Using twisted Alexander polynomials to detect fiberedness

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ABSTRACT. In this paper we discuss how certain algebraic invariants of 3-manifolds can be effectively used in the study of fiberedness and the Thurston norm of links. In particular we use twisted Alexander polynomials to prove that the exterior of a certain graph knot, whose splice diagram is given, is not fibered. Then we consider three 2-component graph links built out of this knot. For these links we use the same technique, involving twisted Alexander polynomials to discuss their fiberedness and Turston norm. This allows us to demonstrate the effectiveness of twisted Alexander polynomials in this context (links in homology spheres different from S^3), where no calculations exist in the literature.

Contents

1. Introduction

In 1990 X. S. Lin introduced a generalization of Alexander polynomials called *twisted* Alexander polynomials for knots [13]. His definition was later generalized to 3-manifolds by B. Jiang and S. Wang [11], M. Wada [17], P. Kirk and C. Livingston [12], and J. Cha [2]. Twisted Alexander polynomials can be used to investigate other properties of 3-manifolds such as fiberedness. In particular S. Friedl and S. Vidussi [8], and J. Cha [2] have used them in relation with fiberedness of 3-manifolds. These polynomials can also be helpful in investigating the Thurston norm, [16].

The main purpose of this paper is to find explicit applications of the relationship between twisted Alexander polynomials and fiberedness. In particular we study a knot K that is included in a homology sphere Σ (different from the 3-dimensional sphere S^3), and three different 2-component links that have K as one of their components. In such cases the Wirtinger presentation can not be used directly to find the fundamental group of the exterior of the knot K or of the aforementioned links.

Our knot K is the result of gluing the exteriors of two right-handed trefoil knots to the 3-component necklace in a special way that is called *splicing*. Using the Wirtinger presentation for the three pieces together with the splicing relations, we calculate the fundamental group of the exterior of the knot K. We use a similar technique for the aforementioned links to calculate the fundamental group of their exterior. We will be using the method of *splice diagrams* as introduced by Eisenbud and Neumann [3] for describing the *graph links* studied in this paper. Using the combinatorial information included in the splice diagram, Eisenbud and Neumann show that K is not fibered. However, the technique they use to show this fact only applies to graph links, whereas we recast this result using twisted Alexander polynomials. The method of this paper can theoretically be applied to any 3-manifold. Although this has been done in some cases (mostly knots in S^3 with few crossings) prior to this work, these techniques have never been applied to knots (or links) which are included in homology spheres different from S^3. We have used the computer program *Knottwister* created by S. Friedl [4]. Knottwister requires the fundamental group of the 3-manifold, N along with a cohomology class ϕ. It uses Fox differential calculus to compute the Alexander polynomial and twisted Alexander polynomials via representations of the fundamental group. The technique used in this paper to determine fiberedness does not depend on the fact that K is a graph knot.

2. Preliminaries

In this section we will introduce graph links and discuss the necessary definitions and concepts to clarify how splice diagrams represent graph links.

The material in this section is a summary of work that appears in [3]. In this work, we follow the definition of [10] for Seifert fibration.

For us, an n-component link is an embedding of a disjoint collection of n copies of S^1 in a homology sphere Σ. (A homology sphere is an n-manifold whose homology groups are the same as the homology groups of the n-sphere, S^n.) The knot K and the links L_α, L_β, and L_γ, which we will introduce in Sections 2.1 and 2.2, are contained in homology spheres different from S^3.

Definition 2.1. A Seifert link is an n-component link $L = (\Sigma, K)$, where $K = S_1 \cup .... \cup S_n \subset \Sigma$, S_i's being copies of S^1, and Σ a homology sphere, whose exterior $\Sigma_0 = \Sigma \setminus \text{int}(\nu(K))$ admits a Seifert fibration, when $\nu(K)$ denotes a neighborhood of K and $\text{int}(\nu(K))$ is its interior.

We know by Lemma 7.1 in [3] that Σ itself must be Seifert fibered and (with one family of exceptions corresponding to the necklaces), the link components are singular or regular fibers of the fibration. (For examples of what we call a necklace, see Figure 9.)

We can specialize the above description of Seifert fibered spaces to obtain homology spheres as follows. We choose F to be S^2 and for any choice of an n-tuple $(\alpha_1, ..., \alpha_n)$ of singular fibers with multiplicity α_i we get a Seifert fibered homology sphere by choosing coefficients β_i to be determined module α_i by the following equation:

$$\sum_1^n \beta_i \alpha_1 ... \widehat{\alpha_i} ... \alpha_n = 1.$$

Each Seifert fibered homology sphere is homeomorphic to $(\varepsilon \Sigma(\alpha_1, ... \alpha_n))$ for some n when $\varepsilon = \pm 1$. For the canonical orientation, $\varepsilon = 1$ and for the opposite orientation, $\varepsilon = -1$ [3]. For example, $\Sigma(p, q, 1, ..., 1)$ is S^3 for all coprime integers p, q and $\Sigma(2, 3, 5)$ is the Poincaré homology sphere. The only case when an α_i may be zero is the case $\Sigma(0, 1, ..., 1)$, which gives S^3 (see [3]).

We can denote a Seifert link as

$$(\varepsilon \Sigma(\alpha_1, ..., \alpha_n), \pm S_1 \cup .... \cup \pm S_m), \ \ m \leq n$$

where the S_i represent singular or regular fibers of the Seifert fibration with their canonical orientation. By allowing the α_i to be negative, we can recast all Seifert links as

$$(\pm \Sigma(\alpha_1, ..., \alpha_n), S_1 \cup .... \cup S_m)), \ \ m \leq n.$$

Now, we describe the splicing as an operation.

