New York Journal of Mathematics

New York J. Math. **19** (2013) 565–582.

Local Calabi and curvature estimates for the Chern–Ricci flow

Morgan Sherman and **Ben Weinkove**

Abstract. Assuming local uniform bounds on the metric for a solution of the Chern–Ricci flow, we establish local Calabi and curvature estimates using the maximum principle.

Contents

1. Introduction

Let $(M, \hat{g})$ be a Hermitian manifold. The *Chern–Ricci flow* starting at $\hat{g}$ is a smooth flow of Hermitian metrics $g = g(t)$ given by

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R^C_{i\bar{j}}, \qquad g_{i\bar{j}}|_{t=0} = \hat{g}_{i\bar{j}}, \tag{1.1}$$

where $R^C_{i\bar{j}} := -\partial_i \partial_{\bar{j}} \log \det g$ is the *Chern–Ricci* curvature of g. If $\hat{g}$ is Kähler, then the Chern–Ricci flow coincides with the Kähler–Ricci flow.

The Chern–Ricci flow was introduced by Gill [11] and further investigated by Tosatti and the second-named author [25, 26]. This flow has many of same properties as the Kähler–Ricci flow. For example: on manifolds with vanishing first Bott–Chern class the Chern–Ricci flow converges to a Chern–Ricci flat metric [11]; on manifolds with negative first Chern class, the Chern–Ricci flow takes any Hermitian metric to the Kähler–Einstein metric [25]; when M is a compact complex surface and $\hat{g}$ is $\partial\overline{\partial}$-closed,

Received May 20, 2013.

2010 *Mathematics Subject Classification.* 53C44, 53C55 (primary).

Key words and phrases. Chern–Ricci flow, Kähler–Ricci flow, Chern connection, Chern curvature, Calabi estimate.

Supported in part by NSF grant DMS-1105373. Part of this work was carried out while the second-named author was a member of the mathematics department of the University of California, San Diego.

ISSN 1076-9803/2013

the Chern–Ricci flow exists until either the volume of the manifold goes to zero or the volume of a curve of negative self-intersection goes to zero [25]; if in addition M is nonminimal with nonnegative Kodaira dimension, the Chern–Ricci flow shrinks exceptional curves in finite time [26] in the sense of Gromov–Hausdorff. These results are closely analogous to results for the Kähler–Ricci flow [3, 10, 23, 20, 21].

In this note, we establish local derivative estimates for solutions of the Chern–Ricci flow assuming local uniform bounds on the metric, generalizing our previous work [18] on the Kähler–Ricci flow. Our estimates are local, so we work in a small open subset of $\mathbb{C}^n$. Write B_r for the ball of radius r centered at the origin in $\mathbb{C}^n$, and fix $T < \infty$. We have the following result (see Section 2 for more details about the notation).

Theorem 1.1. *Fix r with $0 < r < 1$. Let $g(t)$ solve the Chern–Ricci flow (1.1) in a neighborhood of B_r for $t \in [0, T]$. Assume $N > 1$ satisfies*

$$\frac{1}{N}\hat{g} \leq g(t) \leq N\hat{g} \qquad \text{on } B_r \times [0, T]. \tag{1.2}$$

Then there exist positive constants C, α, β depending only on $\hat{g}$ such that:

(i) $|\hat{\nabla} g|^2_g \leq \dfrac{CN^\alpha}{r^2}$ *on $B_{r/2} \times [0, T]$, where $\hat{\nabla}$ is the Chern connection of $\hat{g}$.*

(ii) $|\mathrm{Rm}|^2_g \leq \dfrac{CN^\beta}{r^4}$ *on $B_{r/4} \times [0, T]$, for* Rm *the Chern curvature tensor of g.*

Note that the estimates are independent of the time T and so the results hold also for time intervals $[0, T)$ or $[0, \infty)$. The dependence of the constants on $\hat{g}$ is as follows: up to three derivatives of torsion of $\hat{g}$ and one derivative of the Chern curvature of $\hat{g}$ (see Remarks 3.1 and 4.1). We call the bound (i) a *local Calabi estimate* [2] (see [29] for a similar estimate in the elliptic case).

As a consequence of Theorem 1.1, we have local derivative estimates for g to all orders:

Corollary 1.2. *With the assumptions of Theorem 1.1, for any $\varepsilon > 0$ with $0 < \varepsilon < T$, there exist constants C_m, α_m and γ_m for $m = 1, 2, 3, \ldots$ depending only on $\hat{g}$ and ε such that*

$$|\hat{\nabla}^m_{\mathbb{R}} g|^2_{\hat{g}} \leq \frac{C_m N^{\alpha_m}}{r^{\gamma_m}} \qquad \text{on } B_{r/8} \times [\varepsilon, T],$$

where $\hat{\nabla}_{\mathbb{R}}$ is the Levi-Civita covariant derivative associated to $\hat{g}$.

Note that our assumption (1.2) often holds for the Chern–Ricci flow on compact subsets away from a subvariety. For example, this always occurs for the Chern–Ricci flow on a nonminimal complex surface of nonnegative Kodaira dimension [25, 26]. It has already been shown by Gill [11] that local derivative estimates exist using the method of Evans–Krylov [9, 14] adapted to this setting. The purpose of this note is to give a direct maximum principle proof of Gill's estimates, and in the process identify evolution equations for the Calabi quantity $|\hat{\nabla} g|_g^2$ and the Chern curvature tensor $R_{i\bar{j}k\bar{l}}$, which were previously unknown for this flow. In addition, we more precisely determine the form of dependence on the constants N and r. We anticipate that this may be useful, for example in generalizations of arguments of [21].

In the case when $\hat{g}$ is Kähler, so that $g(t)$ solves the Kähler–Ricci flow, the above result follows from results of the authors in [18]. The more general case we deal with here leads to many more difficulties, arising from the torsion tensors of g and $\hat{g}$. For these reasons, our conclusions here are slightly weaker: for example, we cannot obtain the small values ($\alpha = 3$ and $\beta = 8$) in the estimates of (i) and (ii) that we achieved in [18].

Acknowledgements. The second-named author thanks Valentino Tosatti and Xiaokui Yang for some helpful discussions.

2. Preliminaries

In this section we introduce the basic notions that we will be using throughout the paper. We largely follow notation given in [25]. Given a Hermitian metric g we write ∇ for the *Chern connection* associated to g, which is characterized as follows. Define Christoffel symbols $\Gamma_{ik}^l = g^{\bar{s}l}\partial_i g_{k\bar{s}}$. Let $X = X^l \frac{\partial}{\partial z^l}$ be a vector field and let $a = a_k\, dz^k$ be a $(1,0)$ form. Then

$$\nabla_i X^l = \partial_i X^l + \Gamma_{ir}^l X^r, \quad \nabla_i a_j = \partial_i a_j - \Gamma_{ij}^r a_r. \tag{2.1}$$

We can, in a natural way, extend ∇ to act on any tensor. Note that ∇ makes g parallel: i.e., $\nabla g = 0$. Similarly we let $\hat{\nabla}$ denote the Chern connection associated to $\hat{g}$.

Define the torsion tensor T of g by

$$T_{ij}{}^k = \Gamma_{ij}^k - \Gamma_{ji}^k. \tag{2.2}$$

We note that g is Kähler precisely when $T = 0$. We write

$$T_{\bar{i}\bar{j}}{}^{\bar{k}} := \Gamma_{\bar{i}\bar{j}}^{\bar{k}} - \Gamma_{\bar{j}\bar{i}}^{\bar{k}} := \overline{\Gamma_{ij}^k} - \overline{\Gamma_{ji}^k}$$

for the components of the tensor $\overline{T}$. We lower and raise indices using the metric g. For example, $T^{ij}{}_k = g^{\bar{a}i} g^{\bar{b}j} g_{k\bar{l}} T_{\bar{a}\bar{b}}{}^{\bar{l}}$.

We define the *Chern curvature tensor* of g to be the tensor written locally as

$$R_{i\bar{j}k}{}^l = -\partial_{\bar{j}} \Gamma_{ik}^l. \tag{2.3}$$

Then

$$R_{i\bar{j}k\bar{l}} = -\partial_i \partial_{\bar{j}} g_{k\bar{l}} + g^{\bar{s}r} \partial_i g_{k\bar{s}} \partial_{\bar{j}} g_{r\bar{l}}. \tag{2.4}$$

Again we have lowered an index using the metric g. Note that $\overline{R_{i\bar{j}k\bar{l}}} = R_{j\bar{i}l\bar{k}}$ holds.

