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Asymptotic translation length in the curve complex

Aaron D. Valdivia

Abstract. We prove the minimal pseudo-Anosov translation length in the curve complex behaves like $\frac{1}{\chi(S_{g,n})^2}$ for sequences where $g = rn$ for some $r \in \mathbb{Q}$. We also show that if the genus is fixed as $n \to \infty$ then the behavior is $\frac{1}{|\chi(S_{g,n})|}$. This extends results of Gadre and Tsai and answers a conjecture of theirs in the affirmative.

Contents

1. Introduction

Let $S_{g,n}$ denote a surface with genus g and n punctures. The set of homotopy classes of nontrivial simple closed curves on the surface is denoted by $\mathcal{S}$. The curve complex $\mathcal{C}(S_{g,n})$, written $\mathcal{C}$ when there is no ambiguity, is the simplicial complex with a 0-simplex for each element $c \in \mathcal{S}$ and an n-simplex for each $n-1$ tuple of disjoint elements of $\mathcal{S}$ where each 1-simplex is given length 1.

The mapping class group, $\mathrm{Mod}^+(S_{g,n})$, is the group of isotopy classes of orientation preserving homeomorphisms of the surface $S_{g,n}$. The mapping class group has a natural action on the set $\mathcal{S}$ which gives an action on the curve complex $\mathcal{C}$ as a group of isometries. The elements of $\mathrm{Mod}^+(S_{g,n})$ are either periodic, reducible, or pseudo-Anosov by the Nielson–Thurston

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classification. A pseudo-Anosov mapping class, ϕ, is a mapping class for which there exists a pair of measured singular foliations, $(\mathcal{F}^{\pm}, \mu^{\pm})$ such that $\phi(\mathcal{F}^{\pm}, \mu^{\pm}) = (\mathcal{F}^{\pm}, \lambda^{\pm 1}\mu^{\pm})$ where $\lambda > 1$ is called the dilatation. In this paper we will investigate the pseudo-Anosov elements of the mapping class group in terms of their action on the curve complex. We will be concerned with an invariant called the asymptotic translation length of pseudo-Anosov elements in particular the minimal asymptotic translation length. The asymptotic translation length of a pseudo-Anosov element ϕ is given by

$$l(\phi) = \liminf_{c \in \mathcal{S}} \lim_{n \to \infty} \frac{d(c, \phi^n(c))}{n}.$$

The minimal translation distance for a surface $S_{g,n}$ is

$$L(S_{g,n}) = \min_{\phi \in \mathrm{Mod}^+(S_{g,n})} (l(\phi)).$$

The dilatation λ of a pseudo-Anosov mapping class is an invariant which gives the translation distance in the Teichmuller space of the surface in question. The asymptotic translation distance is the analagous translation in the curve complex. Our results show that the asymptotic bounds differ in the same way for the action on the Teichmuller space and the curve complex.

We write A is asymptotic with B or the asymptotic behavior of A is B by $A \asymp B$, meaning that there is a constant $C > 1$ such that $\frac{B}{C} \leq A \leq BC$.

Our results extend the work of Gadre and Tsai in [GaT11] where they prove the asymptotic behavior of the minimal translation distance for closed surfaces is $\frac{1}{\chi(S_{g,0})^2}$ where $\chi(S_{g,n})$ is the Euler characteristic of the surface $S_{g,n}$. Our first theorem extends this result to the case $g = rn$.

Theorem 1.1. *If $g = rn$ such that $r \in \mathbb{Q}$ then*

$$L(S_{g,n}) \asymp \frac{1}{\chi(S_{g,n})^2}.$$

The proof relies on the lower bound on $L(S_{g,n})$ given in [GaT11] and an upper bound by examples constructed in [Val12]. Furthermore we prove Conjecture 6.2 of [GaT11]. The proof of the conjecture requires a sharper lower bound for $n >> g$.

Theorem 1.2. *For $n >> g$ we have $L(S_{g,n}) \geq \frac{1}{(9K_g+30)|\chi(S_{g,n})|-10n}$ where K_g is a constant depending only on g.*

This lower bound along with another set of examples allows us to give the asymptotic behavior for fixed g with n varying.

