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Operator algebras associated to integral domains

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Abstract. We study operator algebras associated to integral domains. In particular, with respect to a set of natural identities we look at the possible nonselfadjoint operator algebras which encode the ring structure of an integral domain. We show that these algebras give a new class of examples of semicrossed products by discrete semigroups. We investigate the structure of these algebras together with a particular class of representations.

Contents

Recently, in [4] and [9] the notion of a regular C^*-algebra associated to an integral domain was introduced as a generalization of a construction due to Cuntz, [3]. In these papers the authors associate to an integral domain R a C^*-algebra by considering the regular representation of R acting on $\ell^2(R)$. In addition they show that the C^*-algebras so constructed are universal with respect to a collection of identities that encode information about the integral domain.

One can view an integral domain as an additive group together with an action on the additive group given by multiplication by nonzero elements of the ring. This suggests that an important viewpoint for studying operator algebras associated to integral domains is through the use of crossed products. More importantly, since crossed products are well understood many of the significant results can be made brief through the technology of crossed products.

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For this paper we wish to investigate the operator algebras with slightly less restrictive identities imposed by only natural ring-theoretic constraints. This gives rise to operator algebras with a more natural crossed product structure. However since the multiplication in a ring need not act as automorphisms on the additive group, crossed products are not entirely appropriate. To avoid this we use the nonselfadjoint operator algebras where possible. This goes back to a construction of Arveson and Josephson [1] which was generalized by Peters in [11]. This semicrossed product is a nonselfadjoint operator algebra which encodes the same dynamics as the crossed product but does not require that the action on a topological space be via homeomorphisms.

While one may worry that we lose too much information when we lose the $*$-structure of the C^*-algebra, in recent work [5] it was shown that the semicrossed products of Peters in fact encode the action of a continuous self map on a topological space in a manner which is unique up to conjugacy of the map. This is even true when the map is not a homeomorphism and hence unlike with C^*-algebras the nonselfadjoint operator algebras can be used as a topological invariant.

It is these motivating examples which have led us to study the semicrossed product algebras in the context of integral domains. In this paper we have defined the universal operator algebra associated to an integral domain (note the different conditions we require from those of Cuntz and Li). We then study the situation in the case that our integral domain is a field. Here the semicrossed product and the crossed product coincide and we can use standard results for crossed products to prove facts about the algebra. After viewing the case of the integral domain being a field we focus on the situation where this may not be true. Here the semicrossed product technology is necessary, however similar results carry through. After defining the requisite notion of semicrossed product and proving some first results in the context of integral domains we prove some results which show that the algebras thus defined are distinct from the algebras studied by Peters. We then study the question of semisimplicity of the algebra. In the last section we analyze what we call unitary representations of an integral domain R. We show that every such representation factors through a regular unitary representation.

We describe some standard notation we intend to use. If R is an integral domain we write $Q(R)$ for the field of quotients. We write R_+ for the additive group of R. This group is a locally compact discrete group. We denote the Pontryagin dual of a group G by $\widehat{G}$. We refer the reader to [8] for details on the construction of the Pontryagin dual.

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1. Universal algebras of integral domains

Let R be an integral domain. Given a Hilbert space $\mathcal{H}$ we define an *isometric representation of* R to be a collection of isometries

$$\{S_r \in B(\mathcal{H}) : r \in R^\times\}$$

together with unitaries

$$\{U^n \in B(\mathcal{H}) : n \in R\}$$

that satisfy the relations:

(1) $S_r S_t = S_{rt}$ for all $r, t \in R^\times$.
(2) $U^n U^m = U^{m+n}$ for all $m, n \in R$.
(3) $U^n S_r = S_r U^{rn}$ for all $r \in R^\times, n \in R$.

