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Standard deviation is a strongly Leibniz seminorm

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Abstract. We show that standard deviation σ satisfies the Leibniz inequality $\sigma(fg) \leq \sigma(f)\|g\| + \|f\|\sigma(g)$ for bounded functions f, g on a probability space, where the norm is the supremum norm. A related inequality that we refer to as "strong" is also shown to hold. We show that these in fact hold also for noncommutative probability spaces. We extend this to the case of matricial seminorms on a unital C*-algebra, which leads us to treat also the case of a conditional expectation from a unital C*-algebra onto a unital C*-subalgebra.

Contents

Introduction

A seminorm L on a unital normed algebra $\mathcal{A}$ is said to be *Leibniz* if $L(1_{\mathcal{A}}) = 0$ and

$$L(AB) \leq L(A)\|B\| + \|A\|L(B)$$

for all $A, B \in \mathcal{A}$. It is said to be *strongly Leibniz* if further, whenever A is invertible in $\mathcal{A}$ then

$$L(A^{-1}) \leq \|A^{-1}\|^2 L(A).$$

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The latter condition has received almost no attention in the literature, but it plays a crucial role in [19], where I relate vector bundles over compact metric spaces to Gromov–Hausdorff distance. See for example the proofs of Propositions 2.3, 3.1, and 3.4 of [19].

The prototype for strongly Leibniz seminorms comes from metric spaces. For simplicity of exposition we restrict attention here to compact metric spaces. So let (X, d) be a compact metric space, and let $C(X)$ be the algebra of continuous complex-valued functions on X, equipped with the supremum norm $\|\cdot\|_\infty$. For each $f \in C(X)$ let $L(f)$ be its Lipschitz constant, defined by

$$L(f) = \sup\{|f(x) - f(y)|/d(x, y) : x, y \in X \quad \text{and} \quad x \neq y\}. \tag{0.1}$$

It can easily happen that $L(f) = +\infty$, but the set, $\mathcal{A}$, of functions f such that $L(f) < \infty$ forms a dense unital $*$-subalgebra of $C(X)$. Thus $\mathcal{A}$ is a unital normed algebra, and L gives a (finite-valued) seminorm on it that is easily seen to be strongly Leibniz. Furthermore, it is not hard to show [17] that the metric d can be recovered from L. Thus, having L is equivalent to having d.

My interest in this comes from the fact that this formulation suggests how to define “noncommutative metric spaces”. Given a noncommutative normed algebra $\mathcal{A}$, one can *define* a “noncommutative metric” on it to be a strongly Leibniz seminorm on $\mathcal{A}$. There are then important and interesting analytic considerations [17], but we can ignore them for the purposes of the present paper.

For my study of noncommutative metric spaces I have felt a need for more examples and counter-examples that can clarify the variety of phenomena that can occur. While calculating with a simple class of examples (discussed in Section 2) I unexpectedly found that I was looking at some standard deviations. I pursued this aspect, and this paper records what I found.

To begin with, in Section 3 we will see that if (X, μ) is an ordinary probabity measure space and if $A = \mathcal{L}^\infty(X, \mu)$ is the normed algebra of (equivalence classes) of bounded measurable functions on X, and if σ denotes the usual standard deviation of functions, defined by

$$\sigma(f) = \|f - \mu(f)\|_2 = \left(\int \left| f(x) \; - \left(\int f \; d\mu\right)\right|^2 \; d\mu(x)\right)^{1/2},$$

then σ is a strongly Leibniz seminorm on $\mathcal{A}$. I would be surprised if this fact does not appear somewhere in the vast literature on probability theory, but so far I have not been able to find it. However, we will also show that this fact is true for standard deviation defined for noncommutative probability spaces, such as matrix algebras equipped with a specified state, and for corresponding infinite-dimensional algebras (C*-algebras) equipped with a specified state.

In [19] essential use is made of "matricial seminorms" that are strongly Leibniz. By a matricial seminorm on a C*-algebra $\mathcal{A}$ we mean a family $\{L_n\}$ where L_n is a seminorm on the matrix algebra $M_n(\mathcal{A})$ over $\mathcal{A}$ for each natural number n, and the family is coherent in a natural way. I very much want to extend the results of [19] to the noncommutative setting so that I can use them to relate "vector bundles" (i.e., projective modules) over noncommutative algebras such as those studied in [18, 20] that are close for quantum Gromov–Hausdorff distance. For this reason, in Section 4 we begin exploring standard deviation in this matricial setting. In doing this we find that we need to understand a generalization of standard deviation to the setting of conditional expectations from a C*-algebra $\mathcal{A}$ onto a sub-C*-algebra $\mathcal{B}$. That is the subject of Section 5. It leads to the first examples that I know of for Leibniz seminorms that are not strongly Leibniz. That is the subject of Section 6.

We will state many of our results for general unital C*-algebras. But most of our results are already fully interesting for finite-dimensional C*-algebras, that is, unital $*$-subalgebras of matrix algebras, equipped with the operator norm. Thus, readers who are not so familiar with general C*-algebras will lose little if in reading this paper they assume that all of the algebras, and Hilbert spaces, are finite-dimensional.

Very recently I have noticed connections between the topic of this paper and the topic of resistance networks. I plan to explore this connection further and to report on what I find.

1. Sources of strongly Leibniz seminorms

Up to now the only source that I know of for strongly Leibniz seminorms consists of "normed first-order differential calculi". For a unital algebra $\mathcal{A}$, a first-order differential calculus is [10] a bimodule Ω over $\mathcal{A}$ together with a derivation δ from $\mathcal{A}$ to Ω, where the derivation (or Leibniz) property is

$$\delta(AB) = \delta(A)B + A\delta(B)$$

for all $A, B \in \mathcal{A}$. When $\mathcal{A}$ is a normed algebra, we can ask that Ω also be a normed bimodule, so that

$$\|A\omega B\|_\Omega \leq \|A\|\|\omega\|_\Omega\|B\|$$

for all ω in Ω and all $A, B \in \mathcal{A}$. In this case if we set

$$L(A) = \|\delta(A)\|_\Omega$$

for all $A \in \mathcal{A}$, we see immediately that L is a Leibniz seminorm on $\mathcal{A}$. But if A is invertible in $\mathcal{A}$, then the derivation property of δ implies that

$$\delta(A^{-1}) = -A^{-1}\delta(A)A^{-1}.$$

From this it follow that L is strongly Leibniz. For later use we record this as:

Proposition 1.1. *Let $\mathcal{A}$ be a unital normed algebra and let (Ω, δ) be a normed first-order differential calculus over $\mathcal{A}$. Set $L(A) = \|\delta(A)\|_\Omega$ for all $A \in \mathcal{A}$. Then L is a strongly Leibniz seminorm on $\mathcal{A}$.*

Many of the first-order differential calculi that occur in practice are "inner", meaning that there is a distinguished element, ω_0, in Ω such that δ is defined by

$$\delta(A) = \omega_0 A - A\omega_0.$$

Among the normed first-order differential calculi, the ones with the richest structure are the "spectral triples" that were introduced by Alain Connes [7, 6, 8] in order to define "noncommutative Riemannian manifolds" and related structures. In this case $\mathcal{A}$ should be a $*$-algebra. Then a spectral triple for $\mathcal{A}$ consists of a Hilbert space $\mathcal{H}$, a $*$-representation π (or "action" on $\mathcal{H}$ if the notation does not include π) of $\mathcal{A}$ into the algebra $\mathcal{L}(\mathcal{H})$ of bounded operators on $\mathcal{H}$, and a self-adjoint operator D on $\mathcal{H}$ that is often referred to as the "Dirac" operator for the spectral triple. Usually D is an unbounded operator, and the requirement is that for each $A \in \mathcal{A}$ the commutator $[D, \pi(A)]$ should be a bounded operator. (There are further analytical requirements, but we will not need them here.) By means of π we can view $\mathcal{L}(\mathcal{H})$ as a bimodule Ω over $\mathcal{A}$. Then if we set

$$\delta(A) = [D, \pi(A)] = D\pi(A) - \pi(A)D$$

for all $A \in \mathcal{A}$, we obtain a derivation from $\mathcal{A}$ into Ω. It is natural to equip $\mathcal{L}(\mathcal{H})$ with its operator norm. If $\mathcal{A}$ is equipped with a $*$-norm such that π does not increase norms, then (Ω, δ) is clearly a normed first-order differential calculus, and we obtain a strongly Leibniz $*$-seminorm L on $\mathcal{A}$ by setting

$$L(A) = \|[D, \pi(A)]\|.$$

We see that (Ω, δ) is almost inner, with D serving as the distinguished element ω_0, the only obstacle being that D may be unbounded, and so not in Ω.

Part of the richness of spectral triples is that they readily provide matricial seminorms, in contrast to more general normed first-order differential calculi. This will be fundamental to our discussion in Section 4.

