial} & \mathbb{Z}/(p^2) & \twoheadrightarrow & \mathbb{F}_p
\end{array}
$$

with exact rows.

Proof. This is straightforward, except for the exactness of $G^{P/J^2}{}_\bullet$. First note that

$$\mathbb{F}_p \rightarrowtail J/J^2 \xrightarrow{\partial} J^2/J^3 \xrightarrow{\partial} J^3/J^4 \xrightarrow{\partial} \cdots$$

is exact because it can be identified with the super deRham complex $\Omega^n = \mathbb{F}_p\{\mu^\epsilon d\mu_{i_1} \cdots d\mu_{i_n}\}$ with $df = \sum \frac{\partial f}{\partial \mu_k} d\mu_k$ via $v_k = d\mu_k$. Let E_J denote the complex

$$E/J \xrightarrow{\partial} E/J^2 \xrightarrow{\partial} E/J^3 \xrightarrow{\partial} E/J^4 \xrightarrow{\partial} \cdots .$$

Its associated graded with respect to the J-adic filtration is the sum of shifted copies Ω^{k+*} for $k \geq 0$, so one has $H_k(E_J) = \mathbb{F}_p$ for all k. The exactness of $\mathbb{F}_p \hookrightarrow E/J \to \big(\ker \partial : E/J^2 \to E/J^3\big) \twoheadrightarrow \mathbb{F}_p$ is an easy consequence. □

Now let $P(R)Q(\epsilon) \in \Gamma = \mathrm{Hom}_E(\Gamma_*, E)$ denote the dual of $t^R\tau^\epsilon$ with respect to the monomial basis of Γ_*. (One easily verifies that this is indeed the product of $P(R) := P(R)Q(0)$ and $Q(\epsilon) := P(0)Q(\epsilon)$ as suggested by the notation.) We can think of Γ as the set $E\{\{P(R)Q(\epsilon)\}\}$ of infinite sums $\sum a_{R,\epsilon}P(R)Q(\epsilon)$ with coefficients $a_{R,\epsilon} \in E$.

It is important to realize that the $P(R)$ are *not* ∂-cycles: for $p = 2$, for example, one finds that $\partial\tau_n \equiv v_{n-1}^2 t_1 \mod I^3$ which shows that $\partial P^1 \equiv 4Q(0,1) + v_1^2 Q(0,0,1) + \cdots \mod I^3$.

Lemma A.5. *Let $p > 2$. Then $\partial\tau_n \equiv 0 \mod I^3$.*

Proof. The claim is equivalent to $\eta(v_n) \equiv \sum_{0\le k\le n} v_k t_{n-k}^{p^k} \mod I^3$. We leave this as an exercise. □

The following lemma defines $(P/J^2)_\bullet$ and its weak equivalence with $P_\bullet$.

Lemma A.6. *Let $Z\Gamma = \ker \partial : \Gamma \to \Gamma$. There is a commutative diagram*

$$\begin{array}{ccccccccc}
P_\bullet & \quad & A & \rightarrowtail & \Gamma/\partial\Gamma & \xrightarrow{\partial} & Z\Gamma & \twoheadrightarrow & A \\
\downarrow\!\!\downarrow & & \| & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \| \\
P/J^2{}_\bullet & & A & \rightarrowtail & \Gamma/J\Gamma & \xrightarrow{\partial} & \ker(\Gamma/J^2\Gamma \xrightarrow{\partial} \Gamma/J^3\Gamma) & \twoheadrightarrow & A
\end{array}$$

with exact rows.

Proof. Choose $\tilde\tau_k \in \Gamma_*$ with $\tilde\tau_k \equiv \tau_k \mod I$ and $\partial\tilde\tau_k = 0$. Let $X(R;\epsilon) \in \Gamma$ be dual to $t^R\tilde\tau^\epsilon$. Then $\Gamma = \prod_{R,\epsilon} E \cdot X(R;\epsilon)$ and $\partial X(R;\epsilon) = 0$. It follows that the exactness can be checked on the coefficients alone where it was established in Lemma A.4. □

The construction of $T_\bullet$ requires a more explicit understanding of Γ_*/I^2.

