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Lmc-compactification of a semitopological semigroup as a space of e-ultrafilters

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Abstract. Let S be a semitopological semigroup and $\mathcal{CB}(S)$ denote the C^*-algebra of all bounded complex valued continuous functions on S with uniform norm. A function $f \in \mathcal{CB}(S)$ is left multiplicative continuous if and only if $\mathbf{T}_\mu f \in \mathcal{CB}(S)$ for all μ in the spectrum of $\mathcal{CB}(S)$, where $\mathbf{T}_\mu f(s) = \mu(L_s f)$ and $L_s f(x) = f(sx)$ for each $s, x \in S$. The collection of all the left multiplicative continuous functions on S is denoted by $\mathrm{Lmc}(S)$. In this paper, the Lmc-compactification of a semitopological semigroup S is reconstructed as a space of e-ultrafilters. This construction is applied to obtain some algebraic properties of $(\varepsilon, S^{\mathrm{Lmc}})$, such that S^{Lmc} is the spectrum of $\mathrm{Lmc}(S)$, for semitopological semigroups S. It is shown that if S is a locally compact semitopological semigroup, then $S^* = S^{\mathrm{Lmc}} \setminus \varepsilon(S)$ is a left ideal of S^{Lmc} if and only if for each $x, y \in S$, there exists a compact zero set A containing x such that $\{t \in S : yt \in A\}$ is a compact set.

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1. Introduction

It is well known that ultrafilters play a prominent role in the study of algebraic and topological properties of the Stone–Čech compactification βS of a discrete semigroup S. The Stone–Čech compactification βS of a discrete space S can be described as the spectrum of $\mathcal{B}(S)$, where $\mathcal{B}(S)$ is the C^*-algebra of all bounded complex-valued functions on S, or can be defined as the space of all ultrafilters on S (see [3] and [7]).

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When S is a discrete semigroup, $\mathcal{CB}(S)$ will be an m-admissible algebra and as a result, βS will be a semigroup. This semigroup, as the collection of all ultrafilters on S, has a known operation attributed to Glazer. Capability and competence of ultrafilter approach are mentioned clearly in [4], [5], [7] and [14].

Any semigroup compactification of a Hausdorff semitopological semigroup S is determined by the spectrum of a C^*-subalgebra $\mathcal{F}$ of $\mathcal{B}(S)$ containing the constant functions. Also all semigroup compactification of a semitopological semigroup as a collection of z-filters has been described in [12]. This approach sheds a new light on studying this kind of compactifications. With what was done in [12] as a model, some new topics in semigroup compactification are introduced using z-filters in a critical fashion. See [9],[10],[11] and [13]. It seems that the methods presented in [1], [2], [9], [11], [12] and [13] can serve as a valuable tool in the study of semigroup compactifications and also of topological compactifications.

Let X be a completely regular space, $\mathcal{C}(X,\mathbb{R})$ denotes all the real-valued continuous functions on X and $\mathcal{CB}(X,\mathbb{R})$ denotes all the bounded real-valued continuous functions on X. The correspondences between z-filters on X and ideals in $\mathcal{C}(X,\mathbb{R})$, which have been established in [5], are powerful tools in the study of $\mathcal{C}(X,\mathbb{R})$. These correspondences, which also occur in a rudimentary form in $\mathcal{CB}(X,\mathbb{R})$, are inconsequential, as many theorems of [5] become false if $\mathcal{C}(X,\mathbb{R})$ is replaced by $\mathcal{CB}(X,\mathbb{R})$. However, there is another correspondence between a certain class of z-filters on X and ideals in $\mathcal{CB}(X,\mathbb{R})$ that leads to quite analogous theorems to those for $\mathcal{C}(X,\mathbb{R})$. The requisite information is outlined in [5, 2L].

In Section 2, some familiarity with semigroup compactification and Lmc-compactification will be presented. This section also consists of an introduction to z-filters and an elementary external construction of Lmc-compactification as a space of z-filters. Moreover, in this section e-filters and e-ideals will be defined (they are adopted from [5, 2L]).

In Section 3, Lmc-compactification will be reconstructed as a space of e-ultrafilters with a suitable topology, also a binary operation will be defined on e-ultrafilters.

Section 4 concerns some theorems from [7] about the properties of βS which are extended to some properties on S^{Lmc}, for semitopological semigroup S.

2. Preliminary

Let S be a semitopological semigroup (i.e., for each $s \in S$, $\lambda_s : S \to S$ and $r_s : S \to S$ are continuous, where for each $x \in S$, $\lambda_s(x) = sx$ and $r_s(x) = xs$) with a Hausdorff topology, $\mathcal{CB}(S)$ denotes the C^*-algebra of all bounded complex valued continuous functions on S with uniform norm, and $\mathcal{C}(S)$ denotes the algebra of all complex valued continuous functions on S. A semigroup compactification of S is a pair (ψ, X), where X is a compact, Hausdorff, right topological semigroup (i.e., for all $x \in X$, r_x is continuous) and $\psi : S \to X$ is continuous homomorphism with dense image such that, for all $s \in S$, the mapping $x \mapsto \psi(s)x : X \to X$ is continuous, (see Definition 3.1.1 in [3]). Let $\mathcal{F}$ be a C^*-subalgebra of $\mathcal{CB}(S)$ containing the constant functions, then the set of all multiplicative means of $\mathcal{F}$ (the spectrum of $\mathcal{F}$), denoted by $S^{\mathcal{F}}$ and equipped with the Gelfand topology, is a compact Hausdorff topological space. Let $R_s f = f \circ r_s \in \mathcal{F}$ and $L_s f = f \circ \lambda_s \in \mathcal{F}$ for all $s \in S$ and $f \in \mathcal{F}$, and the function

$$s \mapsto (T_\mu f(s)) = \mu(L_s f)$$

is in $\mathcal{F}$ for all $f \in \mathcal{F}$ and $\mu \in S^{\mathcal{F}}$, then $S^{\mathcal{F}}$ under the multiplication $\mu\nu = \mu \circ T_\nu$ $(\mu, \nu \in S^{\mathcal{F}})$, furnished with the Gelfand topology, makes $(\varepsilon, S^{\mathcal{F}})$ a semigroup compactification (called the $\mathcal{F}$-compactification) of S, where $\varepsilon : S \to S^{\mathcal{F}}$ is the evaluation mapping. Also, $\varepsilon^* : \mathcal{C}(S^{\mathcal{F}}) \to \mathcal{F}$ is isometric isomorphism and $\widehat{f} = (\varepsilon^*)^{-1}(f) \in \mathcal{C}(S^{\mathcal{F}})$ for $f \in \mathcal{F}$ is given by $\widehat{f}(\mu) = \mu(f)$ for all $\mu \in S^{\mathcal{F}}$, (for more detail see section 2 in [3]).

Let $\mathcal{F} = \mathcal{CB}(S)$, then $\beta S = S^{\mathcal{CB}(S)}$ is the Stone–Čech compactification of S, where S is a completely regular space.

A function $f \in \mathcal{CB}(S)$ is left multiplicative continuous if and only if

$$\mathbf{T}_\mu f \in \mathcal{CB}(S)$$

for all $\mu \in \beta S = S^{\mathcal{CB}(S)}$. The collection of all left multiplicative continuous functions on S is denoted by $\mathrm{Lmc}(S)$. Therefore,

$$\mathrm{Lmc}(S) = \bigcap \{\mathbf{T}_\mu^{-1}(\mathcal{CB}(S)) : \mu \in \beta S\}$$

is defined and $(\varepsilon, S^{\mathrm{Lmc}})$ is the universal semigroup compactification of S (Definition 4.5.1 and Theorem 4.5.2 in [3]). In general, S can not be embedded in S^{Lmc}. In fact, as it was shown in [6] there is a completely regular Hausdorff semitopological semigroup S, such that the continuous homomorphism ε from S to its Lmc-compactification, is neither one-to-one nor open as a mapping to $\varepsilon(S)$.

The $\mathcal{LUC}$-compactification is the spectrum of the C^*-algebra consisting of all left uniformly continuous functions on semitopological semigroup S; a function $f : S \to \mathbb{C}$ is left uniformly continuous if $s \mapsto L_s f$ is a continuous map from S to the space of bounded continuous functions on S with the uniform norm. Let G be a locally compact Hausdorff topological group, by Theorem 5.7 of chapter 4 in [3] implies that $\mathrm{Lmc}(G) = \mathcal{LUC}(G)$. Also the evaluation map $G \to G^{\mathcal{LUC}}$ is open, (see [3]).

Now, some prerequisite material from [12] are quoted for the description of $(\varepsilon, S^{\mathrm{Lmc}})$ in terms of z-filters. For $f \in \mathrm{Lmc}(S)$, $Z(f) = f^{-1}(\{0\})$ is called zero set, and the collection of all zero sets is denoted by $Z(\mathrm{Lmc}(S))$. For an extensive account of ultrafilters, the readers may refer to [4], [5], [7] and [14].

Definition 2.1. $\mathcal{A} \subseteq Z(\mathrm{Lmc}(S))$ is called a z-filter on $\mathrm{Lmc}(S)$ (or for simplicity z-filter) if:

(i) $\emptyset \notin \mathcal{A}$ and $S \in \mathcal{A}$.
(ii) If $A, B \in \mathcal{A}$, then $A \bigcap B \in \mathcal{A}$.
(iii) If $A \in \mathcal{A}$, $B \in Z(\mathrm{Lmc}(S))$ and $A \subseteq B$ then $B \in \mathcal{A}$.

Because of (iii), (ii) may be replaced by:

(ii′) If $A, B \in \mathcal{A}$, then $A \cap B$ contains a member of $\mathcal{A}$.

A z-ultrafilter is a z-filter which is not properly contained in any other z-filter. The collection of all z-ultrafilters is denoted by $\mathcal{Z}(S)$. For $x \in S$, $\widehat{x} = \{Z(f) : f \in \mathrm{Lmc}(S),\ f(x) = 0\}$ is a z-ultrafilter. The z-filter $\mathcal{F}$ is named converge to the limit $\mu \in S^{\mathrm{Lmc}}$ if every neighborhood of μ contains a member of $\mathcal{F}$. The collection of all z-ultrafilters on $\mathrm{Lmc}(S)$ converge to $\mu \in S^{\mathrm{Lmc}}$ is denoted by $[\mu]$. Let $\mathcal{Q} = \{\widetilde{p} : \widetilde{p} = \cap[\mu]\}$ and define

$$\widetilde{A} = \{\widetilde{p} : A \in \widetilde{p}\}$$

for $A \subseteq S$. Let $\mathcal{Q}$ be equipped with the topology whose basis is

$$\{(\widetilde{A})^c : A \in Z(\mathrm{Lmc}(S))\},$$

and define $\bigcap[\mu] * \bigcap[\nu] = \bigcap[\mu\nu]$. Then $(\mathcal{Q}, *)$ is a (Hausdorff) compact right topological semigroup and $\varphi : S^{\mathrm{Lmc}} \to \mathcal{Q}$ defined by $\varphi(\mu) = \cap[\mu] = \widetilde{p}$, where $\bigcap_{A\in p} \overline{A} = \{\mu\}$, is topologically isomorphism. So $\widetilde{A}$ is equal to $\mathrm{cl}_{S^{\mathrm{Lmc}}} A$ and we denote it by $\overline{A}$, also for simplicity we use x replace $\widehat{x}$. The operation "$\cdot$" on S, extends uniquely to "$*$" on $\mathcal{Q}$. For more discussion and details see [12].