Definition 2.2. Given two links $L = (\Sigma, K)$ and $L' = (\Sigma', K')$, let $S \subset K$ and $S' \subset K'$. Let μ and λ be the standard meridian and longitude of S and μ' and λ' that of S'. Consider $\Sigma'' = (\Sigma \setminus \text{int}(\nu(S))) \cup (\Sigma' \setminus \text{int}(\nu(S')))$, which is formed by identifying λ with μ' and λ' with μ. This operation is well-defined [3]. The link $(\Sigma'', (K \setminus S) \cup (K' \setminus S'))$ is called the splice of the links L and L' along S and S'.

Note that if K and K' have n and m components respectively, K'' has $(n + m) - 2$ components. Using the Mayer–Vietoris sequence, it can be shown that Σ'' is also a homology sphere.

Definition 2.3. A graph link is a link that is the result of splicing two or more Seifert link. a graph knot is a one-component graph link.

Following Eisenbud and Neumann we shall represent graph links by using certain diagrams, which are called "splice diagrams". Splice diagrams encode all the information about graph links. If we consider the minimal version of a splice diagram, there is a one to one correspondence between splice diagrams and graph links. (The concept of minimality of diagrams is discussed in detail in Theorem 8.1 in [3]. Using this theorem, it is straightforward to determine when a diagram is minimal. In what follows, all our splice diagrams will be minimal.)

The building blocks of splice diagrams are Seifert links. The following diagram corresponds to the Seifert link $(\pm\Sigma(\alpha_1, ..., \alpha_n), S_1 \cup ... \cup S_m))$. It is a Seifert fibered homology sphere with the first m fibers removed; S_i is regular if $\alpha_i = 1$, and singular otherwise.
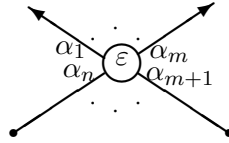

FIGURE 1. Seifert link $(\pm\Sigma(\alpha_1, ..., \alpha_n), S_1 \cup ... \cup S_m))$.

It is worth mentioning that the unknot and the Hopf link have splice diagrams that follow suitable modifications of the same rules. Other than these two exceptions, the splice diagram of every Seifert link is made out of three different parts, which we will explain next.
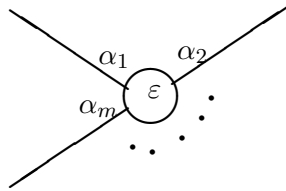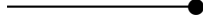
1. Nodes:

FIGURE 2. A node.

Here, $m \geq 3$, the α_i's are pairwise coprime integers, and ε is a sign. If we consider the canonical orientation ε is positive; otherwise it is negative. Each node represents a Seifert link. So if the minimal diagram has k nodes, it is the splice diagram of a graph link that is the result of splicing k Seifert links together.

2. Boundary vertices:

In a splice diagram, these represent *singular* fibers of a fibration that are not components of the link. Splicing cannot happen along these vertices.

3. Arrowhead vertices:

These correspond to actual link components. They are regular fibers (when $\alpha_i = 1$) or singular fibers (when $\alpha_i \neq 1$) of the Seifert fibration of the link being represented. Splicing can happen along these vertices.

As mentioned before, every graph link is the result of splicing two or more Seifert links together to obtain a diagram of the following form.
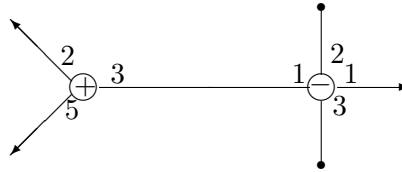
FIGURE 3. Example of a splice diagram.

We will now describe how this is represented in terms of splice diagrams. Recall that we can represent Seifert links $L^{(1)} = (\Sigma^{(1)}, K^{(1)})$ and $L^{(2)} = (\Sigma^{(2)}, K^{(2)})$ by the following diagrams.

FIGURE 4. Seifert links before splicing.

where $S^{(1)}$ and $S^{(2)}$ are components of $K^{(1)}$ and $K^{(2)}$ respectively, along which we do splicing. The graph link that is the result of this splicing is the following.
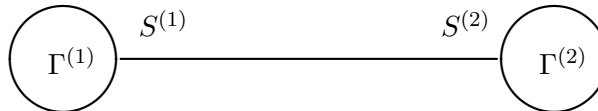
FIGURE 5. The resulting graph link.

As we see here, the components along which splicing happens disappear as arrowhead vertices, and appear as an edge in the diagram of the resulting graph link. When we speak of "the vertices of the diagram" we will include nodes as well as boundary vertices and arrowhead vertices.

In this work, our definition of twisted Alexander polynomial is consistent with that appearing in [6].

3. Main results

The following theorem of C. McMullen shows the ability of the (ordinary) Alexander polynomial to provide information on the Thurston norm and fiberedness for a general 3-manifold N. If $\phi = (m_1, ..., m_n) \in H^1(N; \mathbb{Z})$, then $\text{div}(\phi)$ is the greatest common divisor of $m_1, ..., m_n$.

Theorem 3.1 (McMullen, [14]). *Let N be a compact connected orientable 3-manifold whose boundary (if any) is a union of tori. Then for any $\phi \in H^1(N; \mathbb{Z})$*

$$\deg(\Delta_{N,\phi}) \leq \|\phi\|_T + \begin{cases} 0, & b_1(N) \geq 2 \\ \text{div}(\phi) \cdot (1 + b_3(N)), & b_1(N) = 1. \end{cases}$$

Moreover, if ϕ is fibered $\Delta_{N,\phi}$ is monic and equality holds.