More information about the above sources of strongly Leibniz seminorms can be found in Section 2 of [20]. One can make some trivial modifications of the structures described above, but it would be interesting to have other sources of strongly Leibniz seminorms that are genuinely different.

2. A class of simple examples

In section 7 of [17] I considered the following very simple spectral triple. Let $X = \{1, 2, 3\}$, let $\mathcal{K} = \ell^2(X)$, and let $\mathcal{B} = \ell^\infty(X)$ with its evident action on $\mathcal{K}$ by pointwise multiplication. Let D be the "Dirac" operator on $\mathcal{K}$ whose matrix for the standard basis of $\mathcal{K}$ is

$$\begin{pmatrix} 0 & 0 & \alpha_1 \\ 0 & 0 & \alpha_2 \\ -\bar{\alpha}_1 & -\bar{\alpha}_2 & 0 \end{pmatrix}.$$

(For ease of bookkeeping we prefer to take our "Dirac" operators, here and later, to be skew-adjoint so that the corresponding derivation preserves $*$. This does not change the corresponding seminorm.) Then it is easily calculated that for $f \in \mathcal{B}$, with $f = (f_1, f_2, f_3)$, we have

$$L(f) = \big((f_1 - f_3)^2|\alpha_1|^2 + (f_2 - f_3)^2|\alpha_2|^2\big)^{1/2}.$$

This is quite different from the usual Leibniz seminorm as defined in Equation (0.1) — it looks more like a Hilbert-space norm. This example was shown in [17] to have some interesting properties.

This example can be naturally generalized to the case in which $X = \{1, \dots, n\}$ and we have a vector of constants $\xi = (\alpha_1, \dots, \alpha_{n-1}, 0)$. To avoid trivial complications we will assume that $\alpha_j \neq 0$ for all j. For ease of bookkeeping we will also assume that $\|\xi\|_2 = 1$. It is clear that the last element, n, of X is playing a special role. Accordingly, we set $Y = \{1, \dots, n-1\}$, and we set $\mathcal{A} = \ell^\infty(Y)$, so that $\mathcal{B} = \mathcal{A} \oplus \mathbb{C}$. Let e_n denote the last standard basis vector for $\mathcal{K}$. Thus ξ and e_n are orthogonal unit vectors in $\mathcal{K}$. Then it is easily seen that the evident generalization of the above Dirac operator D can be expressed as:

$$D = \langle \xi, e_n \rangle_c - \langle e_n, \xi \rangle_c,$$

where for any $\xi, \eta \in \mathcal{K}$ the symbol $\langle \xi, \eta \rangle_c$ denotes the rank-one operator on $\mathcal{K}$ defined by

$$\langle \xi, \eta \rangle_c(\zeta) = \xi \langle \eta, \zeta \rangle_{\mathcal{K}}$$

for all $\zeta \in \mathcal{K}$. (We take the inner product on $\mathcal{K}$ to be linear in the second variable, and so $\langle \cdot, \cdot \rangle_c$ is linear in the first variable.)

Our specific unit vector ξ determines a state, μ, on $\mathcal{A}$ by $\mu(A) = \langle \xi, A\xi \rangle$, faithful because of our assumption that $\alpha_j \neq 0$ for all j. Then we see that we can generalize to the situation in which $\mathcal{A}$ is a noncommutative unital C*-algebra and μ is a faithful state on $\mathcal{A}$ (i.e., a positive linear functional on $\mathcal{A}$ such that $\mu(1_{\mathcal{A}}) = 1$, and $\mu(A^*A) = 0$ implies $A = 0$). Let $\mathcal{H} = \mathcal{L}^2(\mathcal{A}, \mu)$ be the corresponding GNS Hilbert space [12, 4, 23] obtained by completing $\mathcal{A}$ for the inner product $\langle A, B \rangle_\mu = \mu(A^*B)$, with its left action of $\mathcal{A}$ on $\mathcal{H}$ and its cyclic vector $\xi = 1_{\mathcal{A}}$.

Let $\mathcal{B} = \mathcal{A} \oplus \mathbb{C}$ as C*-algebra, and let $\mathcal{K} = \mathcal{H} \oplus \mathbb{C}$ with the evident inner product. We use the evident action of $\mathcal{B}$ on the Hilbert space $\mathcal{K}$. Let η be $1 \in \mathbb{C} \subset \mathcal{K}$, so that ξ and η are orthogonal unit vectors in $\mathcal{K}$. We then define a Dirac operator on $\mathcal{K}$, in generalization of our earlier D, by

$$D = \langle \xi, \eta \rangle_c - \langle \eta, \xi \rangle_c.$$

We now find a convenient formula for the corresponding strongly Leibniz seminorm. We write out the calculation in full so as to make clear our conventions. For $(A, \alpha) \in \mathcal{B}$ we have

$$\begin{aligned}[D,(A,\alpha)] &= (\langle \xi, \eta\rangle_c - \langle \eta, \xi\rangle_c)(A,\alpha) - (A,\alpha)(\langle \xi, \eta\rangle_c - \langle \eta, \xi\rangle_c) \\ &= \langle \xi, \bar{\alpha}\eta\rangle_c - \langle \eta, A^*\xi\rangle_c - \langle A\xi, \eta\rangle_c + \langle \alpha\eta, \xi\rangle_c \\ &= -\langle (A - \alpha 1_{\mathcal{A}})\xi, \eta\rangle_c - \langle \eta, (A^* - \bar{\alpha}1_{\mathcal{A}})\xi\rangle_c.\end{aligned}$$

(From now on we will often write just α instead of $\alpha 1_{\mathcal{A}}$.) Because η is orthogonal to $B\xi$ for all $B \in \mathcal{A}$, we see that the two main terms above have orthogonal ranges, as do their adjoints, and so

$$\|[D,(A,\alpha)\| = \|(A-\alpha)\xi\| \vee \|(A^* - \bar{\alpha})\xi\|,$$

where $\vee$ denotes the maximum of the quantities. But ξ determines the state μ, and so for any $C \in \mathcal{A}$ we have $\|C\xi\| = (\mu(C^*C))^{1/2}$. But this is just the norm of C in $\mathcal{L}^2(\mathcal{A}, \mu) = \mathcal{H}$, which we will denote by $\|C\|_\mu$. We have thus obtained:

Theorem 2.1. *Let $\mathcal{A}$ be a unital C^*-algebra, let μ be a faithful state on $\mathcal{A}$, and let $\mathcal{H} = \mathcal{L}^2(\mathcal{A}, \mu)$, with its action of $\mathcal{A}$ and its cyclic vector ξ. Let $\mathcal{K}$ be the Hilbert space $\mathcal{H} \oplus \mathbb{C}$, and let $\mathcal{B}$ be the C^*-algebra $\mathcal{A} \oplus \mathbb{C}$, with its evident representation on $\mathcal{K}$. Let $\eta = 1 \in \mathbb{C} \subset \mathcal{K}$. Define a Dirac operator on $\mathcal{K}$ by*

$$D = \langle \xi, \eta\rangle_c - \langle \eta, \xi\rangle_c$$

as above. Then for any $(A, \alpha) \in \mathcal{B}$ we have

$$L((\mathcal{A},\alpha)) = \|[D,(A,\alpha)]\| = \|(A-\alpha)\|_\mu \vee \|(A^* - \bar{\alpha}\|_\mu.$$

Of course L is a $*$-seminorm which is strongly Leibniz.

3. Standard deviation

There seems to exist almost no literature concerning quotients of Leibniz seminorms, but such literature as does exist [5, 20] recognizes that quotients of Leibniz seminorms may well not be Leibniz. But no specific examples of this seem to be given in the literature, and I do not know of a specific example, though I imagine that such examples would not be very hard to find.

For the class of examples discussed in the previous section there is an evident quotient seminorm to consider, coming from the quotient of $\mathcal{B}$ by its ideal $\mathbb{C}$. This quotient algebra can clearly be identified with $\mathcal{A}$. For L as in Theorem 2.1 let us denote its quotient by $\tilde{L}$, so that

$$\tilde{L}(A) = \inf\{L((A,\alpha)) : \alpha \in \mathbb{C}\}$$

for all $A \in \mathcal{A}$. From the expression for L given in Theorem 2.1 we see that

$$\tilde{L}(A) = \inf\{\|(A-\alpha)\|_\mu \vee \|(A^* - \bar{\alpha}\|_\mu : \alpha \in \mathbb{C}\}.$$

But $\|\cdot\|_\mu$ is the Hilbert space norm on $\mathcal{H}$, and $\|(A-\alpha)\|_\mu$ is the distance from A to an element in the one-dimensional subspace $\mathbb{C} = \mathbb{C}1_{\mathcal{A}}$ of $\mathcal{H}$. The closest element to A in this subspace is just the projection of A into this subspace, which is $\mu(A)$. Furthermore, $\mu(A^*) = \overline{\mu(A)}$, and so by taking $\alpha = \mu(A)$ we obtain:

Proposition 3.1. *The quotient, $\tilde{L}$, of L on $\mathcal{A}$ is given by*

$$\tilde{L}(A) = \|A - \mu(A)\|_\mu \vee \|A^* - \mu(A^*)\|_\mu.$$

If $A^* = A$ then $\tilde{L}(A) = \|A - \mu(A)\|_\mu$.