If the S_r are unitaries for all $r \in R^\times$ then we say that the representation is a *unitary representation.*

We present first two examples:

Example (The regular representation of an integral domain)**.** Let $\mathcal{H}$ be equal to $\ell^2(R)$, with e_q denoting the characteristic function of $\{q\} \subseteq R$. Define operators U^n and S_r as follows:

$$U^n\left(\sum_{q\in R}\zeta_q e_q\right) = \sum_{q\in R}\zeta_q e_{q+n}$$

$$S_r\left(\sum_{q\in R}\zeta_q e_q\right) = \sum_{q\in R}\zeta_q e_{rq}.$$

It is not difficult to see that $\{U^n\}$ and $\{S_r\}$ give rise to an isometric representation of R.

Example (The regular unitary representation of an integral domain)**.** Let $\mathcal{K}$ be equal to $\ell^2(Q(R))$, with e_q denoting the characteristic function of $\{q\} \subseteq Q(R)$. We use the same formulas to define $\widetilde{U^n}$ and $\widetilde{S_r}$. Notice this time however that for all r, $\widetilde{S_r}$ is onto and hence a unitary. (To see this notice that $\widetilde{S_r}(e_{\frac{q}{r}}) = e_q$ for every $q \in Q(R)$).

An important point to notice is that $\mathcal{H} \subseteq \mathcal{K}$ and further $\mathcal{H}$ is an invariant subspace for the collections $\{\widetilde{S_r}\}$ and $\{\widetilde{U^n}\}$. Further $S_r = P_{\mathcal{H}}\widetilde{S_r}|_{\mathcal{H}}$ for all r and $U^n = P_{\mathcal{H}}\widetilde{U^n}|_{\mathcal{H}}$. For this reason we call this representation the *regular unitary representation of* R.

We let $A(R)$ be the norm closed operator algebra generated by unitaries $\{u^n : n \in R\}$ and isometries $\{s_r : r \in R^\times\}$ which is universal for isometric representations of R. We will denote the elements of $A(R)$ with lower case letters to distinguish from a representation of R for which we will use upper case letters.

We notice some initial facts about the algebra $A(R)$.

Lemma 1. *$A(R)$ is unital with $u^0 = s_1$.*

Proof. Let $\{U^r, S_r\}$ be an isometric representation of R. Then notice that U_0 is an idempotent unitary, hence $1 = U_0^* U_0$. Then $U_0 = 1 \cdot U_0 = U_0^* U_0^2 = U_0^* U_0 = 1$. A similar argument yields the same result for S_1. Since this is true for an arbitrary isometric representation of R, the result follows for $A(R)$. $\square$

Lemma 2. *If r is invertible then s_r is a unitary with $s_r^* = s_{r^{-1}}$.*

Proof. This follows from the previous lemma since $S_r S_{r^{-1}} = S_1 = S_{r^{-1}} S_r$, for any isometric representation of R. $\square$

It follows that if R is a field then $A(R)$ is a C^*-algebra. In addition, in this case, the regular representation is a unitary representation. In fact we have the following:

Proposition 1. *R is a field if and only if every isometric representation is a unitary representation.*

Proof. This comes from the fact that for the regular representation $(S_r)^*$ is in the algebra if and only if r is invertible. $\square$

Finally we can see that $A(R)$ is functorial for ring monomorphisms since if there is a ring monomorphism from R_1 to R_2 then any isometric representation R_2 will give rise to an isometric representation of R_1.

Proposition 2. *$A(R)$ is functorial in the sense that if $\pi : R_1 \to R_2$ is a unital ring monomorphism then there is an induced completely contractive representation $\tilde{\pi} : A(R_1) \to A(R_2)$.*

2. The universal C^*-algebra for a field

We now analyze the case where R is a field. Here any isometric representation is, in fact, a unitary representation. We let (R_+) denote the additive group in R. Notice that $R^\times$ acts on $C^*(R_+)$ as $*$-automorphisms via the mapping $\alpha_\lambda(U^n) = U^{\lambda n}$ where U^n is the unitary in $C^*(R_+)$ corresponding to $n \in (R_+)$. This allows us to rewrite $A(R)$ as a crossed product.