The commutation formulas for the Chern connection are given by

$$[\nabla_i, \nabla_{\bar{j}}] X^l = R_{i\bar{j}k}{}^l X^k, \qquad [\nabla_i, \nabla_{\bar{j}}] \overline{X^k} = -R_{i\bar{j}}{}^{\bar{k}}{}_{\bar{l}} \overline{X^l}$$
$$[\nabla_i, \nabla_{\bar{j}}] a_k = -R_{i\bar{j}k}{}^l a_l, \qquad [\nabla_i, \nabla_{\bar{j}}] \overline{a_l} = R_{i\bar{j}}{}^{\bar{k}}{}_{\bar{l}} \overline{a_k}. \tag{2.5}$$

Because g is not assumed to be a Kähler metric the *Bianchi identities* will not necessarily hold for $R_{i\bar{j}k\bar{l}}$. However their failure to hold can be measured with the torsion tensor T defined above:

$$\begin{aligned} R_{i\bar{j}k\bar{l}} - R_{k\bar{j}i\bar{l}} &= -\nabla_{\bar{j}} T_{ik\bar{l}} \\ R_{i\bar{j}k\bar{l}} - R_{i\bar{l}k\bar{j}} &= -\nabla_i T_{\bar{j}\bar{l}k} \\ R_{i\bar{j}k\bar{l}} - R_{k\bar{l}i\bar{j}} &= -\nabla_{\bar{j}} T_{ik\bar{l}} - \nabla_k T_{\bar{j}\bar{l}i} = -\nabla_i T_{\bar{j}\bar{l}k} - \nabla_{\bar{l}} T_{ik\bar{j}} \\ \nabla_p R_{i\bar{j}k\bar{l}} - \nabla_i R_{p\bar{j}k\bar{l}} &= -T_{pi}{}^r R_{r\bar{j}k\bar{l}} \\ \nabla_{\bar{q}} R_{i\bar{j}k\bar{l}} - \nabla_{\bar{j}} R_{i\bar{q}k\bar{l}} &= -T_{\bar{q}\bar{j}}{}^{\bar{s}} R_{i\bar{s}k\bar{l}}. \end{aligned} \tag{2.6}$$

These identities are well-known (see [27] for example). Indeed, it is routine to verify the first line, and the second and third lines follow directly from it. Furthermore the fifth line follows directly from the fourth. For the fourth line we calculate:

$$\nabla_p R_{i\bar{j}k}{}^l = -\nabla_p(\partial_{\bar{j}} \Gamma^l_{ik}) = -\partial_p \partial_{\bar{j}} \Gamma^l_{ik} - \Gamma^l_{pr} \partial_{\bar{j}} \Gamma^r_{ik} + \Gamma^r_{pi} \partial_{\bar{j}} \Gamma^l_{rk} + \Gamma^r_{pk} \partial_{\bar{j}} \Gamma^l_{ir}.$$

Swapping the p and i indices, subtracting, and combining terms, we find

$$\nabla_p R_{i\bar{j}k}{}^l - \nabla_i R_{p\bar{j}k}{}^l = -T_{pi}{}^r R_{r\bar{j}k}{}^l + \partial_{\bar{j}} \left(\partial_i \Gamma^l_{pk} - \partial_p \Gamma^l_{ik} + \Gamma^l_{ir} \Gamma^r_{pk} - \Gamma^l_{pr} \Gamma^r_{ik} \right).$$

Now one checks that the quantity in parentheses vanishes.

We define the Chern–Ricci curvature tensor $R^C_{i\bar{j}}$ by

$$R^C_{i\bar{j}} = g^{\bar{l}k} R_{i\bar{j}k\bar{l}} = -\partial_i \partial_{\bar{j}} \log \det g. \tag{2.7}$$

Note that $\sqrt{-1} R^C_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$ is a real closed (1,1) form. We will suppose that $g = g(t)$ satisfies the *Chern–Ricci flow*:

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R^C_{i\bar{j}}, \quad g_{i\bar{j}}|_{t=0} = \hat{g}_{i\bar{j}}, \tag{2.8}$$

for $t \in [0, T]$ for some fixed positive time T. We will use $\hat{\nabla}$, $\hat{\Gamma}^l_{ik}$, $\hat{T}_{ik}{}^l$, $\hat{R}_{i\bar{j}k\bar{l}}$, etc., to denote the corresponding quantities with respect to the metric $\hat{g}$. Define a real (1,1) form $\omega = \omega(t)$ by $\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$ and similarly for $\hat{\omega}$. From (2.8) we have that

$$\omega = \hat{\omega} + \eta(t) \tag{2.9}$$

for a closed $(1,1)$ form η. Hence

$$T_{ik\bar{l}} = \hat{T}_{ik\bar{l}}. \tag{2.10}$$

Here we raise and lower indices of $\hat{T}$ using the metric $\hat{g}$, in the same manner as for g above. Note that $T_{ik\bar{l}} = g_{r\bar{l}} T_{ik}{}^r = \partial_i g_{k\bar{l}} - \partial_k g_{i\bar{l}}$ and $\hat{T}_{ik\bar{l}} = \hat{g}_{r\bar{l}} \hat{T}_{ik}{}^r = \partial_i \hat{g}_{k\bar{l}} - \partial_k \hat{g}_{i\bar{l}}$.

It is convenient to introduce the tensor ${\Psi_{ik}}^l = \Gamma^l_{ik} - \hat{\Gamma}^l_{ik}$. We raise and lower indices of Ψ using the metric g, and write ${\Psi_{\bar{i}\bar{k}}}^{\bar{l}}$ for the components of $\overline{\Psi}$. We note here that Ψ can be used to switch between the connections ∇ and $\hat{\nabla}$. For example given a tensor of the form ${X_i}^j$ we have

$$\nabla_p {X_i}^j - \hat{\nabla}_p {X_i}^j = -{\Psi_{pi}}^r {X_r}^j + {\Psi_{pr}}^j {X_i}^r. \tag{2.11}$$

Observe that

$$\nabla_{\bar{j}} {\Psi_{ik}}^l = -{R_{i\bar{j}k}}^l + {\hat{R}_{i\bar{j}k}}{}^l. \tag{2.12}$$

We write Δ for the "rough Laplacian" of g, $\Delta = \nabla^{\bar{q}}\nabla_{\bar{q}}$, where $\nabla^{\bar{q}} = g^{\bar{q}p}\nabla_p$. Finally note that we will write all norms $|\cdot|$ with respect to the metric g.

3. Local Calabi estimate

In this section we prove part (i) of Theorem 1.1. We consider the Calabi-type [2, 28] quantity

$$S := |\Psi|^2 = |\hat{\nabla} g|^2. \tag{3.1}$$

Our goal in this section is to uniformly bound S on the set $B_{r/2}$, which we will do using a maximum principle argument. First we compute its evolution. Calculate

$$\begin{aligned}
\Delta S &= g^{\bar{q}p}\nabla_p\nabla_{\bar{q}}\left(g^{\bar{a}i}g^{\bar{b}j}g_{k\bar{c}}{\Psi_{ij}}^k\overline{{\Psi_{ab}}^c}\right)\\
&= g^{\bar{q}p}g^{\bar{a}i}g^{\bar{b}j}g_{k\bar{c}}\nabla_p\left(\nabla_{\bar{q}}{\Psi_{ij}}^k\overline{{\Psi_{ab}}^c} + {\Psi_{ij}}^k\overline{\nabla_q{\Psi_{ab}}^c}\right)\\
&= |\overline{\nabla}\Psi|^2 + |\nabla\Psi|^2 + g^{\bar{a}i}g^{\bar{b}j}g_{k\bar{c}}\Big(\Delta{\Psi_{ij}}^k\overline{{\Psi_{ab}}^c}\\
&\qquad + {\Psi_{ij}}^k\overline{(\Delta{\Psi_{ab}}^c + g^{\bar{q}p}{R_{p\bar{q}a}}^r{\Psi_{rb}}^c + g^{\bar{q}p}{R_{p\bar{q}b}}^r{\Psi_{ar}}^c - g^{\bar{q}p}{R_{p\bar{q}r}}^c{\Psi_{ab}}^r)}\Big)\\
&= |\overline{\nabla}\Psi|^2 + |\nabla\Psi|^2 + 2\mathrm{Re}\left((\Delta{\Psi_{ij}}^k){\Psi^{ij}}_k\right)\\
&\qquad + ({R_p}^p{}_r{}^i{\Psi^{rj}}_k + {R_p}^p{}_r{}^j{\Psi^{ir}}_k - {R_p}^p{}_k{}^r{\Psi^{ij}}_r){\Psi_{ij}}^k.
\end{aligned}$$

From (2.12) we have

$$\Delta{\Psi_{ij}}^k = -\nabla^{\bar{q}}{R_{i\bar{q}j}}^k + \nabla^{\bar{q}}{\hat{R}_{i\bar{q}j}}{}^k. \tag{3.2}$$

For the time derivative of S, first compute (cf. [17] in the Kähler case),

$$\frac{\partial}{\partial t}{\Psi_{ij}}^k = \frac{\partial}{\partial t}\Gamma^k_{ij} = -\nabla_i {(R^C)_j}^k. \tag{3.3}$$

Then

$$\begin{aligned}
\frac{\partial}{\partial t}S &= \frac{\partial}{\partial t}\left(g^{\bar{a}i}g^{\bar{b}j}g_{k\bar{c}}{\Psi_{ij}}^k\overline{{\Psi_{ab}}^c}\right)\\
&= \left(\frac{\partial}{\partial t}g^{\bar{a}i}\right){\Psi_{ij}}^k{\Psi_{\bar{a}}}^j{}_k + \left(\frac{\partial}{\partial t}g^{\bar{b}j}\right){\Psi_{ij}}^k{\Psi^i}_{\bar{b}k}\\
&\qquad + \left(\frac{\partial}{\partial t}g_{k\bar{c}}\right){\Psi_{ij}}^k\Psi^{ij\bar{c}} + 2\mathrm{Re}\left(\left(\frac{\partial}{\partial t}{\Psi_{ij}}^k\right){\Psi^{ij}}_k\right)
\end{aligned}$$

$$= (R^C)^{\bar{a}i}\Psi_{ij}{}^k\Psi_{\bar{a}}{}^j{}_k + (R^C)^{\bar{b}j}\Psi_{ij}{}^k\Psi^i{}_{\bar{b}k}$$
$$- (R^C)_{k\bar{c}}\Psi_{ij}{}^k\Psi^{ij\bar{c}} - 2\mathrm{Re}\left((\nabla_i(R^C)_j{}^k)\Psi^{ij}{}_k\right).$$

