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Crossed products of C*-algebras with the weak expectation property

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Abstract. If α is an amenable action of a discrete group G on a unital C*-algebra $\mathcal{A}$, then the crossed-product C*-algebra $\mathcal{A} \rtimes_\alpha \mathrm{G}$ has the weak expectation property if and only if $\mathcal{A}$ has this property.

Contents

1. Introduction

A weak expectation on a unital C*-subalgebra $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ is a unital completely positive (ucp) linear map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}''$ (the double commutant of $\mathcal{B}$) such that $\phi(b) = b$ for every $b \in \mathcal{B}$. A unital C*-algebra $\mathcal{A}$ has the weak expectation property (WEP) if $\pi(\mathcal{A})$ admits a weak expectation for every faithful representation π of $\mathcal{A}$ on some Hilbert space $\mathcal{H}$. Equivalently, if $\mathcal{A} \subset \mathcal{A}^{**} \subset \mathcal{B}(\mathcal{H}_u)$ denotes the universal representation of $\mathcal{A}$, where $\mathcal{A}^{**}$ is the enveloping von Neumann algebra of $\mathcal{A}$, then $\mathcal{A}$ has WEP if and only if there is a ucp map $\phi : \mathcal{B}(\mathcal{H}_u) \to \mathcal{A}^{**}$ that fixes every element of $\mathcal{A}$. The notion of weak expectation first arose in the work of C. Lance on nuclear C*-algebras [4], where it was shown that every unital nuclear C*-algebra has WEP. Twenty years later E. Kirchberg established a number of important properties and characterisations of the weak expectation property in his penetrating study of exactness [3].

A C*-algebra $\mathcal{A}$ has the quotient weak expectation property (QWEP) if $\mathcal{A}$ is a quotient of a C*-algebra with WEP. The class of C*-algebras with QWEP enjoys a number of permanence properties, many of which are enumerated

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in [6, Proposition 4.1] and originate with Kirchberg [3]. For example, if $\mathcal{A}$ is a unital C*-algebra with QWEP and if α is an amenable action of a discrete group G on $\mathcal{A}$, then the crossed product C*-algebra $\mathcal{A} \rtimes_\alpha \mathrm{G}$ has QWEP [6, Proposition 4.1(vi)].

In contrast to QWEP, the weak expectation property appears to have few permanence properties. For example, $\mathcal{A} \otimes_{\min} \mathcal{B}$ may fail to have WEP if $\mathcal{A}$ and $\mathcal{B}$ have WEP; one such example is furnished by $\mathcal{A} = \mathcal{B} = \mathcal{B}(\mathcal{H})$ [5]. In comparison, if $\mathcal{A}$ and $\mathcal{B}$ are nuclear, then so is $\mathcal{A} \otimes_{\min} \mathcal{B}$, and if $\mathcal{A}$ and $\mathcal{B}$ are exact, then so is $\mathcal{A} \otimes_{\min} \mathcal{B}$ [1, §10.1,10.2].

The purpose of this note is to establish the following permanence result for WEP (Theorem 2.1): *if α is an amenable action of a discrete group* G *on a unital C*-algebra $\mathcal{A}$, then $\mathcal{A} \rtimes_\alpha \mathrm{G}$ has the weak expectation property if and only if $\mathcal{A}$ does.* In this regard, the weak expectation property is consistent with the analogous permanence results for nuclear and exact C*-algebras [1, Theorem 4.3.4].

Before turning to the proof, we note that Lance's definition of WEP requires knowledge of all faithful representations of $\mathcal{A}$. It is advantageous, therefore, to have alternate ways to characterise the weak expectation property. We mention two such ways below.

Theorem 1.1 (Kirchberg's Criterion [3]). *A unital C*-algebra $\mathcal{A}$ has the weak expectation property if and only if $\mathcal{A} \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty) = \mathcal{A} \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)$.*

The second description is useful in cases where one desires to fix a particular faithful representation of $\mathcal{A}$.

Theorem 1.2 (A matrix completion criterion [2]). *If $\mathcal{A}$ is a unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$, then the following statements are equivalent:*

(1) *$\mathcal{A}$ has the weak expectation property.*
(2) *If, given $p \in \mathbb{N}$ and $X_1, X_2 \in \mathcal{M}_p(\mathcal{A})$, there exist strongly positive operators $A, B, C \in \mathcal{M}_p(\mathcal{B}(\mathcal{H}))$ such that $A + B + C = 1$ and*

$$Y = \begin{bmatrix} A & X_1 & 0 \\ X_1^* & B & X_2 \\ 0 & X_2^* & C \end{bmatrix}$$

is strongly positive in $\mathcal{M}_{3p}(\mathcal{B}(\mathcal{H}))$, then there also exist $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{M}_p(\mathcal{A})$ with the same property.