It is well-known that the converse of Theorem 3.1 is not true as we will show for the graph knot K, which has the splice diagram shown in Figure 6.

Proposition 3.2. *The genus of the knot K is 1, it has Alexander polynomial equal to $t^2 - t + 1$ and it is not fibered.*
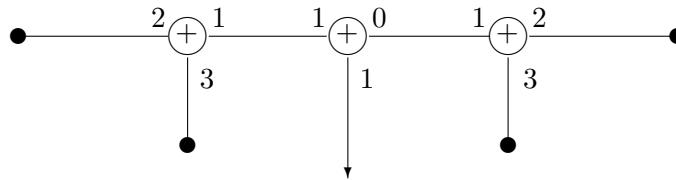

FIGURE 6. Splice diagram of the knot K.

S. Friedl and T. Kim have generalized the result in Theorem 3.1 by considering the collection of twisted Alexander polynomials in the following theorem.

Theorem 3.3 (Friedl–Kim, [5]). *Let N be a 3-manifold different from $S^1 \times D^2$ and $S^1 \times S^2$. Let $\phi \in H^1(N; \mathbb{Z})$ be such that (N, ϕ) fibers over S^1. Then for every representation $\alpha : \pi_1(N) \to \text{GL}_k(\mathbb{Z})$,*

(3.1) $\Delta_{N,\phi}^{\alpha}$ *is monic and* $\deg(\Delta_{N,\phi}^{\alpha}) = k\|\phi\|_T + \deg(\Delta_{N,\phi,0}) + \deg(\Delta_{N,\phi,2})$.

$\Delta_{N,\phi,0}$ *and* $\Delta_{N,\phi,2}$ *are determined by the Alexander modules* $H_0(N; \mathbb{Z}^k[F])$ *and* $H_2(N; \mathbb{Z}^k[F])$.

Theorem 3.3 leads one to believe that the collection of twisted Alexander polynomials gives stronger obstructions to fiberedness. This, in fact is confirmed by the following theorem of S. Friedl and S. Vidussi.

Theorem 3.4 (Friedl–Vidussi, [8])**.** *Let N be a compact connected orientable 3-manifold whose boundary (if any) is a union of tori. Let ϕ be non-trivial in $H^1(N;\mathbb{Z})$. Then if ϕ is not fibered, there is a representation $\alpha : \pi_1(N) \to \mathrm{GL}_k(\mathbb{Z})$ for which the conditions in (3.1) are not satisfied.*

For knots of genus 1 this result has been enhanced to show that there is some representation α for which the twisted Alexander polynomial vanishes, [7]. This result has been further generalized to any 3-manifold pair (N, ϕ), where $\phi \in H^1(N;\mathbb{Z})$, [9].

The proof of Theorem 3.4 is not constructive. We have found explicit representations for the knot K and one 2-component link containing K, for which (3.1) is violated.

Theorem 3.5. *For the representation $\alpha : \pi_1(K) \to S_5 \to \mathrm{GL}_5(\mathbb{Z})$ given in Theorem 5.2, $\Delta^\alpha_{K,\phi}$ is not monic.*

Section 5 is dedicated to the proof of this theorem. In order to find the explicit representation, we will first calculate the fundamental group of the exterior of K and then use the computer program Knottwister written by S. Friedl, [4].

4. Proof of Proposition 3.2

To prove the proposition we use various results from [3]. (More details can be found in [15].)

Proof. As we can see in the diagram in Figure 7 there is one arrowhead vertex, we will call this vertex v_1. Considering the conventions in [3], this knot has 8 vertices. So $n = 1$ and $k = 8$. First we will find l_{ij} for $i = 1$ and $1 < j \leq 8 : l_{12} = l_{13} = l_{14} = l_{15} = 0, l_{16} = 6, l_{17} = 3, l_{18} = 2$.
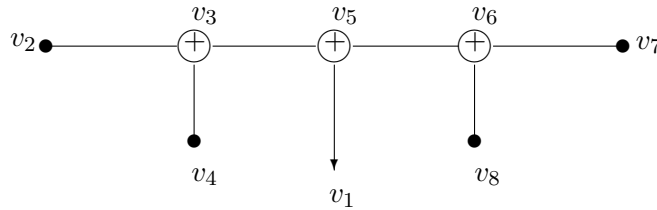
FIGURE 7. Vertices of the knot K.

For boundary vertices and arrowhead vertices, $\delta_i = 1$. For this particular knot each node has 3 arrowhead vertices and/or boundary vertices attached to it. So we have the following values for δ_i where $1 < i \leq 8$:

$$\delta_2 = \delta_4 = \delta_7 = \delta_8 = 1 \quad \text{and} \quad \delta_3 = \delta_5 = \delta_6 = 3.$$

Now we use Theorem 12.1 in [3] to compute the Alexander polynomial:

$$\Delta = (t-1)(t^0-1)^{-1}(t^0-1)(t^0-1)^{-1}(t^0-1)(t^6-1)(t^3-1)^{-1}(t^2-1)^{-1}.$$

Following the convention mentioned in [3] we cancel the terms $(t^0 - 1)$ and $(t^0 - 1)^{-1}$. Doing so we get

$$\Delta = \frac{(t-1)(t^6-1)}{(t^3-1)(t^2-1)} = \frac{t^3+1}{t+1} = t^2 - t + 1.$$