Remark 2.2. If $p, q \in \mathcal{Z}(S)$, then the following statements hold.

(i) If $E \subseteq Z(\mathrm{Lmc}(S))$ has the finite intersection property, then E is contained in a z-ultrafilter.
(ii) If $B \in Z(\mathrm{Lmc}(S))$ and for all $A \in p$, $A \cap B \neq \emptyset$ then $B \in p$.
(iii) If $A, B \in Z(\mathrm{Lmc}(S))$ such that $A \cup B \in p$, then $A \in p$ or $B \in p$.
(iv) Let p and q be distinct z-ultrafilters, then there exist $A \in p$ and $B \in q$ such that $A \cap B = \emptyset$.
(v) Let p be a z-ultrafilter, then there exists $\mu \in S^{\mathrm{Lmc}}$ such that

$$\bigcap_{A\in p} \overline{\varepsilon(A)} = \{\mu\}.$$

(For (i), (ii), (iii) and (iv) see Lemma 2.2 and Lemma 2.3 in [12]. For (v) see Lemma 2.6 in [12]).

In this paper, $\mathbb{R}$ denotes the topological group formed by the real numbers under addition. Also we suppose $\ker(\mu) = \{f \in \mathrm{Lmc}(S) : \mu(f) = 0\}$ for $\mu \in \mathrm{Lmc}(S)^*$. By Theorem 11.5 in [8], M is a maximal ideal of $\mathrm{Lmc}(S)$ if and only if there is $\mu \in S^{\mathrm{Lmc}}$ such that $\ker(\mu) = M$.

Lemma 2.3. *Let S be a Hausdorff semitopological semigroup.*

(1) *If* $f \in \mathrm{Lmc}(S)$ *is real-valued, then* $f^+ = \max\{f, 0\} \in \mathrm{Lmc}(S)$ *and* $f^- = -\min\{f, 0\} \in \mathrm{Lmc}(S)$.
(2) *Let* $f \in \mathrm{Lmc}(S)$. *then* $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ *and* $|f|$ *are all in* $\mathrm{Lmc}(S)$.
(3) *If* f *and* g *are real-valued functions in* $\mathrm{Lmc}(S)$, *then*

$$(f \vee g)(x) = \max\{f(x), g(x)\} \in \mathrm{Lmc}(S),$$

and

$$(f \wedge g)(x) = \min\{f(x), g(x)\} \in \mathrm{Lmc}(S).$$

(4) *Let* $f \in \mathrm{Lmc}(S)$ *and there exists* $c > 0$ *such that* $c < |f(x)|$ *for each* $x \in S$. *Then* $\frac{1}{f} \in \mathrm{Lmc}(S)$.
(5) *Let* M *be a maximal ideal and* $f \in M$, *then* $\overline{f} \in M$.

Proof. For (1), (2) and (3), since $f \mapsto \widehat{f} : \mathrm{Lmc}(S) \to \mathcal{C}(S^{\mathrm{Lmc}})$ is isometrical isomorphism and $|\widehat{f}| \in \mathcal{C}(S^{\mathrm{Lmc}})$ for each $f \in \mathrm{Lmc}(S)$, so we have

$$\begin{aligned} |\widehat{f}|(\varepsilon(x)) &= |\widehat{f}(\varepsilon(x))| \\ &= |\varepsilon(x)(f)| \\ &= |f(x)| \\ &= |f|(x) \end{aligned}$$

for each $x \in S$. Thus, $|\widehat{f}| = \widehat{|f|}$ for each $f \in \mathrm{Lmc}(S)$ and so $|f| \in \mathrm{Lmc}(S)$ for each $f \in \mathrm{Lmc}(S)$.
Now let f and g be real-valued functions, so

$$f \vee g(x) = \frac{|f-g|(x)}{2} + \frac{(f+g)(x)}{2} \in \mathrm{Lmc}(S).$$

In a similar way $f \wedge g$, f^+ and f^- are in $\mathrm{Lmc}(S)$. Pick $f \in \mathrm{Lmc}(S)$, since $\mathrm{Lmc}(S)$ is conjugate closed subalgebra so $\mathrm{Re}(f) = \frac{f+\overline{f}}{2} \in \mathrm{Lmc}(S)$ and $\mathrm{Im}(f) = \frac{f-\overline{f}}{2i} \in \mathrm{Lmc}(S)$.

For (4), let $f \in \mathrm{Lmc}(S)$ and there exists $c > 0$ such that $c < |f(x)|$ for each $x \in S$. So $|\widehat{f}|(\mu) \geq c$ for each $\mu \in S^{\mathrm{Lmc}}$, which implies that $\widehat{\frac{1}{f}} = \frac{1}{\widehat{f}} \in \mathcal{C}(S^{\mathrm{Lmc}})$. Therefore, $\frac{1}{f} \in \mathrm{Lmc}(S)$.

For (5), let M be a maximal ideal in $\mathrm{Lmc}(S)$, so there exists $\mu \in S^{\mathrm{Lmc}}$ such that $M = \ker(\mu) = \{f \in \mathrm{Lmc}(S) : \mu(f) = 0\}$. Now let $f \in M$, so $\mu(f) = \mu(\mathrm{Re}(f)) + i\mu(\mathrm{Im}(f)) = 0$. This implies that $\mu(\mathrm{Re}(f)) = \mu(\mathrm{Im}(f)) = 0$ and so $\mu(\overline{f}) = 0$. Thus, $\overline{f} \in M$. $\square$

For $f \in \mathrm{Lmc}(S)$ and $\epsilon > 0$, we define $E_\epsilon(f) = \{x \in S : |f(x)| \leq \epsilon\}$. Every such set is a zero set. Conversely, every zero set is of this form, $Z(g) = E_\epsilon(\epsilon + |g|)$. For $I \subseteq \mathrm{Lmc}(S)$, we write $E(I) = \{E_\epsilon(f) : f \in I, \epsilon > 0\}$, i.e., $E(I) = \bigcup_{\epsilon > 0} E_\epsilon(I)$. Finally, for any family $\mathcal{A}$ of zero sets, we define

$$E^-(\mathcal{A}) = \{f \in \mathrm{Lmc}(S) : E_\epsilon(f) \in \mathcal{A} \text{ for each } \epsilon > 0\},$$

that is, $E^{-}(\mathcal{A}) = \bigcap_{\epsilon>0} E_{\epsilon}^{\leftarrow}(\mathcal{A})$, where

$$E_{\epsilon}^{\leftarrow}(\mathcal{A}) = \{f \in \mathrm{Lmc}(S) : E_{\epsilon}(f) \in \mathcal{A}\}.$$

Lemma 2.4. *For any family $\mathcal{A}$ of zero sets,*

$$E(E^{-}(\mathcal{A})) = \bigcup_{\epsilon>0} \{E_{\epsilon}(f) : f \in \mathrm{Lmc}(S),\ E_{\delta}(f) \in \mathcal{A} \textit{ for all } \delta > 0\} \subseteq \mathcal{A}.$$

The inclusion may be proper, when $\mathcal{A}$ is a z-filter.

Proof. Let $\mathcal{A}$ be a family of zero sets, so

$$E^{-}(\mathcal{A}) = \{f \in \mathrm{Lmc}(S) : E_{\epsilon}(f) \in \mathcal{A} \text{ for all } \epsilon > 0\},$$

and thus,

$$\begin{aligned} E(E^{-}(\mathcal{A})) &= \{E_{\epsilon}(f) : f \in E^{-}(\mathcal{A}), \epsilon > 0\} \\ &= \bigcup_{\epsilon>0} \{E_{\epsilon}(f) : f \in E^{-}(\mathcal{A})\} \\ &= \bigcup_{\epsilon>0} \{E_{\epsilon}(f) : E_{\delta}(f) \in \mathcal{A} \text{ for all } \delta > 0\} \\ &\subseteq \mathcal{A}. \end{aligned}$$

Finally, let $M_0 = \{f \in \mathrm{Lmc}((\mathbb{R}, +)) : f(0) = 0\}$, then M_0 is a maximal ideal in $\mathrm{Lmc}((\mathbb{R}, +))$ and $\mathcal{A} = \{Z(f) : f \in M_0\}$ is a z-filter. Define $g(x) = |x| \wedge 1$ for each $x \in \mathbb{R}$, then $g \in M_0$ and so $\{0\} = Z(g) \in \mathcal{A}$. Since

$$\begin{aligned} E(E^{-}(\mathcal{A})) &= \bigcup_{\epsilon>0} \{E_{\epsilon}(f) : E_{\delta}(f) \in \mathcal{A} \text{ for all } \delta > 0\} \\ &\subseteq \mathcal{A}, \end{aligned}$$

pick $f \in \mathrm{Lmc}((\mathbb{R}, +))$ such that $E_{\epsilon}(f) \in E(E^{-}(\mathcal{A}))$ for each $\epsilon > 0$. Since f is continuous so for each $\epsilon > 0$ there exists $\delta > 0$ such that $f((-\delta, \delta)) \subseteq (-\epsilon, \epsilon)$; therefore, $(-\delta, \delta) \subseteq E_{\epsilon}(f)$. This implies that $E(E^{-}(\mathcal{A}))$ is a collection of uncountable zero sets. But $\{0\} \in \mathcal{A}$ is finite and so $\{0\} \notin E(E^{-}(\mathcal{A}))$. Therefore, $E(E^{-}(\mathcal{A})) \neq \mathcal{A}$. □

Definition 2.5. Let $\mathcal{A}$ be a z-filter. Then $\mathcal{A}$ is called an e-filter if

$$E(E^{-}(\mathcal{A})) = \mathcal{A}.$$

Hence, $\mathcal{A}$ is an e-filter if and only if, whenever $Z \in \mathcal{A}$, there exist $f \in \mathrm{Lmc}(S)$ and $\epsilon > 0$ such that $Z = E_{\epsilon}(f)$ and $E_{\delta}(f) \in \mathcal{A}$ for every $\delta > 0$.