Therefore

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) S = & - |\overline{\nabla}\Psi|^2 - |\nabla\Psi|^2 + \left(R^{\bar{r}i}{}_p{}^p - R_p{}^{p\bar{r}i}\right)\Psi_{ij}{}^k\Psi_{\bar{r}}{}^j{}_k \\ & + \left(R^{\bar{r}j}{}_p{}^p - R_p{}^{p\bar{r}j}\right)\Psi_{ij}{}^k\Psi^i{}_{\bar{r}k} - \left(R_{k\bar{r}p}{}^p - R_p{}^p{}_{k\bar{r}}\right)\Psi_{ij}{}^k\Psi^{ij\bar{r}} \\ & - 2\mathrm{Re}\left[\left(\nabla_i R_j{}^k{}_p{}^p + \Delta\Psi_{ij}{}^k\right)\Psi^{ij}{}_k\right].\end{aligned}$$

By (2.6) we can re-write the terms involving a difference in curvature using the torsion tensor T. For the term in square brackets we compute, using (3.2) and again (2.6) that

$$\begin{aligned}\nabla_i R_j{}^k{}_p{}^p + \Delta\Psi_{ij}{}^k = & \nabla_i\left(R_p{}^p{}_j{}^k + \nabla_j T^{pk}{}_p + \nabla^p T_{pj}{}^k\right) - \nabla^{\bar{q}} R_{i\bar{q}j}{}^k + \nabla^{\bar{q}}\hat{R}_{i\bar{q}j}{}^k \\ = & \left(\nabla_p R_i{}^p{}_j{}^k - T_{ip}{}^r R_r{}^p{}_j{}^k + \nabla_i\nabla_j T^{pk}{}_p + \nabla_i\nabla^p T_{pj}{}^k\right) \\ & - \nabla^{\bar{q}} R_{i\bar{q}j}{}^k + \nabla^{\bar{q}}\hat{R}_{i\bar{p}j}{}^k \\ = & - T_{ip}{}^r R_r{}^p{}_j{}^k + \nabla_i\nabla_j T^{pk}{}_p + \nabla_i\nabla^p T_{pj}{}^k + \nabla^{\bar{q}}\hat{R}_{i\bar{q}j}{}^k.\end{aligned}$$

Hence S satisfies the following evolution equation

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - \Delta\right) S = \\ & - |\overline{\nabla}\Psi|^2 - |\nabla\Psi|^2 \\ & + \left(\nabla_r T_{\bar{q}}{}^{i\bar{q}} + \nabla_{\bar{q}} T^{\bar{q}}{}_r{}^i\right)\Psi_{ij}{}^k\Psi^{rj}{}_k + \left(\nabla_r T_{\bar{q}}{}^{j\bar{q}} + \nabla_{\bar{q}} T^{\bar{q}}{}_r{}^j\right)\Psi_{ij}{}^k\Psi^{ir}{}_k \\ & - \left(\nabla_k T_{\bar{q}}{}^{r\bar{q}} + \nabla_{\bar{q}} T^{\bar{q}}{}_k{}^r\right)\Psi_{ij}{}^k\Psi^{ij}{}_r \\ & - 2\mathrm{Re}\left[\left(\nabla_i\nabla_j T^{pk}{}_p + \nabla_i\nabla_{\bar{q}} T^{\bar{q}}{}_j{}^k - T_{ip}{}^r R_r{}^p{}_j{}^k + g^{\bar{q}p}\nabla_p\hat{R}_{i\bar{q}j}{}^k\right)\Psi^{ij}{}_k\right].\end{aligned} \tag{3.4}$$

There are similar calculations to (3.4) in the literature which generalize Calabi's argument [2, 28]: in the elliptic Hermitian case [6, 29]; in the case of the Kähler–Ricci flow (see also [18]) in [3, 17]; and in other settings [27, 24, 22].

For the remainder of this section we will write C for a constant of the form CN^α for C and α depending only on $\hat{g}$. Our goal is to show that $S \leq C/r^2$. The constant C will be used repeatedly and may change from line to line, and we may at times use C' or C_1, etc.

We would like to bound the right-hand side of (3.4). First, from (2.10) and (2.11) we have, for example,

$$\nabla_{\bar{a}} T_{ij}{}^k = g^{\bar{l}k}(\hat{\nabla}_{\bar{a}}\hat{T}_{ij\bar{l}} - \Psi_{\bar{a}\bar{l}}{}^{\bar{r}}\hat{T}_{ij\bar{r}}). \tag{3.5}$$

This and similar calculations show that the third and fourth lines of (3.4) can be bounded by $C(S^{3/2}+1)$. Next we address the terms in the last line of the evolution equation for S.

- Building on (3.5) we find

$$\begin{aligned}\nabla_a\nabla_b T_{\bar i\bar j}{}^{\bar k} &= g^{\bar k l}\left(\nabla_a(\hat\nabla_b \hat T_{\bar i\bar j l} - \Psi_{bl}{}^r \hat T_{\bar i\bar j r})\right)\\ &= g^{\bar k l}\Big(\hat\nabla_a\hat\nabla_b \hat T_{\bar i\bar j l} - \Psi_{ab}{}^r\hat\nabla_r \hat T_{\bar i\bar j l} - \Psi_{al}{}^r\hat\nabla_b\hat T_{\bar i\bar j r}\\ &\qquad -(\nabla_a\Psi_{bl}{}^r)\hat T_{\bar i\bar j r} - \Psi_{bl}{}^r\hat\nabla_a \hat T_{\bar i\bar j r} + \Psi_{bl}{}^r\Psi_{ar}{}^s\hat T_{\bar i\bar j s}\Big),\end{aligned} \tag{3.6}$$

  and hence $|\nabla_i\nabla_j T^{pk}{}_p|$ can be bounded by $C(S+|\nabla\Psi|+1)$.
- Similarly,

$$\begin{aligned}&\nabla_a\nabla_{\bar b} T_{ij}{}^k\\ &= g^{\bar l k}\nabla_a(\hat\nabla_{\bar b}\hat T_{ij\bar k} - \Psi_{\overline{bk}}{}^{\bar q}\hat T_{ij\bar q})\\ &= g^{\bar l k}\Big(\hat\nabla_a\hat\nabla_{\bar b}\hat T_{ij\bar k} - \Psi_{ai}{}^p\hat\nabla_{\bar b}\hat T_{pj\bar k} - \Psi_{aj}{}^p\hat\nabla_{\bar b}\hat T_{ip\bar k} - (\nabla_a\Psi_{\overline{bk}}{}^{\bar q})\hat T_{ij\bar q}\\ &\quad -\Psi_{\overline{bk}}{}^{\bar q}(\hat\nabla_a\hat T_{ij\bar q} - \Psi_{ai}{}^p\hat T_{pj\bar q} - \Psi_{aj}{}^p\hat T_{ip\bar q})\Big),\end{aligned} \tag{3.7}$$

  and so $|\nabla_i\nabla_{\bar q} T^{\bar q}{}_j{}^k|$ can be bounded by $C(S+|\overline\nabla\Psi|+1)$.
- Next, using (2.10) and (2.12):

$$T_{ip}{}^r R_r{}^p{}_j{}^k = g^{\bar s r}g^{\bar q p}\hat T_{ip\bar s}\left(\hat R_{r\bar q j}{}^k - \nabla_{\bar q}\Psi_{rj}{}^k\right),$$

  so we can bound $|T_{ip}{}^r R_r{}^p{}_j{}^k|$ by $C(|\overline\nabla\Psi|+1)$.
- Finally, compute

$$\nabla_p\hat R_{i\bar q j}{}^k = \hat\nabla_p\hat R_{i\bar q j}{}^k - \Psi_{pi}{}^r\hat R_{r\bar q j}{}^k - \Psi_{pj}{}^r\hat R_{i\bar q r}{}^k + \Psi_{pr}{}^k\hat R_{i\bar q j}{}^r.$$

  So $|g^{\bar q p}\nabla_p\hat R_{i\bar q j}{}^k|$ can be bounded by $C(S^{1/2}+1)$.

Putting this all together we arrive at the bound

$$\left(\frac{\partial}{\partial t}-\Delta\right)S \le C(S^{3/2}+1) - \frac12(|\overline\nabla\Psi|^2+|\nabla\Psi|^2). \tag{3.8}$$

We note here the bounds:

$$|\nabla \mathrm{tr}_{\hat g} g|^2 \le CS \tag{3.9}$$

$$|\nabla S|^2 \le 2S(|\overline\nabla\Psi|^2+|\nabla\Psi|^2). \tag{3.10}$$

The first follows from $\nabla_p\left(\hat g^{\bar j i}g_{i\bar j}\right) = \hat\nabla_p\left(\hat g^{\bar j i}g_{i\bar j}\right) = \hat g^{\bar j i}\hat\nabla_p g_{i\bar j}$ and the second follows from $|\nabla S|^2 = \left|\nabla|\Psi|^2\right|\left|\overline\nabla|\Psi|^2\right| \le 2|\Psi|^2\left(|\nabla\Psi|^2+|\overline\nabla\Psi|^2\right)$. Furthermore from [25, Proposition 3.1] (see also [6] in the elliptic case), we also have the following evolution equation for $\mathrm{tr}_{\hat g} g$:

$$
\begin{aligned}
(3.11)\qquad & \left(\frac{\partial}{\partial t} - \Delta\right) \mathrm{tr}_{\hat{g}} g = \\
& - g^{\bar{j}p} g^{\bar{q}i} \hat{\nabla}_k g_{i\bar{j}} \hat{\nabla}^k g_{p\bar{q}} - 2\,\mathrm{Re}\left(g^{\bar{j}i} \hat{T}_{ki}{}^{p} \hat{\nabla}^k g_{p\bar{j}} \right) \\
& + g^{\bar{j}i} \left(\hat{\nabla}_i \hat{T}_{\bar{j}}{}^{k\bar{q}} - \hat{R}_i{}^{k\bar{q}}{}_{\bar{j}} \right) g_{k\bar{q}} - g^{\bar{j}i} \left(\hat{\nabla}_i \hat{T}_{\bar{j}\bar{q}}{}^{\bar{q}} + \hat{\nabla}^k \hat{T}_{ik\bar{j}} \right) \\
& + g^{\bar{j}i} \hat{T}_{\bar{j}}{}^{k\bar{q}} \hat{T}_{ik}{}^{p} (\hat{g} - g)_{p\bar{q}}.
\end{aligned}
$$