To find the genus of the knot, we calculate the Thurston norm of the class $\phi = (1) \in H^1(\Sigma \setminus (\nu(K), \mathbb{Z}) \simeq \mathbb{Z}$. By Theorem 11.1 in [3],

$$\|\phi\|_T = \|(1)\|_T = \sum_{j=2}^{8} (\delta_j - 2)|l_{1j}| = 1.$$

So this knot has genus equal to 1 as claimed since $\|\phi\|_T = 2g - 1$. It remains to show it is not fibered. To show this, we use Theorem 11.2 in [3], which assets that if some of the terms in the summation are zero, as in our case, then K is not fibered. $\qquad\square$

5. Proof of the main theorem

5.1. The fundamental group. To find the explicit representation α we first need to calculate the fundamental group of its exterior. For a knot in S^3 one can use the Wirtinger presentation of any blackboard projection of the knot to compute its fundamental group. Given that the knot K is contained in a homology sphere Σ, this method is not directly available, because we do not have access to any blackboard presentation. The route we will follow uses instead the Seifert–van Kampen theorem and the decomposition of the knot exterior into three components reflected by the splice diagram of K given in Figure 6.

From now on, for the sake of simplicity, when we talk about the fundamental group of the exterior of a link or a knot L, we will call it the fundamental group of L. We will follow this convention in our notation as well. For example we will denote the fundamental group of the exterior of the knot K as $\pi_1(K)$ instead of $\pi_1(\Sigma \setminus (\nu(K)))$.

Lemma 5.1. *The exterior of the knot K has the following fundamental group:*

$$\pi_1(K) = \langle x, y, s, t, b \mid xyx = yxy, stbst = bstb, xs = sx, xt = tx,$$
$$s = x^{-1}yx^2yx^{-3}, x = (st)^{-1}b(st)^2b(st)^{-3}\rangle.$$

Proof. First we will look at the three building blocks of the splice diagram. If we separate the middle node from the rest, we get the following splice diagram.

The three-component necklace that this splice diagram represents is the one in Figure 9. The arrowhead vertex with weight 0 is the main loop and the ones with weight 1 are the two hanging loops. We will call the main loop N_0, the loop hanging on the left N_1 and the one hanging on the right N_2. The following is its projection.
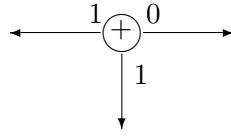
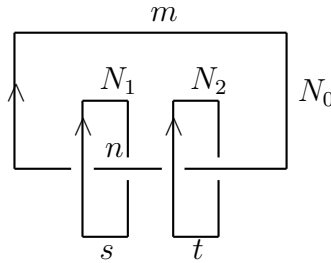FIGURE 8. Splice diagram of the 3-component necklace.
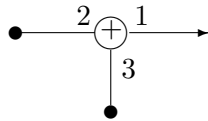
FIGURE 9. 3-component necklace.

FIGURE 10. Splice diagram of the trefoil on the left.

To avoid making the diagrams busy we put the names of the meridians on the arc and will not include the actual meridians in pictures. For this necklace, let $\mu(N_1) = s$, and $\mu(N_2) = t$ be the meridians of N_1 and N_2. Also since N_0 is made of two arcs m and n, we can choose as meridian of this component either m or n. Using the Wirtinger presentation for links we see that the (simplified) fundamental group of this link is

$$\pi_1(N) = \langle n, s, t \mid ns = sn, nt = tn \rangle.$$

The node on the left is the $(2, 3)$ cable on the unknot, as we can read from its splice diagram (Proposition 7.3 in [3]). Hence it represents the right-handed trefoil knot with the canonical orientation. We will call it T_L. The diagram in Figure 10 shows the node on the left separated from the rest.

Considering the projection of the right-handed trefoil shown in Figure 11, we can use the Wirtinger presentation for knots to calculate the fundamental group. Doing so will give us the following (simplified) fundamental group:

$$\pi_1(T_L) = \langle x, y \mid xyx = yxy \rangle.$$

For this knot, we will choose the meridian to be $\mu(T_L) = x$. Then by the details discussed in Remark 3.13 of [1], the longitude will be

$$\lambda(T_L) = zxyx^{-3} = x^{-1}yx^2yx^{-3}.$$
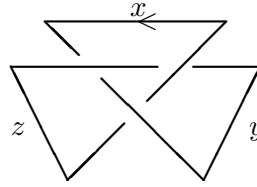
FIGURE 11. The trefoil knot on the left, T_L.

Splicing on the left we identify the longitude of T_L with the meridian of N_1 and the meridian of T_L with the longitude of N_1. Doing so will yield the relations $s = x^{-1}yx^2yx^{-3}$ and $x = n$ respectively.

Since the node on the right is another copy of the right-handed trefoil knot, we will call it T_R. This knot has the fundamental group

$$\pi_1(T_R) = \langle a, b \mid aba = bab \rangle.$$

If we choose its meridian to be a, then the longitude is $caba^{-3} = a^{-1}ba^2ba^{-3}$. The splicing on the right happens along the N_0 component of the necklace with meridian $\mu(N_0) = n$ and longitude $\lambda(N_0) = st$. Hence after splicing on the right we will have the relations $st = a$ and $n = a^{-1}ba^2ba^{-3}$.