By strongly positive one means a positive operator A for which there is a real $\delta > 0$ such that $A \geq \delta 1$.

Chapters 2 and 4 of the book of Brown and Ozawa [1] shall form our main reference for facts concerning amenable groups, amenable actions, and reduced crossed products.

2. The main result

Theorem 2.1. *If α is an amenable action of a discrete group* G *on a unital* C^**-algebra* $\mathcal{A}$*, then* $\mathcal{A} \rtimes_\alpha \mathrm{G}$ *has the weak expectation property if and only if* $\mathcal{A}$ *does.*

Proof. We begin with two preliminary observations that are independent of whether $\mathcal{A}$ has WEP or not.

The first observation is that, because α is an amenable action of G on $\mathcal{A}$, the C*-algebra $\mathcal{A} \rtimes_\alpha \mathrm{G}$ coincides with the reduced crossed product C*-algebra $\mathcal{A} \rtimes_{\alpha,\mathrm{r}} \mathrm{G}$ [1, Theorem 4.3.4(1)]. The second observation is that if $\iota : \mathrm{G} \to \mathrm{Aut}(\mathcal{B})$ denotes the trivial action of G on a unital C*-algebra $\mathcal{B}$, then the action $\alpha \otimes_{\max} \iota$ of G on $\mathcal{A} \otimes_{\max} \mathcal{B}$ is amenable. (The action $\alpha \otimes_{\max} \iota$ of G on $\mathcal{A} \otimes_{\max} \mathcal{B}$ satisfies $\alpha \otimes_{\max} \iota(g)[a \otimes b] = \alpha_g(a) \otimes b$ for all $g \in \mathrm{G}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$ [8, Remark 2.74].)

To prove this second fact, using the properties that define α as an amenable action [1, pp. 124–125], let $\{T_i\}_i$ denote a net of finitely supported positive-valued functions $T_i : \mathrm{G} \to \mathcal{Z}(\mathcal{A})$ (the centre of $\mathcal{A}$) such that $\sum_{g \in \mathrm{G}} T_i(g)^2 = 1$ and

$$\lim_i \left(\left\| \sum_{g \in \mathrm{G}} \left[\alpha_g(T_i(s^{-1}g)) - T_i(g) \right]^* \left[\alpha_g(T_i(s^{-1}g)) - T_i(g) \right] \right\|^2 \right) \to 0$$

for all $s \in \mathrm{G}$. Define finitely supported positive-valued functions

$$\tilde{T}_i : \mathrm{G} \to \mathcal{Z}\left(\mathcal{A} \otimes_{\max} \mathcal{B}\right)$$

by $\tilde{T}_i(g) = T_i(g) \otimes_{\max} 1_{\mathcal{B}}$. Then $\sum_{g \in \mathrm{G}} \tilde{T}_i(g)^2 = 1_{\mathcal{A} \otimes_{\max} \mathcal{B}}$ and the limiting equation above holds with T_i replaced with $\tilde{T}_i$ and α replaced with $\alpha \otimes_{\max} \iota$. Hence, the action $\alpha \otimes_{\max} \iota$ of G on $\mathcal{A} \otimes_{\max} \mathcal{B}$ is amenable.

Assume now that $\mathcal{A}$ has the weak expectation property. By Kirchberg's Criterion (Theorem 1.1),

$$\mathcal{A} \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty) = \mathcal{A} \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty).$$