But for $A^* = A$ the term $\|A - \mu(A)\|_\mu$ is exactly the standard deviation of A for the state μ, as used in quantum mechanics, for example on page 56 of [24]. When one expands the inner product used to define this term, one quickly obtains, by a well-known calculation,

$$\|A - \mu(A)\|_\mu = (\mu(A^*A) - |\mu(A)|^2)^{1/2}, \tag{3.2}$$

which is frequently more useful for calculations of the standard deviation. I have not seen the standard deviation defined for non-self-adjoint operators, but in view of all of the above, it seems reasonable to define it as follows:

Definition 3.3. Let $\mathcal{A}$ be a unital C*-algebra and let μ be a state on $\mathcal{A}$. We define the *standard deviation* with respect to μ, denoted by σ^μ, by

$$\sigma^\mu(A) = \|A - \mu(A)\|_\mu \vee \|A^* - \mu(A^*)\|_\mu.$$

for all $A \in \mathcal{A}$.

As is natural here, we have not required μ to be faithful. For simplicity of exposition we will nevertheless continue to require that μ be faithful as we proceed. But with some further arguing this requirement can be dropped.

When I noticed this connection with standard deviation, I said to myself that surely the standard deviation fails the Leibniz inequality, thus giving an example of a Leibniz seminorm L that has a quotient seminorm $\tilde{L}$ that is not Leibniz. This expectation was reinforced when I asked several probabilists if they had ever heard of the standard deviation satisfying the Leibniz inequality, and they replied that they had not. But when I tried to find specific functions for which the Leibniz inequality failed, I failed. Eventually I found the following simple but not obvious proof that the Leibniz inequality does hold. The proof depends on using the original form of the definition of standard deviation rather than the often more convenient form given in Equation (3.2). Define a seminorm, L_0, on $\mathcal{A}$ by

$$L_o(A) = \|A - \mu(A)\|_\mu.$$

We begin with:

Proposition 3.4. *Let notation be as above. Then L_0 is a Leibniz seminorm on $\mathcal{A}$.*

Proof. Let $A, B \in \mathcal{A}$. Since $\mu(AB)$ is the closest point in $\mathbb{C}$ to AB for the Hilbert-space norm of $\mathcal{H}$, we will have

$$\begin{aligned} L_0(AB) = \|AB - \mu(AB)\|_\mu &\le \|AB - \mu(A)\mu(B)\|_\mu \\ &\le \|AB - A\mu(B)\|_\mu + \|A\mu(B) - \mu(A)\mu(B)\|_\mu \\ &\le \|A\|_{\mathcal{A}}\|B - \mu(B)\|_\mu + \|A - \mu(A)\|_\mu |\mu(B)|, \end{aligned}$$

and since $|\mu(B)| \le \|B\|_{\mathcal{A}}$, we obtain the Leibniz inequality. □

Note that L_0 need not be a $*$-seminorm. Because the maximum of two Leibniz seminorms is again a Leibniz seminorm according to Proposition 1.2iii of [20], we obtain from the the definition of σ^μ given in Definition 3.3 and from the above proposition:

Theorem 3.5. *Let notation be as above. The standard deviation seminorm, σ^μ, is a Leibniz $*$-seminorm.*

This leaves open the question as to whether L_o and σ^μ are strongly Leibniz. I was not able to adapt the above techniques to show that they are. But in conversation with David Aldous about all of this (for ordinary probability spaces), he showed me the "independent copies trick" for expressing the standard deviation. (As a reference for its use he referred me to the beginning of the proof of Proposition 1 of [9]. I have so far not found this trick discussed in an expository book or article.) A few hours after that conversation I realized that this trick fit right into the normed first-order differential calculus framework described in Section 1. But when adapted to the noncommutative setting it seems to work only when μ is a tracial state (in which case $\sigma^\mu = L_0$). The "trick" goes as follows. Let $\Omega = \mathcal{A} \otimes \mathcal{A}$ (with the minimal C*-tensor-product norm [4, 13]), which is in an evident way an $\mathcal{A}$-bimodule. Set $\nu = \mu \otimes \mu$, which is a state on $\Omega = \mathcal{A} \otimes \mathcal{A}$ as C*-algebra. Thus ν determines an inner product on Ω whose norm makes Ω into a normed bimodule (because μ is tracial). Let $\omega_0 = 1_{\mathcal{A}} \otimes 1_{\mathcal{A}}$. Then for $A \in \mathcal{A}$ we have

$$\begin{aligned} \|\omega_0 A - A\omega_0\|_\nu^2 &= \langle 1_{\mathcal{A}} \otimes A - A \otimes 1_{\mathcal{A}}, 1_{\mathcal{A}} \otimes A - A \otimes 1_{\mathcal{A}} \rangle_\nu \\ &= \mu(A^*A) - \mu(A)\mu(A^*) - \mu(A^*)\mu(A) + \mu(A^*A) \\ &= 2(\mu(A^*A) - |\mu(A)|^2) = 2\|A - \mu(A)\|_\mu^2. \end{aligned}$$

From Proposition 1.1 we thus obtain:

Proposition 3.6. *Let notation be as above, and assume that μ is a tracial state. Then L_0, and so σ^μ, is a strongly Leibniz seminorm on $\mathcal{A}$.*

But by a different path we can obtain the general case for σ^μ (but not for L_0):

Theorem 3.7. *Let notation be as above (without assuming that μ is tracial). The standard deviation seminorm, σ^μ, is a strongly Leibniz $*$-seminorm.*

Proof. Let E be the orthogonal projection from $\mathcal{H} = L^2(\mathcal{A}, \mu)$ onto its subspace $\mathbb{C}1_{\mathcal{A}}$. Note that for $A \in \mathcal{A} \subseteq \mathcal{H}$ we have $E(A) = \mu(A)$. We use E as a Dirac operator, and we let L^E denote the corresponding strongly Leibniz seminorm, defined by

$$L^E(A) = \|[E, A]\|,$$

where we use the natural action of $\mathcal{A}$ on $\mathcal{H}$, and the norm is that of $\mathcal{L}(\mathcal{H})$.

Let $\mathcal{H}_0$ be the kernel of E, which is just the closure of $\{B - \mu(B) : B \in \mathcal{A}\}$, and is the orthogonal complement of $\mathbb{C}1_{\mathcal{A}}$. Notice that

$$[E, A](1_{\mathcal{A}}) = \mu(A) - A,$$

while if $B \in \mathcal{H}_0 \cap \mathcal{A}$ then

$$[E, A](B) = \mu(AB).$$

Thus $[E, A]$ takes $\mathbb{C}1_{\mathcal{A}}$ to $\mathcal{H}_0$ and $\mathcal{H}_0$ to $\mathbb{C}1_{\mathcal{A}}$. We also see that the norm of $[E, A]$ restricted to $\mathbb{C}1_{\mathcal{A}}$ is $\|A - \mu(A)\|_\mu$.

Notice next that $L^E(A) = L^E(A - \mu(A))$, so we only need consider A such that $\mu(A) = 0$, that is, $A \in \mathcal{H}_0$. For such an A we see from above that the norm of the restriction of $[E, A]$ to $\mathcal{H}_0$ is no larger than $\|A^*\|_\mu$. But because $A \in \mathcal{H}_0$ we have $[E, A](A^*) = \mu(AA^*) = \|A^*\|_\mu^2$. Thus the norm of the restriction of $[E, A]$ to $\mathcal{H}_0$ is exactly $\|A^*\|_\mu$. Putting this all together, we find that

$$\|[E, A]\| = \|A - \mu(A)\|_\mu \vee \|A^* - \mu(A^*)\|_\mu = \sigma^\mu(A)$$

for all $A \in \mathcal{A}$. Then from Proposition 1.1 we see that σ^μ is strongly Leibniz as desired. □

We remark that for every Leibniz $*$-seminorm its null-space (where it takes value 0) is a $*$-subalgebra, and that the null-space of σ^μ is the subalgebra of A's such that $\mu(A^*A) = \mu(A^*)\mu(A)$. When such an A is self-adjoint one says that μ is "definite" on A — see Exercise 4.6.16 of [12].

The above theorem leaves open the question as to whether L_0 is strongly Leibniz when μ is not tracial. I have not been able to answer this question. Computer calculations lead me to suspect that it is strongly Leibniz when $\mathcal{A}$ is finite-dimensional. We will see in Section 6 some examples of closely related Leibniz seminorms that fail to be strongly Leibniz.

I had asked Jim Pitman about the "strongly" part of the strongly Leibniz property for the case of standard deviation on ordinary probability spaces, and he surmised that it might be generalized in the following way, and Steve Evans quickly produced a proof. For later use we treat the case of complex-valued functions, with σ^μ defined as L_0.