Given the fundamental groups of each of the building blocks, along with the relations due to the splicing, the Seifert–van Kampen Theorem states that the fundamental group of the knot K is:

$$\pi_1(K) = \langle x, y, n, s, t, a, b \mid xyx = yxy, aba = bab, ns = sn, nt = tn, x = n,$$
$$s = x^{-1}yx^2yx^{-3}, st = a, n = a^{-1}ba^2ba^{-3} \rangle.$$

Simplifying this group, we get:

$$\pi_1(K) = \langle x, y, s, t, b \mid xyx = yxy, stbst = bstb, sx = xs, xt = tx,$$
$$s = x^{-1}yx^2yx^{-3}, x = (st)^{-1}b(st)^2b(st)^{-3} \rangle. \qquad \square$$

5.2. Finding an explicit representation α that shows K is not fibered. In this section, using the above presentation of $\pi_1(K)$ we find an explicit representation of $\pi_1(K) \to \mathrm{GL}_5(\mathbb{Z})$ for which the twisted Alexander polynomial is not monic. To do so, we use the computer program Knottwister.

Theorem 5.2. *For the representation $\alpha : \pi_1(K) \to S_5 \to \mathrm{GL}_5(\mathbb{Z})$ given by*

$$\alpha(a) = (15234), \alpha(b) = (13524), \alpha(n) = (14523), \alpha(s) = (12345)$$
$$\alpha(t) = (15234), F\alpha(x) = (14523), \alpha(y) = (34125),$$

$\Delta_{K,\phi}^\alpha$ *is not monic. (Here, one-line permutation notation is used.)*

Proof. Knottwister takes the fundamental group of K along with a cohomology class ϕ as the input data. For knots, ϕ can be chosen to be the abelianization map $\phi : \pi_1(K) \to \mathbb{Z}$. To identify explicitly the abelianization

map ϕ we add the commutator relations to the fundamental group found in Lemma 5.1. Then the map ϕ is given explicitly as:

$$\phi(x) = \phi(y) = \phi(s) = 0 \quad \text{and} \quad \phi(b) = \phi(t) = 1.$$

It can be easily checked that α is a homomorphism, meaning that it respects the relations of the fundamental group. The ordinary Alexander polynomial is $t^2 - t + 1$, which is identical to that of the trefoil knot. However, Knottwister gives the twisted Alexander polynomial $\Delta_{K,\phi}^{\alpha}$ with coefficients modulo p for different prime numbers. The twisted Alexander polynomial given by this particular representation α over $\mathbb{F}_5[t^{\pm 1}]$, $\mathbb{F}_7[t^{\pm 1}]$, $\mathbb{F}_{11}[t^{\pm}]$, $\mathbb{F}_{13}[t^{\pm 1}]$, $\mathbb{F}_{17}[t^{\pm 1}]$, $\mathbb{F}_{19}[t^{\pm 1}]$, $\mathbb{F}_{23}[t^{\pm 1}]$ and $\mathbb{F}_{29}[t^{\pm 1}]$ is equal to 0. Since the twisted Alexander polynomial associated with any one of these representations vanishes, it is not monic. $\qquad\square$

We can conclude from the previous theorem and Theorem 3.3 that the knot K is not fibered. Clearly, having the polynomial vanish over any of the fields above would be sufficient to show it is not monic. However, the fact that it vanishes over all these fields is a strong evidence that it is indeed 0. Since the genus of K is 1 as we saw in Proposition 3.2, this observation is consistent with the enhanced version of Theorem 3.4 appearing in [7].

6. 2-component links containing K

In this section we discuss three 2-component links that contain the knot K as a component. These links are the result of adding an arrowhead vertex to the three nodes of the splice diagram of K.

6.1. The link L_α. First we put the second arrowhead vertex on the last node. The following is the splice diagram of this 2-component link. From now on we call this link L_α. Since this link contains the knot K as a component, we can denote it as $L_\alpha = K_\alpha \bigcup K$, when K_α is the new component of the link.
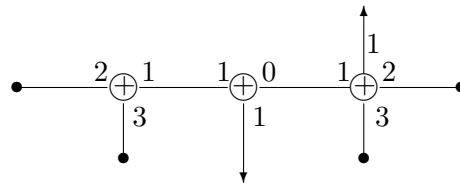

FIGURE 12. Splice diagram of the link L_α.

Using the theorems in [3], we can easily prove the following proposition. The proof is similar to that of 3.2 and hence is omitted.

Proposition 6.1. *The 2-component link L_α in Figure 12 has the following properties:*

1. *Its multivariable Alexander polynomial is:*

$$\Delta_{L_\alpha}(t_1, t_2) = (t_1^{12} - t_1^6 + 1)(t_1^4 t_2^4 + t_1^2 t_2^2 + 1)(t_1^3 t_2^3 + 1).$$

2. *For a general $\phi = (p, q)$, the Thurston norm is:*

$$\|\phi\|_T = 7|p + q| + 12|p|.$$

3. *If N is the exterior of the link, the pairs $(N, (0, 1))$ and $(N, (1, -1))$ are not fibered.*

Remark 6.2. From Proposition 6.1, we can observe that for the class $\phi = (1, -1)$ the single variable Alexander polynomial is

$$\Delta_{L_\alpha, \phi} = 6(t - 1)(t^{12} - t^6 + 1).$$

Even though $\deg(\Delta_{L_\alpha, \phi}) = \|\phi\|_T + 1$, the polynomial is not monic. So Theorem 3.1 states that this class is not fibered. However, for $\phi = (0, 1)$, we have the following ordinary Alexander polynomial:

$$\Delta_{L_\alpha, \phi} = (t - 1)(t^4 + t^2 + 1)(t^3 + 1) = (t^6 - 1)(t^2 - t + 1).$$

In this case the Alexander polynomial is monic and $\deg(\Delta_{L_\alpha, \phi}) = 8$. According to Theorem 3.1 this result is compatible with fiberedness, but we showed in Proposition 6.1 that it is not fibered.