Theorem 1.3. *Fixing $g > 1$, as $n \to \infty$ the minimal translation length has behavior,*

$$L(S_{g,n}) \asymp \frac{1}{|\chi(S_{g,n})|}.$$

The lower bound, Theorem 1.2, mirrors Tsai's lower bound for dilatations of pseudo-Anosov mapping classes in [Tsa09]. We will use the nesting lemma of [MM99] to give a lower bound but this will require a positive measure on the train track associated to the pseudo-Anosov. The problem of finding a positive measure reduces to finding an iterate of a given pseudo-Anosov whose action on homology has trace larger than two. Lemma 2.2 gives us bounds for the iterate that has the required trace and the proof will require us to use symmetric and power symmetric polynomials.

The rest of the paper is organized as follows. In Section 2 we will discuss background material including train tracks, Lefschetz numbers, and symmetric polynomials. In Section 3 we will give a give the proof of our lower bound, Theorem 1.2, and in Section 4 we will provide examples for the upper bounds of Theorems 1.1 and 1.3 and will then finish the proofs of these theorems.

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2. Background

2.1. Train tracks. A *train track*, σ, is a one dimensional CW complex embedded in a surface $S_{g,n}$ with a some extra conditions attached to it. The vertices are called *switches* and the edges are called *branches*. Each branch is embedded smoothly in $S_{g,n}$ and there is a definable tangent direction at each switch for all branches meeting at that switch. Choosing a tangent direction at each switch we can then define *incoming* and *outgoing* branches at each switch. We refer the reader to [PH92] for a more detailed treatment of train tracks.

A *train route* is an immersed path on σ where at each switch the path passes from an outgoing branch to an incomming branch or vice versa. We say that a train track σ_1 is *carried* by a track σ_2, or $\sigma_1 < \sigma_2$ if there is a homotopy $f : S_{g,n} \times \mathbb{I} \to S_{g,n}$ of $S_{g,n}$ such that $f(\sigma_1, 0) = \sigma_1$, $f(\sigma_1, 1) \subset \sigma_2$ and each train route is taken to another train route. A smooth simple closed curve γ is said to be carried by σ if the homotopy $f : S_{g,n} \times \mathbb{I} \to S_{g,n}$ $f(\gamma, 0) = \gamma$ and $f(\gamma, 1)$ is a train route.

To each train track σ with n branches we can also associate the set of n-tuples, called *measures*, of nonnegative numbers w_i, called *weights*. Further we require that at each switch the sum of weights on the incoming branches is equal to the sum weights on the outgoing branches. We denote this set of n-tuples by $P(\sigma)$. If there is a measure on σ which is positive then σ is called recurrent. The set of all positive measures is $\mathrm{int}(P(\sigma))$.

A train track is called *large* if all the complementary regions are polygons or once punctured polygons. Every pseudo-Anosov ϕ has a large train track, σ, such that $\phi(\sigma) < \sigma$, this train track is called an *invariant* train track.

The carrying induces a transition matrix M that records the train route that each branch is taken to under the pseudo-Anosov followed by the carrying map. The Bestvina Handel Algorithm [BH95] gives one such train track, σ, and transition matrix, M, associated to the mapping class ϕ. Each of the branches of σ fall into one of two catagories, either *real* or *infinitesimal*. There are at most $9|\chi(S_{g,n})|$ real braches and at most $24|\chi(S_{g,n})| - 8n$ infinitesimal branches [GaT11] (cf [BH95]).

The infinitesimal branches are permuted by the mapping class while the real branches stretch over the rest of the train track. The transition matrix, M, has the form

$$M = \begin{pmatrix} A & B \\ 0 & M_{\mathcal{R}} \end{pmatrix}.$$

Here A is a permutation matrix corresponding to how the infinitesimal edges are permuted. On the other hand there is a positive integer m such that $M_{\mathcal{R}}^m$ is a positive matrix. The matrix, $M_{\mathcal{R}}$, is called the Markov partition matrix for the pseudo-Anosov, ϕ, and keeps track of the transistion between the real edges.