Let $\iota : \mathrm{G} \to \mathrm{Aut}\left(\mathrm{C}^*(\mathbb{F}_\infty)\right)$ denote the trivial action of G on $\mathrm{C}^*(\mathbb{F}_\infty)$. Thus, $\alpha \otimes_{\max} \iota$ is an amenable action. Hence,

$$\begin{aligned}
(\mathcal{A} \rtimes_\alpha \mathrm{G}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty) &= (\mathcal{A} \rtimes_{\alpha,\mathrm{r}} \mathrm{G}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty) \\
&= (\mathcal{A} \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty)) \rtimes_{\alpha \otimes_{\max} \iota, \mathrm{r}} \mathrm{G} \\
&= (\mathcal{A} \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)) \rtimes_{\alpha \otimes_{\max} \iota, \mathrm{r}} \mathrm{G} \\
&= (\mathcal{A} \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)) \rtimes_{\alpha \otimes_{\max} \iota} \mathrm{G} \\
&= (\mathcal{A} \rtimes_\alpha \mathrm{G}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty),
\end{aligned}$$

where the final equality holds by [8, Lemma 2.75]. Another application of Kirchberg's Criterion implies that $\mathcal{A} \rtimes_\alpha \mathrm{G}$ has WEP.

Conversely, assume that $\mathcal{A} \rtimes_\alpha \mathrm{G}$ has the weak expectation property and that $\mathcal{A} \rtimes_{\alpha,\mathrm{r}} \mathrm{G}$ is represented faithfully on a Hilbert space $\mathcal{H}$. Thus,

$$\mathcal{A} \subset \mathcal{A} \rtimes_{\alpha,\mathrm{r}} \mathrm{G} = \mathcal{A} \rtimes_\alpha \mathrm{G} \subset \mathcal{B}(\mathcal{H})$$

also represents $\mathcal{A}$ faithfully on $\mathcal{H}$. Let $\mathcal{E} : \mathcal{A} \rtimes_{\alpha,\mathrm{r}} \mathrm{G} \to \mathcal{A}$ denote the canonical conditional expectation of $\mathcal{A} \rtimes_{\alpha,\mathrm{r}} \mathrm{G}$ onto $\mathcal{A}$ [1, Proposition 4.1.9]. We now use the criterion of Theorem 1.2 for WEP.

Suppose that $p \in \mathbb{N}$, $X_1, X_2 \in \mathcal{M}_p(\mathcal{A})$, and $A, B, C \in \mathcal{M}_p(\mathcal{B}(\mathcal{H}))$ are such that $A + B + C = 1$ and the matrix

$$Y \;=\; \begin{bmatrix} A & X_1 & 0 \\ X_1^* & B & X_2 \\ 0 & X_2^* & C \end{bmatrix} \in \mathcal{M}_{3p}(\mathcal{B}(\mathcal{H}))$$

is strongly positive. Because $\mathcal{A} \subset \mathcal{A} \rtimes_\alpha \mathrm{G}$ and because $\mathcal{A} \rtimes_\alpha \mathrm{G}$ has WEP, there are, by Theorem 1.2, $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{M}_p(\mathcal{A} \rtimes_\alpha \mathrm{G})$ such that

$$\tilde{Y} \;=\; \begin{bmatrix} \tilde{A} & X_1 & 0 \\ X_1^* & \tilde{B} & X_2 \\ 0 & X_2^* & \tilde{C} \end{bmatrix} \in \mathcal{M}_{3p}(\mathcal{A} \rtimes_\alpha \mathrm{G})$$

is strongly positive and $\tilde{A} + \tilde{B} + \tilde{C} = 1$. Because ucp maps preserve strong positivity, the matrix

$$(\mathcal{E} \otimes \mathrm{id}_{\mathcal{M}_3})[\tilde{Y}] \;=\; \begin{bmatrix} \mathcal{E}(\tilde{A}) & X_1 & 0 \\ X_1^* & \mathcal{E}(\tilde{B}) & X_2 \\ 0 & X_2^* & \mathcal{E}(\tilde{C}) \end{bmatrix} \in \mathcal{M}_{3p}(\mathcal{A})$$

is strongly positive and the diagonal elements sum to $1 \in \mathcal{M}_{3p}(\mathcal{A})$. Thus, $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ satisfies the criterion of Theorem 1.2 for WEP. □

3. A direct proof in the case of amenable groups

The proof of Theorem 2.1 relies on the criteria for WEP given by Theorems 1.1 and 1.2, which seem far removed from the defining condition of Lance and thereby making the argument of Theorem 2.1 somewhat indirect. The purpose of this section is to present a more conceptual proof in the case of amenable discrete groups using Lance's definition of WEP directly together with basic facts about amenable groups and C*-algebras.