6.2. Fundamental group of the exterior of L_α. In order to use twisted Alexander polynomials to discuss the fiberedness of L_α, we need to calculate the fundamental group of its exterior.

Lemma 6.3. *The fundamental group of the exterior of L_α is:*

$$\begin{aligned}
\pi_1(L_\alpha) = \langle c, d, e, f, g, h, i, j, k, l, o, p, q, r, u, v, w, a, x, y, n, s, t \mid & xyx = yxy, \\
& ns = sn, nt = tn, s = x^{-1}yx^2yx^{-3}, e = st, gd = cg, ve = dv, \\
& cf = ec, pg = fp, vh = gv, wi = hw, aj = ia, ek = je, rc = kr, \\
& eo = le, rp = or, gq = pg, vr = qv, cu = rc, pv = up, hw = vh, \\
& ia = wi, jl = aj \rangle.
\end{aligned}$$

Proof. Again, we need to decompose the link over its three nodes. For the node on the left and the one in the middle the calculations are identical to those of the knot K. For L_α, the node on the right before splicing is shown in Figure 13.
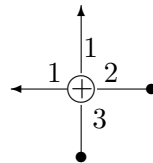

FIGURE 13. Splice diagram of the link D on the right.

The splice diagram in Figure 13 represents a 2-component link as it has two arrowhead vertices. It is the $(2, 3)$ cable on the right-handed trefoil (see

Proposition 7.3 in [3]). Hence each component is a copy of the right-handed trefoil knot, such that they have linking number 6. We call this 2-component link D. The blackboard projection of the link D is shown in Figure 14. We only need to discuss the splicing relations on the right, as the ones on the left are identical to those of K. As for K, splicing on the right happens along the main loop of the necklace, N_0. If we choose to splice along the outer trefoil of D, and choose its meridian to be $\mu(D) = e$, the longitude will be $\lambda(D) = cpvwxergve^{-3}$. Hence the splicing relations are:

$$n = cpvwxergve^{-3}, \text{and } e = st.$$

Therefore, considering the fundamental groups of the three building blocks of L_α and the relations that result from splicing, we see that the fundamental group of the exterior of L_α is:

$$\begin{aligned}
\pi_1(L_\alpha) = \langle c, d, e, f, g, h, i, j, k, l, o, p, q, r, u, v, w, a, x, y, n, s, t \mid xyx = yxy, \\
ns = sn, nt = tn, s = x^{-1}yx^2yx^{-3}, e = st, gd = cg, ve = dv, \\
cf = ec, pg = fp, vh = gv, wi = hw, aj = ia, ek = je, rc = kr, \\
eo = le, rp = or, gq = pg, vr = qv, cu = rc, pv = up, hw = vh, \\
ia = wi, jl = aj \rangle.
\end{aligned}$$

$\square$

6.3. Finding representations for $\pi_1(L_\alpha)$ in two cases.

Since for all knots the abelianization of their fundamental group is isomorphic to $\mathbb{Z}$, if one cohomology class is fibered, all are. However, it is possible that for the same link some cohomology classes are fibered and others are not. Now we will show that two different cohomology classes for L_α are not fibered.

In the following theorem we will find explicit representations for which $\Delta^\alpha_{N,\phi}$ is not monic, when N is the exterior of L_α and ϕ is one of the classes $(0, 1)$ or $(1, -1)$. Consequently by Theorem 3.3, the pair (N, ϕ) is not fibered for either ϕ.

Theorem 6.4. *Let N be the exterior of L_α. For $\phi_1 = (0, 1)$ and $\phi_2 = (1, -1)$, there are corresponding representations*

$$\alpha_1, \alpha_2 : \pi_1(N) \to S_5 \to \mathrm{GL}_5(\mathbb{Z})$$

such that $\Delta^{\alpha_1}_{N,\phi_1}$ and $\Delta^{\alpha_2}_{N,\phi_2}$ are not monic.

Proof. First we need to understand what ϕ_1 does as a map. We add all the commutator relations to the fundamental group in Lemma 6.3. This will result in the following relations:

$$c = d = e = f = g = h = i = j = k = t$$
$$o = l = p = q = r = u = v = w = a$$
$$s = 1, x = y = n = v^6.$$

As expected for a 2-component link, the abelianization of $\pi_1(L_\alpha)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. We can see from the splice diagram of this link that the
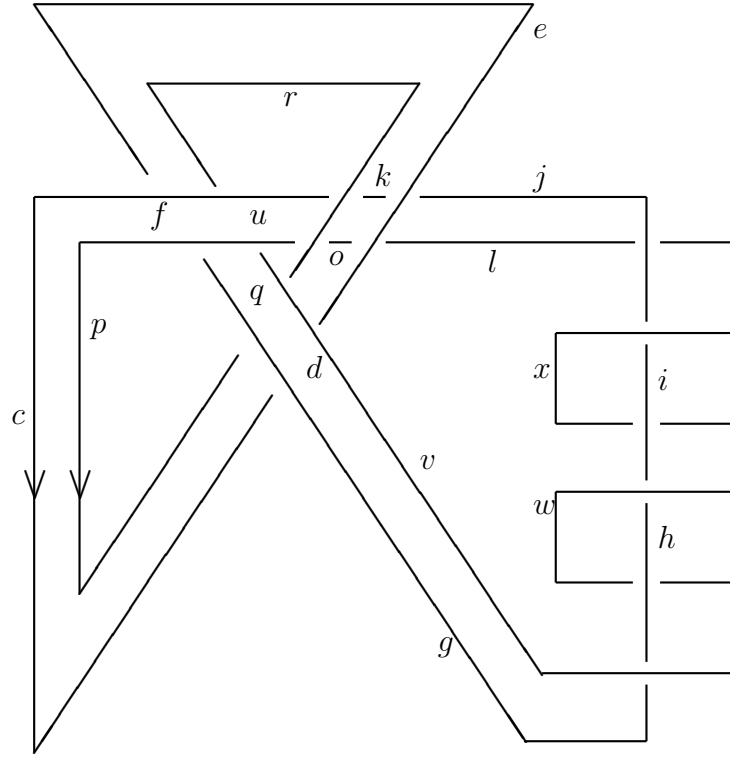
FIGURE 14. The link D.