(Here $\hat{\nabla}^k = \hat{g}^{\bar{l}k} \hat{\nabla}_{\bar{l}}$ and we have raised indices on the tensor $\widehat{\mathrm{Rm}}$ using $\hat{g}$). This generalizes the second order evolution inequality for the Kähler–Ricci flow [3] (cf. [28, 1]). Hence we have the estimate

$$
\left(\frac{\partial}{\partial t} - \Delta\right) \mathrm{tr}_{\hat{g}} g \le -\frac{S}{C_0} + C(S^{1/2} + 1), \tag{3.12}
$$

for a uniform positive constant C_0 (in fact we can take $C_0 = N$).

We now would like to show that the evolution inequalities (3.8, 3.12) imply a uniform bound on $S = |\hat{\nabla} g|^2$ on $\overline{B_{r/2}} \times [0, T]$. Choose a smooth cutoff function ρ which is supported in B_r and is identically 1 on $\overline{B_{r/2}}$. We may assume that $|\nabla \rho|^2, |\Delta \rho|$ are bounded by C/r^2. Let K be a large uniform constant, to be specified later, which is at least large enough so that

$$
\frac{K}{2} \le K - \mathrm{tr}_{\hat{g}} g \le K.
$$

Let A denote another large positive constant to be specified later. We will use a maximum principle argument with the function (cf. [5])

$$
f = \rho^2 \frac{S}{K - \mathrm{tr}_{\hat{g}} g} + A \mathrm{tr}_{\hat{g}} g
$$

to show that S is bounded on $B_{r/2}$.

Suppose that the maximum of f on $\overline{B_r} \times [0, T]$ occurs at a point (x_0, t_0). We assume for the moment that $t_0 > 0$ and that x_0 does not lie in the boundary of $\overline{B}_r$. We wish to show that at (x_0, t_0), S is bounded from above by a uniform constant C. Hence we may assume without loss of generality that $S > 1$ at (x_0, t_0). In particular, we have

$$
\begin{aligned}
(3.13)\qquad \left(\frac{\partial}{\partial t} - \Delta\right) S &\le C S^{3/2} - \frac{1}{2}(|\overline{\nabla} \Psi|^2 + |\nabla \Psi|^2), \\
\left(\frac{\partial}{\partial t} - \Delta\right) \mathrm{tr}_{\hat{g}} g &\le -\frac{S}{2C_0} + C.
\end{aligned}
$$

We compute at (x_0, t_0),

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right) f = \\
&A\left(\frac{\partial}{\partial t}-\Delta\right)\operatorname{tr}_{\hat{g}} g + (-\Delta(\rho^2))\frac{S}{K-\operatorname{tr}_{\hat{g}} g} \\
&+\rho^2\frac{S}{(K-\operatorname{tr}_{\hat{g}} g)^2}\left(\frac{\partial}{\partial t}-\Delta\right)\operatorname{tr}_{\hat{g}} g+\rho^2\frac{1}{K-\operatorname{tr}_{\hat{g}} g}\left(\frac{\partial}{\partial t}-\Delta\right) S \\
&-4\mathrm{Re}\left[\rho\frac{S}{(K-\operatorname{tr}_{\hat{g}} g)^2}\nabla\operatorname{tr}_{\hat{g}} g\cdot\overline{\nabla}\rho\right]-4\mathrm{Re}\left[\rho\frac{1}{K-\operatorname{tr}_{\hat{g}} g}\nabla\rho\cdot\overline{\nabla}S\right] \\
&-2\mathrm{Re}\left[\rho^2\frac{1}{(K-\operatorname{tr}_{\hat{g}} g)^2}\nabla\operatorname{tr}_{\hat{g}} g\cdot\overline{\nabla}S\right]-\frac{2\rho^2 S}{(K-\operatorname{tr}_{\hat{g}} g)^3}|\nabla\operatorname{tr}_{\hat{g}} g|^2.
\end{aligned}
$$

But since a maximum occurs at (x_0, t_0) we have $\overline{\nabla} f = 0$ at this point, and hence

$$
2\rho\overline{\nabla}\rho\frac{S}{K-\operatorname{tr}_{\hat{g}} g}+\rho^2\frac{\overline{\nabla}S}{K-\operatorname{tr}_{\hat{g}} g}+\rho^2\frac{S\overline{\nabla}\operatorname{tr}_{\hat{g}} g}{(K-\operatorname{tr}_{\hat{g}} g)^2}+A\overline{\nabla}\operatorname{tr}_{\hat{g}} g=0.
$$

Then at (x_0, t_0),

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right) f = \\
&A\left(\frac{\partial}{\partial t}-\Delta\right)\operatorname{tr}_{\hat{g}} g + (-\Delta(\rho^2))\frac{S}{K-\operatorname{tr}_{\hat{g}} g}+\rho^2\frac{S}{(K-\operatorname{tr}_{\hat{g}} g)^2}\left(\frac{\partial}{\partial t}-\Delta\right)\operatorname{tr}_{\hat{g}} g \\
&+\rho^2\frac{1}{K-\operatorname{tr}_{\hat{g}} g}\left(\frac{\partial}{\partial t}-\Delta\right) S-4\mathrm{Re}\left[\rho\frac{1}{K-\operatorname{tr}_{\hat{g}} g}\nabla\rho\cdot\overline{\nabla}S\right]+\frac{2A|\nabla\operatorname{tr}_{\hat{g}} g|^2}{K-\operatorname{tr}_{\hat{g}} g}.
\end{aligned}
$$

Making use of (3.9, 3.10, 3.13) and Young's inequality, we obtain at (x_0, t_0),

$$
\begin{aligned}
0\le\left(\frac{\partial}{\partial t}-\Delta\right) f &\le \left(-\frac{A}{2C_0}S+CA\right)+\left(\frac{CS}{r^2K}\right)+\left(-\frac{\rho^2}{2K^2C_0}S^2+\frac{C\rho^2}{K^2}S\right) \\
&\quad+\left(-\frac{\rho^2}{2K}(|\overline{\nabla}\Psi|^2+|\nabla\Psi|^2)+\frac{\rho^2}{4K^2C_0}S^2+C\rho^2 S\right) \\
&\quad+\left(\frac{\rho^2}{4K}(|\overline{\nabla}\Psi|^2+|\nabla\Psi|^2)+\frac{C}{Kr^2}S\right)+\frac{CA}{K}S \\
&\le -\frac{A}{2C_0}S+CA+\frac{C'}{r^2}S+\frac{CA}{K}S.
\end{aligned}
$$

Now pick $K \ge 4C_0C$ so that at (x_0, t_0),

$$
0\le-\frac{A}{4C_0}S+CA+\frac{C'}{r^2}S.
$$

Then choose $A=\frac{8C'C_0}{r^2}$ so that at (x_0, t_0),

$$
\frac{C'}{r^2}S\le CA,
$$

giving a uniform upper bound for S. It follows that f is bounded from above by Cr^{-2} for a uniform C. Hence S on $\overline{B_{r/2}}$ is bounded above by Cr^{-2}.

It remains to deal with the cases when $t_0 = 0$ or x_0 lies on the boundary of $\overline{B_r}$. In either case we have $f(x_0, t_0) \leq A\mathrm{tr}_{\hat{g}} g(x_0, t_0) \leq Cr^{-2}$ and the same bound holds.

Remark 3.1. Tracing through the argument, one can see that the constants only depend on uniform bounds for the torsion and curvature of $\hat{g}$, and one and two derivatives (with respect to $\hat{\nabla}$ or $\overline{\hat{\nabla}}$) of torsion and one derivative of curvature.

4. Local curvature bound

In this section we prove part (ii) of Theorem 1.1. As in the previous section, we write C for a constant of the form CN^{γ} for some uniform C, γ. We compute in the ball $\overline{B_{r/2}}$ on which we already have the bound $S \leq C/r^2$.