Proposition 3.8. *Let (X, μ) be an ordinary probability space, and let f be a complex-valued function in $\mathcal{L}^\infty(X, \mu)$. For any complex-valued Lipschitz function F defined on a subset of $\mathbb{C}$ containing the range of f we will have*

$$\sigma^\mu(F \circ f) \le \mathrm{Lip}(F)\sigma^\mu(f)$$

where $\mathrm{Lip}(F)$ *is the Lipschitz constant of* F*.*

Proof. (Evans) By the independent copies trick mentioned before Proposition 3.6 we have

$$\begin{aligned}(\sigma^\mu(F\circ f))^2 &= (1/2)\int |F(f(x)) - F(f(y))|^2 \, d\mu(x)\, d\mu(y) \\ &\le (1/2)(\mathrm{Lip}(F))^2 \int |f(x)-f(y)|^2 \, d\mu(x)\, d\mu(y) \\ &= (\mathrm{Lip}(F))^2(\sigma^\mu(f))^2. \end{aligned}$$ □

We can use this to obtain the corresponding noncommutative version:

Theorem 3.9. *Let* $\mathcal{A}$ *be a unital* C^**-algebra and let* μ *be a state on* $\mathcal{A}$*. Let* $A \in \mathcal{A}$ *be normal, that is,* $A^*A = AA^*$*. Then for any complex-valued Lipschitz function* F *defined on the spectrum of* A *we have*

$$\sigma^\mu(F(A)) \le \mathrm{Lip}(F)\sigma^\mu(A),$$

where $F(A)$ *is defined by the continuous functional calculus for normal operators, and* $\mathrm{Lip}(F)$ *is the Lipschitz constant of* F*.*

Proof. Let $\mathcal{B}$ be the C*-subalgebra of $\mathcal{A}$ generated by A and $1_{\mathcal{A}}$. Then $\mathcal{B}$ is commutative because A is normal, and so [12, 4, 23] $\mathcal{B}$ is isometrically $*$-algebra isomorphic to $C(\Sigma)$ where Σ is the spectrum of A (so Σ is a compact subset of $\mathbb{C}$) and $C(\Sigma)$ is the C*-algebra of continuous complex-valued functions on Σ. (This is basicly the spectral theorem for normal operators.) Under this isomorphism A corresponds to the function $f(z) = z$ for $z \in \Sigma \subset \mathbb{C}$. Then $F(A)$ corresponds to the function $F = F \circ f$ restricted to Σ. The state μ restricts to a state on $C(\Sigma)$, giving a probability measure on Σ. Then the desired inequality becomes

$$\sigma^\mu(F) \le \mathrm{Lip}(F)\sigma^\mu(f).$$

But this follows immediately from Proposition 3.8. □

It would be reasonable to state the content of Proposition 3.8 and Theorem 3.9 as saying that σ^μ satisfies the "Markov" property, in the sense used for example in discussing Dirichlet forms.

It is easily checked that for a compact metric space (X, d) and with L defined by Equation (0.1) one again has $L(F\circ f) \le \mathrm{Lip}(F)L(f)$ for $f \in C(X)$ and F defined on the range of f, so that L satisfies the Markov property. But it is not clear to me what happens already for the case of f in the C*-algebra $C(X, M_n)$ for $n \ge 2$, with $f^* = f$ or f normal, and with F defined on the spectrum of f, and with the operator norm of M_n replacing the absolute value in Equation (0.1). A very special case that is crucial to [19] is buried in the proof of Proposition 3.3 of [19] . It would be very interesting to know what other classes of strongly Leibniz seminorms satisfy the Markov property for the continuous functional calculus for normal elements in the way given by Theorem 3.9.

We remark that by considering the function $F(z) = z^{-1}$ Theorem 3.9 gives an independent proof of the "strongly" property of σ^μ for normal elements of $\mathcal{A}$, but not for general elements. Consequently, if L_0 fails to be strongly Leibniz it is because the failure is demonstrated by some nonnormal invertible element of $\mathcal{A}$.

Let $\mathcal{A} = M_n$, the algebra of $n \times n$ complex matrices, for some n, and let $S(A)$ be the state space of $\mathcal{A}$, that is, the set of all states on $\mathcal{A}$. In this setting Audenaert proved in Theorem 9 of [2] that for any $A \in \mathcal{A}$ we have

$$\max\{\|A - \mu(A)\|_\mu : \mu \in S(A)\} = \min\{\|A - \alpha\| : \alpha \in \mathbb{C}\}.$$

In [14] the left-hand side is called the "maximal deviation" of A. A slightly simpler proof of Audenaert's theorem is given in Theorem 3.2 of [3]. I thank Franz Luef for bringing [3] to my attention, which led me to [2]. We now generalize Audenaert's theorem to any unital C*-algebra.

Theorem 3.10. *Let $\mathcal{A}$ be a unital C*-algebra. For any $A \in \mathcal{A}$ set*

$$\Delta(A) = \min\{\|A - \alpha\| : \alpha \in \mathbb{C}\}.$$

Then for any $A \in \mathcal{A}$ we have

$$\Delta(A) = \max\{\|A - \mu(A)\|_\mu : \mu \in S(A)\}.$$

Proof. For any $\mu \in S(\mathcal{A})$ and any $A \in \mathcal{A}$ we have $\mu(A^*A) \le \|A\|^2$, and so $\mu(A^*A) - |\mu(A)|^2 \le \|A\|^2$. Consequently $\|A - \mu(A)\|_\mu \le \|A\|$. But the left-hand side takes value 0 on $1_\mathcal{A}$, and so $\|A - \mu(A)\|_\mu \le \|A - \alpha\|$ for all $\alpha \in \mathbb{C}$. Consequently we have

$$\sup\{\|A - \mu(A)\|_\mu : \mu \in S(A)\} \le \Delta(A).$$

Thus it suffices to show that for any given $A \in \mathcal{A}$ there exists a $\mu \in S(A)$ such that $\|A - \mu(A)\|_\mu = \Delta(A)$. By the proof of Theorem 3.2 of [21] there is a $*$-representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, and two unit-length vectors ξ and η that are orthogonal, such that $\langle \eta, \pi(A)\xi \rangle = \Delta(A)$. For notational simplicity we omit π in the rest of the proof. Let μ be the state of $\mathcal{A}$ determined by ξ, that is, $\mu(B) = \langle \xi, B\xi \rangle$ for all $B \in \mathcal{A}$. Decompose $A\xi$ as

$$A\xi = \alpha\xi \; + \beta\eta \; + \gamma\zeta$$

where ζ is a unit vector orthogonal to ξ and η. Note that $\beta = \langle \eta, A\xi \rangle = \Delta(A)$. Then

$$\begin{aligned}\mu(A^*A) - |\mu(A)|^2 &= \langle A\xi, A\xi \rangle - |\langle \xi, A\xi \rangle|^2 \\ &= |\alpha|^2 + |\beta|^2 + |\gamma|^2 - |\alpha|^2 \\ &= |\beta|^2 + |\gamma|^2 = (\Delta(A))^2 + |\gamma|^2.\end{aligned}$$

Thus $\|A - \mu(A)\|_\mu = \Delta(A)$ as desired (and $\gamma = 0$). □

It would be interesting to have a generalization of Theorem 3.10 to the setting of a unital C*-algebra and a unital C*-subalgebra (with the subalgebra replacing $\mathbb{C}$ above) along the lines of Theorem 3.1 of [21], or to the setting of conditional expectations discussed in Section 5.

We remark that Theorem 3.2 of [21] asserts that Δ (denoted there by L) is a strongly Leibniz seminorm. We have seen (Proposition 3.4) that each seminorm $A \mapsto \|A - \mu(A)\|_\mu$ is Leibniz. This is consistent with the fact that the supremum of a family of Leibniz seminorms is again Leibniz (Proposition 1.2iii of [20]). Note that Δ is a $*$-seminorm even though $A \mapsto \|A - \mu(A)\|_\mu$ need not be. This is understandable since $|\langle \eta, C^*\xi\rangle| = |\langle \xi, C\eta\rangle|$ and we can apply the above reasoning with μ replaced by the state determined by η.