If a train track, σ, is large then a *diagonal extesion* of σ is a train track which contains σ as a subset and the branches not in σ meet switches at the cusps of complementary regions of σ. We denote the set of all diagonal extensions, which is finite, by $E(\sigma)$ and

$$P(E(\sigma)) = \cup_{\sigma_i \in E(\sigma)} P(\sigma_i).$$

By $\operatorname{int} P(E(\sigma))$ we mean all measures that are positive on the braches of σ.

In their investigation of the geometry of the curve complex Masur and Minsky [MM99] give a nesting behavior for the measures on train tracks of a surface. In [GaT11] Gadre and Tsai prove that the requirement that the train track be birecurrent can be replaced by only recurrent, giving the following theorem.

Theorem 2.1 ([GaT11] cf. [MM99])**.** *Given a large recurrent train track σ*

$$\mathcal{N}_1(\operatorname{int} P(E(\sigma))) \subset P(E(\sigma)).$$

Where $\mathcal{N}_1(X)$ is the 1-neighborhood of X.

2.2. Lefschetz numbers. We will now review the definition and basic properties of Lefschetz numbers. A more detailed discussion can be found in [BT82] and [GP74].

If X is a compact oriented manifold and $\phi : X \to X$ is a continuous map then the graph of ϕ is given as the set

$$graph(\phi) = \{(x, \phi(x)) | x \in X\} \subset X \times X.$$

The diagonal of $X \times X = \Delta$ and the algebraic intersection number $i(\Delta, graph(\phi))$ is the *global Lefschetz number* also denoted $Lf(\phi)$. The Lefschetz number is invariant up to homotopy and can be computed by the trace formula,

$$\sum_{i=0}^{n}(-1)^i \mathrm{Tr}(f_i^*),$$

where f_i^* is the induced map on the homology group $H_i(X, \mathbb{R})$.

Since the Lefschetz number of a mapping class is a homotopy invariant the Lefschetz number of $\phi : S_{g,n} \to S_{g,n}$ can be computed by forgetting the marked points.

Lemma 2.1 ([Tsa09])**.** *If a mapping class ϕ is the identity or multitwist(after forgetting the marked points) then*

$$Lf(\phi) = 2 - 2g.$$

We would like to have some similar statement about the Lefschetz numbers for pseudo-Anosov mapping classes as well. The next section will discuss symmetric polynomials and finish with Lemma 2.2 which begins to addresses this problem.

2.3. Symmetric polynomials. Here we will review the definitons of symmetric polynomials and power symmetric polynomials and develop a few key components of our proof for the new lower bound in Theorem 1.2. For a more complete discussion of symmetric ploynomials we refer the reader to [Mac95].

Definition 2.1. A partition is an n-tuple of nonnegative integers $\lambda = (\lambda_1, \dots, \lambda_n)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Each λ_i is called a part of λ. The length of λ is denoted $l(\lambda) = n$, and the weight is the sum of the components, $|\lambda| = \sum_i \lambda_i$.

The power symmetric polynomials, p_k, for N variables, $x_1, \dots, x_N$ are defined as

$$p_k(x) = \sum_{i=1}^{N} x_i^k.$$

Furthermore we can define for any partition λ the polynomial $p_\lambda = p_{\lambda_1} \dots p_{\lambda_n}$.

A symmetric polynomial in N variables is a polynomial that is invariant under the action of the symmetric group S_N on those N variables. These polynomials are generated by the elementary symmetric polynomials e_n, the sum of all products of n distinct variables and we additionally define $e_0 = 1$. For $n > 0$ we have

$$e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \dots x_{i_n},$$

and $e_n = 0$ for $n > N$.

The generating function, $E(t)$, for e_n gives a way to relate the elementry symmetric polynomials to polynomials of 1 variable.

$$E(t) = \sum_{n\geq 0} e_n t^n = \prod_{n\geq 1}(1 + x_i t)$$

Lastly we will need Newton's formula,

$$ne_n = \sum_{r=1}^{n}(-1)^{r-1}p_r e_{n-r},$$

in conjunction with another formula for e_n given by (2.14') in [Mac95].

$$e_n = \sum_{|\lambda|=n} \epsilon_\lambda z_\lambda^{-1} p_\lambda$$

Here $\epsilon_\lambda = (-1)^{|\lambda|-l(\lambda)}$ and $z_\lambda = \prod_{i\geq 1} i^{m_i} m_i!$ where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i.