In what follows, λ shall denote the left regular representation of G on the Hilbert space $\ell^2(\mathrm{G})$ and e denotes the identity of G. Two properties that an amenable group G is well known to have are:

(i) $\mathcal{A} \rtimes_\alpha \mathrm{G} = \mathcal{A} \rtimes_{\alpha,\mathrm{r}} \mathrm{G}$, for every unital C*-algebra $\mathcal{A}$.

(ii) G admits a Følner net—namely a net $\{F_i\}_{i\in\Lambda}$ of finite subsets $F_i \subset \mathrm{G}$ such that, for every $g \in \mathrm{G}$,

$$\lim_i \frac{|F_i \cap gF_i|}{|F_i|} \;=\; 1\,.$$

(In fact the second property above characterises amenable groups.)

Theorem 3.1. *If α is an action of an amenable discrete group* G *on a unital C^*-algebra $\mathcal{A}$, then $\mathcal{A} \rtimes_\alpha \mathrm{G}$ has the weak expectation property if and only if $\mathcal{A}$ does.*

Proof. Assume first that $\mathcal{A} \rtimes_\alpha \mathrm{G}$ has the weak expectation property. To show that $\mathcal{A}$ has WEP, it is sufficient to show that if $\mathcal{A}$ is represented faithfully as a unital C*-subalgebra of $\mathcal{B}(\mathcal{K})$, for some Hilbert space $\mathcal{K}$, and if $\pi_u^{\mathcal{A}} : \mathcal{A} \to \mathcal{B}(\mathcal{H}_u^{\mathcal{A}})$ is the universal representation of $\mathcal{A}$, then there a ucp map $\omega : \mathcal{B}(\mathcal{K}) \to \mathcal{A}^{**}$ such that $\omega(a) = \pi_u^{\mathcal{A}}(a)$ for every $a \in \mathcal{A}$.

To this end, let $\mathcal{A} \rtimes_\alpha \mathrm{G} \subset \mathcal{B}(\mathcal{H}_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}})$ be the universal representation of $\mathcal{A} \rtimes_\alpha \mathrm{G}$. Because $\mathcal{A}$ is unital, $\mathcal{A}$ is a unital C*-subalgebra of $\mathcal{A} \rtimes_\alpha \mathrm{G}$. Hence,

$$\mathcal{A} \subset \mathcal{A} \rtimes_\alpha \mathrm{G} \subset (\mathcal{A} \rtimes_\alpha \mathrm{G})^{**} \subset \mathcal{B}(\mathcal{H}_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}})$$

and we therefore, on the one hand, consider $\mathcal{A}$ as a unital C*-subalgebra of $\mathcal{B}(\mathcal{K})$, where $\mathcal{K} = \mathcal{H}_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}}$. On the other hand,

$$\begin{aligned}\mathcal{A} \subset \mathcal{A} \rtimes_\alpha \mathrm{G} = \mathcal{A} \rtimes_{\alpha,\mathrm{r}} \mathrm{G} &\subset \mathcal{B}(\mathcal{H}_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}}) \otimes_{\min} \mathrm{C}^*_{\mathrm{r}}(\mathrm{G}) \\ &\subset \mathcal{B}(\mathcal{H}_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}}) \mathbin{\overline{\otimes}} \mathcal{B}\left(\ell^2(\mathrm{G})\right) \\ &\subset \mathcal{B}\left(\mathcal{K} \otimes \ell^2(\mathrm{G})\right),\end{aligned}$$

where $\overline{\otimes}$ denotes the von Neumann algebra tensor product, yields another faithful representation of $\mathcal{A} \rtimes_\alpha \mathrm{G}$—in this case, as a unital C*-subalgebra of $\mathcal{B}\left(\mathcal{K} \otimes \ell^2(\mathrm{G})\right)$. Let $(\mathcal{A} \rtimes_\alpha \mathrm{G})''$ denote the double commutant of $\mathcal{A} \rtimes_\alpha \mathrm{G}$ in $\mathcal{B}\left(\mathcal{K} \otimes \ell^2(\mathrm{G})\right)$.