Lemma 2.6. *Let I be a subset of* $\mathrm{Lmc}(S)$. *Then,*

$$I \subseteq E^{-}(E(I)) = \{f \in \mathrm{Lmc}(S) : E_{\epsilon}(f) \in E(I) \textit{ for all } \epsilon > 0\}.$$

The inclusion may be proper, even when I is an ideal.

Proof. By Definition

$$I \subseteq E^{-}(E(I)) = \{f \in \mathrm{Lmc}(S) : E_\epsilon(f) \in E(I) \text{ for all } \epsilon > 0\}.$$

Finally, let I be the ideal of all functions in $\mathrm{Lmc}((\mathbb{R},+))$ that vanish on a neighborhood of 0. Pick $g(x) = |x| \wedge 1$ in $\mathrm{Lmc}((\mathbb{R},+))$ that vanishes precisely at 0. Since for each $\epsilon > 0$, $E_\epsilon(g) = E_{\frac{\epsilon}{2}}((g \vee \frac{\epsilon}{2}) - \frac{\epsilon}{2})$ and $(g \vee \frac{\epsilon}{2}) - \frac{\epsilon}{2} \in I$, then $E_\epsilon(g) \in E(I)$ for each $\epsilon > 0$, and so $g \in E^{-}(E(I))$ but $g \notin I$. This completes the proof. □

Definition 2.7. Let I be an ideal of $\mathrm{Lmc}(S)$. I is called an e-ideal if $E^{-}(E(I)) = I$.

Hence, I is an e-ideal if and only if, whenever $E_\epsilon(f) \in E(I)$ for all $\epsilon > 0$, then $f \in I$.

Lemma 2.8. *The following statements hold.*

(1) *The intersection of e-ideals is an e-ideal.*
(2) *If I is an ideal in $\mathrm{Lmc}(S)$, then $E(I)$ is an e-filter.*
(3) *If $\mathcal{A}$ is any z-filter, then $E^{-}(\mathcal{A})$ is an e-ideal in $\mathrm{Lmc}(S)$.*
(4) *$I \subseteq J \subseteq \mathrm{Lmc}(S)$ implies $E(I) \subseteq E(J)$, and $\mathcal{A} \subseteq \mathcal{B} \subseteq Z(\mathrm{Lmc}(S))$ implies $E^{-}(\mathcal{A}) \subseteq E^{-}(\mathcal{B})$.*
(5) *If J is an e-ideal, then $I \subseteq J$ if and only if $E(I) \subseteq E(J)$. If $\mathcal{A}$ is an e-filter, then $\mathcal{A} \subseteq \mathcal{B}$ if and only if $E^{-}(\mathcal{A}) \subseteq E^{-}(\mathcal{B})$.*
(6) *If $\mathcal{A}$ is any e-filter, then $E^{-}(\mathcal{A})$ is an e-ideal. Let I be an ideal in $\mathrm{Lmc}(S)$, then $E^{-}(E(I))$ is the smallest e-ideal containing I. In particular, every maximal ideal in $\mathrm{Lmc}(S)$ is an e-ideal.*
(7) *For any z-filter $\mathcal{A}$, $E(E^{-}(\mathcal{A}))$ is the largest e-filter contained in $\mathcal{A}$.*

Proof. (1) Suppose that $\{I_\alpha\}$ is a collection of e-ideals and $I = \bigcap_\alpha I_\alpha$. Let $E_\epsilon(f) \in E(I)$ for each $\epsilon > 0$, then $E_\epsilon(f) \in E(I_\alpha)$ for each $\epsilon > 0$, so $f \in I_\alpha$ for each α. This implies $f \in I$.

(2) Let $E_\epsilon(f) = \emptyset$ for some $\epsilon > 0$ and $f \in I$, then $\epsilon \leq |f(x)| \leq M$ for some $M > 0$ and for each $x \in S$. So $\frac{1}{f} \in \mathrm{Lmc}(S)$ and $1 = f\frac{1}{f} \in I$. This is a contradiction and so $\emptyset \notin E(I)$.

Let $f' \in \mathrm{Lmc}(S)$, $f \in I$ be a nonnegative function and $E_\epsilon(f) \subseteq Z(f')$, then $g(x) = |f'(x)| + \frac{\epsilon}{\epsilon \vee |f(x)|} \in \mathrm{Lmc}(S)$. Now

$$|f(x)|g(x) = |f'(x)f(x)| + \frac{\epsilon|f(x)|}{\epsilon \vee |f(x)|},$$

so $x \in Z(f')$ implies that $|f(x)g(x)| = \frac{\epsilon|f(x)|}{\epsilon \vee |f(x)|} \leq \epsilon$. Hence $Z(f') \subseteq E_\epsilon(fg)$. If $x \in E_\epsilon(fg)$, then

$$|f'(x)f(x)| \leq |f'(x)f(x)| + \frac{\epsilon|f(x)|}{\epsilon \vee |f(x)|} = |f(x)g(x)| \leq \epsilon$$

and if $x \notin Z(f')$ then $\epsilon < |f(x)|$ and $|g(x)f(x)| > \epsilon$. Therefore this implies $E_\epsilon(fg) \subseteq Z(f')$, and so $E_\epsilon(fg) = Z(f')$.

Suppose that $E_\epsilon(f), E_\delta(g) \in E(I)$ for some $f, g \in I$ and $\epsilon, \delta > 0$. Let $\gamma = \epsilon \wedge \delta \wedge \frac{1}{2}$, then

$$E_{\frac{\gamma^2}{4}}(f\overline{f} + g\overline{g}) \subseteq E_\gamma(f) \cap E_\gamma(g) \subseteq E_\epsilon(f) \cap E_\delta(g),$$

thus $E_\epsilon(f) \cap E_\delta(g) \in E(I)$.

Now let $Z \in E(I)$, so there exists $f \in I$ such that $Z = E_\epsilon(f)$ for some $\epsilon > 0$. By definition of $E(I)$, $E_\delta(f) \in E(I)$ for each $\delta > 0$, so $E(I)$ is an e-filter.

(3) Let $f, g \in E^-(\mathcal{A})$. Since $E_{\epsilon/2}(|f|) \cap E_{\epsilon/2}(|g|) \subseteq E_\epsilon(|f-g|)$; therefore, $E_\epsilon(f-g) \in \mathcal{A}$ for each $\epsilon > 0$. Thus, $f - g \in E^-(\mathcal{A})$. Let $f \in E^-(\mathcal{A})$, $g \in \mathrm{Lmc}(S)$ and $M = \|g\| + 1$. Hence, $E_{\frac{\epsilon}{M}}(f) \subseteq E_\epsilon(fg)$ and $fg \in E^-(\mathcal{A})$. Now let $E_\epsilon(f) \in E^-(\mathcal{A})$ for each $\epsilon > 0$. Definition of $E^-(\mathcal{A})$ implies that $f \in E^-(\mathcal{A})$. Thus, $E^-(\mathcal{A})$ is an e-ideal.

(4) This can easily be checked.

(5) It is obvious that if $I \subseteq J$ then $E(I) \subseteq E(J)$ by (4).

Conversely. If $f \in I$, then $E_\epsilon(f) \in E(I)$ for each $\epsilon > 0$, so $E_\epsilon(f) \in E(J)$. Since J is an e-filter, so $f \in J$. If $\mathcal{A} \subseteq \mathcal{B}$, then $E^-(\mathcal{A}) \subseteq E^-(\mathcal{B})$. Since $\mathcal{A}$ is an e-filter, then $\mathcal{A} = E(E^-(\mathcal{A})) \subseteq E(E^-(\mathcal{B})) \subseteq \mathcal{B}$.

(6) Let $I = E^-(\mathcal{A}) = \{f \in \mathrm{Lmc}(S) : \forall \epsilon > 0,\ E_\epsilon(f) \in \mathcal{A}\}$; thus, $\mathcal{A}$ is an e-filter, and $\mathcal{A} = E(E^-(\mathcal{A})) = E(I)$. This implies $I = E^-(\mathcal{A}) = E^-(E(I))$ and so I is an e-ideal. Let $I \subseteq \mathrm{Lmc}(S)$ be an ideal, then $J = E^-(E(I))$ is an e-ideal (by (3) and (4)), so $I \subseteq J$. Let $I \subseteq K \subseteq J$ and K be an e-ideal, then $E(I) \subseteq E(K) \subseteq E(J) = E(I)$ and $E(K) = E(I)$. Thus, $J = E^-(E(I)) = E^-(E(K)) = K$, and this implies that J is the smallest e-ideal containing I.

Finally, every maximal ideal in $\mathrm{Lmc}(S)$ is an e-ideal. For this, let M be a maximal ideal in $\mathrm{Lmc}(S)$. Then, $E^-(E(M))$ is an e-ideal, $M \subseteq E^-(E(M))$ and M is maximal so, $M = E^-(E(M))$.

(7) Let $\mathcal{A}$ be a z-filter, then $E^-(\mathcal{A})$ is an ideal in $\mathrm{Lmc}(S)$, so $E(E^-(\mathcal{A}))$ is an e-filter and $\mathcal{B} = E(E^-(\mathcal{A})) \subseteq \mathcal{A}$. Now let $\mathcal{U}$ be an e-filter such that $\mathcal{B} \subseteq \mathcal{U} \subseteq \mathcal{A}$, then $E^-(\mathcal{U}) = E^-(\mathcal{A})$. Hence, $\mathcal{B} \subseteq \mathcal{A}$ is an e-filter. □

A maximal e-filter is called an e-ultrafilter. Zorn's Lemma implies that every e-filter is contained in an e-ultrafilter. Because, if $\mathcal{Y}$ is a chain of e-filters, then it is also a chain of z-filters and so $\cup\mathcal{Y}$ is a z-filter. It is sufficient to show $\cup\mathcal{Y}$ is an e-filter. Let $Z \in \cup\mathcal{Y}$, then there exists $Y \in \mathcal{Y}$, such that $Z \in Y$. Since Y is an e-ideal, so there exist $f \in \mathrm{Lmc}(S)$ and $\varepsilon > 0$ such that $Z = E_\epsilon(f)$ and $\{E_\delta(f) : \delta > 0\} \subseteq Y$. Thus, there exist $f \in \mathrm{Lmc}(S)$ and $\epsilon > 0$ such that $Z = E_\epsilon(f)$ and $\{E_\delta(f) : \delta > 0\} \subseteq \cup\mathcal{Y}$. Therefore, $\cup\mathcal{Y}$ is an e-filter.

Theorem 2.9. *If M is a maximal ideal in* $\mathrm{Lmc}(S)$*, then $E(M)$ is an e-ultrafilter, and if $\mathcal{A}$ is an e-ultrafilter, then $E^-(\mathcal{A})$ is a maximal ideal in* $\mathrm{Lmc}(S)$*.*

Proof. Let M be a maximal ideal, so $E(M)$ is an e-filter (Lemma 2.8(2)). Suppose that there exists an e-filter $\mathcal{U}$ such that $E(M) \subseteq \mathcal{U}$, then $M = E^{-}(E(M)) \subseteq E^{-}(\mathcal{U})$ and so $E(M) = E(E^{-}(\mathcal{U})) = \mathcal{U}$, by Lemma 2.8(7). Thus, $E(M)$ is an e-ultrafilter.