Proposition 3. *Let R be a field, then $A(R) \cong C^*(R_+) \rtimes R^\times$.*

Proof. We begin by noting (see [2, II.10.3.10]) that since $C^*(R_+)$ is unital and $R^\times$ is discrete we have $C^*(R_+) \subset C^*(R_+) \rtimes R^\times$ via a representation π_0 and there is a natural map $\rho_0 : R^\times \to C^*(R, \times) \rtimes R^\times$. Together (π_0, ρ_0) give rise to a covariant representation of the triple $(C^*(R_+), R^\times, \alpha)$.

Now analyzing the covariance conditions that define $C^*(R_+) \rtimes R^\times$ we see that $\rho_0(r)\pi_0(n)\rho_0(r)^* = \pi_0(rn)$. For each n, $\pi_0(n)$ is a unitary and for each r, $\rho_0(r)$ is a unitary and hence the natural covariant representation $\pi_0 \rtimes \rho_0$ gives rise to a unitary representation of R, and hence there is a completely contractive representation $\iota : A(R) \to C^*(R_+) \rtimes R^\times$.

Next notice that any unitary representation of $A(R)$ gives rise to a covariant representation of $(C^*(R_+), R^\times, \alpha)$ and hence $\|x\| \le \|\iota(x)\|$ so that ι is faithful. □

Since $C^*(R_+)$ and $R^\times$ are both abelian we have that the universal norm on the crossed product $C^*(R_+) \rtimes R^\times$ coincides with the reduced norm [12, Theorem 7.13]. In addition, we have that the algebra $A(R)$ is nuclear [12, Corollary 7.18], when R is a field. We next analyze the regular representation of $A(R)$ where R is a field.

Proposition 4. *Let R be a field, then the regular representation of $A(R)$ is faithful.*

Proof. This follows from analyzing the regular representation of $C^*(R_+) \rtimes_\alpha R^\times$, which is faithful since $R^\times$ is amenable. In effect, we take the left regular representation of $C^*(R_+)$ acting on $\ell^2(R_+)$ and add to this the action of $R^\times$ via $*$-automorphisms. This is exactly the construction of the regular representation of R and hence the two coincide. □

Other facts about $A(R)$ can also be explained using the crossed product machinery. To do this we remind the reader that the Pontryagin dual of a locally compact abelian group G is the abelian group

$$\widehat{G} = \{\varphi : G \to \mathbb{T}\}$$

with identity denoted id_G the trivial homomorphism $\mathrm{id}_G(g) = 1$.

Proposition 5. *For a field R the algebra $A(R)$ is not simple.*

Proof. Notice that $C^*(R_+) = C(\widehat{R_+})$ where $(\widehat{R_+})$ is the Pontryagin dual of the locally compact abelian group (R_+). Notice that the $*$-automorphisms α_λ induce a homeomorphism $\widehat{\alpha_\lambda}$ on $(\widehat{R_+})$ which has a fixed point for each λ; in particular, $\widehat{\alpha_\lambda}(\mathrm{id}_{R_+}) = \mathrm{id}_{R_+}$ for all λ. It follows that there is a nontrivial invariant ideal in $C^*(R_+)$ for the action by $R^\times$ and hence, see [12, Section 3.5] there is a nontrivial induced ideal in $C^*(R_+) \rtimes R^\times$. □

We can, however completely describe the ideal structure of $A(R)$ in the case of a field.

Theorem 1. *$A(R)$ is $*$-isomorphic to $\mathbb{C} \oplus A$ where A is a simple C^*-algebra. In fact, A is $*$-isomorphic to $C_0((\widehat{R_+}) \setminus \{\widehat{0}\}) \rtimes R^\times$.*

Proof. We use the nontrivial invariant ideal from the previous proposition. In particular, since $C^*(R_+) = C(\widehat{R_+})$ let π be the multiplicative linear function given on $C(\widehat{R_+})$ by evaluation at $\widehat{0}$. Further, if we let $\widehat{\alpha_\lambda}$ be the induced homeomorphism on $(\widehat{R_+})$ given by the $*$-automorphism α_λ for all $\lambda \in R^\times$. The induced representation, π is a multiplicative linear functional and hence has range $\mathbb{C}$. Hence, $A(R) \cong \mathbb{C} \oplus \ker \pi$. We now wish to describe π. So let $\sigma : C_0(\widehat{R_+}) \to C_0((\widehat{R_+}) \setminus \{\mathrm{id}_{R_+}\})$ be the restriction mapping.