Anyway, we obtain:

Corollary 3.11. *With notation as above, for every $A \in \mathcal{A}$ we have*

$$\max\{\sigma^\mu(A) : \mu \in S(\mathcal{A})\} = \Delta(A).$$

It is easy to see that the supremum of a family of Markov seminorms is again Markov. We thus obtain:

Corollary 3.12. *With notation as above, the seminorm Δ is Markov.*

4. Matricial seminorms

Let us now go back to the setting of Section 2, with $\mathcal{B} = \mathcal{A} \oplus \mathbb{C}$ and $\mathcal{K} = \mathcal{H} \oplus \mathbb{C}$, where $\mathcal{H} = \mathcal{L}^2(\mathcal{A}, \mu)$. As suggested near the end of Section 1, the Dirac operator D defined on $\mathcal{K}$ in Section 2 will define a matricial seminorm $\{L_n\}$ on $\mathcal{B}$ (more precisely an $\mathcal{L}^\infty$-matricial seminorm, but we do not need the definition [15] of that here). This works as follows. Each $M_n(\mathcal{B})$ has a unique C*-algebra norm coming from its evident action on $\mathcal{K}^n$. Then D determines a Dirac operator D_n on $\mathcal{K}^n$, namely the $n \times n$ matrix with D's on the diagonal and 0's elsewhere. Notice that for any $B \in M_n(\mathcal{B})$ the effect of taking the commutator with D_n is simply to take the commutator with D of each entry of B. For any $B \in M_n(\mathcal{B})$ we then set $L_n(B) = \|[D_n, B]\|$. Each L_n will be strongly Leibniz.

It is known [22, 15] that if $\mathcal{B}$ is any C*-algebra with a ($\mathcal{L}^\infty$-) matricial seminorm $\{\mathcal{L}_n\}$, and if $\mathcal{I}$ is a closed two-sided ideal in $\mathcal{B}$, then we obtain a ($\mathcal{L}^\infty$-) matricial seminorm on $\mathcal{B}/\mathcal{I}$ by taking the quotient seminorm of L_n on $M_n(\mathcal{B})/M_n(\mathcal{I})$ for each n. We apply this to the class of examples that we have been discussing, with $\mathcal{I} = \mathbb{C} \subset \mathcal{B} = \mathcal{A} \oplus \mathbb{C}$. We denote the quotient seminorm of L_n by $\tilde{L}_n$. Our main question now is whether each $\tilde{L}_n$ is Leibniz, or even strongly Leibniz.

To answer this question we again first need a convenient expression for the norm of $[D_n, B]$. From our calculations preceding Theorem 2.1, for $\{(A_{jk}, \alpha_{jk})\} \in M_n(\mathcal{B})$ its commutator with D_n will have as entries (dropping the initial minus sign)

$$(A_{jk} - \alpha_{jk})\langle \xi, \eta\rangle_c + \langle \eta, \xi\rangle_c(A_{jk} - \alpha_{jk}).$$

If we let V denote the element of $M_n(\mathcal{L}(\mathcal{K}))$ having $\langle \xi, \eta \rangle_c$ in each diagonal entry and 0's elsewhere, and if we let G be the matrix $\{A_{jk} - \alpha_{jk}\}$, viewed as an operator on $\mathcal{K}^n$ that takes $\mathbb{C}^n$ to 0, then the matrix of commutators can be written as $GV + V^*G$. Now G carries $\mathcal{K}^n$ into $\mathcal{H}^n \subset \mathcal{K}^n$ and so GV carries $\mathbb{C}^n \subset \mathcal{K}^n$ into $\mathcal{H}^n$ and carries $\mathcal{H}^n$ to 0. Similarly V^*G carries $\mathcal{H}^n$ into $\mathbb{C}^n$ and $\mathbb{C}^n$ to 0. It follows that

$$\|GV + V^*G\| = \|GV\| \vee \|V^*G\|.$$

But $\|V^*G\| = \|G^*V\|$. Thus we basically just need to unwind the definitions and obtain a convenient expression for $\|GV\|$.

Now in an evident way GV, as an operator from $\mathbb{C}^n$ to $\mathcal{H}^n$, is given by the matrix $\{\langle G_{jk}\xi, \eta\rangle_c\}$. But because $\eta = 1 \in \mathbb{C} \subset \mathcal{K}$, we see that for $\beta \in \mathbb{C}^n$ we have $GV(\beta) = \{(\sum_k G_{jk}\beta_k)\xi\}$, an element of $\mathcal{H}^n$. Then

$$\|GV(\beta)\|^2 = \sum_j \left\| \left(\sum_k G_{jk}\beta_k \right) \xi \right\|^2.$$

But $\mathcal{H} = \mathcal{L}^2(\mathcal{A}, \mu)$ and $\xi = 1_{\mathcal{A}} \in \mathcal{H}$, and so for each j we have

$$\begin{aligned}
\left\| \left(\sum_k G_{jk}\beta_k \right) \xi \right\|^2 &= \left\| \sum_k G_{jk}\beta_k \right\|_\mu^2 \\
&= \left\langle \sum_k G_{jk}\beta_k, \sum_\ell G_{j\ell}\beta_\ell \right\rangle_\mu \\
&= \sum_{k,\ell} \bar{\beta}_k \mu(G_{jk}^* G_{j,\ell})\beta_\ell.
\end{aligned}$$

Thus

$$\|GV(\beta)\|^2 = \sum_j \langle \beta,\ \{\mu(G_{jk}^* G_{j\ell})\}\beta\rangle = \langle \beta, \{\mu(G^*G)_{k\ell}\}\beta\rangle.$$

From this it is clear that

$$\|GV\| = \|\{\mu(G^*G)_{k\ell}\}\|,$$

where now the norm on the right side is that of M_n. View $M_n(\mathcal{A})$ as $M_n \otimes \mathcal{A}$, and set

$$\boldsymbol{E}_n^\mu = \mathrm{id}_n \otimes \mu$$

where id_n is the identity map of M_n onto itself, so that $\boldsymbol{E}_n^\mu$ is a linear map from $M_n(\mathcal{A})$ onto M_n. Then

$$\|\{\mu(G^*G)_{k\ell}\}\| = \|\boldsymbol{E}_n^\mu(G^*G)\|.$$

For any $H \in M_n(\mathcal{A})$ set

$$\|H\|_{\boldsymbol{E}} = \|\boldsymbol{E}_n^\mu(H^*H)\|^{1/2}.$$

The conclusion of the above calculations can then be formulated as:

Proposition 4.1. *With notation as above, we have*

$$L_n((A,\alpha)) = \|A-\alpha\|_{\boldsymbol{E}} \vee \|A^*-\bar{\alpha}\|_{\boldsymbol{E}}$$

for all $(A,\alpha) \in M_n(\mathcal{B})$.

Now $\boldsymbol{E}_n^\mu$ is an example of a "conditional expectation", as generalized to the noncommutative setting [4, 13] (when we view M_n as the subalgebra $M_n \otimes 1_{\mathcal{A}}$ of $M_n(\mathcal{A})$). Thus to study the quotient, $\tilde{L}_n$, of L_n we are led to explore our themes in the setting of general conditional expectations.

5. Conditional expectations

Let $\mathcal{A}$ be a unital C*-algebra and let $\mathcal{D}$ be a unital C*-subalgebra of $\mathcal{A}$ (so $1_{\mathcal{A}} \in \mathcal{D}$). We recall [4, 13] that a *conditional expectation* from $\mathcal{A}$ to $\mathcal{D}$ is a bounded linear projection, $\boldsymbol{E}$, from $\mathcal{A}$ onto $\mathcal{D}$ which is positive, and has the property that for $A \in \mathcal{A}$ and $C, D \in \mathcal{D}$ we have

$$\boldsymbol{E}(CAD) = C\boldsymbol{E}(A)D.$$

(This latter property is often called the "conditional expectation property".) It is known [4, 13] that conditional expectations are of norm 1, and in fact are completely positive. One says that $\boldsymbol{E}$ is "faithful" if $\boldsymbol{E}(A^*A) = 0$ implies that $A = 0$. For simplicity of exposition we will assume that our conditional expectations are faithful. Given a conditional expectation $\boldsymbol{E}$, one can define a $\mathcal{D}$-valued inner product on $\mathcal{A}$ by

$$\langle A, B\rangle_{\boldsymbol{E}} = \boldsymbol{E}(A^*B)$$

for all $A, B \in \mathcal{A}$. (See section 2 of [16], and [4].) From this we get a corresponding (ordinary) norm on $\mathcal{A}$, defined by

$$\|A\|_{\boldsymbol{E}} = (\|\boldsymbol{E}(A^*A)\|_{\mathcal{D}})^{1/2}.$$

Actually, to show that this is a norm one needs a suitable generalization of the Cauchy-Schwartz inequality, for which see Proposition 2.9 of [16], or [4]. From the conditional expectation property one sees that for $A, B \in \mathcal{A}$ and $D \in \mathcal{D}$ one has

$$\langle A, BD\rangle_{\boldsymbol{E}} = \langle A, B\rangle_{\boldsymbol{E}} D.$$

Accordingly, one should view $\mathcal{A}$ as a right $\mathcal{D}$-module. Since it is evident that $(\langle A, B\rangle_{\boldsymbol{E}})^* = \langle B, A\rangle_{\boldsymbol{E}}$, we also have $\langle AD, B\rangle_{\boldsymbol{E}} = D^*\langle A, B\rangle_{\boldsymbol{E}}$. It follows that $\|AD\|_{\boldsymbol{E}} \le \|A\|_{\boldsymbol{E}}\|D\|_{\mathcal{D}}$. When $\mathcal{A}$ is completed for the norm $\|\cdot\|_{\boldsymbol{E}}$, the above operations extend to the completion, and one obtains what is usually called a right Hilbert $\mathcal{D}$-module [4].