Now let $\mathcal{E}$ be an e-ultrafilter, then $E^{-}(\mathcal{E})$ is an ideal in $\mathrm{Lmc}(S)$ (Lemma 2.8(3)). Let J be a maximal ideal such that $E^{-}(\mathcal{E}) \subseteq J$, then J is an e-ideal and so $E(E^{-}(\mathcal{E})) \subseteq E(J)$. Since $\mathcal{E}$ is an e-ultrafilter, so $\mathcal{E} = E(E^{-}(\mathcal{E}))$ and $E^{-}(\mathcal{E}) = E^{-}(E(J)) = J$. This implies that $E^{-}(\mathcal{E})$ is maximal. $\square$

The correspondence $M \mapsto E(M)$ is one to one from the set of all maximal ideals in $\mathrm{Lmc}(S)$ onto the set of all e-ultrafilters.

Theorem 2.10. *The following property characterizes an ideal M in $\mathrm{Lmc}(S)$ as a maximal ideal: given $f \in \mathrm{Lmc}(S)$, if $E_\epsilon(f)$ meets every member of $E(M)$ for each $\epsilon > 0$, then $f \in M$.*

Proof. Let M be a maximal ideal and $f \in \mathrm{Lmc}(S)$. Let $E_\epsilon(f)$ meet every member of $E(M)$ for each $\epsilon > 0$. So $E(M) \cup \{E_\epsilon(f) : \epsilon > 0\}$ has the finite intersection property, and so there exists a z-ultrafilter $\mathcal{A}$ containing it. By Lemma 2.8 and Theorem 2.9,

$$M = E^{-}(\mathcal{A}) = \{g \in \mathrm{Lmc}(S) : E_\epsilon(g) \in \mathcal{A} \text{ for each } \epsilon > 0\}.$$

This implies that $f \in M$.

Now let M be an ideal in $\mathrm{Lmc}(S)$ with the following property: given $f \in \mathrm{Lmc}(S)$, if $E_\epsilon(f)$ meets every member of $E(M)$ for each $\epsilon > 0$, then $f \in M$. We show that M is a maximal ideal. Let $f \in \mathrm{Lmc}(S) \setminus M$ and so some $E_\epsilon(f)$ fails to meet some member of $E(M)$. Therefore, there exist $g \in M$ and $\delta > 0$ such that $E_\epsilon(f) \cap E_\delta(g) = \emptyset$. Pick $\gamma = \min\{\delta^2, \epsilon^2, 1\}$, then $E_\gamma(f\overline{f} + g\overline{g}) \subseteq E_\epsilon(f) \cap E_\delta(g)$, so $f\overline{f} + g\overline{g}$ is invertible and generated ideal by $M \cup \{f\}$ is equal with $\mathrm{Lmc}(S)$. This implies M is a maximal ideal. $\square$

Let $\mathcal{A}$ and $\mathcal{B}$ be z-ultrafilters. It is said that $\mathcal{A} \sim \mathcal{B}$ if and only if $E(E^{-}(\mathcal{A})) = E(E^{-}(\mathcal{B}))$. It is obvious that $\sim$ is an equivalence relation. The equivalence class of $\mathcal{A} \in \mathcal{Z}(S)$ is denoted by $[\mathcal{A}]$.

Lemma 2.11. *Let $\mathcal{A}$ be a z-ultrafilter, then:*

(a) *Let $Z(f) \in \mathcal{A}$ for some $f \in \mathrm{Lmc}(S)$, then $f \in E^{-}(\mathcal{A})$.*
(b) *$E^{-}(\mathcal{A})$ is a maximal ideal.*
(c) *$E(E^{-}(\mathcal{A}))$ is an e-ultrafilter.*
(d) *Let Z be a zero set that meets every member of $E(E^{-}(\mathcal{A}))$, then there exists $\mathcal{B} \in [\mathcal{A}]$, such that $Z \in \mathcal{B}$.*

Proof. (a) By Remark 2.2(v), pick $\mu \in S^{\mathrm{Lmc}}$ such that $\bigcap_{A \in \mathcal{A}} \overline{\varepsilon(A)} = \{\mu\}$. Now let $Z(f) \in \mathcal{A}$, then $\mu \in \overline{\varepsilon(Z(f))}$ and so there exists a net $\{\varepsilon(x_\alpha)\} \subseteq \varepsilon(A)$ such that $lim_\alpha \varepsilon(x_\alpha) = \mu$. Since

$$\mu(f) = lim_\alpha \varepsilon(x_\alpha)(f) = lim_\alpha f(x_\alpha) = 0,$$

so $f \in \ker(\mu)$. It is obvious $Z(f) \subseteq E_\epsilon(f)$ for each $\epsilon > 0$ and so $E_\epsilon(f) \in \mathcal{A}$ for each $\epsilon > 0$. This implies $f \in E^-(\mathcal{A})$.

(b) By (a), there exists $\mu \in S^{\mathrm{Lmc}}$ such that $\ker(\mu) \subseteq E^-(\mathcal{A})$. Since $\ker(\mu)$ is a maximal ideal in $\mathrm{Lmc}(S)$ and also by Lemma 2.8(3), so $\ker(\mu) = E^-(\mathcal{A})$.

(c) Since $E^-(\mathcal{A})$ is a maximal ideal, so $E(E^-(\mathcal{A}))$ is an e-ultrafilter by Theorem 2.9.

(d) Let Z be a zero set that meets every member of $E(E^-(\mathcal{A}))$. Then, $\{Z\} \cup E(E^-(\mathcal{A}))$ has the finite intersection property. Hence there exists some z-ultrafilter $\mathcal{B}$ containing $\{Z\} \cup E(E^-(\mathcal{A}))$. Since $E(E^-(\mathcal{A}))$ is an e-ultrafilter contained in $\mathcal{B}$, so by (b), $E^-(\mathcal{B})$ is a maximal ideal and by Lemma 2.8(4), $E^-(\mathcal{A}) \subseteq E^-(\mathcal{B})$. Thus by Theorem 2.9, $E^-(\mathcal{B}) = E^-(\mathcal{A})$ and so $E(E^-(\mathcal{B})) = E(E^-(\mathcal{A}))$. Therefore, there exists $\mathcal{B} \in [\mathcal{A}]$ such that $Z \in \mathcal{B}$. $\square$

Remark 2.12. Since $(\mathbb{R}, +)$ is a locally compact topological group, by Theorem 5.7 of Chapter 4 in [3],

$$\mathrm{Lmc}(\mathbb{R}) = \{f \in \mathcal{CB}(\mathbb{R}) : t \mapsto f \circ \lambda_t : \mathbb{R} \to \mathcal{CB}(\mathbb{R}) \text{ is norm continuous.}\}.$$

Let $\mathcal{C}_\circ(\mathbb{R}) = \{f \in \mathcal{CB}(\mathbb{R}) : lim_{x\to\pm\infty} f(x) = 0\}$, then $\mathcal{C}_\circ(\mathbb{R})$ is an ideal of $\mathrm{Lmc}(\mathbb{R})$. Let M be a maximal ideal in $\mathrm{Lmc}(\mathbb{R})$ which contains $\mathcal{C}_\circ(\mathbb{R})$. It is obvious that $f(x) = e^{-x^2} sin(x)$ and $g(x) = e^{-x^2} cos(\pi x)$ belong to $C_\circ(\mathbb{R})$. Then $Z(f) = \{k\pi : k \in \mathbb{Z}\}$, $Z(g) = \{\frac{2k+1}{2} : k \in \mathbb{Z}\}$, and $E(M) \cup \{Z(f)\}$ and $E(M) \cup \{Z(g)\}$ have the finite intersection property. So there exist z-ultrafilters $\mathcal{A}$ and $\mathcal{B}$ such that $E(M) \cup \{Z(f)\} \subseteq \mathcal{A}$ and also $E(M) \cup \{Z(g)\} \subseteq \mathcal{B}$. Since $E(M)$ is an e-ultrafilter so there exist at least two distinct z-ultrafilters containing $E^-(\mathcal{A})$. Thus:

(i) It is not necessary the collection of all z-ultrafilters containing an e-ultrafilter be a single set.
(ii) Let $\mathcal{A}$ be a z-ultrafilter. Then there exists a zero-set Z such that Z meets every member of $E(E^-(\mathcal{A}))$ and $Z \notin \mathcal{A}$.

3. Space of e-ultrafilters

In this section we will define a topology on the set of all e-ultrafilters of a semitopological semigroup S, and establish some of the properties of the resulting space. Also, the operation of the semitopological semigroup has been extended to the set of all e-ultrafilters.

Definition 3.1. Let S be a Hausdorff semitopological semigroup.

(a) The collection of all e-ultrafilters is denoted by $\mathcal{E}(S)$, i.e.,

$$\mathcal{E}(S) = \{p : p \text{ is an } e\text{-ultrafilter}\}.$$

(b) Define $A^\dagger = \{p \in \mathcal{E}(S) : A \in p\}$ for each $A \in Z(\mathrm{Lmc}(S))$.
(c) Define $e(a) = \{E_\epsilon(f) : f(a) = 0, \epsilon > 0\}$ for each $a \in S$.

(d) It is said that $\mathcal{A} \subset Z(\mathrm{Lmc}(S))$ has the e-finite intersection property if and only if $E(E^{-}(\mathcal{A}))$ has the finite intersection property.

Pick $\varepsilon(a) \in S^{\mathrm{Lmc}}$ for some $a \in S$, then

$$\begin{aligned}\ker(\varepsilon(a)) &= \{f \in \mathrm{Lmc}(S) : \varepsilon(a)(f) = 0\} \\ &= \{f \in \mathrm{Lmc}(S) : f(a) = 0\}\end{aligned}$$

is a maximal ideal and by Theorem 2.9,

$$E^{-}(\ker(\varepsilon(a))) = \{E_\epsilon(f) : f(a) = 0,\ \forall \epsilon > 0\} = e(a)$$

is an e-ultrafilter.

Lemma 3.2. *Let $A, B \in Z(\mathrm{Lmc}(S))$ and $f, g \in \mathrm{Lmc}(S)$. Then:*

(1) $(A \cap B)^{\dagger} = A^{\dagger} \cap B^{\dagger}$.
(2) $(A \cup B)^{\dagger} \supseteq A^{\dagger} \cup B^{\dagger}$.
(3) *Pick $x \in S$ and $\epsilon > 0$. Then $\lambda_x^{-1}(E_\epsilon(f)) = E_\epsilon(Lxf)$.*
(4) $E_{\epsilon\wedge\delta}(|f| \vee |g|) \subseteq E_\epsilon(f) \cap E_\delta(g)$ *and* $E_\epsilon(|f| \vee |g|) = E_\epsilon(f) \cap E_\epsilon(g)$, *for each* $\delta, \epsilon > 0$.