Lemma A.7. *For a family (x_k) let $\Phi_{p^n}(x_k) \in \mathbb{F}_p[x_k]$ be defined by $\sum x_k^{p^n} - (\sum x_k)^{p^n} = p\Phi_{p^n}(x_k)$. Then modulo I^2 one has*

$$\Delta t_n \equiv \sum_{n=a+b} t_a \otimes t_b^{p^a} + \sum_{0<k\le n} v_k \Phi_{p^k}\left(t_a \otimes t_b^{p^a} \,\middle|\, a+b = n-k\right).$$

Let $w_k = -\partial(\mu_0\mu_k) = v_k\mu_0 - p\mu_k$. Then

$$\Delta\tau_n \equiv 1 \otimes \tau_n + \sum_{n=a+b} \tau_a \otimes t_b^{p^a} + \sum_{0<k\le n} w_k \Phi_{p^k}\left(t_a \otimes t_b^{p^a} \,\middle|\, a+b = n-k\right).$$

Furthermore,

$$\eta_R(v_n) \equiv \sum_{0\le k\le n} v_k t_{n-k}^{p^k},$$

$$\eta_R(w_n) \equiv -p\tau_n + \sum_{1\le k<n} w_k t_{n-k}^{p^k} + \sum_{0\le k\le n} v_k t_{n-k}^{p^k}\tau_0,$$

Proof. The v_k are defined by $pm_n = \sum_{n=a+b} m_a v_b^{p^a}$ and it follows easily that $v_n \equiv pm_n$ modulo $I^2 \cdot H_*(\mathrm{EBP})$. Recall that $\eta_R(m_n) = \sum_{n=a+b} m_a t_b^{p^a}$

and that Δt_n can be computed from $(\eta_R \otimes \mathrm{id})\eta_R(m_n) = \Delta\eta_R(m_n)$. Inductively, this gives

$$\begin{aligned}\Delta t_n &= \sum_{n=a+b} t_a \otimes t_b^{p^a} + \sum_{0<k\le n} m_k \left(-\Delta t_{n-k}^{p^k} + \sum_{n-k=a+b} t_a^{p^k} \otimes t_b^{p^{k+a}} \right) \\ &\equiv \sum_{n=a+b} t_a \otimes t_b^{p^a} + \sum_{0<k\le n} v_k \Phi_{p^k} \left(t_a \otimes t_b^{p^a} \,\middle|\, a+b = n-k \right)\end{aligned}$$

as claimed. The formula for $\Delta\tau_n$ now follows with Lemma A.1. We leave the computation of $\eta_R(v_n)$ and $\eta_R(w_n)$ to the reader. □

Let $S_\bullet = G^T{}_\bullet$ and recall that

$$\begin{aligned} S_0 &= \mathbb{Z}/p^2 + \mathbb{F}_p\{v_k, w_k \,|\, k \ge 1\} \subset E/J^2, \\ S_1 &= \mathbb{F}_p\{1, \mu_k, \mu_0\mu_k\} \subset E/J. \end{aligned}$$

We now define

$$\begin{aligned} T_0 &= S_0\{\{P(R)Q(\epsilon)\}\} \subset \Gamma/J^2\Gamma, \\ T_1 &= S_1\{\{P(R)Q(\epsilon)\}\} \subset \Gamma/J\Gamma. \end{aligned}$$

Lemma A.8. *This defines a crossed algebra $T_\bullet \subset (P/J^2)_\bullet$ as claimed in Theorem A.3.*