Further, if $\lambda \in R^\times$ then $\widehat{\alpha_\lambda}$, the homeomorphism on $\widehat{R_+}$ induced by the automorphism α_λ, then the range of σ is invariant under $\widehat{\alpha_\lambda}$ and hence there is a map $\tau : C_0((\widehat{R_+}) \setminus \{if_{R_+}\}) \rtimes R^\times \to \ker \pi$. But since $R^\times$ acts transitively on $(\widehat{R_+}) \setminus \{\mathrm{id}_{R_+}\}$ the crossed product $C_0((\widehat{R_+}) \setminus \{\mathrm{id}_{R_+}\}) \rtimes R^\times$ is simple and hence the map τ must be an isomorphism. □

3. Semicrossed products for discrete semigroups

The preceding construction suggests that for non-field integral domains the crossed product may be replaced by a semicrossed product. We quickly outline the relevant construction referring to [11] for motivation and to [7] for more information about this semicrossed product.

Given a compact Hausdorff space X we say that a semigroup S acts on X via continuous maps if for each $s \in S$ there is a continuous map $\tau_s : X \to X$ with $\tau_s \circ \tau_t = \tau_{st}$. If S is unital with identity 0 we will assume that τ_0 is the identity map. Say that a pair (π, S_t) is an isometric covariant representation of (X, S, τ_s) if π is a representation of $C(X)$ on a Hilbert space $\mathcal{H}$ and for each $t \in S$, S_t is an isometry in $B(\mathcal{H})$ such that $S_t\pi(f(x)) = \pi(f(\tau_t(x)))S_t$ for all $x \in X$.

It is not hard to see that given the triple (X, S, τ_s), there is a nontrivial isometric covariant representation. The construction follows in the same manner as in [11], we only outline the idea here. Let $\mathcal{H} = \ell^2(X, S)$ where this latter Hilbert space is sequences indexed over elements of S with entries from x, with canonical basis $\{e_s\}$. Define $\pi : C(X) \to B(\mathcal{H})$ by $\pi(f(x)) = (f(\tau_s(x)))_{s\in S}$. Then set $S_t(e_s) = e_{st}$ and extend by linearity. Then (π, S_t) is an isometric covariant representation of (X, S, τ_s).

We say that the universal operator algebra generated by all isometric covariant representations of (X, S, τ_s) is the semicrossed product of X by S via τ. We denote this algebra as $C(X) \rtimes_\tau S$.

As examples notice that if α is a single endomorphism of a C^*-algebra then we are in the situation described in Peter's original work [11], where the semigroup is $\mathbb{Z}^+$. For an example on the opposite end of the spectrum we can view the semicrossed product of [6] as a semicrossed product where the monoid is the free monoid on n generators.

Returning to an integral domain R, we let α_r be the $*$-endomorphism of $C^*(R_+)$ induced by the group endomorphism given by left multiplication by r. This gives a map from $R^\times$ into the set of $*$-endomorphisms of $C^*(R_+)$. Notice that since we are in an integral domain each of these endomorphisms is injective. However they are only surjective when r is a unit.

Proposition 6. *The algebra $A(R)$ is completely isometrically isomorphic to the algebra $C^*(R_+) \rtimes_\alpha R^\times$.*

Proof. We will show that any isometric representation of R gives rise to an isometric covariant representation of the pair $(C^*(R_+) \rtimes_\alpha R^\times)$ and vice-versa, hence the two algebras will be completely isometrically isomorphic.