Let $\Delta_{\mathbb{R}} = \frac{1}{2} g^{\bar{q}p}(\nabla_p \nabla_{\bar{q}} + \nabla_{\bar{q}} \nabla_p)$. First we need an evolution equation for the curvature tensor. We begin with

$$\frac{\partial}{\partial t} R_{i\bar{j}k}{}^{l} = \frac{\partial}{\partial t}\left(-\partial_{\bar{j}} \Gamma^l_{ik}\right) = -\partial_{\bar{j}} \frac{\partial}{\partial t}\left(\Gamma^l_{ik}\right) = -\partial_{\bar{j}}(-\nabla_i (R^C)_k{}^l) = \nabla_{\bar{j}} \nabla_i R_k{}^l{}_p{}^p$$

and therefore,

$$\frac{\partial}{\partial t} R_{i\bar{j}k\bar{l}} = -R_{q\bar{l}p}{}^{p} R_{i\bar{j}k}{}^{q} + \nabla_{\bar{j}} \nabla_i R_{k\bar{l}p}{}^{p}. \tag{4.1}$$

Now, computing in coordinates where g is the identity, we find

$$\begin{aligned}
\Delta_{\mathbb{R}} R_{i\bar{j}k\bar{l}} =& \frac{1}{2}(\nabla_p \nabla_{\bar{p}} + \nabla_{\bar{p}} \nabla_p) R_{i\bar{j}k\bar{l}} \\
=& \nabla_p \nabla_{\bar{p}} R_{i\bar{j}k\bar{l}} + \frac{1}{2}(R_{p\bar{p}i\bar{q}} R_{q\bar{j}k\bar{l}} - R_{p\bar{p}q\bar{j}} R_{i\bar{q}k\bar{l}} + R_{p\bar{p}k\bar{q}} R_{i\bar{j}q\bar{l}} - R_{p\bar{p}q\bar{l}} R_{i\bar{j}k\bar{q}}) \\
=& \nabla_p (\nabla_{\bar{j}} R_{i\bar{p}k\bar{l}} - T_{\bar{p}\bar{j}q} R_{i\bar{q}k\bar{l}}) \\
& + \frac{1}{2}(R_{p\bar{p}i\bar{q}} R_{q\bar{j}k\bar{l}} - R_{p\bar{p}q\bar{j}} R_{i\bar{q}k\bar{l}} + R_{p\bar{p}k\bar{q}} R_{i\bar{j}q\bar{l}} - R_{p\bar{p}q\bar{l}} R_{i\bar{j}k\bar{q}}) \\
=& \nabla_{\bar{j}} \nabla_p R_{i\bar{p}k\bar{l}} - R_{p\bar{j}i\bar{q}} R_{q\bar{p}k\bar{l}} + R_{p\bar{j}q\bar{p}} R_{i\bar{q}k\bar{l}} \\
& - R_{p\bar{j}k\bar{q}} R_{i\bar{p}q\bar{l}} + R_{p\bar{j}q\bar{l}} R_{i\bar{p}k\bar{q}} - \nabla_p (T_{\bar{p}\bar{j}q} R_{i\bar{q}k\bar{l}}) \\
& + \frac{1}{2}(R_{p\bar{p}i\bar{q}} R_{q\bar{j}k\bar{l}} - R_{p\bar{p}q\bar{j}} R_{i\bar{q}k\bar{l}} + R_{p\bar{p}k\bar{q}} R_{i\bar{j}q\bar{l}} - R_{p\bar{p}q\bar{l}} R_{i\bar{j}k\bar{q}}) \\
=& \nabla_{\bar{j}} (\nabla_i R_{p\bar{p}k\bar{l}} - T_{pi\bar{q}} R_{q\bar{p}k\bar{l}}) - R_{p\bar{j}i\bar{q}} R_{q\bar{p}k\bar{l}} + R_{p\bar{j}q\bar{p}} R_{i\bar{q}k\bar{l}} \\
& - R_{p\bar{j}k\bar{q}} R_{i\bar{p}q\bar{l}} + R_{p\bar{j}q\bar{l}} R_{i\bar{p}k\bar{q}} - \nabla_p (T_{\bar{p}\bar{j}q} R_{i\bar{q}k\bar{l}}) \\
& + \frac{1}{2}(R_{p\bar{p}i\bar{q}} R_{q\bar{j}k\bar{l}} - R_{p\bar{p}q\bar{j}} R_{i\bar{q}k\bar{l}} + R_{p\bar{p}k\bar{q}} R_{i\bar{j}q\bar{l}} - R_{p\bar{p}q\bar{l}} R_{i\bar{j}k\bar{q}}) \\
=& \nabla_{\bar{j}} \nabla_i (R_{k\bar{l}p\bar{p}} - \nabla_p T_{\bar{p}\bar{l}k} - \nabla_{\bar{l}} T_{pk\bar{p}}) - \nabla_{\bar{j}} (T_{pi\bar{q}} R_{q\bar{p}k\bar{l}}) - R_{p\bar{j}i\bar{q}} R_{q\bar{p}k\bar{l}} \\
& + R_{p\bar{j}q\bar{p}} R_{i\bar{q}k\bar{l}} - R_{p\bar{j}k\bar{q}} R_{i\bar{p}q\bar{l}} + R_{p\bar{j}q\bar{l}} R_{i\bar{p}k\bar{q}} - \nabla_p (T_{\bar{p}\bar{j}q} R_{i\bar{q}k\bar{l}}) \\
& + \frac{1}{2}(R_{p\bar{p}i\bar{q}} R_{q\bar{j}k\bar{l}} - R_{p\bar{p}q\bar{j}} R_{i\bar{q}k\bar{l}} + R_{p\bar{p}k\bar{q}} R_{i\bar{j}q\bar{l}} - R_{p\bar{p}q\bar{l}} R_{i\bar{j}k\bar{q}}).
\end{aligned}$$

Hence

$$\left(\frac{\partial}{\partial t}-\Delta_{\mathbb{R}}\right)R_{i\bar{j}k\bar{l}}= \tag{4.2}$$
$$- R_{q\bar{l}p\bar{p}}R_{i\bar{j}k\bar{q}}+R_{p\bar{j}i\bar{q}}R_{q\bar{p}k\bar{l}}-R_{p\bar{j}q\bar{p}}R_{i\bar{q}k\bar{l}}+R_{p\bar{j}k\bar{q}}R_{i\bar{p}q\bar{l}}-R_{p\bar{j}q\bar{l}}R_{i\bar{p}k\bar{q}}$$
$$-\frac{1}{2}(R_{p\bar{p}i\bar{q}}R_{q\bar{j}k\bar{l}}-R_{p\bar{p}q\bar{j}}R_{i\bar{q}k\bar{l}}+R_{p\bar{p}k\bar{q}}R_{i\bar{j}q\bar{l}}-R_{p\bar{p}q\bar{l}}R_{i\bar{j}k\bar{q}})$$
$$+\nabla_p(\hat{T}_{\overline{p}\overline{j}q}R_{i\bar{q}k\bar{l}})+\nabla_{\bar{j}}(\hat{T}_{pi\bar{q}}R_{q\bar{p}k\bar{l}})+\nabla_{\bar{j}}\nabla_i(\nabla_p T_{\bar{p}\bar{l}k}+\nabla_{\bar{l}}T_{pk\bar{p}}).$$

To estimate this, we first compute

$$\nabla_p(\hat{T}_{\overline{p}\overline{j}q}R_{i\bar{q}k\bar{l}})=(\hat{\nabla}_p\hat{T}_{\overline{p}\overline{j}q}-\Psi_{pq\bar{r}}\hat{T}_{\overline{p}\overline{j}r})R_{i\bar{q}k\bar{l}}+\hat{T}_{\overline{p}\overline{j}q}\nabla_p R_{i\bar{q}k\bar{l}},$$

and this is bounded by $C(|\mathrm{Rm}|/r+|\nabla\mathrm{Rm}|)$. Using the fact that

$$R_{i\bar{j}k}{}^{l}=-\nabla_{\bar{j}}\Psi_{ik}{}^{l}+\hat{R}_{i\bar{j}k}{}^{l}$$

we have

$$|\mathrm{Rm}|\leq|\overline{\nabla}\Psi|+C, \tag{4.3}$$

and hence

$$|\nabla_p(\hat{T}_{\overline{p}\overline{j}q}R_{i\bar{q}k\bar{l}})|\leq C\left(|\nabla\mathrm{Rm}|+\frac{|\overline{\nabla}\Psi|}{r}+\frac{1}{r}\right). \tag{4.4}$$

Similarly for the term $\nabla_{\bar{j}}(\hat{T}_{pi\bar{q}}R_{q\bar{p}k\bar{l}})$.

The last two terms of (4.2) involve three derivatives of torsion. We claim that

$$|\overline{\nabla}\nabla\nabla\overline{T}|,\ |\overline{\nabla}\nabla\overline{\nabla}T|\leq C\left(|\nabla\mathrm{Rm}|+\frac{|\nabla\Psi|+|\overline{\nabla}\Psi|}{r}+\frac{1}{r^3}\right). \tag{4.5}$$

Indeed, applying $\nabla_{\overline{c}}$ to (3.6), we have

(4.6)
$$\nabla_{\overline{c}}\nabla_a\nabla_b T_{\overline{i}\overline{j}}{}^{\overline{k}}=g^{\overline{k}l}\Big(\hat{\nabla}_{\overline{c}}\hat{\nabla}_a\hat{\nabla}_b\hat{T}_{\overline{i}\overline{j}l}-\Psi_{\overline{c}\overline{i}}{}^{\overline{q}}\hat{\nabla}_a\hat{\nabla}_b\hat{T}_{\overline{q}\overline{j}l}-\Psi_{\overline{c}\overline{j}}{}^{\overline{q}}\hat{\nabla}_a\hat{\nabla}_b\hat{T}_{\overline{i}\overline{q}l}$$
$$-\nabla_{\overline{c}}(\Psi_{ab}{}^{r}\hat{\nabla}_r\hat{T}_{\overline{i}\overline{j}l})-\nabla_{\overline{c}}(\Psi_{al}{}^{r}\hat{\nabla}_b\hat{T}_{\overline{i}\overline{j}r})-\nabla_{\overline{c}}(\Psi_{bl}{}^{r}\hat{\nabla}_a\hat{T}_{\overline{i}\overline{j}r})$$
$$-\nabla_{\overline{c}}(\nabla_a\Psi_{bl}{}^{r}\hat{T}_{\overline{i}\overline{j}r})+\nabla_{\overline{c}}(\Psi_{bl}{}^{r}\Psi_{ar}{}^{s}\hat{T}_{\overline{i}\overline{j}s})\Big).$$