Proof. The proofs are routine. □

Since $(A \cap B)^{\dagger} = A^{\dagger} \cap B^{\dagger}$ for each $A, B \in Z(\mathrm{Lmc}(S))$, so the sets $A^{\dagger}$ are closed under finite intersection. Consequently, $\{A^{\dagger} : A \in Z(\mathrm{Lmc}(S))\}$ forms a base for an open topology on $\mathcal{E}(S)$.

Theorem 3.3.

(1) *Pick $f \in \mathrm{Lmc}(S)$ and $\epsilon > 0$, then $\mathrm{int}_S(A) = e^{-1}(A^{\dagger})$, and so $e : S \to \mathcal{E}(S)$ is continuous.*
(2) *Pick $p \in \mathcal{E}(S)$ and $A \in Z(\mathrm{Lmc}(S))$, then the following statements are equivalent:*
   (i) $p \in \mathrm{cl}_{\mathcal{E}(S)}(e(A))$.
   (ii) *For each $B \in p$, $\mathrm{int}_S(B) \cap A \neq \emptyset$.*
   (iii) *For each $B \in p$, $B \cap A \neq \emptyset$.*
   (iv) *There exists a z-ultrafilter $\mathcal{A}_p$ containing p such that $A \in \mathcal{A}_p$.*
(3) *Pick $A, B \in Z(\mathrm{Lmc}(S))$ such that $p \in \mathrm{cl}_{\mathcal{E}(S)}(e(A)) \cap \mathrm{cl}_{\mathcal{E}(S)}(e(B))$ and $p \cup \{A, B\}$ has the finite intersection property, then*

$$p \in \mathrm{cl}_{\mathcal{E}(S)}(e(A \cap B)).$$

(4) $\{\mathrm{cl}_{\mathcal{E}(S)}(e(A)) : A \in Z(\mathrm{Lmc}(S))\}$ *is a base for closed subsets of $\mathcal{E}(S)$.*
(5) *$\mathcal{E}(S)$ is a compact Hausdorff space.*
(6) *$e(S)$ is a dense subset of $\mathcal{E}(S)$.*

Proof. (1) Let $p \in A^{\dagger}$, so there exist $f \in E^{-}(p)$ and $\epsilon > 0$ such that $E_\epsilon(f) = A$ and $E_\delta(f) \in p$ for each $\delta > 0$. Pick $x_\circ \in \mathrm{int}_S(A)$, then $|f(x_\circ)| < \epsilon$ or $|f(x_\circ)| = \epsilon$.

If $\delta = |f(x_\circ)| < \epsilon$, then $E_{\epsilon-\delta}(|f| \vee \delta - \delta) = E_\epsilon(f)$, $x_\circ \in E_{\epsilon-\delta}(|f| \vee \delta - \delta)$ and $x_\circ \in E_\eta(|f| \vee \delta - \delta)$ for each $\eta > 0$. Thus,

$$e(x_\circ) \in E_{\epsilon-\delta}(|f| \vee \delta - \delta)^\dagger = E_\epsilon(f)^\dagger = A^\dagger.$$

If $|f(x_\circ)| = \epsilon$, so there exists a neighborhood U such that $x_\circ \in U \subseteq A$. Since $\mathrm{Lmc}(S)$ and $C(S^{\mathrm{Lmc}})$ are isometrically isomorphism, pick $g \in \mathrm{Lmc}(S)$ such that $g(U) = \{0\}$, $g(A^c) = \{\|f\|\}$ and $g(S) \subseteq [0, \|f\|]$. Define $h = |f| \wedge g$, then $E_\epsilon(h) = E_\epsilon(f) = A$ and $|h(x_\circ)| = 0 < \epsilon$. It is obvious that $E_\delta(f) \subseteq E_\delta(h)$ for each $0 < \delta < \epsilon$ and $E_\epsilon(f) \subseteq E_\delta(h)$ for each $\epsilon < \delta$. Therefore $E_\delta(h) \in p$ for each $\delta > 0$ and $|h(x_\circ)| = 0 < \epsilon$. So by previous case, $e(x_\circ) \in E_\epsilon(h)^\dagger = A^\dagger$. Thus $x_\circ \in e^{-1}(A^\dagger)$ and so $\mathrm{int}_S(A) \subseteq e^{-1}(A^\dagger)$.

Now pick $e(x) \in A^\dagger$, so there exist $\epsilon > 0$ and $f \in \mathrm{Lmc}(S)$ such that $E_\epsilon(f) = A$, and so $E_\delta(f) \in e(x)$ for any $\delta > 0$. Therefore, $f(x) = 0$ and $x \in E_\epsilon(f)$ for each $\epsilon > 0$. Thus, $e^{-1}(A^\dagger) = \mathrm{int}_S(A)$.

(2) (i) $\Leftrightarrow$ (ii): Since $p \in \mathrm{cl}_{\mathcal{E}(S)}(e(A))$ if and only if $B^\dagger \cap e(A) \neq \emptyset$ for any $B \in p$, if and only if $e^{-1}(B^\dagger \cap e(A)) \neq \emptyset$ for any $B \in p$, if and only if

$$\mathrm{int}_S(B) \cap A = e^{-1}(B^\dagger) \cap e^{-1}(e(A)) \neq \emptyset$$

for any $B \in p$, by item (1).

It is obvious that (iii) and (iv) are equivalent and (ii) implies (iii).

(iii) $\Rightarrow$ (ii): Let for some $B \in p$, $B \cap A \neq \emptyset$ and $\mathrm{int}_S(B) \cap A = \emptyset$. Since $B \in p$ so there exist $f \in \mathrm{Lmc}(S)$ and $\epsilon > 0$ such that $B = E_\epsilon(f)$, $E_\delta(f) \in p$ for each $\delta > 0$ and

$$E_{\frac{\epsilon}{2}}(f) \cap A \subseteq \mathrm{int}_S(B) \cap A = \emptyset.$$

This is a contradiction.

(3) Let $p \cup \{A, B\}$ has the finite intersection property, so $p \cup \{A \cap B\}$ has the finite intersection property. Let $\mathcal{A}_p$ be a z-ultrafilter containing $p \cup \{A \cap B\}$ and hence item (2), implies that $p \in \mathrm{cl}_{\mathcal{E}(S)}(e(A \cap B))$.

(4) It suffices to show that $\{(\mathrm{cl}_{\mathcal{E}(S)}(e(A)))^c : A \in Z(\mathrm{Lmc}(S))\}$ is a base for open subsets of $\mathcal{E}(S)$. Let U be an open subset containing $p \in \mathcal{E}(S)$. Since $\{A^\dagger : A \in Z(\mathrm{Lmc}(S))\}$ forms a base for an open topology on $\mathcal{E}(S)$, so there exist $f \in \mathrm{Lmc}(S)$ and $\epsilon > 0$ such that $p \in E_\epsilon(f)^\dagger \subseteq U$ and $E_\delta(f) \in p$ for each $\delta > 0$. Now pick $0 < \gamma < \min\{\frac{\epsilon}{2}, \|f\|\}$, and define $g(x) = \|f\| - |f(x)|$. Then $g \in \mathrm{Lmc}(S)$ and $(E_{\|f\|-\gamma}(g))^c \subseteq E_\gamma(f)$, so

$$(\mathrm{cl}_{\mathcal{E}(S)}(E_{\|f\|-\gamma}(g)))^c \subseteq \mathrm{cl}_{\mathcal{E}(S)}((E_{\|f\|-\gamma}(g))^c) \subseteq \mathrm{cl}_{\mathcal{E}(S)}(E_\gamma(f)).$$

Hence, there exists $\delta > 0$ such that $(E_{\|f\|-\gamma}(g) \cap E_\gamma(f)) \bigcap E_\delta(f) = \emptyset$, and

$$E_{\|f\|-\gamma}(g) \cap E_\delta(f) = \emptyset.$$

This implies $p \notin \mathrm{cl}_{\mathcal{E}(S)} E_{\|f\|-\gamma}(g)$ and so

$$p \in (\mathrm{cl}_{\mathcal{E}(S)}(E_{\|f\|-\gamma}(g))^c \subseteq E_\epsilon(f)^\dagger.$$

This shows that $\{(\mathrm{cl}_{\mathcal{E}(S)}(e(A)))^c : A \in Z(\mathrm{Lmc}(S))\}$ is a base for open subsets of $\mathcal{E}(S)$.

(5) Suppose that p and q are distinct elements of $\mathcal{E}(S)$, then $E^-(p)$ and $E^-(q)$ are maximal ideals, by Theorem 2.9. Pick $f \in E^-(p)\backslash E^-(q)$. So by Theorem 2.10, there exist $\epsilon > 0$ and $A \in q = E(E^-(q))$, such that $E_\epsilon(f) \cap A = \emptyset$. Since $A \in q = E(E^-(q))$, pick $\delta > 0$ and $g \in E^-(q)$ such that $A = E_\delta(g)$ and for all $\gamma > 0$, $E_\gamma(g) \in q$. Then $E_\epsilon(f) \cap E_\delta(g) = \emptyset$. Now let $B = E_\epsilon(f)$, then $A \in p$, $B \in q$ and $A \cap B = \emptyset$. Thus $A^\dagger \cap B^\dagger = \emptyset$, $p \in A^\dagger$ and $q \in B^\dagger$, and so $\mathcal{E}(S)$ is Hausdorff.

Define $\eta : p \mapsto E(E^-(p)) : \mathcal{Z}(S) \to \mathcal{E}(S)$. By Lemma 2.11, if $p \in \mathcal{Z}(S)$, then $E(E^-(p)) \in \mathcal{E}(S)$ so η is well defined. Now let p be an e-ultrafilter, so there exists a z-ultrafilter $\mathcal{A}$ containing p. By Lemma 2.11, $p = E(E^-(\mathcal{A}))$. This implies η is onto. For each $A \in Z(\mathrm{Lmc}(S))$, we have

$$\begin{aligned}
\eta^{-1}(\mathrm{cl}_{\mathcal{E}(S)}(e(A))) &= \{p \in \mathcal{Z}(S) : \eta(p) \in \mathrm{cl}_{\mathcal{E}(S)}(e(A))\} \\
\text{By Theorem 3.3(2)} \quad &= \{\, p \in \mathcal{Z}(S) : \forall B \in \eta(p), \quad B \cap A \neq \emptyset\} \\
\text{By Theorem 3.3(2)} \quad &= \{\, p \in \mathcal{Z}(S) : \eta(p) \cup \{A\} \subseteq p\} \\
&= \{\, p \in \mathcal{Z}(S) : A \in p\} \\
&= \widehat{A}.
\end{aligned}$$

Since $\{\mathrm{cl}_{\mathcal{E}(S)}(e(A)) : A \in Z(\mathrm{Lmc}(S))\}$ is a base for closed subsets of $\mathcal{E}(S)$, so η is continuous. Since $\mathcal{Z}(S)$ is compact by Lemma 2.8 in [12], so $\mathcal{E}(S)$ is also compact.