In this setting we can imitate much of what we did earlier. Accordingly, set $\mathcal{B} = \mathcal{A} \oplus \mathcal{D}$. On $\mathcal{B}$ we can define a seminorm L by

$$L_0((A, D)) = \|A - D\|_{\boldsymbol{E}}.$$

(Note that L_0 need not be a $*$-seminorm.) To see that L_0 is Leibniz, we should first notice that for any $A, B \in \mathcal{A}$ since $B^*A^*AB \le \|A\|^2 B^*B$ and $\boldsymbol{E}$ is positive, we have $\boldsymbol{E}(B^*A^*AB) \le \|A\|^2 \boldsymbol{E}(B^*B)$, so that

$$\|AB\|_{\boldsymbol{E}} \le \|A\|_{\mathcal{A}} \|B\|_{\boldsymbol{E}}. \tag{5.1}$$

We can now check that L_0 is Leibniz. For $A, B \in \mathcal{A}$ and $C, D \in \mathcal{D}$ we have

$$\begin{aligned} L_0((A,C)(B,D)) &= \|AB - CD\|_{\boldsymbol{E}} \le \|AB - AD\|_{\boldsymbol{E}} + \|AD - CD\|_{\boldsymbol{E}} \\ &\le \|A\|_{\mathcal{A}} \|B - D\|_{\boldsymbol{E}} + \|A - C\|_{\boldsymbol{E}} \|D\|_{\mathcal{A}} \\ &\le \|(A,C)\|_{\mathcal{B}} L_0((B,D)) + L_0((A,C)) \|(B,D)\|_{\mathcal{B}}, \end{aligned}$$

as desired. Furthermore, L_0 is strongly Leibniz, for if A^{-1} and D^{-1} exist, then

$$\begin{aligned} L_0((A,D)^{-1}) &= \|A^{-1} - D^{-1}\|_{\boldsymbol{E}} = \|A^{-1}(D - A)D^{-1}\|_{\boldsymbol{E}} \\ &\le \|A^{-1}\|_{\mathcal{A}} \|A - D\|_{\boldsymbol{E}} \|D^{-1}\|_{\mathcal{A}} \le \|(A,D)^{-1}\|_{\mathcal{B}}^2 L_0((A,D)), \end{aligned}$$

as desired. Since L_0 need not be a $*$-norm, we will also want to use $L_0(A) \vee L_0(A^*)$. Then it is not difficult to put the above considerations into the setting of the spectal triples mentioned in Section 1, along the lines developed in Section 2. But we do not need to do this here.

We can now consider the quotient, $\tilde{L}_0$, of L_0 on the quotient of $\mathcal{B}$ by its ideal $\mathcal{D}$, which we naturally identify with $\mathcal{A}$, in generalization of what we did in Section 3. Thus we set

$$\tilde{L}_0(A) = \inf\{L_0(A - D) : D \in \mathcal{D}\}.$$

But we can argue much as one does for Hilbert spaces to obtain:

Proposition 5.2. *For every $A \in \mathcal{A}$ we have*

$$\tilde{L}_0(A) = \|A - \boldsymbol{E}(A)\|_{\boldsymbol{E}}.$$

Proof. Suppose first that $\boldsymbol{E}(A) = 0$. Then for any $D \in \mathcal{D}$

$$\begin{aligned} (L_0(A - D))^2 &= \|\boldsymbol{E}((A - D)^*(A - D))\|_{\mathcal{D}} \\ &= \|\boldsymbol{E}(A^*A) - D^*\boldsymbol{E}(A) - \boldsymbol{E}(A^*)D + D^*D\|_{\mathcal{D}} \\ &= \|\boldsymbol{E}(A^*A) + D^*D\|_{\mathcal{D}} \ge \|\boldsymbol{E}(A^*A)\|_{\mathcal{D}}. \end{aligned}$$

Thus 0 is a (not necessarily unique) closest point in $\mathcal{D}$ to A for the norm $\|\cdot\|_{\boldsymbol{E}}$. Thus $\tilde{L}_0(A) = \|A\|_{\boldsymbol{E}}$. For general A note that $\boldsymbol{E}(A - \boldsymbol{E}(A)) = 0$. From the above considerations it follows that $\boldsymbol{E}(A)$ is a closest point in $\mathcal{D}$ to A. □

Note that again this expression for $\tilde{L}_0$ need not be a $*$-seminorm. In view of the discussion in Section 3 it is appropriate to make:

Definition 5.3. With notation as above, for $A \in \mathcal{A}$ set

$$\sigma^{\boldsymbol{E}}(A) = \tilde{L}_0(A) \vee \tilde{L}_0(A^*) = \|A - \boldsymbol{E}(A)\|_{\boldsymbol{E}} \vee \|A^* - \boldsymbol{E}(A^*)\|_{\boldsymbol{E}},$$

and call it the *standard deviation* of A *with respect to* $\boldsymbol{E}$.

We can now argue much as we did in the proof of Proposition 3.4 to obtain:

Proposition 5.4. *With notation as above, both $\tilde{L}_0$ and $\sigma^{\boldsymbol{E}}$ are Leibniz seminorms.*

Proof. Let $A, B \in \mathcal{A}$. By the calculation in the proof of Proposition 5.2 we know that $\boldsymbol{E}(A)\boldsymbol{E}(B)$ is no closer to AB for the norm $\|\cdot\|_{\boldsymbol{E}}$ than is $\boldsymbol{E}(AB)$. Thus

$$\begin{aligned}\tilde{L}_0(AB) &= \|AB - \boldsymbol{E}(AB)\|_{\boldsymbol{E}} \le \|AB - \boldsymbol{E}(A)\boldsymbol{E}(B)\|_{\boldsymbol{E}} \\ &\le \|A(B - \boldsymbol{E}(B))\|_{\boldsymbol{E}} + \|(A - \boldsymbol{E}(A))\boldsymbol{E}(B)\|_{\boldsymbol{E}} \\ &\le \|A\|_{\mathcal{A}}\tilde{L}_0(B) + \tilde{L}_0(A)\|B\|_{\mathcal{A}},\end{aligned}$$

where we have used Proposition 5.2 and, implicitly, the conditional expectation property. Thus $\tilde{L}_0$ is Leibniz. As mentioned earlier, the maximum of two Leibniz seminorms is again Leibniz, and so $\sigma^{\boldsymbol{E}}$ too is Leibniz. □

This leaves open the question as to whether $\tilde{L}_0$ and $\sigma^{\boldsymbol{E}}$ are strongly Leibniz. We will try to imitate the proof of Theorem 3.7. We have mentioned earlier that $\mathcal{A}$, equipped with its $\mathcal{D}$-valued inner product and completed for the corresponding norm, is a right Hilbert $\mathcal{D}$-module. If Z is any right Hilbert $\mathcal{D}$-module, the appropriate corresponding linear operators on Z are the bounded adjointable right $\mathcal{D}$-module endomorphisms (as in Definition 2.3 of [16], or in [4]), that is, the norm-bounded endomorphisms T for which there is another such endomorphism, T^*, such that $\langle y, Tz\rangle_{\boldsymbol{E}} = \langle T^*y, z\rangle_{\boldsymbol{E}}$ for all $y, z \in Z$. (This is not automatic.) These endomorphisms form a C*-algebra for the operator norm.

For our situation of $\mathcal{A}$ equipped with the $\mathcal{D}$-valued inner product given by $\boldsymbol{E}$, the operators that we are about to use all carry $\mathcal{A}$ into itself, and so we do not need to form the completion, as long as we check that the operators are norm-bounded and have adjoints. We will denote the algebra of such operators by $\mathcal{L}^\infty(\mathcal{A}, \boldsymbol{E})$, in generalization of our earlier $\mathcal{L}^\infty(\mathcal{A}, \mu)$. It is a unital pre-C*-algebra.

Each $A \in \mathcal{A}$ determines an operator in $\mathcal{L}^\infty(\mathcal{A}, \boldsymbol{E})$ via the left regular representation. We denote this operator by $\hat{A}$. The proof that $\hat{A}$ is norm-bounded is essentially Equation (5.1). It is easily checked that the adjoint of $\hat{A}$ is $(A^*)\hat{}$, and that in this way we obtain a $*$-homomorphism from $\mathcal{A}$ into $\mathcal{L}^\infty(\mathcal{A}, \boldsymbol{E})$. Because $\boldsymbol{E}$ is faithful, this homomorphism will be injective, and so isometric.