Using the vector state τ on $\mathcal{B}\left(\ell^2(\mathrm{G})\right)$ defined by $\tau(x) = \langle x\delta_e, \delta_e\rangle$ together with the identity map $\mathrm{id}_{\mathcal{B}(\mathcal{K})} : \mathcal{B}(\mathcal{H}_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}}) \to \mathcal{B}(\mathcal{H}_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}})$, we obtain a normal ucp map

$$\psi = \mathrm{id}_{\mathcal{B}(\mathcal{K})} \overline{\otimes} \tau : \mathcal{B}(\mathcal{K}) \mathbin{\overline{\otimes}} \mathcal{B}\left(\ell^2(\mathrm{G})\right) \to \mathcal{B}(\mathcal{K}).$$

If $\mathcal{E} : \mathcal{A} \rtimes_{\alpha,\mathrm{r}} \mathrm{G} \to \mathcal{A}$ denotes the conditional expectation of $\mathcal{A} \rtimes_{\alpha,\mathrm{r}} \mathrm{G}$ onto $\mathcal{A}$ whereby $\mathcal{E}\left(\sum_g a_g \lambda_g\right) = a_e$, then, using the identification $\mathcal{A} \rtimes_\alpha \mathrm{G} = \mathcal{A} \rtimes_{\alpha,\mathrm{r}} \mathrm{G}$, the restriction of ψ to $(\mathcal{A} \rtimes_\alpha \mathrm{G})''$ is a normal extension of $\rho \circ \mathcal{E}$, where $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is the faithful representation of $\mathcal{A} \subset \mathcal{B}\left(\mathcal{K} \otimes \ell^2(\mathrm{G})\right)$ as a unital C*-subalgebra of $\mathcal{B}(\mathcal{K})$. That is, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A} \rtimes_\alpha \mathrm{G} & \xrightarrow{\mathcal{E}} & \mathcal{A} \\ \big\downarrow & & \big\downarrow \rho \\ (\mathcal{A} \rtimes_\alpha \mathrm{G})'' & \xrightarrow[\psi]{} & \mathcal{B}(\mathcal{K})\,. \end{array}$$

Because ψ is normal, the range of $\psi_{|(\mathcal{A}\rtimes_\alpha \mathrm{G})''}$ is determined by

$$\psi\left((\mathcal{A} \rtimes_\alpha \mathrm{G})''\right) = \overline{(\psi(\mathcal{A} \rtimes_\alpha \mathrm{G}))}^{\mathrm{SOT}} = \overline{(\rho(\mathcal{A}))}^{\mathrm{SOT}}.$$

In other words, the range of $\psi_{|(\mathcal{A}\rtimes_\alpha \mathrm{G})''}$ is the strong-closure of the C*-subalgebra $\mathcal{A}$ of $\mathcal{A} \rtimes_\alpha \mathrm{G}$ in the enveloping von Neumann algebra $(\mathcal{A} \rtimes_\alpha \mathrm{G})^{**}$

of $\mathcal{A} \rtimes_\alpha \mathrm{G}$. Therefore, by [7, Corollary 3.7.9], there is an isomorphism $\theta : \overline{(\rho(\mathcal{A}))}^{\mathrm{SOT}} \to \mathcal{A}^{**}$ such that $\pi_u^{\mathcal{A}} = \theta_{|\rho(\mathcal{A})}$.

Now let $\pi_0 : (\mathcal{A} \rtimes_\alpha \mathrm{G})^{**} \to (\mathcal{A} \rtimes_\alpha \mathrm{G})''$ be the normal epimorphism that extends the identity map of $\mathcal{A} \rtimes_\alpha \mathrm{G}$. Because $\mathcal{A} \rtimes_\alpha \mathrm{G}$ has WEP, there is a ucp map $\phi_0 : \mathcal{B}(\mathcal{H}_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}}) \to (\mathcal{A} \rtimes_\alpha \mathrm{G})^{**}$ that fixes every element of $\mathcal{A} \rtimes_\alpha \mathrm{G}$. Hence, if $\omega = \theta \circ \psi_{|(\mathcal{A}\rtimes_\alpha \mathrm{G})''} \circ \pi_0 \circ \phi_0$, then ω is a ucp map of $\mathcal{B}(\mathcal{K}) \to \mathcal{A}^{**}$ for which $\omega(a) = \pi_u^{\mathcal{A}}(a)$ for every $a \in \mathcal{A}$. That is, $\mathcal{A}$ has WEP.