(6) By (4), e is continuous. Also,

$$\begin{aligned}
\overline{e(S)} &= \{p \in \mathcal{E}(S) : \forall\ B \in p, \ \ B^\dagger \cap e(S) \neq \emptyset\} \\
&= \{p \in e(S) : \forall\ B \in p, \ \ B \cap S \neq \emptyset\} \\
&= \mathcal{E}(S).
\end{aligned}$$

□

Definition 3.4. Let $\mathcal{A}$ be an e-filter. Then $\widehat{\mathcal{A}} = \{p \in \mathcal{E}(S) : \mathcal{A} \subseteq p\}$.

Theorem 3.5.

(a) *If $\mathcal{A}$ is an e-filter, then $\widehat{\mathcal{A}}$ is a closed subset of $\mathcal{E}(S)$.*
(b) *Let $\mathcal{A}$ be an e-filter and $A \in Z(\mathrm{Lmc}(S))$. Then, $A \in \mathcal{A}$ if and only if $\widehat{\mathcal{A}} \subseteq A^\dagger$.*
(c) *Suppose that $A \subseteq \mathcal{E}(S)$ and $\mathcal{A} = E(E^-(\cap A))$, then $\mathcal{A}$ is an e-filter and $\widehat{\mathcal{A}} = \mathrm{cl}_{\mathcal{E}(S)}A$.*

Proof. (a) Pick $p \in \mathrm{cl}_{\mathcal{E}(S)}\widehat{\mathcal{A}}$, so $A^\dagger \cap \widehat{\mathcal{A}} \neq \emptyset$, for each $A \in p$. Hence, $\mathcal{A} \cup \{A\}$ has the e-finite intersection property for each $A \in p$. This implies that $\mathcal{A} \cup p \subseteq p$ and so $p \in \widehat{\mathcal{A}}$.

(b) It is easy to verify the assertion.

(c) By assumption, $\mathcal{A}$ is an e-filter (by Lemma 2.8). Further, for each $p \in A$, $\mathcal{A} \subseteq p$ implies that $A \subseteq \widehat{\mathcal{A}}$, thus by (a), $\mathrm{cl}_{\mathcal{E}(S)}A \subseteq \widehat{\mathcal{A}}$.

To see that $\widehat{\mathcal{A}} \subseteq \mathrm{cl}_{\mathcal{E}(S)} A$, let $p \notin \mathrm{cl}_{\mathcal{E}(S)} A$. Then, there exist $B \in p$ and $C \in Z(\mathrm{Lmc}(S))$ such that $\mathrm{cl}_{\mathcal{E}(S)} A \subseteq C^\dagger$ and $B^\dagger \cap C^\dagger = \emptyset$. Hence, $\widehat{\mathcal{A}} \subseteq C^\dagger$ and this implies $p \notin \widehat{\mathcal{A}}$. $\square$

Definition 3.6. Suppose that $p, q \in \mathcal{E}(S)$ and $A \in Z(\mathrm{Lmc}(S))$. Then, $A \in p + q$ if there exist $\epsilon > 0$ and $f \in \mathrm{Lmc}(S)$ such that $A = E_\epsilon(f)$ and $E_\delta(q, f) = \{x \in S : \lambda_x^{-1}(E_\delta(f)) \in q\} \in p$ for each $\delta > 0$.

Theorem 3.7. *Let $p, q \in \mathcal{E}(S)$, then $p + q$ is an e-ultrafilter.*

Proof. It is obvious that $\emptyset \notin p + q$ and $S \in p + q$. Let $A \in p + q$, then there exist $\epsilon > 0$ and $f \in \mathrm{Lmc}(S)$ such that $A = E_\epsilon(f)$ and for each $\delta > 0$, $E_\delta(q, f) = \{x \in S : \lambda_x^{-1}(E_\delta(f)) \in q\} \in p$. Let $A, B \in p + q$; therefore, there exist $\delta, \epsilon > 0$ and $f, g \in \mathrm{Lmc}(S)$ such that $A = E_\epsilon(f)$ and $B = E_\delta(g)$. So

$$\begin{aligned} A \cap B &= E_\epsilon(f) \cap E_\delta(g) \\ &\supseteq E_{\epsilon\wedge\delta}(f) \cap E_{\epsilon\wedge\delta}(g) \\ &= E_{\epsilon\wedge\delta}(|f| \vee |g|), \end{aligned}$$

and

$$\begin{aligned} E_\gamma(q, |f| \vee |g|) &= \{x \in S : \lambda_x^{-1}(E_\gamma(|f| \vee |g|)) \in q\} \\ &= \{x \in S : E_\gamma(|L_x f| \vee |L_x g|) \in q\} \\ &= \{x \in S : E_\gamma(L_x f) \cap E_\gamma(L_x g) \in q\} \\ &= E_\gamma(q, f) \cap E_\gamma(q, g). \end{aligned}$$

Since $E_\gamma(q, f), E_\gamma(q, g) \in p$, so $E_\gamma(q, |f| \vee |g|) = E_\gamma(q, f) \cap E_\gamma(q, g) \in p$. Thus, $E_{\delta\wedge\epsilon}(|f| \vee |g|) \in p + q$ and so $A \cap B \in p + q$.

Now pick $A \in p + q$ and $B \in Z(\mathrm{Lmc}(S))$ such that $A \subseteq B$. So $A \in p + q$ implies that there exist $\epsilon > 0$ and $f \in \mathrm{Lmc}(S)$ such that $E_\epsilon(f) = A$ and $E_\delta(q, f) \in p$ for each $\delta > 0$. For $B \in Z(\mathrm{Lmc}(S))$, so there exists $g \in \mathrm{Lmc}(S)$ such that $Z(g) = B$. Now define $u(x) = g(x) + \frac{\epsilon}{|f(x)| \vee \epsilon}$. Clearly, $h = \frac{u}{\|u\|} \in \mathrm{Lmc}(S)$, $Z(g) = E_\epsilon(fh)$ and $L_x f \in E^-(q)$ for each $x \in E_\delta(q, f)$ and $\delta > 0$. This implies $L_x f L_x h \in E^-(q)$ for each $x \in E_\delta(q, f)$, and so $E_\gamma(L_x f L_x h) \in q$ for each $\gamma > 0$. Thus, $E_\delta(q, f) \subseteq E_\delta(q, fh)$ and $E_\delta(q, fh) \in p$ for each $\delta > 0$; therefore, $Z(g) = E_\epsilon(fh) \in p + q$. So $p + q$ is an e-filter.

Now, it is proved that $p + q$ is an e-ultrafilter. Let $E^-(p) = \ker(\mu)$ and $E^-(q) = \ker(\nu)$ for $\mu, \nu \in S^{\mathrm{Lmc}}$. It is claimed that $E^-(p + q) = \ker(\mu\nu)$, thus $p + q$ is an e-ultrafilter. Pick $f \in \ker(\mu\nu)$, so $T_\nu f \in \ker(\mu)$ and for each $\epsilon > 0$,

$$\begin{aligned} E_\epsilon(T_\nu f) &= \{x \in S : |T_\nu f(x)| \le \epsilon\} \\ &= \{x \in S : |\nu(L_x f)| \le \epsilon\} \\ &= \{x \in S : |\widehat{L_x f}(\nu)| \le \epsilon\} \\ &\in p. \end{aligned}$$

It is obvious that $\{t \in S : |\widehat{L_x f}(t)| \leq \epsilon\} = \{t \in S : |L_x f(t)| \leq \epsilon\} = E_\epsilon(L_x f)$. Pick $\epsilon > 0$. For each $x \in E_{\frac{\epsilon}{2}}(T_\nu f)$, $E_{\frac{\epsilon}{2}}((|L_x f| \vee \frac{\epsilon}{2}) - \frac{\epsilon}{2}) \subseteq E_\epsilon(L_x f)$, and $E_{\frac{\epsilon}{2}}((|L_x f| \vee \frac{\epsilon}{2}) - \frac{\epsilon}{2}) \in E(\ker(\nu)) = q$, so

$$E_\epsilon(T_\nu f) \subseteq \{x \in S : E_\epsilon(L_x f) \in q\} = E_\epsilon(q, f).$$

Thus, $E_\epsilon(f) \in p + q$ for each $\epsilon > 0$, and so $f \in E^-(p+q)$. Therefore $\ker(\mu\nu) \subseteq E^-(p+q)$ and this completes the proof. □

Theorem 3.8. *$\mathcal{E}(S)$ and S^{Lmc} are topologically isomorphic.*

Proof. M is a maximal ideal of $\mathrm{Lmc}(S)$ if and only if there is a $\mu \in S^{\mathrm{Lmc}}$ such that $\ker(\mu) = M$. Thus, $\gamma : \mu \mapsto E(\ker(\mu)) : S^{\mathrm{Lmc}} \to \mathcal{E}(S)$ is well defined and surjective. By Theorem 3.3(4), $\{\mathrm{cl}_{\mathcal{E}(S)}(e(A)) : A \in Z(\mathrm{Lmc}(S))\}$ is a base for closed subsets of $\mathcal{E}(S)$, pick $A \in Z(\mathrm{Lmc}(S))$ then

$$\begin{aligned}
\gamma^{-1}(\mathrm{cl}_{\mathcal{E}(S)} e(A)) &= \{\mu \in S^{\mathrm{Lmc}} : E(\ker(\mu)) \in \mathrm{cl}_{\mathcal{E}(S)} e(A)\} \\
&= \{\mu \in S^{\mathrm{Lmc}} : \forall B \in E(\ker(\mu)),\ \ B^\dagger \cap e(A) \neq \emptyset\} \\
&= \{\mu \in S^{\mathrm{Lmc}} : \forall f \in \ker(\mu),\ \forall\, \delta > 0,\ \ E_\delta(f) \cap A \neq \emptyset\} \\
&= \{\mu \in S^{\mathrm{Lmc}} : \forall f \in \ker(\mu),\ \forall \delta > 0,\ \exists x_\delta \in A \cap E_\delta(f)\} \\
&= \mathrm{cl}_{S^{\mathrm{Lmc}}}(A).
\end{aligned}$$

So γ is continuous. Since, $\gamma : S^{\mathrm{Lmc}} \to \mathcal{E}(S)$ is a surjective continuous function, and S^{Lmc} is a compact space; therefore, γ is homeomorphism. Now pick $\mu, \nu \in S^{\mathrm{Lmc}}$, then

$$\begin{aligned}
\gamma(\mu\nu) &= E(\ker(\mu\nu)) && \text{(see the proof of Theorem 3.7)} \\
&= E(\ker(\mu)) + E(\ker(\nu)) \\
&= \gamma(\mu) + \gamma(\nu).
\end{aligned}$$

Therefore, γ is homomorphism and thus $\mathcal{E}(S)$ and S^{Lmc} are topologically isomorphic. □

By Theorem 3.8, S^{Lmc} could be described as a space of e-ultrafilters, i.e., $S^{\mathrm{Lmc}} = \{E(\ker(\mu)) : \mu \in S^{\mathrm{Lmc}}\}$.