So let $\{U^n : n \in R\}$ and $\{S_r \in R^\times\}$ be an isometric representation of R. Then the map $n \mapsto U^n$ gives rise to a representation of $C^*(R_+)$, call it π. Further the isometries S_r will satisfy $S_r U^n S_r^* = U^{rn}$ and hence the pair $(\pi, \{S_r\})$ will be an isometric covariant representation of $(C^*(R,+), R^\times, \alpha)$.

Let $(\pi, \{S_r\})$ be an isometric covariant representation of $(C^*(R_+), R^\times, \alpha)$. Then $\{\pi(u^n)\}$ is a collection of unitaries and $\{S_r\}$ is a collection of isometries that trivially satisfy the first two conditions of an isometric representation of R. Further $S_r U^n S_r^* = U^{rn}$ so that $S_r U^n = U^{rn} S_r$ for all $r \in R^\times$ so that we have an isometric representation of R. $\square$

Notice that in the case of the algebra $A(R)$ for an integral domain R the semigroup $R^\times$ will always be commutative with no torsion. In addition the semigroup $R^\times$ can be viewed as a spanning cone for the group $Q(R)^\times$ (see [10, Page 60] for the definition of a spanning cone for a group).

We can actually improve our characterization of $A(R)$ as a semicrossed product by looking at a more tractable semigroup. Let $U(R)$ denote the group of units in R. Now $R^\times$ is a commutative monoid which contains $U(R)$ as a normal submonoid. We let $M(R)$ denote the monoid $R^\times/U(R)$. Notice that $R^\times \subseteq Q(R)^\times$ and further that $M(R) \subseteq Q(R)^\times/U(R)$, this latter group we call $G(R)$.

For $u \in U(R)$ let $\alpha_u : C^*(R_+) \to C^*(R_+)$ be the $*$-automorphism induced by the automorphism of (R_+) that corresponds to left multiplication by u. Next for $r \in M(R)$ define a covariant representation of the triple $(C^*(R_+), U(R), \alpha)$ by $\beta_r(U^n) = U^{nr}$, $\rho(S_u) = S_u$. This covariant representation induces a $*$-endomorphism of $C^*(R_+) \rtimes_\alpha U(R)$. Hence, β gives rise to a map from $M(R)$ into the set of $*$-endomorphisms of $C^*(R_+) \rtimes_\alpha U(R)$.

Theorem 2. *The algebra $A(R)$ is completely isometrically isomorphic to the algebras $(C^*(R_+) \rtimes_\alpha U(R)) \rtimes_\beta M(R)$, and the diagonal algebra $A(R) \cap A(R)^*$ is isomorphic to $(C^*(R_+) \rtimes_\alpha U(R))$.*

Proof. We will show that any isometric representation of R gives rise to an isometric covariant representation of the pair

$$(C^*(R_+) \rtimes_\alpha U(R), M(R))$$

and vice-versa, hence the two algebras will be completely isometrically isomorphic.

So let $\{U^n : n \in R\}$ and $\{S_r : r \in R^\times\}$ be an isometric representation of R. Then define a covariant representation of $(C^*(R_+), U(R), \alpha)$ by $n \mapsto U^n$ and $r \mapsto S_r$ for all $r \in U(R)$. This yields a representation π of $C^*(R_+) \rtimes_\alpha U(R)$. Next notice that $S_\lambda \pi(U^n) = \pi(U^{\lambda n}) S_\lambda$ for all $\lambda \in M(R)$, and $S_\lambda S_r = S_r S_\lambda$ for all $r \in U(R), \lambda \in M(R)$. Hence we have an isometric covariant representation of $(C^*(R_+) \rtimes_\alpha U(R), M(R), \beta)$.

Finally we take an isometric covariant representation (π, S) of $(C^*(R_+) \rtimes_\alpha U(R), M(R), \beta)$. Define $U^n = \pi(n)$ and $S_r = \pi(S_r)$ if $r \in U(R)$, else $S_r = S_r$. This gives rise to an isometric representation of R.