Perhaps more surprising is that $\boldsymbol{E}$ too acts as an operator in $\mathcal{L}^\infty(\mathcal{A}, \boldsymbol{E})$. (See Proposition 3.3 of [16].) By definition $\boldsymbol{E}$ is a right $\mathcal{D}$-module endomorphism. For any $A \in \mathcal{A}$ we have

$$\langle \boldsymbol{E}(A), \boldsymbol{E}(A)\rangle_{\boldsymbol{E}} = \boldsymbol{E}(\boldsymbol{E}(A^*)\boldsymbol{E}(A)) = \boldsymbol{E}(A^*)\boldsymbol{E}(A).$$

But $\boldsymbol{E}(A^*)\boldsymbol{E}(A) \leq \boldsymbol{E}(A^*A)$ by the calculation (familiar for the variance, and related to Equation (3.2) above) that

$$0 \leq \boldsymbol{E}((A^* - \boldsymbol{E}(A^*))(A - \boldsymbol{E}(A))) = \boldsymbol{E}(A^*A) - \boldsymbol{E}(A^*)\boldsymbol{E}(A).$$

Thus $\|\boldsymbol{E}(A)\|_{\boldsymbol{E}} \leq \|A\|_{\boldsymbol{E}}$, so that $\boldsymbol{E}$ is a norm-bounded operator. Furthermore, for $A, B \in \mathcal{A}$ we have

$$\begin{aligned}\langle A, \boldsymbol{E}(B)\rangle_{\boldsymbol{E}} &= \boldsymbol{E}(A^*\boldsymbol{E}(B)) = \boldsymbol{E}(A^*)\boldsymbol{E}(B)\\ &= \boldsymbol{E}(\boldsymbol{E}(A^*)B) = \langle \boldsymbol{E}(A), B\rangle_{\boldsymbol{E}},\end{aligned}$$

so that $\boldsymbol{E}$ is "self-adjoint". When we view $\boldsymbol{E}$ as an element of $\mathcal{L}^\infty(\mathcal{A}, \boldsymbol{E})$ we will denote it by $\hat{\boldsymbol{E}}$.

Let us now use $\hat{\boldsymbol{E}}$ as a "Dirac operator" to obtain a strongly Leibniz $*$-seminorm, $L^{\boldsymbol{E}}$, on $\mathcal{A}$. Thus $L^{\boldsymbol{E}}$ is defined by

$$L^{\boldsymbol{E}}(A) = \|[\hat{\boldsymbol{E}}, \hat{A}]\|,$$

where the norm here is that of $\mathcal{L}^\infty(\mathcal{A}, \boldsymbol{E})$. We now unwind the definitions to obtain a more convenient expression for $L^{\boldsymbol{E}}$. Notice that $\hat{\boldsymbol{E}}^2 = \hat{\boldsymbol{E}}$. Now if $\mathcal{A}$ is any unital algebra and if $a, e \in \mathcal{A}$ with $e^2 = e$, then because $[a, \cdot]$ is a derivation of $\mathcal{A}$, we find that $e[a, e]e = 0$. Similarly we see that

$$(1 - e)[a, e](1 - e) = 0.$$

Let $\mathcal{Y}$ be the kernel of $\hat{\boldsymbol{E}}$, so that it consists of the elements of $\mathcal{A}$ of the form $A - \boldsymbol{E}(A)$. Note that $\mathcal{Y}$ and $\mathcal{D}$ are "orthogonal" for $\langle\cdot,\cdot\rangle_{\boldsymbol{E}}$, and that $\mathcal{A} = \mathcal{Y} \oplus \mathcal{D}$. The calculations just above show that $[\hat{\boldsymbol{E}}, \hat{A}]$ carries $\mathcal{D}$ into $\mathcal{Y}$ and $\mathcal{Y}$ into $\mathcal{D}$. From this it follows that

$$\|[\hat{\boldsymbol{E}}, \hat{A}]\| = \|\hat{\boldsymbol{E}}[\hat{\boldsymbol{E}}, \hat{A}](I - \hat{\boldsymbol{E}})\| \vee \|(I - \hat{\boldsymbol{E}})[\hat{\boldsymbol{E}}, \hat{A}]\hat{\boldsymbol{E}}\|$$

for all $A \in \mathcal{A}$, where I is the identity operator on $\mathcal{A}$. But note that

$$(|\hat{\boldsymbol{E}}[\hat{\boldsymbol{E}}, \hat{A}](I - \hat{\boldsymbol{E}}))^* = -(I - \hat{\boldsymbol{E}})[\hat{\boldsymbol{E}}, \hat{A}^*]\hat{\boldsymbol{E}}.$$

Thus we basically only need a convenient expression for $\|(I - \hat{\boldsymbol{E}})[\hat{\boldsymbol{E}}, \hat{A}]\hat{\boldsymbol{E}}\|$, and the latter is equal to $\|[\hat{\boldsymbol{E}}, \hat{A}]|_{\mathcal{D}}\|$.

Now for $D \in \mathcal{D}$ we have

$$\begin{aligned}\|[\hat{\boldsymbol{E}}, \hat{A}](D)\|_{\boldsymbol{E}} &= \|\boldsymbol{E}(AD) - A\boldsymbol{E}(D)\|_{\boldsymbol{E}} = \|(\boldsymbol{E}(A) - A)D\|_{\boldsymbol{E}}\\ &\leq \|A - \boldsymbol{E}(A)\|_{\boldsymbol{E}}\|D\|_{\mathcal{A}}.\end{aligned}$$

From this and the result when $D = 1_{\mathcal{A}}$ we see that

$$\|[\hat{\boldsymbol{E}}, \hat{A}]|_{\mathcal{D}}\| = \|A - \boldsymbol{E}(A)\|_{\boldsymbol{E}} = \tilde{L}_0(A).$$

It follows that

$$L^{\boldsymbol{E}}(A)) = \|A - \boldsymbol{E}(A)\|_{\boldsymbol{E}} \vee \|A^* - \boldsymbol{E}(A^*)\|_{\boldsymbol{E}} = \sigma^{\boldsymbol{E}}(A)$$

for all $A \in \mathcal{A}$. In view of what was said in Section 1 about first-order differential calculi, we have thus obtained:

Theorem 5.5. *For notation as above, $\sigma^{\boldsymbol{E}}$ is a strongly Leibniz $*$-seminorm.*

We can immediately apply this to the matricial setting of Section 4. For that setting and any n we have $\boldsymbol{E} = \boldsymbol{E}_n^\mu$. Then, in the notation of the present setting, the conclusion of Proposition 4.1 is again that

$$L_n((A,\alpha)) = \|A - \alpha\|_{\boldsymbol{E}} \vee \|A^* - \bar{\alpha}\|_{\boldsymbol{E}}$$

for all $(A,\alpha) \in M_n(\mathcal{B})$. Note that for the present situation, the L of the earlier part of this section is given exactly by $L_0((A,\alpha)) = \|A - \alpha\|_{\boldsymbol{E}}$. Then from Proposition 5.2 we see that

$$\tilde{L}_n(A) = \|A - \boldsymbol{E}(A)\|_{\boldsymbol{E}} \vee \|A^* - \boldsymbol{E}(A^*)\|_{\boldsymbol{E}}$$

for any $A \in \mathcal{A}$. And the right-hand side is just the corresponding standard deviation, which we will denote by $\sigma_n^{\boldsymbol{E}}$. Then from Theorem 5.5 we obtain:

Theorem 5.6. *Let $\mathcal{A}$ be a unital C^*-algebra and let μ be a faithful state on $\mathcal{A}$. For each natural number n let $\boldsymbol{E}_n^\mu$ be the corresponding conditional expectation from $M_n(\mathcal{A})$ onto $M_n \subset M_n(\mathcal{A})$, and let $\|\cdot\|_{\boldsymbol{E}_n^\mu}$ be the associated norm. Then the standard deviation σ_n^μ on $M_n(\mathcal{A})$ defined by*

$$\sigma_n^\mu(A) = \|A - \boldsymbol{E}_n^\mu(A)\|_{\boldsymbol{E}_n^\mu} \vee \|A^* - \boldsymbol{E}_n^\mu(A^*)\|_{\boldsymbol{E}_n^\mu}$$

for all $A \in M_n(\mathcal{A})$ is a strongly Leibniz $*$-seminorm. The family $\{\sigma_n^\mu\}$ is a strongly Leibniz ($\mathcal{L}^\infty$)-matricial $*$-seminorm on $\mathcal{A}$.