Lemma 3.9. *Let $A \in Z(\mathrm{Lmc}(S))$ and $x \in S$. Then $A \in e(x) + p$ if and only if $\lambda_x^{-1}(A) \in p$.*

Proof. Pick $A \in e(x) + q$, so there exist $\epsilon > 0$ and $f \in \mathrm{Lmc}(S)$ such that $A = E_\epsilon(f)$ and $E_\delta(q, f) = \{t \in S : \lambda_t^{-1}(E_\delta(f)) \in q\} \in e(x)$ for each $\delta > 0$ and $\lambda_x^{-1}(E_\delta(f)) \in q$ for each $\delta > 0$. This implies $\lambda_x^{-1}(A) \in p$.
Conversely, let $\lambda_x^{-1}(A) \in p$, so there exist $\epsilon > 0$ and $f \in \mathrm{Lmc}(S)$ such that $A = E_\epsilon(f)$ and $\lambda_x^{-1}(A) \in p$. Thus $E_\delta(L_x f) = \lambda_x^{-1}(E_\delta(f)) \in p$ for each $\delta > 0$, and $L_x f \in E^-(p) = \ker(\mu)$ for some $\mu \in S^{\mathrm{Lmc}}$. Clearly, $\mu(L_x f) = 0$ and so $\varepsilon(x)\mu(f) = 0$. This implies $A \in E(\ker(\varepsilon(x)\mu)) = e(x) + p$. □

Definition 3.10. Let $\mathcal{A}$ and $\mathcal{B}$ be e-filters, and pick $A \in Z(\mathrm{Lmc}(S))$. Then $A \in \mathcal{A} + \mathcal{B}$ if there exist $\epsilon > 0$ and $f \in \mathrm{Lmc}(S)$ such that $E_\epsilon(f) = A$ and $E_\delta(\mathcal{B}, f) = \{x \in S : \lambda_x^{-1}(E_\delta(f) \in \mathcal{B})\} \in \mathcal{A}$ for each $\delta > 0$.

Lemma 3.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be e-filters. Then $\mathcal{A} + \mathcal{B}$ is an e-filter.*

Proof. See Theorem 3.7. □

4. Applications

In this section, as an application, we consider the semigroup $S^* = S^{\mathrm{Lmc}} \setminus S$ and work out some conditions characterizing when S^* is a left ideal of S^{Lmc}. The results of this section are found in [7], when S is a discrete semigroup.

Theorem 4.1. *Pick $p, q \in \mathcal{E}(S)$ and let $f \in \mathrm{Lmc}(S)$. Then $E_\epsilon(f) \in p + q$ for each $\epsilon > 0$ if and only if for each $\epsilon > 0$ there exist $B_\epsilon \in p$ and an indexed family $< C_{\epsilon,s} >_{s \in B_\epsilon}$ in q such that $\bigcup sC_{\epsilon,s} \subseteq E_\epsilon(f)$.*

Proof. Let $E_\epsilon(f) \in p + q$ for each $\epsilon > 0$. Pick $\epsilon > 0$, $x \in B_\epsilon = E_\epsilon(q, f)$ and let $C_{\epsilon,x} = E_\epsilon(L_x f) = \lambda_x^{-1}(E_\epsilon(f))$. For each $x \in B_\epsilon$, $C_{\epsilon,x} \in q$ and so $\bigcup_{x \in B_\epsilon} xC_{\epsilon,x} \subseteq E_\epsilon(f)$.

Conversely, by hypothesis for each $\epsilon > 0$, there exist $B_\epsilon \in p$ and an indexed family $< C_{\epsilon,s} >_{s \in B_\epsilon}$ in q such that $\bigcup_{s \in B_\epsilon} sC_{\epsilon,s} \subseteq E_\epsilon(f)$. Then for each $s \in B_\epsilon$, $C_{\epsilon,s} \subseteq \lambda_s^{-1}(E_\epsilon(f)) = E_\epsilon(L_s f)$ and so $E_\epsilon(L_s f) \in q$, for each $s \in B_\epsilon$. Thus, $B_\epsilon \subseteq \{t \in S : E_\epsilon(L_t f) \in q\} = E_\epsilon(q, f) \in p$, and $E_\epsilon(f) \in p + q$ for each $\epsilon > 0$. □

Theorem 4.2. *Let $\mathcal{A} \subseteq Z(\mathrm{Lmc}(S))$ has the e-finite intersection property. If for each $A \in E(E^-(\mathcal{A}))$ and $x \in A$, there exists $B \in E(E^-(\mathcal{A}))$ such that $xB \subseteq A$, then $\bigcap_{A \in E(E^-(\mathcal{A}))} \overline{\varepsilon(A)}$ is a subsemigroup of S^{Lmc}.*

Proof. Let $T = \bigcap_{A \in E(E^-(\mathcal{A}))} \overline{\varepsilon(A)}$. Since $E(E^-(\mathcal{A}))$ has the e-finite intersection property, so $T \neq \emptyset$. Pick $p, q \in T$ and let $A \in E(E^-(\mathcal{A}))$. Given $x \in A$, there is some $B \in E(E^-(\mathcal{A}))$ such that $xB \subseteq A$. Therefore, there exist $f, g \in \mathrm{Lmc}(S)$ such that $B = E_\delta(g)$, $A = E_\epsilon(f)$ and $E_\gamma(g), E_\gamma(f) \in p \cap q$ for each $\gamma > 0$, so $xE_\delta(g) \subseteq E_\epsilon(f)$ and $E_\delta(g) \subseteq \lambda_x^{-1}(E_\epsilon(f)) = E_\epsilon(L_x f)$. Since $B \in p \cap q$ thus $A \subseteq \{t \in S : E_\epsilon(L_t f) \in q\} = E_\epsilon(q, f)$, and $A = E_\epsilon(f) \in p + q$. □

Definition 4.3.

(a) $A \subseteq S$ is an unbounded set if $\overline{\varepsilon(A)} \cap S^* \neq \emptyset$.
(b) A sequence $\{x_n\}$ is unbounded if $\overline{\varepsilon(\{x_n : n \in \mathbb{N}\})} \cap S^* \neq \emptyset$.

Lemma 4.4. *Let $\{x_n\}$ and $\{y_n\}$ be unbounded sequences in S. Let $p, q \in S^*$, $q \in \overline{\varepsilon(\{x_n : n \in \mathbb{N}\})}$ and $p \in \overline{\varepsilon(\{y_n : n \in \mathbb{N}\})}$, then*

$$p + q \in \overline{\varepsilon(\{y_k x_n : k < n,\ k, n \in \mathbb{N}\})}.$$

Proof. It is obvious that for each $A \in q$, $\varepsilon(\{x_n : n \in \mathbb{N}\}) \cap A^\dagger \neq \emptyset$ and for each $B \in p$, $\varepsilon(\{y_n : n \in \mathbb{N}\}) \cap B^\dagger \neq \emptyset$. Now let $C \in p + q$, then there exist $\epsilon > 0$ and $f \in \mathrm{Lmc}(S)$ such that $C = E_\epsilon(f)$ and for each $\delta > 0$, $E_\delta(q, f) \in p$. Pick $\delta > 0$ and let $x \in E_\delta(q, f)$, then

$$\varepsilon(\lambda_x^{-1}(E_\delta(f)) \cap \{x_n : n \in \mathbb{N}\})$$

and

$$\varepsilon(E_\delta(q, f) \cap \{y_n : n \in \mathbb{N}\})$$

are unbounded, by Theorem 3.3(4). Hence for each $y_k \in E_\delta(q, f)$,

$$\varepsilon(\lambda_{y_k}^{-1}(E_\delta(f)) \cap \{x_n : n \in \mathbb{N}\})$$

and so

$$\varepsilon(\{y_k x_n : k, n \in \mathbb{N},\ k < n\} \cap E_\delta(f))$$

are unbounded, by Theorem 3.3(4). This implies $\varepsilon(\{y_k x_n : k, n \in \mathbb{N}\}) \cap C^\dagger \neq \emptyset$ and $p + q \in \overline{\varepsilon(\{y_k x_n : k < n,\ k, n \in \mathbb{N}\})}$. □

Theorem 4.5. *Suppose that S is a σ-compact commutative semigroup, then S^{Lmc} is not commutative if and only if there exist unbounded sequences $\{x_n\}$ and $\{y_n\}$ such that*

$$\overline{\varepsilon(\{x_k y_n : k < n,\ k, n \in \mathbb{N}\})} \cap \overline{\varepsilon(\{y_k x_n : k < n,\ k, n \in \mathbb{N}\})} = \emptyset.$$

Proof. Necessity. Since S is σ-compact, so there exists a sequence $\{F_n\}_{n=1}^\infty$ of compact subsets of S such that $F_n \subseteq F_{n+1}$ and $S = \bigcup_{n=1}^\infty F_n$. Now pick p and q in S^* such that $p+q \neq q+p$. Then, there exist $A \in p+q$ and $B \in q+p$ such that $\overline{\varepsilon(A)} \cap \overline{\varepsilon(B)} = \emptyset$. So, there exist $\gamma, \epsilon > 0$ and $f, g \in \mathrm{Lmc}(S)$ such that $E_\epsilon(f) = A$ and $E_\gamma(g) = B$. Pick $0 < \delta < \epsilon \wedge \gamma$, let $A_1 = E_\delta(q, f)$ and $B_1 = E_\delta(p, g)$. Then, $A_1 \in p$ and $B_1 \in q$. Choose $x_1 \in A_1$ and $y_1 \in B_1$. Inductively given $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$, choose x_{n+1} and y_{n+1} such that

$$\varepsilon(x_{n+1}) \in \varepsilon\left({A_1}^\dagger \cap \left(\bigcap_{k=1}^n \lambda_{y_k}^{-1}(E_\delta(g))\right) \cap F_n^c\right)$$

and

$$\varepsilon(y_{n+1}) \in \varepsilon\left({B_1}^\dagger \cap \left(\bigcap_{k=1}^n \lambda_{y_k}^{-1}(E_\delta(f))\right) \cap F_n^c\right).$$

Then $\{x_n\}$ and $\{y_n\}$ are unbounded sequences,

$$\varepsilon(\{y_k x_n : k, n \in \mathbb{N},\ k < n\}) \subseteq \varepsilon(A)$$

and

$$\varepsilon(\{x_k y_n : k, n \in \mathbb{N},\ k < n\}) \subseteq \varepsilon(B).$$

Sufficiency. Now let there exist two unbounded sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\overline{\varepsilon(\{x_k y_n : k < n,\ k, n \in \mathbb{N}\})} \cap \overline{\varepsilon(\{y_k x_n : k < n,\ k, n \in \mathbb{N}\})} = \emptyset.$$

Pick $p \in \overline{\varepsilon(\{x_n : n \in \mathbb{N}\})} \cap S^*$ and $q \in \overline{\varepsilon(\{y_n : n \in \mathbb{N}\})} \cap S^*$. Then by Lemma 4.4,

$$q + p \in \overline{\varepsilon(\{y_k x_n : k < n,\ k, n \in \mathbb{N}\})}$$

and

$$p + q \in \overline{\varepsilon(\{x_k y_n : k < n,\ k, n \in \mathbb{N}\})}. \qquad \square$$

Definition 4.6. A semitopological semigroup S is topologically weak left cancellative if for all $u \in S$ there exists a compact zero set A such that $\varepsilon(u) \in A^\dagger$ and $\lambda_v^{-1}(A)$ is a compact set for each $v \in S$.