The last result follows from Corollary 2 of [7]. □

Notice that if R is not a field $U(R)$ does not act transitively on the nonunital subalgebra of $C^*(R_+)$, as in the case of a field. In fact we have the following fact.

Corollary 1. *The diagonal is isomorphic to $\mathbb{C} \oplus A$ where*

$$A \cong C_0((\widehat{R_+}) \setminus \mathrm{id}_{R_+}) \times U(R).$$

Further, A is simple if and only if R is a field.

Proof. That A is not simple follows from the fact that $U(R)$ does not act transitively on $C_0(\widehat{C^*(G,+)} \setminus \mathrm{id}_{R_+})$ unless $R^\times = U(R)$. □

We now prove some other facts about the relationship between the integral domain R and the structure of the algebra $A(R)$.

Proposition 7. *$A(R) \cong C(X) \rtimes S$, where S is a monoid with no nontrivial invertible elements if and only if the identity of R is the only unit. Further if $U(R) = \{1\}$ then $M(R)$ will not be finitely generated.*

Proof. If the identity of R is the only unit, then we have

$$C^*(R_+) \times U(R) \cong C^*(R_+)$$

and $M(R)$ has no nontrivial invertible elements, else $M(R) \cap U(R) \neq \{1\}$.

Notice that if $A(R) \cong C(X) \times S$ where S is a monoid with no nontrivial invertible elements then the diagonal algebra $A(R) \cap A(R)^* = C(X)$. However, if $x \in U(R)$ with $x \neq 1$, then $x \notin M(R)$ and hence

$$S_x \in C^*(R_+) \times U(R) = A(R) \cap A(R)^*.$$

But notice that $S_x U^n \neq U^n S_x$ unless $x = 1$ and hence $C^*(R_+) \times U(R)$ is not commutative.

Now assume $U(R) = \{1\}$ and $M(R)$ is generated by $\{x_1, x_2, \ldots, x_n\}$. Then $1 + x_1 x_2 \cdots x_n \notin U(S)$ so

$$1 + x_1 x_2 \cdots x_n = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

with at least one $k_j \neq 0$. We will assume without loss of generality that $k_1 \neq 0$. Then $1 = x_1(x_2 x_3 \cdots x_n - x_1^{k_1 - 1} x_2^{k_2} \cdots x_n^{k_n})$ which implies that x_1 is a unit yielding a contradiction. □

As a corollary we have the following.

Proposition 8. *If R is a unique factorization domain and $A(R) \cong A \times \mathbb{Z}^+$ where A is a C^*-algebra, then A is not commutative and $U(R)$ is not trivial.*

Proof. We will assume that A is commutative and hence $U(R)$ is trivial. In particular $A = C^*(R_+) = C_0(\widehat{R_+})$. Now let x_1 and x_2 be two irreducible elements of $M(R)$, then define two two-dimensional nest representations of $A(R)$ by

$$\begin{aligned}\pi_i(f) &= \begin{bmatrix} f(\widehat{x_i}) & 0 \\ 0 & f(\mathrm{id}_{R_+}) \end{bmatrix}, \text{ for } f \in C_0(\widehat{R_+}) \\ \pi_i(S_{x_i}) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \pi_i(S_r) &= 0 \text{ for } r \neq x_i.\end{aligned}$$

Here $\widehat{x_i}$ is a homomorphism that sends every irreducible other than x_i to 1 and x_i to $\lambda \neq 1$. It follows from the description of the two-dimensional nest representations of $A \times \mathbb{Z}^+$, see [5], that $x_1 = x_2$ which contradicts the fact that $M(R)$ must be infinitely generated. □

It follows that if R is a unique factorization domain $A(R)$ is never a semicrossed product in the sense of [11] and hence this collection of algebras presents a unique type of semicrossed product.