The first three terms on the right-hand side are bounded by $C(\sqrt{S}+1)$ and hence by C/r. Next compute

$$\nabla_{\overline{c}}(\Psi_{ab}{}^{r}\hat{\nabla}_r\hat{T}_{\overline{i}\overline{j}l})=(\nabla_{\overline{c}}\Psi_{ab}{}^{r})\hat{\nabla}_r\hat{T}_{\overline{i}\overline{j}l}+\Psi_{ab}{}^{r}\hat{\nabla}_{\overline{c}}\hat{\nabla}_r\hat{T}_{\overline{i}\overline{j}l}-\Psi_{ab}{}^{r}\Psi_{\overline{c}\overline{i}}{}^{\overline{q}}\hat{\nabla}_r\hat{T}_{\overline{q}\overline{j}l}$$
$$-\Psi_{ab}{}^{r}\Psi_{\overline{c}\overline{j}}{}^{\overline{q}}\hat{\nabla}_r\hat{T}_{\overline{i}\overline{q}l}, \tag{4.7}$$

which is bounded by $C|\overline{\nabla}\Psi|+C\sqrt{S}+CS$ and hence by $C(|\overline{\nabla}\Psi|+1/r^2)$. The same bound holds for the other two terms on the second line of (4.6).

For the third line, compute

$$\nabla_{\overline{c}}(\nabla_a\Psi_{bl}{}^{r}\hat{T}_{\overline{i}\overline{j}r})=\left(\nabla_a\nabla_{\overline{c}}\Psi_{bl}{}^{r}+R_{a\overline{c}b}{}^{p}\Psi_{pl}{}^{r}+R_{a\overline{c}l}{}^{p}\Psi_{bp}{}^{r}-R_{a\overline{c}p}{}^{r}\Psi_{bl}{}^{p}\right)\hat{T}_{\overline{i}\overline{j}r}$$
$$+(\nabla_a\Psi_{bl}{}^{r})\left(\hat{\nabla}_{\overline{c}}\hat{T}_{\overline{i}\overline{j}r}-\Psi_{\overline{c}\overline{i}}{}^{\overline{q}}\hat{T}_{\overline{q}\overline{j}r}-\Psi_{\overline{c}\overline{j}}{}^{\overline{q}}\hat{T}_{\overline{i}\overline{q}r}\right),$$

and using the fact that $\nabla_{\overline{c}}\Psi_{bl}{}^{r} = -R_{b\overline{c}l}{}^{r} + \hat{R}_{b\overline{c}l}{}^{r}$ we obtain

$$\begin{aligned}\nabla_{\overline{c}}(\nabla_a \Psi_{bl}{}^{r}\hat{T}_{\overline{i}\overline{j}r}) = \big(&- \nabla_a R_{b\overline{c}l}{}^{r} + \hat{\nabla}_a \hat{R}_{b\overline{c}l}{}^{r} - \Psi_{ab}{}^{p}\hat{R}_{p\overline{c}l}{}^{r} - \Psi_{al}{}^{p}\hat{R}_{b\overline{c}p}{}^{r} \\ &+ \Psi_{ap}{}^{r}\hat{R}_{b\overline{c}l}{}^{p} + R_{a\overline{c}b}{}^{p}\Psi_{pl}{}^{r} + R_{a\overline{c}l}{}^{p}\Psi_{bp}{}^{r} - R_{a\overline{c}p}{}^{r}\Psi_{bl}{}^{p}\big)\hat{T}_{\overline{i}\overline{j}r} \\ &+ (\nabla_a \Psi_{bl}{}^{r})\left(\hat{\nabla}_{\overline{c}}\hat{T}_{\overline{i}\overline{j}r} - \Psi_{\overline{c}\overline{i}}{}^{\overline{q}}\hat{T}_{\overline{q}\overline{j}r} - \Psi_{\overline{c}\overline{j}}{}^{\overline{q}}\hat{T}_{\overline{i}\overline{q}r}\right).\end{aligned}$$

It follows that

$$|\nabla_{\overline{c}}(\nabla_a \Psi_{bl}{}^{r}\hat{T}_{\overline{i}\overline{j}r})| \leq C\left(|\nabla \mathrm{Rm}| + \frac{|\mathrm{Rm}|}{r} + \frac{|\nabla\Psi|}{r} + \frac{1}{r}\right). \tag{4.8}$$

Finally,

$$\begin{aligned}\nabla_{\overline{c}}(\Psi_{bl}{}^{r}\Psi_{ar}{}^{s}\hat{T}_{\overline{i}\overline{j}s}) = (-R_{b\overline{c}l}{}^{r} + \hat{R}_{b\overline{c}l}{}^{r})\Psi_{ar}{}^{s}\hat{T}_{\overline{i}\overline{j}s} + \Psi_{bl}{}^{r}(-R_{a\overline{c}r}{}^{s} + \hat{R}_{a\overline{c}r}{}^{s})\hat{T}_{\overline{i}\overline{j}s} \\ + \Psi_{bl}{}^{r}\Psi_{ar}{}^{s}(\hat{\nabla}_{\overline{c}}\hat{T}_{\overline{i}\overline{j}s} - \Psi_{\overline{c}\overline{i}}{}^{\overline{q}}\hat{T}_{\overline{q}\overline{j}s} - \Psi_{\overline{c}\overline{j}}{}^{\overline{q}}\hat{T}_{\overline{i}\overline{q}s}),\end{aligned}$$

giving

$$|\nabla_{\overline{c}}(\Psi_{bl}{}^{r}\Psi_{ar}{}^{s}\hat{T}_{\overline{i}\overline{j}s})| \leq C\left(\frac{|\mathrm{Rm}|}{r} + \frac{1}{r^3}\right). \tag{4.9}$$

Putting together (4.6, 4.7, 4.8, 4.9), and making use of (4.3), we obtain

$$|\overline{\nabla}\nabla\nabla\overline{T}| \leq C\left(|\nabla \mathrm{Rm}| + \frac{|\nabla\Psi| + |\overline{\nabla}\Psi|}{r} + \frac{1}{r^3}\right),$$

and the bound for $|\overline{\nabla}\nabla\overline{\nabla}T|$ follows similarly. This completes the proof of the claim (4.5).

From (4.4) and the claim we just proved, since the second and third lines of (4.2) are of the order $|\mathrm{Rm}|^2$, we have the bound

$$\left|\left(\frac{\partial}{\partial t} - \Delta_{\mathbb{R}}\right)\mathrm{Rm}\right| \leq C\left(|\mathrm{Rm}|^2 + |\nabla \mathrm{Rm}| + \frac{|\nabla\Psi| + |\overline{\nabla}\Psi|}{r} + \frac{1}{r^3}\right). \tag{4.10}$$

Now

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)|\mathrm{Rm}|^2 = {}& g^{\overline{j}b}g^{\overline{c}k}g^{\overline{l}d}(R^C)^{\overline{a}i}R_{i\overline{j}k\overline{l}}\overline{R_{a\overline{b}c\overline{d}}} \\ &+ g^{\overline{a}i}g^{\overline{c}k}g^{\overline{l}d}(R^C)^{\overline{j}b}R_{i\overline{j}k\overline{l}}\overline{R_{a\overline{b}c\overline{d}}} \\ &+ g^{\overline{a}i}g^{\overline{j}b}g^{\overline{l}d}(R^C)^{\overline{c}k}R_{i\overline{j}k\overline{l}}\overline{R_{a\overline{b}c\overline{d}}} \\ &+ g^{\overline{a}i}g^{\overline{j}b}g^{\overline{c}k}(R^C)^{\overline{l}d}R_{i\overline{j}k\overline{l}}\overline{R_{a\overline{b}c\overline{d}}} \\ &+ 2\mathrm{Re}\left[g^{\overline{a}i}g^{\overline{j}b}g^{\overline{c}k}g^{\overline{l}d}\left((\frac{\partial}{\partial t} - \Delta_{\mathbb{R}})R_{i\overline{j}k\overline{l}}\right)\overline{R_{a\overline{b}c\overline{d}}}\right] \\ &- 2|\nabla \mathrm{Rm}|^2.\end{aligned} \tag{4.11}$$

This together with (4.10) and (4.3) implies

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)|\mathrm{Rm}|^2 \\
&\leq C\left(|\mathrm{Rm}|^2+|\mathrm{Rm}|^3+|\nabla \mathrm{Rm}|\cdot|\mathrm{Rm}|+\frac{(|\nabla\Psi|+|\overline{\nabla}\Psi|)|\mathrm{Rm}|}{r}+\frac{|\mathrm{Rm}|}{r^3}\right) \\
&\leq C\left(|\mathrm{Rm}|^3+\frac{1}{r}+\frac{|\nabla\Psi|^2+|\overline{\nabla}\Psi|^2}{r}+\frac{|\mathrm{Rm}|}{r^3}\right)-|\nabla \mathrm{Rm}|^2.
\end{aligned} \tag{4.12}
$$