Theorem 4.7.

(a) *Let S be a locally compact noncompact Hausdorff semitopological semigroup and let S^* be a closed left ideal of S^{Lmc}. Then S is topologically weak left cancellative.*

(b) *Let S be a topologically weak left cancellative locally compact noncompact Hausdorff semitopological semigroup. Then S^* is a left ideal of S^{Lmc}.*

(c) *Let S be a locally compact noncompact Hausdorff semitopological semigroup and let S^* be a closed subset of S^{Lmc}. Then S^* is a left ideal of S^{Lmc} if and only if S is topologically weak left cancellative.*

Proof. (a) Pick $x, y \in S$ such that for each compact zero set $A \in Z(\mathrm{Lmc}(S))$, $\varepsilon(x) \in A^\dagger$ and $B_A = \lambda_y^{-1}(A)$ is noncompact. Pick $p_A \in S^* \cap \overline{\varepsilon(B_A)}$ so $\varepsilon(y) + p_A \in \varepsilon(A)$. Now let

$$\mathcal{U} = \{A \in Z(\mathrm{Lmc}(S)) : \varepsilon(x) \in A^\dagger \text{ and } A \text{ is compact}\},$$

then $\{p_A\}_{A \in \mathcal{U}}$ is a net, $\varepsilon(y) + p_A \to \varepsilon(x)$, and $\varepsilon(x) \in \overline{S^*} = S^*$. So this is a contradiction.

(b) Since S is noncompact so $S^* \neq \emptyset$. Pick $p \in S^*$, $q \in S^{\mathrm{Lmc}}$ and let $q + p = \varepsilon(x) \in \varepsilon(S)$. Let $A \in Z(\mathrm{Lmc}(S))$ be a compact set and $\varepsilon(x) \in A^\dagger$. Then $A \in q + p$ and there exist $f \in \mathrm{Lmc}(S)$ and $\epsilon > 0$ such that $E_\epsilon(f) = A$ and $E_\delta(p, f) \in q$ for each $\delta > 0$. Now pick $y \in E_\epsilon(p, f)$ then $\lambda_y^{-1}(A) \in p$, so $\lambda_y^{-1}(A)$ is not compact and this is a contradiction.

(c) This can easily be verified. $\square$

Corollary 4.8. *Let G be a locally compact non compact Hausdorff topological group. Then G^* is a left ideal of $G^{\mathcal{LUC}}$.*

Proof. Let G be a locally compact non compact Hausdorff topological group, so $\varepsilon(G)$ is an open subset of $G^{\mathcal{LUC}}$, and hence G^* is closed. Now by Theorem 4.7, proof is completed. $\square$

Theorem 4.9. *Let S be a locally compact semitopological semigroup. The following statements are equivalent:*

(a) *S^* is right ideal of S^{Lmc}.*

(b) *Given any zero compact subset A of S, any sequence $\{z_n\}$ in S, and any unbounded sequence $\{x_n\}$ in S, there exists a $n < m$ in $\mathbb{N}$ such that $x_n \cdot z_m \notin A$.*

Proof. (a) implies (b). Suppose that $\{x_n \cdot z_m : n, m \in \mathbb{N} \text{ and } n < m\} \subseteq A$. Pick $p \in \overline{\varepsilon(\{z_m : m \in \mathbb{N}\})}$ and $q \in S^* \cap \overline{\varepsilon(\{x_n : n \in \mathbb{N}\})}$, which we can do, since $\{x_n : n \in \mathbb{N}\}$ is unbounded. Thus $q + p \in \overline{\varepsilon(A)} = \varepsilon(A) \subseteq \varepsilon(S)$, is a contradiction.

(b) implies (a). Since $S^* \neq \emptyset$, pick $p \in S^{\mathrm{Lmc}}$ and $q \in S^*$ such that $q + p = \varepsilon(a) \in \varepsilon(S)$ for some $a \in S$, so there exists a compact set $A \in Z(\mathrm{Lmc}(S))$ such that $\varepsilon(a) \in A^{\dagger}$. Hence there exist $\epsilon > 0$ and $f \in \mathrm{Lmc}(S)$ such that $E_\epsilon(f) = A$ and $E_\delta(f) \in \varepsilon(a)$, for each $\delta > 0$. Then for each $1/n < \epsilon$,

$$E_{1/n}(p, f) = \{s \in S : \lambda_s^{-1}(E_{1/n}(f)) \in p\} \in q,$$

choose an unbounded sequence $\{x_n\}$ such that $x_n \in E_{1/n}(p, f)$. Inductively choose a sequence $\{z_m\}$ in S such that for each $m \in \mathbb{N}$,

$$z_m \in \bigcap_{n=1}^{m} \lambda_{x_n}^{-1}(E_{1/n}(f))$$

(which one can do) since $\bigcap_{n=1}^{m} \lambda_{x_n}^{-1}(E_{1/n}(f)) \in p$. Then for each $n < m$ in $\mathbb{N}$, $x_n \cdot z_m \in E_{1/n}(f) \subseteq E_\epsilon(f) = A$, is a contradiction. □

Examples 1.

(a) Let S be a discrete semigroup. If S is either right or left cancellative, then $S^* = \beta S \setminus S$ is a subsemigroup of βS, (See Corollary 4.29 in [7]). This is not true for a left cancellative semitopological semigroup S. Let $(S = (1, +\infty), +)$ with the natural topology. Then S^* is not subsemigroup. Pick $p, q \in \mathrm{cl}_{S^{\mathrm{Lmc}}}(1, 2]$, thus there exist nets $\{x_\alpha\}$ and $\{y_\beta\}$ in $(1, 2]$ such that $x_\alpha \to p$, $y_\beta \to q$ and $x_\alpha + y_\beta \in [2, 4]$. Hence $p + q \in [2, 4]$ and so S^* is not subsemigroup. Also, S^* is not a left ideal and so S is not topologically weak left cancellative.

(b) $(S = [1, +\infty), +)$ with the natural topology is a topologically weak left cancellative, thus S^* is a left ideal of S^{Lmc}.

References

[1] Alaste, Tomi. Representation of spectrums of C^*-algebra of bounded functions in terms of filters. 2013. arXiv:1302.1340.

[2] Alaste, Tomi. $\mathcal{U}$-filters and uniform compactification. *Studia Math.* **211** (2012), no. 3, 215–229. MR3002443, Zbl 06113314, doi: 10.4064/sm211-3-3.

[3] Berglund, John F.; Junghenn, Hugo D.; Milnes, Paul. Analysis on semigroups. Function spaces, compactifications, representations. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. *John Wiley & Sons, Inc., New York*, 1989. xiv+334 pp. ISBN: 0-471-61208-1. MR0999922 (91b:43001), Zbl 0727.22001.

[4] Comfort, W. W.; Negrepontis, S. The theory of ultrafilters. Die Grundlehren der mathematischen Wissenschaften, Band 211. *Springer-Verlag, New York-Heidelberg*, 1974. x+482 pp. MR0396267 (53 #135), Zbl 0298.02004.

[5] Gillman, Lonard; Jerison, Meyer. Rings of continuous functions. The University Series in Higher Mathematics. *D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York*, 1960. ix+300 pp. MR116199 (22 #6994).

[6] Hindman, Neil; Milnes, Paul. The $\mathcal{LMC}-$compactification of a topologized semigroup. *Czechoslovak Math. J.* **38(113)** (1988), no. 1, 103–119. MR0925945 (89c:22008).

[7] Hindman, Neil; Strauss, Dona. Algebra in the Stone–Čech compactification. Theory and applications. de Gruyter Expositions in Mathematics, 27. *Walter de Gruyter & Co., Berlin*, 1998. xiv+485 pp. ISBN: 3-11-015420-X. MR1642231 (99j:54001), Zbl 0918.22001.

[8] Rudin, Walter. Functional analysis. Second edition. International Series in Pure and Applied Mathematics. *McGraw-Hill, Inc., New York*, 1991. xviii+424 pp. ISBN: 0-07-054236-8. MR1157815 (92k:46001), Zbl 0867.46001.

[9] Tootkaboni, Mohammad Akbari. Quotient space of $\mathcal{LMC}$-compactification as a space of $z-$filters. *Acta Math Sci. Ser. B Engl. Ed.* **32** (2012), no. 5, 2010–2020. MR2960766, Zbl 06179108, arXiv:1005.0083.

[10] Tootkaboni, M.A.; Sales, A. Bagheri. Comfort order on *Lmc*-compactification. To appear.

[11] Tootkaboni, M.A.; Eslami, Z. Density in locally compact topological groups. Preprint.

[12] Tootkaboni, M.A.; Riazi, A. Ultrafilters on semitopological semigroups. *Semigroup Forum* **70** (2005), no. 3, 317–328. MR2148146 (2006d:22005), Zbl 1078.22001, doi: 10.1007/s00233-003-0015-y.

[13] Tootkaboni, M. A.; Vishki, H. R. E. Filters and semigroup compactification properties. *Semigroup Forum* **78** (2009), no. 2, 349–359. MR2486647 (2009m:22005), Zbl 1165.22003, doi: 10.1007/s00233-008-9118-9.

[14] Zelenyuk, Yevhen G. Ultrafilters and topologies on groups. de Gruyter Expositions in Mathematics, 50. *Walter de Gruyter GmbH & Co. KG, Berlin*, 2011. viii+219 pp. ISBN: 978-3-11-020422-3. MR2768144 (2012c:22002), Zbl 1215.22001 doi: 10.1007/BF00971407.

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