Notice also that if R_1 and R_2 are integral domains with $A(R_1) \cong A(R_2)$ we are not in a position to decide that $R_1 \cong R_2$. Since the diagonal is not, in general, a commutative algebra the two ingredients for such isomorphism results in [5] (the maximal ideal space, and the collection of two dimensional nest representations) will require some generalization before being applied in this context. We expect the details of such a generalization will be taken up in future work.

4. Failure of semisimplicity

We now consider the question of semisimplicity of the algebra $A(R)$. To address this we will need to need some information about $C^*(R_+)$ which we will obtain by considering the Pontryagin dual and its properties, we refer the reader to [8] for proofs of the relevant facts. Recall that if G is a discrete, locally compact, abelian Hausdorff group, then $\widehat{G}$ is a connected, compact, Hausdorff space.

Next recall that if H is an open and closed subgroup of G then there is a natural open continuous homomorphism of $\widehat{G}$ onto $\widehat{H}$ with compact kernel $K_H := \{\varphi \in \widehat{G} : \varphi|_H = \mathrm{id}_H\}$. Further the kernel of this homomorphism is trivial if and only if $H = G$. Now by Proposition 6 $A(R) \cong C^*(R_+) \rtimes_\alpha R^\times$. We have already used the fact that $C^*(R_+) \cong C(\widehat{R_+})$ and notice that the action of $r \in R^\times$ on $C^*(R_+)$ given by $r : s \mapsto rs$ induces an action $\widehat{r}$ on $C(\widehat{R_+})$ given by $\widehat{r}(\varphi)(s) = \varphi(rs)$. We are now in a position to discuss semisimplicity of $A(R)$.

Theorem 3. *The algebra $A(R)$ is semisimple if and only if R is a field.*

Proof. Clearly if R is a field then since $A(R)$ is a C^*-algebra then $A(R)$ is semisimple. For the converse we will assume that R is not a field and show that $A(R)$ has a nilpotent ideal (similar to the idea in [6]).

So let $r \in R^{\times}$ be a nonzero nonunit. Then notice that $H_r := \{rx : x \in R\}$ is a normal, open and closed, proper subgroup in (R_+). Now consider a nontrivial homomorphism φ in $\widehat{R_+}$ such that $\varphi|_{H_r} = 1$. Notice that $\widehat{r}(\varphi) = \mathrm{id}_{R_+}$. Now since $\widehat{R_+}$ is Hausdorff there exists open sets U and V such that $\varphi \in U$ and $\mathrm{id} \in V$ with $U \cap V = \emptyset$. Now let f be a nonzero function in $C(\widehat{R_+})$ with support in $\widehat{\alpha}^{-1}(V) \cap U$ such that $f(\varphi) \neq 0$. Now consider the nonzero element $S_r f \in A(R)$. Notice that for any $A \in A(R)$ we have $(AS_r f)^2 = AS_r f AS_r f = ABS_r^2 (f \circ \widehat{r}) f = 0$. It follows that the ideal generated by $S_r f$ is a nontrivial nilpotent ideal and hence it must be contained in the radical of $A(R)$, so that $A(R)$ is not semisimple. $\square$

5. Unitary representations of $\boldsymbol{R}$

Finally we wish to analyze the unitary representations of $A(R)$.

Lemma 3. *There is a canonical completely contractive representation*

$$i : A(R) \to C^*(Q(R)).$$

Proof. As $R \subseteq Q(R)$ the inclusion map provides an isometric representation of R inside $C^*(Q(R))$ and hence the induced map on $A(R)$ is completely contractive. $\square$

Proposition 9. *Let $\pi : A(R) \to B(H)$ be a unitary representation. There exists a $*$-representation $\tau_\pi : C^*(Q(R)) \to C^*(\pi(A(R))$ which is onto and satisfies $\tau_\pi \circ i(x) = \pi(x)$ for all $x \in A(R)$.*

Proof. Since π is a unitary representation we know that $\pi(s_r) = T_r$ is a unitary for all $r \in R^{\times}$. For all $\left[\frac{p}{q}\right] \in Q(R)^{\times}$ we define $\tilde{\pi}\big(s_{[\frac{p}{q}]}\big) = T_p T_q^*$. We also define $\tilde{\pi}\big(u^{[\frac{p}{q}]}\big) = T_q V^p T_q^*$. We need only show that the unitaries $T_p T_q^*$ and $T_q V^p T_q^*$ satisfy the relations for $Q(R)$ and hence the induced representation τ_π will be the required $*$-representation.