Conversely, assume that $\mathcal{A}$ has the weak expectation property and that $\mathcal{A}$ is (represented faithfully as) a unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Thus, we consider $\mathcal{A}$ and $\mathcal{A} \rtimes_\alpha \mathrm{G}$ faithfully represented via

$$\mathcal{A} \subset \mathcal{A} \rtimes_\alpha \mathrm{G} = \mathcal{A} \rtimes_{\alpha,\mathrm{r}} \mathrm{G} \subset \mathcal{B}\left(\mathcal{H} \otimes \ell^2(\mathrm{G})\right).$$

Note that $\mathfrak{u} : \mathrm{G} \to \mathcal{B}(\mathcal{H}_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}})$ whereby $\mathfrak{u}(g) = \pi_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}}(1 \otimes \lambda_g)$ is a unitary representation of G such that $(1 \otimes \lambda) \times \pi$ is the regular (covariant) representation associated with the dynamical system $(\mathcal{A}, \alpha, \mathrm{G})$.

Let $\pi_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}} : \mathcal{A} \rtimes_\alpha \mathrm{G} \to \mathcal{B}(\mathcal{H}_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}})$ be the universal representation of $\mathcal{A} \rtimes_\alpha \mathrm{G}$ and define $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_u^{\mathcal{A}})$ by $\pi = \pi_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}}{}_{|\mathcal{A}\rtimes_\alpha \mathrm{G}}$. Because π is a faithful representation of $\mathcal{A}$ and $\mathcal{A}$ has WEP, there is a ucp map

$$\phi_0 : \mathcal{B}(\mathcal{H}) \to \pi(\mathcal{A})'' \subset \pi_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}}(\mathcal{A} \rtimes_\alpha \mathrm{G})''$$

such that $\phi_0(\pi(a)) = \pi(a)$ for every $a \in \mathcal{A}$.

As in [1, Proposition 4.5.1], if $F \subset \mathrm{G}$ is a finite nonempty subset and if $p_F \in \mathcal{B}(\ell^2(\mathrm{G}))$ is the projection with range $\mathrm{Span}\{\delta_f : f \in F\}$, then $p_F \mathcal{B}(\ell^2(\mathrm{G}))p_F$ is isomorphic to the matrix algebra $\mathcal{M}_n$ for $n = |F|$, and so we obtain a ucp map $\phi_F : \mathcal{B}(\mathcal{H} \otimes \ell^2(\mathrm{G}) \to \mathcal{B}(\mathcal{H}) \otimes \mathcal{M}_n$ defined by $\phi_F(x) = (1 \otimes p_F)x(1 \otimes p_F)$. Next, let $\{e_{f,h}\}_{f,h\in F}$ denote the matrix units of $\mathcal{M}_n$ and define an action β of G on $\pi(\mathcal{A})''$ by $\beta_g(y) = \mathfrak{u}(g)y\mathfrak{u}(g)^*$, for $y \in \pi(\mathcal{A})''$. Observe that $\pi(\mathcal{A})'' \rtimes_\beta \mathrm{G} \subset \pi_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}}(\mathcal{A} \rtimes_\alpha \mathrm{G})''$.

The linear map $\psi_F : \pi(\mathcal{A})'' \otimes \mathcal{M}_n \to \mathcal{A} \rtimes_\beta \mathrm{G}$ for which

$$\psi_F(y \otimes e_{f,h}) = |F|^{-1}\beta_f(y)\mathfrak{u}(fh^{-1}),$$

for $y \in \pi(\mathcal{A})''$, is a ucp map by the proof of [1, Lemma 4.2.3]. Hence, $\theta_F := \psi_F \circ (\phi_0 \otimes \mathrm{id}_{\mathcal{M}_n}) \circ \phi_F$ is a ucp map $\mathcal{B}\left(\mathcal{H} \otimes \ell^2(\mathrm{G})\right) \to \pi_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}}(\mathcal{A} \rtimes_\alpha \mathrm{G})''$.

Hence, if $\{F_i\}_i$ is a Følner net in G and if

$$\theta_i : \mathcal{B}\left(\mathcal{H} \otimes \ell^2(\mathrm{G})\right) \to \pi_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}}(\mathcal{A} \rtimes_\alpha \mathrm{G})''$$

is the ucp map constructed above, for each i, then the net $\{\theta_i\}_i$ admits a cluster point θ relative to the point-ultraweak topology. Now, for every $i \in \Lambda$, $a\lambda_g \in \mathcal{A} \rtimes_{\alpha,\mathrm{r}} \mathrm{G}$, and $\xi, \eta \in \mathcal{H}^{\mathcal{A}\rtimes_\alpha \mathrm{G}}$,

$$\begin{aligned}
\left|\langle\left(\theta(a\lambda_g) - \pi_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}}(a\lambda_g)\right)\xi, \eta\rangle\right| &\leq |\langle\left(\theta(a\lambda_g) - \theta_{F_i}(a\lambda_g)\right)\xi, \eta\rangle| \\
&\quad + \left|\langle\left(\theta_{F_i}(a\lambda_g) - \pi_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}}(a\lambda_g)\right)\xi, \eta\rangle\right| \\
&= \left|\left(1 - \frac{|F_i \cap gF_i|}{|F_i|}\right)\langle\pi_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}}(a\lambda_g)\xi, \eta\rangle\right|.
\end{aligned}$$