We can obtain from a formula for p_{N+1} in terms of the polynomials $p_1 \ldots p_n$ using Newtons formula for $n = N + 1$ and the formula for e_n.

$$p_{N+1} = \sum_{r=1}^{N} \sum_{|\lambda|=N+1-r} (-1)^{2N+1-l(\lambda)} z_\lambda^{-1} p_\lambda p_r$$

Since we are concerned with the action a pseudo-Anosov has on homology we are considering reciprocal polynomials, $p(x) = x^n p(x^{-1})$, where n is the degree of $p(x)$.

Lemma 2.2 ([Ago13])**.** *Consider the monic, reciprocal, degree N polynomial $q(x) = \prod_{i=1}^{N}(x - \mu_i) \in \mathbb{R}[x]$ where $\mu_i \in \mathbb{R}$ are the roots of $q(x)$. If $p_k(\mu) \leq \delta$ where $\delta > 0$ for all $k \leq N(N + 1)$ then the polynomial $q(x)$ has bounded coefficients.*

Proof. We have the following cases. Case one is $p_1 \ldots p_N \leq \delta$ and $p_k < -R$ for some $R >> 0$ and $1 \leq k \leq N$. Case two is when $-R < p_1 \ldots p_N \leq \delta$.

We will address the case one first. We observe that there is a term $(-1)^{N+1} p_1^{N+1}$ in the expression for p_{N+1}.

Now assume that $p_1 \ldots p_N < 0$ and $p_1 < -R$ for some $R >> 0$. Then each term is positive since there are $l(\lambda) + 1$ power symmetric factors in each term and a factor of $(-1)^{l(\lambda)-1}$. Then we see that $p_{N+1} > \frac{R^{N+1}}{N!} > \delta$ for large enough R.

If some subset of $p_2 \ldots p_N \leq \delta$ are positive and $p_1 < -R$ then the negative terms come from terms with an odd number positive power symmetric factors. The group of terms with j positive power symmetric factors can be paired with a term which replaces each positive power symmetric factor p_{q_j} with $p_1^{q_j}$, this term is positive and dominates the negative terms if R is large enough.

If instead $p_k < -R$ for some $1 < k \leq N$ then from above we have that $p_{k(k+1)} > \delta$ and $k(k + 1) \leq N(N + 1)$.

Now we address case two. Each monic, reciprocal polynomial of degree N can be written with the elementary symmetric polynomials through the generating function as $\prod_1^N(\mu_i t - 1) = \sum_0^N e_i t^i$. In turn we can write each elementary symmetric polynomial as function of the power symmetric polynomials of whose values we have restricted to a compact set. Therefore the polynomial $q(x)$ has bounded coefficients.

□

3. The lower bound

The proof of the lower bound for curve complex translation length follows Tsai and Gadre's proof for their lower bound but includes elements of Tsai's proof of the lower bound on minimal dilitation in order to acheive a lower bound when $n >> g$.

Lemma 3.1 ([Tsa09]). *For any pseudo-Anosov mapping class $\phi \in Mod(S_{g,n})$ equipped with a Markov partition, if $Lf(\phi) < 0$ then there exists a rectangle R of the Markov partition such that R and $\phi(R)$ intersect(i.e. there is a positive entry on the diagonal of the Markov partition matrix).*

Lemma 3.2 ([Tsa09, Lemma 3.2]). *Given $\phi \in Mod(S_{g,n})$ let $\hat{\phi} \in Mod(S_{g,0})$ be the mapping class induced by forgetting the marked points. Then there exists a constant $0 < \alpha \leq F(g)$ such that $\hat{\phi}^\alpha$ satisfies one of the following.*

(1) $\hat{\phi}^\alpha$ *is pseudo-Anosov on a connected subsurface.*
(2) $\hat{\phi}^\alpha = \mathrm{id}$.
(3) $\hat{\phi}^\alpha$ *is a multitwist.*

Here, the upper bound on α, $F(g)$, is a function only of g.