Notice first that $T_p T_q = T_q T_p$ and $T_p^* T_q^* = T_q^* T_p^*$ since π is a unitary representation of $A(R)$. It then follows that

$$\begin{aligned} T_p T_q^* &= T_q^* T_q T_p T_q^* \\ &= T_q^* T_p T_q T_q^* \\ &= T_q^* T_p \end{aligned}$$

for all $q, p \in R^{\times}$. It follows that $\tilde{\pi}\big(s_{[\frac{p_1}{q_1}]}\big)\tilde{\pi}\big(s_{[\frac{p_2}{q_2}]}\big) = \tilde{\pi}\big(s_{[\frac{p_1 p_2}{q_1 q_2}]}\big)$ for all $[\frac{p_1}{q_1}]$ and $[\frac{p_2}{q_2}]$ in $Q(R)^{\times}$.

Next notice that $V^p T_q = T_q V^{pq}$ and $T_q^* V^p = V^{pq} T_q^*$ and hence

$$\begin{aligned} T_{q_1} V^{p_1} T_{q_1}^* T_{q_2} V^{p_2} T_{q_2}^* &= T_{q_1} T_{q_2} V^{p_1 q_2} V^{q_1 p_2} T_{q_1}^* T_{q_2}^* \\ &= T_{q_1 q_2} V^{p_1 q_2 + q_1 p_2} T_{q_1 q_2}. \end{aligned}$$

In other words $\tilde{\pi}\big(u^{[\frac{p_1}{q_1}]}\big)\tilde{\pi}\big(u^{[\frac{p_2}{q_2}]}\big) = \tilde{\pi}\big(u^{[\frac{p_1}{q_1}]+[\frac{p_2}{q_2}]}\big)$.

Next we have that

$$\begin{aligned}\tilde{\pi}\big(u^{[\frac{p_1}{q_1}]}\big)\tilde{\pi}\big(s_{[\frac{p_2}{q_2}]}\big) &= T_{q_1}V^{p_1}T_{q_1}^*T_{p_2}T_{q_2}^* \\ &= T_{p_2}T_{q_2}^*T_{q_1}T_{q_2}V^{p_1p_2}T_{q_2}^*T_{q_1}^* \\ &= T_{p_2}T_{q_2}^*T_{q_1q_2}V^{p_1p_2}T_{q_1q_2}^* \\ &= \tilde{\pi}\big(s_{[\frac{p_2}{q_2}]}\big)\tilde{\pi}\big(u^{[\frac{p_1}{q_1}][\frac{p_2}{q_2}]}\big).\end{aligned}$$

Hence the C^*-algebra generated by the T_p and V^n satisfies the relations for $Q(R)$. We call the induced representation τ_π.

Finally we note that $\tau_\pi \circ i(s_p) = \tau_\pi\big(s_{[\frac{p}{1}]}\big) = T_pT_1^* = T_p$ for all $p \in Q^\times$ and $\tau_\pi \circ i(u^n) = \tau_\pi\big(u^{[\frac{n}{1}]}\big) = T_1V_nT_1^* = V_n$ for all $n \in R$ and hence $\tau_\pi \circ i(x) = \pi(x)$ for all $x \in A(R)$. □

It would follow that if every isometric representation of R dilated to a unitary representation (as for example the regular representation does), then we could identify the C^*-envelope of $A(R)$ as a crossed product, since the canonical representation i would be completely isometric. We do not think this is likely since this does not even work in the case of $C(X) \rtimes_\alpha \mathbb{Z}^+$ where α is a non-surjective continuous mapping, see [6].

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