6. Leibniz seminorms that are not strongly Leibniz

Let us return now to the case of a general conditional expectation $\boldsymbol{E} : \mathcal{A} \to \mathcal{D}$. We saw in Proposition 5.4 that the seminorm $\tilde{L}_0$ on $\mathcal{A}$ defined by $\tilde{L}_0(A) = \|A - \boldsymbol{E}(A)\|_{\boldsymbol{E}}$ is a Leibniz seminorm. So we can ask whether it too is strongly Leibniz. We will now show that it need not be. One evening while at a conference I began exploring this question. It occurred to me to consider what happens to unitary elements of $\mathcal{A}$. If U is a unitary element of $\mathcal{A}$ and if $\tilde{L}_0$ is strongly Leibniz, then we will have

$$\tilde{L}_0(U^{-1}) \le \tilde{L}_0(U) \quad \text{and} \quad \tilde{L}_0(U) \le \tilde{L}_0(U^{-1})$$

so that $\tilde{L}_0(U^{-1}) = \tilde{L}_0(U)$. Since $U^{-1} = U^*$, we would thus have $\tilde{L}_0(U^*) = \tilde{L}_0(U)$. If $\tilde{L}_0$ is a $*$-seminorm, then this is automatic. But $\tilde{L}_0$ may not be a $*$-seminorm. Now

$$\begin{aligned}\tilde{L}_0(U) &= \|U - \boldsymbol{E}(U)\|_{\boldsymbol{E}} = \|\boldsymbol{E}((U^* - \boldsymbol{E}(U^*))(U - \boldsymbol{E}(U))\|_{\mathcal{A}}^{1/2} \\ &= \|1_{\mathcal{A}} - \boldsymbol{E}(U^*)\boldsymbol{E}(U)\|_{\mathcal{A}}^{1/2}.\end{aligned}$$

So the question becomes whether $\|1_{\mathcal{A}} - \boldsymbol{E}(U^*)\boldsymbol{E}(U)\|_{\mathcal{A}}$ can be different from $\|1_{\mathcal{A}} - \boldsymbol{E}(U)\boldsymbol{E}(U^*)\|_{\mathcal{A}}$. But $\|1_{\mathcal{A}} - \boldsymbol{E}(U)\boldsymbol{E}(U^*)\|_{\mathcal{A}}$ is equal to $1 - m$ where m is the smallest point in the spectrum of $\boldsymbol{E}(U)\boldsymbol{E}(U^*)$. Now the spectrum of $\boldsymbol{E}(U^*)\boldsymbol{E}(U)$ is equal to that of $\boldsymbol{E}(U)\boldsymbol{E}(U^*)$ except possibly for the value 0. (See Proposition 3.2.8 of of [12].) Thus the question becomes: Is there an example of a conditional expectation $\boldsymbol{E} : \mathcal{A} \to \mathcal{D}$ and a unitary element U of $\mathcal{A}$ such that $\boldsymbol{E}(U)\boldsymbol{E}(U^*)$ is invertible but $\boldsymbol{E}(U^*)\boldsymbol{E}(U)$ is not invertible?

The next day I asked this question of several attendees of the conference who had some expertise is such matters. The following morning, shortly before I was to give a talk on the topic of this paper, Sergey Neshveyev gave me the following example (which I have very slightly reformulated).

Example 6.1. Suppose that one can find a unital C*-algebra $\mathcal{D}$ containing two partial isometries S and T and two unitary operators V and W such that, for $R = S + T$, we have:

i) R^*R is invertible but RR^* is not invertible.
ii) $S^* = VTW$.

Then let $\mathcal{A} = M_2(\mathcal{D})$, and define a unitary operator U in $\mathcal{A}$ by

$$U = \begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T & (1 - TT^*)^{1/2} \\ -(1 - T^*T)^{1/2} & T^* \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & 1 \end{pmatrix}.$$

(See the solution of Problem 222 of [11].) Let τ denote the normalized trace, i.e., the tracial state, on M_2, and let $\boldsymbol{E} = \tau \otimes \mathrm{id}$ where id is the identity map on $\mathcal{A}$. Then $\boldsymbol{E}$ is a conditional expectation from $\mathcal{A}$ onto $\mathcal{D}$, where $\mathcal{D}$ is identified with $I_2 \otimes \mathcal{D}$ in $M_2 \otimes \mathcal{D} = \mathcal{A}$. Then

$$\boldsymbol{E}(U) = (S^* + T^*)/2 = R^*/2.$$

Consequently $\boldsymbol{E}(U)\boldsymbol{E}(U^*)$ is invertible but $\boldsymbol{E}(U^*)\boldsymbol{E}(U)$ is not invertible, as desired.

It remains to show that there exist operators S, T, V, W satisfying the properties listed above. Let $\mathcal{H} = \ell^2(\mathbb{Z})$ with its standard orthonormal basis $\{e_n\}$, and let $\mathcal{D} = \mathcal{L}(\mathcal{H})$. Let B denote the right bilateral shift operator on $\mathcal{H}$, so $Be_n = e_{n+1}$ for all n. Let J be the unitary operator determined by $Je_n = e_{-n}$ for all n, and let P be the projection determined by $Pe_n = e_n$ if $n \geq 0$ and 0 otherwise. Set $S = JBP$ and $T = BPJ$, and set $R = S + T$. It is easily checked that $R^*Re_n = e_n$ if $n \neq 0$ while $R^*Re_0 = 2e_0$, so that R^*R is invertible, but $R^*e_0 = 0$ so that RR^* is not invertible, as desired. Furthermore, if we set $V = B^{-1}$ and $W = B$, then it is easily checked that $S^* = VTW$ as desired.

The above example provides the first Leibniz seminorm L that I know of that is not strongly Leibniz, and so can not be obtained from a normed first-order differential calculus. But motivated by the above example we can obtain simpler examples, which are not so closely related to conditional expectations.

Example 6.2. Let $\mathcal{A}$ be a unital C*-algebra, and let P be a projection in $\mathcal{A}$ (with $P^* = P$). Let $P^\perp = 1_{\mathcal{A}} - P$. Define γ on $\mathcal{A}$ by

$$\gamma(A) = P^\perp AP$$

for all $A \in \mathcal{A}$. Then γ is usually not a derivation, but we have

$$\begin{aligned}\gamma(AB) &= P^\perp ABP - P^\perp APBP + P^\perp APBP \\ &= P^\perp A(P^\perp BP) + (P^\perp AP)BP = \gamma(A\gamma(B) + \gamma(A)B))\end{aligned}$$

for all $A, B \in \mathcal{A}$. Now set

$$L(A) = \|\gamma(A)\|$$

for all $A \in \mathcal{A}$. Because γ is norm nonincreasing, it is clear from the above calculation that L is a Leibniz seminorm. It is also clear that L may not be a *-seminorm. We remark that if L is restricted to any unital C*-subalgebra of $\mathcal{A}$, without requiring that P be in that subalgebra, we obtain again a Leibniz seminorm on that subalgebra.

We can ask whether L is strongly Leibniz. The following example shows that it need not be. Much as in Example 6.1, we use the fact that if L is strongly Leibniz then for any unitary element U in $\mathcal{A}$ we must have $L(U^*) = L(U)$.

Let $\mathcal{H} = \ell^2(\mathbb{Z})$ with its standard orthonormal basis $\{e_n\}$, and let $\mathcal{A} = \mathcal{L}(\mathcal{H})$. Let U denote the right bilateral shift operator on $\mathcal{H}$, so $Ue_n = e_{n+1}$ for all n, and let P be the projection determined by $Pe_n = e_n$ if $n \geq 0$ and 0 otherwise. Then it is easily seen that $P^\perp UP = 0$ while $P^\perp U^* Pe_0 = e_{-1}$. Thus $L(U) = 0$ while $L(U^{-1}) = 1$.

We now show that if $P\mathcal{A}P$ is finite dimensional, or at least has a finite faithful trace, then $L(U^*) = L(U)$ for any unitary element U of $\mathcal{A}$. Notice that

$$\|P^\perp UP\|^2 = \|PU^*P^\perp UP\| = \|P - PU^*PUP\| = 1 - m$$

where m is the minimum of the spectrum of PU^*PUP inside $P\mathcal{A}P$. On applying this also with U replaced by U^*, we see, much as in Example 6.1, that $L(U) \neq L(U^*)$ exactly if one of PU^*PUP and $PUPU^*P$ is invertible in $P\mathcal{A}P$ and the other is not. This can not happen if $P\mathcal{A}P$ has a finite faithful trace. But this does not prove that L is strongly Leibniz in that case.

For the general case of this example, if we set

$$L_s(A) = \max\{L(A), L(A^*)\},$$

then, much as earlier, L_s will be a Leibniz $*$-seminorm. But in fact, L_s will be strongly Leibniz. This is because

$$[P, A] = PAP^\perp - P^\perp AP,$$

so that

$$\|[P, A]\| = \|PAP^\perp\| \vee \|P^\perp AP\| = L_s(A).$$

This is all closely related to the Arveson distance formula [1], as shown to me by Erik Christensen at the time when I developed Theorem 3.2 of [21].

But the above examples depend on the fact that L is not a $*$-seminorm. It would be interesting to have examples of Leibniz $*$-seminorms that are not strongly Leibniz. It would also be interesting to have examples for which $\mathcal{A}$ is finite-dimensional. (Note that right after Proposition 1.2 of [20] there is an example of a Leibniz $*$-seminorm that is not strongly Leibniz, but this example depends crucially on the Leibniz seminorm taking value $+\infty$ on some elements.)

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