To show $|\mathrm{Rm}|^2$ is locally uniformly bounded we will use an argument similar to the previous section. Let ρ now denote a cutoff function which is identically 1 on $\overline{B_{r/4}}$, and supported in $B_{r/2}$. From the previous section we know that S is bounded by C/r^2 on $B_{r/2}$. As before we can assume $|\nabla\rho|^2$ and $|\Delta\rho|$ are bounded by C/r^2. Let $K=C_1/r^2$ where C_1 is a constant to be determined later, and is at least large enough so that $\frac{K}{2}\leq K-S\leq K$. Let A denote a constant to be specified later. We will apply the maximum principle argument to the quantity

$$
f=\rho^2\frac{|\mathrm{Rm}|^2}{K-S}+AS.
$$

As in the previous section, we calculate at a point (x_0,t_0) where a maximum of f is achieved, and we first assume that $t_0>0$ and that x_0 does not occur at the boundary of $\overline{B_{r/2}}$. We use the fact that $\nabla f=0$ at this point, giving us

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)f =&A(\frac{\partial}{\partial t}-\Delta)S+(-\Delta(\rho^2))\frac{|\mathrm{Rm}|^2}{K-S}+\rho^2\frac{|\mathrm{Rm}|^2}{(K-S)^2}(\frac{\partial}{\partial t}-\Delta)S \\
&+\rho^2\frac{1}{K-S}(\frac{\partial}{\partial t}-\Delta)|\mathrm{Rm}|^2-4\mathrm{Re}\left(\frac{1}{K-S}\rho\nabla\rho\cdot\overline{\nabla}|\mathrm{Rm}|^2\right) \\
&+\frac{2A|\nabla S|^2}{K-S}.
\end{aligned}
$$

Our goal is to show that at (x_0,t_0), we have $|\mathrm{Rm}|^2\leq C/r^4$. Hence without loss of generality, we may assume that $1/r+|\mathrm{Rm}|/r^3\leq C|\mathrm{Rm}|^3$ and hence (4.12) becomes

$$
\left(\frac{\partial}{\partial t}-\Delta\right)|\mathrm{Rm}|^2\leq C\left(|\mathrm{Rm}|^3+\frac{Q}{r}\right)-|\nabla\mathrm{Rm}|^2,
$$

where for convenience we are writing $Q=|\nabla\Psi|^2+|\overline{\nabla}\Psi|^2$. For later purposes, recall from (4.3) that $|\mathrm{Rm}|^2\leq Q+C$ and from (3.10) that $|\nabla S|^2\leq 2SQ$.

Also note that $\left|\nabla|\mathrm{Rm}|^2\right|\leq 2|\mathrm{Rm}||\nabla\mathrm{Rm}|$. By (3.8) we find that on $B_{r/2}$ we have

$$
\left(\frac{\partial}{\partial t}-\Delta\right)S\leq\frac{C}{r^3}-\frac{1}{2}Q.
$$

Using these, we find at (x_0, t_0),

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) f \leq & \left(\frac{CA}{r^3} - \frac{AQ}{2}\right) + \left(\frac{C|\mathrm{Rm}|^2}{Kr^2}\right) + \left(\frac{C\rho^2|\mathrm{Rm}|^2}{K^2r^3} - \frac{\rho^2|\mathrm{Rm}|^2Q}{2K^2}\right) \\ & + \left(\frac{C\rho^2|\mathrm{Rm}|^3}{K} + \frac{C\rho^2Q}{Kr} - \frac{\rho^2}{K}|\nabla\mathrm{Rm}|^2\right) \\ & + \left(\frac{\rho^2|\nabla\mathrm{Rm}|^2}{2K} + C\frac{|\mathrm{Rm}|^2}{Kr^2}\right) + \left(\frac{8ASQ}{K}\right).\end{aligned}$$

First choose C_1 in the definition of K to be sufficiently large so that

$$\frac{8ASQ}{K} \leq \frac{AQ}{4},$$

where we use the fact that $S \leq C/r^2$. Next observe that

$$\frac{C\rho^2|\mathrm{Rm}|^3}{K} \leq \frac{\rho^2|\mathrm{Rm}|^2Q}{2K^2} + C'\rho^2|\mathrm{Rm}|^2,$$

and hence

$$\left(\frac{\partial}{\partial t} - \Delta\right) f \leq \frac{CA}{r^3} - \frac{AQ}{4} + C''Q + C.$$

Now we may choose A sufficiently large so that $A \geq 8C''$ and we obtain at (x_0, t_0),

$$Q \leq \frac{C}{r^3},$$

which implies that $|\mathrm{Rm}|^2 \leq C/r^3$ at this point. It follows that at (x_0, t_0), f is bounded from above by C/r^2. The same bound holds if x_0 lies in the boundary of $\overline{B_{r/2}}$ or if $t_0 = 0$. Hence on $\overline{B_{r/4}}$ we obtain

$$|\mathrm{Rm}|^2 \leq \frac{C}{r^4},$$

as required. This completes the proof of Theorem 1.1.

Remark 4.1. In addition to the dependence discussed in Remark 3.1, the constants also depend on *three* derivatives of the torsion of $\hat{g}$, with respect to $\hat{\nabla}$ or $\overline{\hat{\nabla}}$.

5. Higher order estimates

In this last section, we prove Corollary 1.2 by establishing the estimates for $|\hat{\nabla}^m_{\mathbb{R}} g|^2_{\hat{g}}$ for $m = 2, 3, \ldots$. For this part, we essentially follow the method of Gill [11] (cf. [4, 7, 8, 16] in the Kähler case), but since the setting here is slightly more general, we briefly outline the argument. In this section, we say that a quantity is uniformly bounded if it can be bounded by $CN^\alpha r^{-\gamma}$ for uniform C, α, γ.

We work on the ball $B_{r/4}$, and assume the bounds established in Theorem 1.1. As in [25], define reference tensors $(\hat{g}_t)_{i\bar{j}} = \hat{g}_{i\bar{j}} - t\hat{R}^C_{i\bar{j}}$, where $\hat{R}^C_{i\bar{j}}$ is the Chern–Ricci curvature of $\hat{g}$. For each fixed $x \in M$, let $\varphi = \varphi(x,t)$ solve

$$\frac{\partial \varphi}{\partial t} = \log \frac{\det g(t)}{\det \hat{g}}, \quad \varphi|_{t=0} = 0.$$

Then $g_{i\bar{j}} = (\hat{g}_t)_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi$ is the solution of the Chern–Ricci flow starting at $\hat{g}$.

Consider the first order differential operator $D = \frac{\partial}{\partial x^\gamma}$, where x^γ is a real coordinate. Applying D to the equation for φ, we have

$$\frac{\partial}{\partial t}(D\varphi) = g^{\bar{j}i} D g_{i\bar{j}} - \hat{g}^{\bar{j}i} D(\hat{g})_{i\bar{j}} = g^{\bar{j}i}\partial_i \partial_{\bar{j}}(D\varphi) + g^{\bar{j}i} D(\hat{g}_t)_{i\bar{j}} - \hat{g}^{\bar{j}i} D\hat{g}_{i\bar{j}}.$$

Hence, working in real coordinates, the function $u = D(\varphi)$ satisfies a linear parabolic PDE of the form

$$\partial_t u = a^{\alpha\beta}\partial_{x^\alpha}\partial_{x^\beta} u + f, \tag{5.1}$$

where $A = (a^{\alpha\beta})$ is a real $2n \times 2n$ positive definite symmetric matrix whose largest and smallest eigenvalues Λ and λ satisfy

$$C^{-1} \leq \lambda \leq \Lambda \leq C, \tag{5.2}$$

for a uniform positive constant C.

Moreover, the entries of A are uniformly bounded in the $C^{\delta/2,\delta}$ parabolic norm for $0 < \delta < 1$. Indeed our Calabi-type estimate from part (i) of Theorem 1.1,

$$|\hat{\nabla} g|^2 \leq C,$$

implies that the Riemannian metric g_R associated to g is bounded in the C^1 norm in the space direction. On the other hand,

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R^C_{i\bar{j}} = -g^{\bar{l}k} R_{i\bar{j}k\bar{l}}.$$

From the curvature bound of Theorem 1.1, we know that $g^{\bar{l}k}R_{i\bar{j}k\bar{l}}$ is uniformly bounded for any fixed i, j. It follows that $\frac{\partial}{\partial t}(g_R)_{\alpha\beta}$ is also uniformly bounded for any fixed α, β. Thus we see that each entry $a^{\alpha\beta}$ in the matrix A has uniform bounds in one space and one time derivative. This implies that $a^{\alpha\beta}$ is uniformly bounded in the $C^{\delta/2,\delta}$ parabolic norm for any $0 < \delta < 1$.

Next, note that $u = \frac{\partial \varphi}{\partial x^\gamma}$ in (5.1) is bounded in the C^0 norm since $g(t)$ is uniformly bounded and hence $|\sqrt{-1}\partial\bar{\partial}\varphi|_{C^0}$ is uniformly bounded. Moreover, f in (5.1) is uniformly bounded in the $C^{\delta/2,\delta}$ norm.

We can then apply Theorem 8.11.1 in [15] to (5.1) to see that u is bounded in the parabolic $C^{1+\delta/2,2+\delta}$ norm on a slightly smaller parabolic domain: $[\varepsilon', T] \times B_{r'}$ for any ε' and r' with $0 < \varepsilon' < \varepsilon$ and $r/8 < r' < r/4$. Tracing through the argument in [15], one can check that the estimates we obtain indeed are of the desired form.