Lemma 3.3. *In either of cases (1), (2), or (3) we have $Lf(\hat{\phi}^{\alpha q}) < 0$ and $\alpha q \leq K_g$ where K_g is a constant depending only on g.*

Proof. We address cases (2) and (3) first. If $\hat{\phi}^\alpha$ is the identity or a multitwist map then so is $\hat{\phi}^{\alpha q}$ and so $Lf(\hat{\phi}^{\alpha q}) < 0$ by Lemma 2.1.

In case (1) we have ϕ^α is a pseudo-Anosov mapping class on a connected subsurface S_{g_0,n_0} such that $2g_0 + n_0 < 2g$. Therefore the action of ϕ^α on $H_1(S_{g_0,n_0}, \mathbb{Z})$ is given by a matrix A of dimension at most $2g \times 2g$. Lemma 2.2 tells us there are finitely many monic reciprocal polynomials with roots $\mu = (\mu_1 \ldots \mu_{2g_0+n_0})$ such that $p_k(\mu) \leq 2$ for some $k < 2g(2g+1)$. Let that finite set of polynomials be C. If the characteristic polynomial of ϕ^n never leaves the set of polynomials, C, for all n then the roots are periodic and some iterate, m, of ϕ will have action on homology with all eigenvalues equal to 1. Therefore $2 - tr(\phi^m) = 2 - (2g_0 + n_0) < 0$.

Otherwise some iterate leaves the finite set, so there is a constant $|C| + 1$ depending only on g such that $\mathrm{Tr}(A^{|C|+1}) > 2$ and therefore $L(\phi^{\alpha(|C|+1)}) < 0$. □

This gives a positive diagonal entry in the Markov partition's transition matrix for the mapping class $\phi^{\alpha q}$ by Lemma 3.1. The number αq is only dependent on the genus of the surface in question and so there is a bound $\alpha q \leq K_g$.

Proposition 3.1 ([GaT11, Lemma 5.2 case 2])**.** *If $\sigma_0 \in E(\tau)$ and $\mu \in P(\sigma_0)$ then in at most $j \leq 6|\chi(S_{g,n})| - 2n$ iterates $\phi^j(\mu)$ is positive on some real branch of $\sigma_j \in E(\tau)$.*

Lemma 3.4. *There exists a positive integer $k \leq (9K_g + 30)|\chi(S_{g,n})| - 10n$ such that $\phi^k(\mu)$ is positive on every branch of τ where $\mu \in P(\sigma_0)$ and $\sigma_0 \in E(\tau)$.*

Proof. By Lemma 3.3 above we see that $\phi^{\alpha q}$ has a positive entry on the diagonal of the transition matrix for the Markov partition. By Proposition 2.4 of [Tsa09] we see that the Markov partition matrix for $\phi^{\alpha q r}$ is positive for some $r \leq 9|\chi(S_{g,n})|$ and by Proposition 3.1 we require at most $6|\chi(S_{g,n})| - 2n$ iterates to be positive on a real branch. Therefore in

$$(9\alpha q + 6)|\chi(S_{g,n})| - 2n \leq (9K_g + 6)|\chi(S_{g,n})| - 2n$$

iterations we will be positive on all real branches of τ. Since there are at most $24|\chi(S_{g,n})| - 8n$ infinitesimal branches we require an aditional $24|\chi(S_{g,n})| - 8n$ iterations to be positive on every branch. □

Proof of Theorem 1.2. let $\phi : S_{g,n} \to S_{g,n}$ be a pseudo-Anosov with invariant train track σ. Then by Lemma 3.4 there is an iterate

$$k \leq 9K_g|\chi(S_{g,n})| + 30|\chi(S_{g,n})| - 10n$$

such that given a measure μ on $\sigma_0 \in E(\sigma)$, $\phi^k(\mu) \in \text{int}(P(E(\sigma)))$ giving the inclusion $\phi(PE((\sigma))) \subset \text{int}(P(E(\sigma)))$. Then using the nesting lemma (Theorem 2.1) we get the sequence of inclusions

$$\begin{aligned} P(\sigma_{i+1}) &\subset \text{int}(P(E(\sigma_i))) \subset \mathcal{N}_1(\text{int}(P(E(\sigma_i)))) \subset \cdots \\ \text{int}(P(E(\sigma_2))) &\subset \mathcal{N}_1(P(E(\sigma_2))) \subset \text{int}(P(E(\sigma))) \subset \mathcal{N}_1(\text{int}(P(E(\sigma)))) \\ &\subset P(E(\sigma)). \end{aligned}$$