two components that survive are one of the hanging loops of the necklace, N_2, and the trefoil knot inside the link D. These are the arrowhead vertices in the splice diagram. Hence ϕ_1 is the homomorphism that sends v to 0 and t to 1. Again, Knottwister takes the fundamental group of L_α from Lemma 6.3 along with the homomorphism ϕ_1 as an input. In multiplicative notation ϕ_1 is the following map:

$$\phi_1(c) = \phi_1(d) = \phi_1(e) = \phi_1(f) = \phi_1(g) = \phi_1(h) = \phi_1(i) = \phi_1(j) = \phi_1(k)$$
$$= \phi_1(t) = 1$$
$$\phi_1(a) = \phi_1(l) = \phi_1(o) = \phi_1(p) = \phi_1(q) = \phi_1(r) = \phi_1(u) = \phi_1(v) = \phi_1(w)$$
$$= \phi_1(s) = \phi_1(x) = \phi_1(y) = \phi_1(n) = 0.$$

Knottwister gives the following representation

$$\alpha_1 : \pi_1(N) \to S_5 \to \mathrm{GL}_5(\mathbb{Z}),$$

when the elements in S_5 are written in on-line permutation form:

$$
\begin{array}{llllllll}
a & \mapsto & (13245) & c & \mapsto & (23415) & d & \mapsto & (45321) & e & \mapsto & (24351) \\
f & \mapsto & (32514) & g & \mapsto & (13524) & h & \mapsto & (14532) & i & \mapsto & (15234) \\
j & \mapsto & (13524) & k & \mapsto & (31425) & l & \mapsto & (14325) & n & \mapsto & (45312) \\
o & \mapsto & (21345) & p & \mapsto & (21345) & q & \mapsto & (42315) & r & \mapsto & (21345) \\
s & \mapsto & (12345) & t & \mapsto & (24351) & u & \mapsto & (42315) & v & \mapsto & (14325) \\
w & \mapsto & (15342) & x & \mapsto & (45312) & y & \mapsto & (42513).
\end{array}
$$

For this twist the twisted Alexander polynomial $\Delta^{\alpha_1}_{N,\phi_1}$ vanishes over $\mathbb{F}_7[t^{\pm 1}]$, $\mathbb{F}_{11}[t^{\pm 1}]$, $\mathbb{F}_{13}[t^{\pm 1}]$, $\mathbb{F}_{17}[t^{\pm 1}]$, $\mathbb{F}_{19}[t^{\pm 1}]$, $\mathbb{F}_{23}[t^{\pm 1}]$ and $\mathbb{F}_{29}[t^{\pm 1}]$. Since the twisted Alexander polynomial vanishes over these finite fields, it cannot be monic.

Now, we do the same for $\phi_2 = (1, -1)$. Using multiplicative notation, ϕ_2 can be viewed as the map that acts as follows on the generators of $\pi_1(L_\alpha)$:

$$
\phi_2(c) = \phi_2(d) = \phi_2(e) = \phi_2(f) = \phi_2(g) = \phi_2(h) = \phi_2(i) = \phi_2(j) = \phi_2(k)
$$
$$
= \phi_2(t) = -1
$$
$$
\phi_2(a) = \phi_2(l) = \phi_2(o) = \phi_2(p) = \phi_2(q) = \phi_2(r) = \phi_2(u) = \phi_2(v) = \phi_2(w)
$$
$$
= 1
$$
$$
\phi_2(s) = 0
$$
$$
\phi_2(x) = \phi_2(y) = \phi_2(n) = 6.
$$

Given this information, Knottwister gives us the following representation α_2 (in one-line permutation form):

$$
\alpha_2 : \pi_1(M) \to S_5 \to \mathrm{GL}_5(\mathbb{Z})
$$

$$
\begin{array}{llllllll}
a & \mapsto & (24513) & c & \mapsto & (45132) & d & \mapsto & (35124) & e & \mapsto & (23154) \\
f & \mapsto & (21534) & g & \mapsto & (24513) & h & \mapsto & (45213) & i & \mapsto & (53214) \\
j & \mapsto & (54231) & k & \mapsto & (51432) & l & \mapsto & (45213) & n & \mapsto & (12345) \\
o & \mapsto & (41523) & p & \mapsto & (54231) & q & \mapsto & (25431) & r & \mapsto & (54123) \\
s & \mapsto & (12345) & t & \mapsto & (23154) & u & \mapsto & (54231) & v & \mapsto & (54231) \\
w & \mapsto & (54123) & x & \mapsto & (12345) & y & \mapsto & (12345).
\end{array}
$$

For this representation, the twisted Alexander polynomial, $\Delta^{\alpha_2}_{N,\phi_2}$, vanishes over $\mathbb{F}_5[t^{\pm 1}]$ and all of the fields previously mentioned for $\Delta^{\alpha_1}_{N,\phi_1}$. Hence neither twisted Alexander polynomial is monic as claimed. $\square$

Again, by Theorem 3.3, the pairs $(N, (0, 1))$ and $(N, (1, -1))$ are not fibered.