Now apply D to the equality $g_{i\bar{j}}(t) = (\hat{g}_t)_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi$. We get

$$Dg_{i\bar{j}} = D(\hat{g}_t)_{i\bar{j}} + \partial_i \partial_{\bar{j}} u,$$

where we recall that $D = \partial/\partial x^\gamma$ for some γ. Since we have bounds for u in $C^{1+\delta/2,2+\delta}$ this implies that $\partial_i \partial_{\bar{j}} u$ is bounded in $C^{\delta/2,\delta}$. Since $D(\hat{g}_t)_{i\bar{j}}$ is uniformly bounded in all norms we get that $Dg_{i\bar{j}}$ is uniformly bounded in $C^{\delta/2,\delta}$ for all i, j. Since $D = \partial/\partial x^\gamma$ and γ was an arbitrary index, it follows that $\partial_\gamma a^{\alpha\beta}$ is uniformly bounded in $C^{\delta/2,\delta}$ for all α, β, γ. We have a similar estimate for $\partial_\gamma f$. Now apply Theorem 8.12.1 in [15] (with $k = 1$) to see that, for any α, $\partial_\alpha u$ is uniformly bounded in $C^{1+\delta/2,2+\delta}$ on a slightly smaller parabolic domain. This means that $D^\alpha \varphi$ is uniformly bounded in $C^{1+\delta/2,2+\delta}$ for any multi-index $\alpha \in \mathbb{R}^{2n}$ with $|\alpha| \leq 2$.

We can then iterate this procedure and obtain the required C^k bounds for $g(t)$ for all k. This completes the proof of the corollary.

Remark 5.1. In [18], we showed how to obtain higher derivative estimates for curvature using simple maximum principle arguments (following [13, 19]). However, in the case of the Chern–Ricci flow, there are difficulties in using this approach because of torsion terms that need to be controlled. An alternative method to proving the estimates in this section may be to generalize the work of Gill on the Kähler–Ricci flow [12]. This could give an "elementary" maximum principle proof, but the technical difficulties in carrying this out seem to be substantial.

References

[1] Aubin, Thierry. Équations du type Monge-Ampère sur les variétés kählériennes compactes. *Bull. Sci. Math.* (2) **102** (1978), no. 1, 63–95. MR0494932 (81d:53047), Zbl 0374.53022.

[2] Calabi, Eugenio. Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens. *Michigan Math. J.* **5** (1958), 105–126. MR0106487 (21 #5219), Zbl 0113.30104, doi: 10.1307/mmj/1028998055.

[3] Cao, Huai Dong. Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds. *Invent. Math.* **81** (1985), no. 2, 359–372. MR0799272 (87d:58051), Zbl 0574.53042, doi: 10.1007/BF01389058.

[4] Chau, Albert. Convergence of the Kähler-Ricci flow on noncompact Kähler manifolds. *J. Differential Geom.* **66** (2004), no. 2, 211–232. MR2106124 (2005g:53118), Zbl 1082.53070.

[5] Cheng, S. Y.; Yau, S. T. Differential equations on Riemannian manifolds and their geometric applications. *Comm. Pure Appl. Math.* **28** (1975), no. 3, 333–354. MR0385749 (52 #6608), Zbl 0312.53031, doi: 10.1002/cpa.3160280303.

[6] Cherrier, Pascal. Équations de Monge-Ampère sur les variétés hermitiennes compactes. *Bull. Sci. Math.* (2) **111** (1987), no. 4, 343–385. MR0921559 (89d:58131), Zbl 0629.58028.

[7] Chow, Bennett; Knopf, Dan. The Ricci flow: an introduction. Mathematical Surveys and Monographs, 110. *American Mathematical Society, Providence, RI*, 2004. xii+325 pp. ISBN: 0-8218-3515-7. MR2061425 (2005e:53101), Zbl 1086.53085.

[8] Chow, Bennett; Lu, Peng; Ni, Lei. Hamilton's Ricci flow. Graduate Studies in Mathematics, 77. *American Mathematical Society, Providence, RI, Science Press,*

New York, 2006. xxxvi+608 pp. ISBN: 978-0-8218-4231-7; 0-8218-4231-5. MR2274812 (2008a:53068), Zbl 1118.53001.

[9] Evans, Lawrence C. Classical solutions of fully nonlinear, convex, second order elliptic equations. *Comm. Pure Appl. Math.* **25** (1982), no. 3, 333–363. MR0649348 (83g:35038), Zbl 0469.35022, doi: 10.1002/cpa.3160350303.

[10] Feldman, Mikhail; Ilmanen, Tom; Knopf, Dan. Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons. *J. Differential Geometry* **65** (2003), no. 2, 169–209. MR2058261 (2005e:53102), Zbl 1069.53036.

[11] Gill, Matt. Convergence of the parabolic complex Monge-Ampère equation on compact Hermitian manifolds. *Comm. Anal. Geom.* **19** (2011), no. 2, 277–303. MR2835881 (2012h:32047), Zbl 1251.32035, arXiv:1009.5756.

[12] Gill, Matt. Collapsing of products along the Kähler-Ricci Flow. Preprint, 2012. arXiv:1203.3781.

[13] Hamilton, Richard S. Three-manifolds with positive Ricci curvature. *J. Differential Geom.* **17** (1982), no. 2, 255–306. MR0664497 (84a:53050), Zbl 0504.53034.

[14] Krylov, N. V. Boundedly inhomogeneous elliptic and parabolic equations. *Izv. Akad. Nauk. SSSR Ser. Mat.* **46** (1982), no. 3, 487–523, 670. MR0661144 (84a:35091). English translation in *Math. USSR Izv.* **20** (1983), no. 3, 459–492. Zbl 0529.35026, doi: 10.1070/IM1983v020n03ABEH001360.

[15] Krylov, N. V. Lectures on elliptic and parabolic equations in Hölder spaces. Graduate Studies in Mathematics, 12. *American Mathematical Society, Providence, RI*, 1996. xii+164 pp. ISBN: 0-8218-0569-X. MR1406091 (97i:35001), Zbl 0865.35001.

[16] Phong, D. H.; Song, Jian; Sturm, Jacob; Weinkove, Ben. On the convergence of the modified Kähler-Ricci flow and solitons. *Comment. Math. Helv.* **86** (2011), no. 1, 91–112. MR2745277 (2012d:53218), Zbl 1210.53066, doi: 10.4171/CMH/219.

[17] Phong, D. H.; Sesum, Natasa; Sturm, Jacob. Multiplier ideal sheaves and the Kähler-Ricci flow. *Comm. Anal. Geom.* **15** (2007), no. 3, 613–632. MR2379807 (2009a:32037), Zbl 1143.53064, arXiv:math/0611794.

[18] Sherman, Morgan; Weinkove, Ben. Interior derivative estimates for the Kähler-Ricci flow. *Pacific J. Math.* **257** (2012), no. 2, 491–501. MR2972475, Zbl 1262.53056, arXiv:1107.1853, doi: 10.2140/pjm.2012.257.491.

[19] Shi, Wan-Xiong. Deforming the metric on complete Riemannian manifolds. *J. Differential Geom.* **30** (1989), no. 1, 223–301. MR1001277 (90i:58202), Zbl 0676.53044.

[20] Song, Jian; Weinkove, Ben. The Kähler-Ricci flow on Hirzebruch surfaces. *J. Reine Angew. Math.* **659** (2011), 141–168. MR2837013 (2012g:53142), Zbl 1252.53080, arXiv:0903.1900, doi: 10.1515/crelle.2011.071.

[21] Song, Jian; Weinkove, Ben. Contracting exceptional divisors by the Kähler-Ricci flow. *Duke Math. J.* **162**, (2013), no. 2, 367–415. MR3018957, Zbl 1266.53063, arXiv:1003.0718.

[22] Streets, Jeffrey; Tian, Gang. Regularity results for pluriclosed flow. Preprint, 2010. arXiv:1008.2794.

[23] Tian, Gang; Zhang, Zhou. On the Kähler-Ricci flow on projective manifolds of general type. *Chinese Ann. Math. Ser. B* **27** (2006), no. 2, 179–192. MR2243679 (2007c:32029), Zbl 1102.53047, doi: 10.1007/s11401-005-0533-x.

[24] Tosatti, Valentino. Adiabatic limits of Ricci-flat Kähler metrics. *J. Differential Geom.* **84** (2010), no. 2, 427–453. MR2652468 (2011m:32039), Zbl 1208.32024, arXiv:0905.4718.

[25] Tosatti, Valentino; Weinkove, Ben. On the evolution of a Hermitian metric by its Chern-Ricci form. Preprint, 2011. arXiv:1201.0312.

[26] Tosatti, Valentino; Weinkove, Ben. The Chern-Ricci flow on complex surfaces. Preprint, 2012. arXiv:1209.2662.

[27] Tosatti, Valentino; Weinkove, Ben; Yau, Shing-Tung. Taming symplectic forms and the Calabi-Yau equation. *Proc. Lond. Math. Soc.* (3) **97** (2008), no. 2, 401–424. MR2439667 (2009h:32032), Zbl 1153.53054, arXiv:math/0703773, doi: 10.1112/plms/pdn008.

[28] Yau, Shing Tung. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. *Comm. Pure Appl. Math.* **31** (1978), no. 3, 339–411. MR0480350 (81d:53045), Zbl 0369.53059, doi: 10.1002/cpa.3160310304.

[29] Zhang, Xi; Zhang, Xiangwen. Regularity estimates of solutions to complex Monge-Ampère equations on Hermitian manifolds. *J. Funct. Anal.* **260** (2011), no. 7, 2004–2026. MR2756147 (2011m:32074), Zbl 1215.53038, arXiv:1007.2627, doi: 10.1016/j.jfa.2010.12.024.

Department of Mathematics, California Polytechnic State University, San Luis Obispo, CA 93407
sherman1@calpoly.edu

Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208
weinkove@math.northwestern.edu

This paper is available via http://nyjm.albany.edu/j/2013/19-29.html.