Then if we choose a curve $\gamma \in \mathcal{C}(S_{g,n}) \backslash P(E(\sigma))$ we have $\phi^{ik}(\gamma) \in P(E(\sigma_i))$ but not in $P(E(\sigma_{i+1}))$. We then have $d_{\mathcal{C}}(\gamma, \phi^{ik}(\gamma)) \geq i$ giving

$$l(\phi^k) = \liminf_{i\to\infty} \frac{d_{\mathcal{C}}(\gamma, \phi^{ik}(\gamma))}{i} \geq \liminf_{i\to\infty} \frac{i}{i} = 1.$$

Then using the formula $l(\phi^n) = nl(\phi)$ we get

$$l(\phi) \geq \frac{1}{k}.$$ □

This gives us a better lower bound for $n >> g$, the lower bound for Theorem 1.3.

4. Upper bounds by example and asymptotic behavior

In this section we will describe 2 types of examples for the upper bound. The first are the examples defined in [Val12] for rational rays defined by $g = rn$ for $r \in \mathbb{Q}$. The second are a series of examples giving upper bounds for rays with fixed g. The second set of examples are the ones we use to answer Conjecture 6.2 of [GaT11].

The first set of examples we will consider are called Penner sequences. These examples are generalize the examples Penner uses to give asymptotic conditions for the minimal dilatation on closed surfaces. Before defining a Penner sequence we will need to build some notation. First we pick a surface $S_{g,n,b}$ with $2g-2+n > 0$ where g is the genus, n is the number of fixed points, and b is the number of boundary components. Let Σ be homeomorphic to the surface $S_{g,n,b}$ and then consider Σ_i to be a homeomorphic copy of Σ for each integer i with homeomorphism

$$h_i : S_{g,n} \to \Sigma_i.$$

We then pick two disjoint homeomorphic subsets of the boundary components, a^+ and a^-, on Σ which gives homeomorphic copies on Σ_i, a_i^+ and a_i^-. Then we have orientation reversing homeomorphisms

$$\iota_i : a_i^+ \to a_{i+1}^-.$$

We can then construct a surface F_∞ which is the collection of the Σ_i with identifications made corresponding to the ι_j. There is homeomorphism of F_∞ to itself given by

$$\rho(x) = h_{i+1}(h_i^{-1}(x))$$

where $x \in \Sigma_i$. We can also define the surfaces $F_m = F_\infty/\rho^m$ and the quotient map $\pi_m : F_\infty \to F_m$. The map ρ then pushes forward to a map $\rho_m : F_m \to F_m$ which is periodic on F_m. After the construction F_m may have boundary components or punctures that are left invariant by the action of ρ_m this may be filled in by points or discs.

We then make a choice of 2 multicurves C and D on Σ_1 and multicurve $\gamma \subset \Sigma_1 \cup \Sigma_2$ such that

$$\{\rho^n(C \cup \gamma)\}_{n=-\infty}^{\infty}$$

is a multicurve and

$$\{\rho^n(C \cup \gamma \cup D)\}_{n=-\infty}^{\infty}$$

fills F_∞ and intersects efficeintly. Last given the semigroup $R(C^+, D^-)$ generated by positive Dehn twists about curves in C and negative Dehn twists about curves in D we pick a pseudo-Anosov word $\omega \in R(C^+, D^-)$. By pseudo-Anosov word we mean that it is pseudo-Anosov on Σ_1.