6.4. Links L_β and L_γ. In this section we briefly discuss the 2-component link that results from adding an arrowhead vertex to the middle node, L_β, and the one that results from adding it to the first node, L_γ. We use the theorems in [3] to conclude the following propositions.

Proposition 6.5. *For the link L_β the following are true.*

    1. *The Alexander polynomial vanishes.*

2. *The Thurston norm of the class $\phi = (p, q)$ on L_β is $|p + q|$.*
3. *No cohomology class ϕ on L_β is fibered.*

Proposition 6.6. *The link L_γ has the following properties:*

1. *The Alexander polynomial vanishes.*
2. *The Thurston norm for a class $\phi = (p, q)$ on this link is $7|p| + |6p + q|$.*
3. *No class ϕ on this link is fibered.*

We can use similar techniques to find the fundamental groups of these links. We have discussed the three "building blocks" of L_γ already. For the link L_β, notice that the middle node gives the splice diagram of a 4-component necklace. The following propositions give the fundamental groups of the exteriors of these two links.

Proposition 6.7. *The fundamental group of L_β is the following:*

$$\pi_1(L_\beta) = \langle x, y, a, b, s, r, t, n \mid aba = bab, xyx = yxy, nr = rn, nt = tn,$$
$$ns = sn, x = n, s = x^{-1}yx^2yx^{-3}, a = rst, n = a^{-1}ba^2ba^{-3}\rangle.$$

Proposition 6.8. *The fundamental group of L_γ is the following group:*

$$\pi_1(L_\gamma) = \langle a, b, n, s, t, c, d, e, f, g, h, i, j, k, o, l, p, q, r, u, v, w \mid$$
$$gd = cg, ve = dv, cf = ec, pg = fp, vh = gv, wi = hw, xj = ix,$$
$$ek = je, rc = kr, eo = le, rp = or, gq = pg, vr = qv, cu = rc,$$
$$pv = up, hw = vh, ix = wi, jl = xj, aba = bab, ns = sn,$$
$$nt = tn, a = st, n = a^{-1}ba^2ba^{-3}, e = n, s = cpvwxergve^{-3}\rangle.$$

7. A "secondary" polynomial, $\widetilde{\Delta}_1^\alpha(t)$

Since the ordinary Alexander polynomial is 0 for L_β and L_γ, we may not use Theorem 3.1 to get a useful bound for the Thurston norm. From now on we will only be concerned with the single-variable version of the twisted Alexander polynomial for simplicity. Also, since $\mathbb{F}[t^{\pm 1}]$ is a principal ideal domain, we replace $\mathbb{Z}[t^{\pm 1}]$ by $\mathbb{F}[t^{\pm 1}]$ in the definition of the Alexander module where $\mathbb{F} = \mathbb{F}_p$ is a field. As a result, we have the following isomorphism:

$$H_1(N, \mathbb{F}^k[t^{\pm 1}]) \cong \mathbb{F}[t^{\pm 1}]^r \oplus \bigoplus_{j=1}^m \mathbb{F}[t^{\pm 1}]/(p_j(t))$$

for $p_1(t), ..., p_m(t) \in \mathbb{F}[t^{\pm 1}]$. The type of polynomials we will examine are defined by:

$$\widetilde{\Delta}_{N,\phi}^\alpha := \prod_{j=1}^m p_j(t)$$

regardless of the rank r. Not much is known about these polynomials.

S. Friedl and T. Kim have proved the following theorem that relates these polynomials to the Thurston norm in [5].

Theorem 7.1 (Friedl–Kim, [5]). *Let $L = L_1 \cup L_2 \cup .... \cup L_m$ be a link with m components. Denote its meridian by $\mu_1, ..., \mu_m$. Let $\phi \in H^1(X(L); \mathbb{Z})$ be primitive and dual to a meridian μ_i, when $X(L)$ denotes the exterior of L. Hence $\phi(\mu_i) = 1$ for some i and $\phi(\mu_j) = 0$ for $j \neq i$. Then*

$$\|\phi\|_T \geq \frac{1}{k} \deg(\widetilde{\Delta}_1^\alpha(t)) - 1.$$

Here, k is the size of the representation α.

Theorem 7.1 will help us improve the bound of the Thurston norm for the class $(0, 1)$ for both L_β and L_γ. Recall from Section 2.2 that for L_β the Thurston norm of a general cohomology class (p, q) is $|p + q|$. So for this link, $\|(0, 1)\|_T = 1$. In this case Knottwister computes the $\widetilde{\Delta}_1^\alpha(t)$ to be $1 - t + t^2$ over $\mathbb{F}_{13}$ when α is trivial (so $k = 1$). Therefore, for the pair $(L_\beta, (0, 1))$ we get

$$\|(0, 1)\|_T \geq 2 - 1 = 1$$

which is a sharp bound.

Now we consider the same cohomology classes on L_γ. We know from our calculations in section 2.2 that for this link, $\|\phi\|_T = \|(p, q)\|_T = 7|p| + |6p + q|$. So for this link $\|(0, 1)\|_T = 1$. Knottwister yields the $\widetilde{\Delta}_1^\alpha(t) = 1 - t + t^2$ over $\mathbb{F}_{13}$ again, when α is trivial, which is again a sharp bound.

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