Because θ is a cluster point of $\{\theta_i\}_i$, we deduce that $\theta(a\lambda_g) = \pi_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}}(a\lambda_g)$. Hence, by continuity, $\theta : \mathcal{B}\left(\mathcal{H} \otimes \ell^2(\mathrm{G})\right) \to \pi_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}}(\mathcal{A} \rtimes_\alpha \mathrm{G})''$ is a ucp map for that extends the identity map on $\pi_u^{\mathcal{A}\rtimes_\alpha \mathrm{G}}(\mathcal{A} \rtimes_\alpha \mathrm{G})$, which proves that $\mathcal{A} \rtimes_\alpha \mathrm{G}$ has the weak expectation property. $\square$

4. Remarks

The two proofs given in Theorems 2.1 and 3.1 of the implication $\mathcal{A} \rtimes_\alpha \mathrm{G}$ *has WEP* $\Rightarrow$ $\mathcal{A}$ *has WEP* depend only on the equality $\mathcal{A} \rtimes_\alpha \mathrm{G} = \mathcal{A} \rtimes_{\alpha,\mathrm{r}} \mathrm{G}$ rather than on the amenability of the action α or the group G itself.

The arguments to establish Theorems 2.1 and 3.1 depend crucially on the fact that $\mathcal{A}$ is a unital C*-algebra, and it would be of interest to know to what extent such results remain true for nonunital C*-algebras.

References

[1] Brown, Nathanial P.; Ozawa, Narutaka. C^*-algebras and finite-dimensional approximations. Graduate Studies in Mathematics, 88. *American Mathematical Society, Providence, RI*, 2008. xvi+509 pp. ISBN: 978-0-8218-4381-9; 0-8218-4381-8. MR2391387 (2009h:46101), Zbl 1160.46001.

[2] Farenick, Douglas.; Kavruk, Ali S.; Paulsen, Vern I. C*-algebras with the weak expectation property and a multivariable analogue of Ando's theorem on the numerical radius. To appear in *J. Operator Theory.* arXiv:1107.0418.

[3] Kirchberg, Eberhard. On nonsemisplit extensions, tensor products and exactness of group C^*-algebras. *Invent. Math.* **112** (1993), no. 3, 449–489. MR1218321 (94d:46058), doi: 10.1007/BF01232444.

[4] Lance, Christopher. On nuclear C^*-algebras. *J. Functional Analysis* **12** (1973), 157–176. MR0344901 (49 #9640), Zbl 0252.46065, doi: 10.1016/0022-1236(73)90021-9.

[5] Ozawa, Narutaka. An application of expanders to $\mathbb{B}(l_2) \otimes \mathbb{B}(l_2)$. *J. Funct. Anal.* **198** (2003), no. 2, 499–510. MR1964549 (2004d:46065), Zbl 1041.46041, arXiv:math/0110151, doi: 10.1016/S0022-1236(02)00107-6.

[6] Ozawa, Narutaka. About the QWEP conjecture. *Internat. J. Math.* **15** (2004), no. 5, 501–530. MR2072092 (2005b:46124), Zbl 1056.46051, arXiv:math/0306067, doi: 10.1142/S0129167X04002417.

[7] Pedersen, Gert K. C^*-algebras and their automorphism groups. London Mathematical Society Monographs, 14. *Academic Press Inc., London-New York*, 1979. ix+416 pp. ISBN: 0-12-549450-5. MR548006 (81e:46037), Zbl 0416.46043.

[8] Williams, Dana P. Crossed products of C^*-algebras, Mathematical Surveys and Monographs, 134. *American Mathematical Society, Providence, RI*, 2007. xvi+528 pp. ISBN: 978-0-8218-4242-3; 0-8218-4242-0. MR2288954 (2007m:46003), Zbl 1119.46002.

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