Proof. Lemma A.7 shows that $(S_0, S_0[t_k, \tau_k])$ is a sub Hopf algebroid of $(E/J^2, \Gamma_*/J^2)$ with $\Gamma_*/J^2 = E/J^2 \otimes_{S_0} S_0[t_k, \tau_k]$. Therefore

$$T_0 = \mathrm{Hom}_{S_0}(S_0[t_k, \tau_k], S_0) \hookrightarrow \mathrm{Hom}_{E/J^2}(\Gamma_*/J^2, E/J^2) = \Gamma/J^2$$

is the inclusion of a subalgebra. By Lemma A.5, T_0 is actually contained in $(P/J^2)_0 = \ker\partial : \Gamma/J^2 \to \Gamma/J^3$. The remaining details are left to the reader. □

To prove the theorem it only remains to establish the weak equivalence $B_\bullet \to T_\bullet$. Recall that B_0 is the free $\mathbb{Z}/p^2$-algebra on generators Q_0 and P^k, $k \ge 1$. We can therefore define a multiplicative $\mathfrak{p}_0 : B_0 \to T_0$ via $Q_0 \mapsto Q(1)$ and $P^k \mapsto P(k)$.

Lemma A.9. *There is a weak equivalence $\mathfrak{p} : B_\bullet \to T_\bullet$ that extends $\mathfrak{p}_0$.*

Proof. The multiplication on Γ_* dualizes to a coproduct $\Delta_\Gamma : \Gamma \to \Gamma \mathbin{\widetilde{\otimes}_E} \Gamma$ were $\widetilde{\otimes}_E$ denotes a suitably completed tensor product. This turns Γ into a topological Hopf algebra over E. We define the completed folding product $(P \mathbin{\widehat{\otimes}_E} P)_\bullet$ as the pullback

$$\begin{array}{ccccccc} A\otimes A & \rightarrowtail & \big(\Gamma \mathbin{\widetilde{\otimes}_E} \Gamma\big)/\mathrm{im}\,\partial_\otimes & \xrightarrow{\partial_\otimes} & \ker\partial_\otimes & \twoheadrightarrow & A\otimes A \\ \| & & \uparrow & & \uparrow & & \| \\ A\otimes A & \rightarrowtail & (P \mathbin{\widehat{\otimes}_E} P)_1 & \xrightarrow{\partial_\otimes} & P_0 \mathbin{\widetilde{\otimes}_E} P_0 & \twoheadrightarrow & A\otimes A \end{array}$$

where $\partial_\otimes = \partial \otimes \mathrm{id} + \mathrm{id} \otimes \partial$ is the differential on $\Gamma \mathbin{\widetilde{\otimes}_E} \Gamma$. Δ_Γ then restricts to a coproduct $\Delta_\bullet : P_\bullet \to (P \mathbin{\widehat{\otimes}_E} P)_\bullet$. Note that Δ_1 is bilinear and symmetric, since this is true for Δ_Γ. By restriction we get a $\Delta_\bullet : T_\bullet \to (T \mathbin{\widehat{\otimes}_S} T)_\bullet$ where the right hand side is given by

$$(T \mathbin{\widehat{\otimes}_S} T)_0 = S_0\{\{P(R_1)Q(\epsilon_1) \otimes P(R_2)Q(\epsilon_2)\}\} \subset (P \mathbin{\widehat{\otimes}_E} P)_1/J^2,$$
$$(T \mathbin{\widehat{\otimes}_S} T)_1 = S_1\{\{P(R_1)Q(\epsilon_1) \otimes P(R_2)Q(\epsilon_2)\}\} \subset (P \mathbin{\widehat{\otimes}_E} P)_1/J.$$

Let $\mathfrak{p}^*T_\bullet$ be the pullback of $T_\bullet$ along $B_0 \to T_0$. It inherits a secondary Hopf algebra structure from $T_\bullet$. This structure has $L = S = 0$ since the same is true for $P_\bullet$. Baues's Uniqueness Theorem thus implies $B_\bullet \cong \mathfrak{p}^*T_\bullet$. □

Remark A.10. It seems to be an interesting challenge to relate our 2-primary models to BP and the theory of formal group laws. For $p = 2$ the constructions of the Appendix can only produce a model for an associated graded algebra of the Steenrod algebra. We hope to come back to these questions in the future.

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