Definition 4.1. A Penner sequence is a sequence of mapping classes ϕ_m such that for some choice of $\omega \in R(C^+, D^-)$ and γ

$$\phi_m = \rho_m d_{\pi_m(\gamma)} \pi_m(\omega).$$

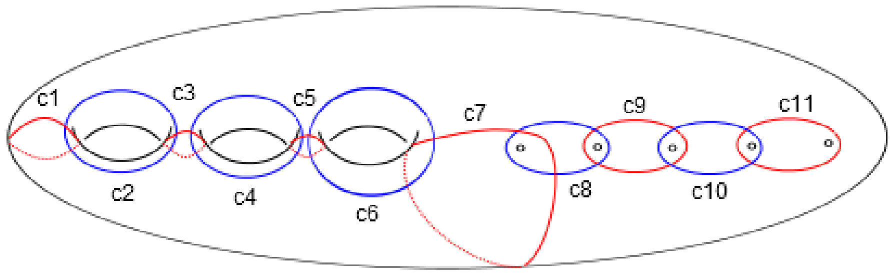

FIGURE 1. Curves c_i for the mapping class $\psi_{3,5}$.

Mapping classes of this form are all pseudo-Anosov [Val12] (cf. [Pen88]) and there is a sequence for any sequence of surfaces with $g = rn$ for some $r \in \mathbb{Q}$.

In [Val12] the train track transition matrix for these mapping classes is also given. Let M be the train track transition matrix for the pseudo-Anosov ϕ_m. Then each mapping class ϕ_m^m has the following form as a block matrix where the nth block corresponds to the measures induced by

$$\rho_m^n(\pi_m(C \cup D \cup \gamma)).$$

$$M^m = \begin{pmatrix} A & D & 0 & 0 & . & 0 & F \\ B & E & G & 0 & . & 0 & F^2 \\ 0 & F & H & G & . & 0 & 0 \\ . & 0 & F & H & . & 0 & . \\ . & . & 0 & F & . & 0 & . \\ . & . & . & 0 & . & G & . \\ . & . & . & . & . & H & G \\ C & 0 & 0 & 0 & . & F & H \end{pmatrix}$$

Where M^m is a $m \times m$ block matrix of $r \times r$ blocks. Given a standard basis vector e_i such that $r(n-1) < i \leq rn$ and $n \neq 1$ or m, $M^m e_i$ is the sum of basis vectors $\{e_j\}$ such that $r(n-2) < j \leq r(n+1)$. This means that if we pick a standard basis vector e_k such that if $m > 3$ is even $r(\lfloor \frac{m}{2} \rfloor - 1) < k \leq \lfloor \frac{m}{2} \rfloor$ then we have $M^{\lfloor \frac{m}{2} \rfloor - 1} e_k$ is zero in the last r entries. If $m > 3$ is odd then we pick e_k such that $r\lfloor \frac{m}{2} \rfloor < k \leq \lceil \frac{m}{2} \rceil$ and we also then get that $M^{\lfloor \frac{m}{2} \rfloor - 1} e_k$ is zero in the last r entries. Using this fact we see that

$$d_C(e_k, \phi_m^{m(\lfloor \frac{m}{2} \rfloor - 1)}(e_k)) \leq 2$$

Therefore we have $l(\phi_m^{m(\lfloor \frac{m}{2} \rfloor - 1)}) \leq 2$ and

$$l(\phi_m) \leq \frac{2}{m(\lfloor \frac{m}{2} \rfloor - 1)} \leq \frac{4}{m^2 - 2m}.$$

This gives an upper bound for all rational rays through the origin finishing the proof of Theorem 1.1.

The second set of examples is simpler. If you consider the curves in Figure 1 with numerical labeling $c_1 \dots c_{2g+n}$ from left to right we can easliy see that the mapping class, $\psi_{g,n}$ defined by a positive or negative Dehn twist about each curve c_i starting with c_1 and ending with c_{2g+n} where we perform a positive twist about each odd curve and a negative twist about each even curve then we see that $d_C(\psi^n_{g,n}(c_{2g+n}), c_{2g+n}) = 2$ giving $l(\psi_{g,n}) \leq \frac{2}{n}$. This gives the upper bound for Theorem 1.3 which completes the proof.

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Florida Southern College; 111 Lake Hollingsworth Drive; Lakeland, FL 33801-5698
aaron.david.valdivia@gmail.com

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