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Dense domains, symmetric operators and spectral triples

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Abstract. This article is about erroneous attempts to weaken the standard definition of unbounded Kasparov module (or spectral triple). This issue has been addressed previously, but here we present concrete counterexamples to claims in the literature that Fredholm modules can be obtained from these weaker variations of spectral triple. Our counterexamples are constructed using self-adjoint extensions of symmetric operators.

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1. Introduction

In this note we show, by counterexample, that weaker definitions of unbounded Kasparov module, and so spectral triple, may not yield KK or K-homology classes. In particular, we consider counterexamples arising from extensions of symmetric operators. These counterexamples address errors both in [4, pp 164-165] and subsequent errors in [3]. These issues have previously been raised by Hilsum in [8, Section 4], where it is shown that the

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definition in [4, pp 164-165] leads to contradictions in index theory.[1] It is the purpose of this note to present a set of elementary, concrete counterexamples which avoid the need to appeal to index theory, while shedding further light on the fine structure of unbounded K-homology.

The principal requirement of any definition of unbounded Kasparov module is that it defines a KK-class. This requirement constrains how far the definition can be extended. The work of Baaj–Julg, [1], provides sufficient conditions for this to be guaranteed. Different conditions apply to the definition of relative Fredholm modules, which can be obtained from symmetric operators, as shown by [2].

The definition of spectral triple that does give a well defined Fredholm module reads as follows (see [1, 7] and Section 2 of the present paper):

Definition 1.1. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by a Hilbert space $\mathcal{H}$, a $*$-subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ acting on $\mathcal{H}$, and a densely defined unbounded self-adjoint operator $\mathcal{D}$ such that:

(1) $a \cdot \operatorname{dom} \mathcal{D} \subset \operatorname{dom} \mathcal{D}$ for all $a \in \mathcal{A}$, so that $[\mathcal{D}, a]$ is densely defined. Moreover, $[\mathcal{D}, a]$ is bounded on $\operatorname{dom} \mathcal{D}$ and so extends to a bounded operator in $\mathcal{B}(\mathcal{H})$ for all $a \in \mathcal{A}$.

(2) $a(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{K}(\mathcal{H})$ for all $a \in \mathcal{A}$.

We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is even if in addition there is a $\mathbb{Z}_2$-grading such that $\mathcal{A}$ is even and $\mathcal{D}$ is odd. This means there is an operator γ such that $\gamma = \gamma^*$, $\gamma^2 = \mathrm{Id}_{\mathcal{H}}$, $\gamma a = a\gamma$ for all $a \in \mathcal{A}$ and $\mathcal{D}\gamma + \gamma\mathcal{D} = 0$. Otherwise we say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is odd.

It is asserted in [4, pp 164–165] that condition (1) of the definition may be weakened to:

($1'$) There is a subspace Y of $\operatorname{dom} \mathcal{D}$ such that Y is dense in $\mathcal{H}$, $a \cdot Y \subset \operatorname{dom} \mathcal{D}$, and $[\mathcal{D}, a]$ is bounded on Y.

Moreover [4, Proposition 17.11.3] asserts that condition ($1'$) ensures that $(\mathcal{A}, \mathcal{H}, \mathcal{D}(1 + \mathcal{D}^2)^{-1/2})$ is a Fredholm module. Our first and fourth counterexamples prove that this is false, by showing that if the algebra $\mathcal{A}$ does not preserve the domain of $\mathcal{D}$, then the commutators $[\mathcal{D}(1+\mathcal{D}^2)^{-1/2}, a]$ need not be compact, even when $(1 + \mathcal{D}^2)^{-1/2}$ is compact.

In [3, Theorems 1.2, 1.3, 6.2], the authors assert that a Fredholm module can be obtained from any self-adjoint extension of a symmetric operator $\mathcal{D}$ satisfying certain spectral-triple-like conditions, [3, Definition 1.1, Definition 6.3]. In particular, condition ($1'$) is invoked to handle commutators with algebra elements not preserving the domain of the operator $\mathcal{D}$. They further claim that the resulting K-homology class is independent of the particular self-adjoint extension. Both these claims are false, as our counterexamples show.

[1] Hilsum shows [8, Example 10.3] that some of the topological results in [3] relying on earlier errors are nonetheless valid.

To obtain counterexamples to [3] and [4], we also consider self-adjoint extensions of symmetric operators. To address both the cases of finite and infinite deficiency indices, we need two examples. It might be thought that by restricting to one or other of these two cases one could justify weakening the definition of spectral triple. Our counterexamples show that this is not the case.

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2. From spectral triple to Fredholm module

The idea of the (hard part of the) proof that a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ defines a Fredholm module, due originally to Baaj and Julg, [1], is to write, for $a \in \mathcal{A}$,

$$[\mathcal{D}(1+\mathcal{D}^2)^{-1/2}, a] = [\mathcal{D}, a](1+\mathcal{D}^2)^{-1/2} + \mathcal{D}[(1+\mathcal{D}^2)^{-1/2}, a]. \tag{2.1}$$

As we want to show that the left hand side is compact, the aim is to show that both terms on the right are compact. For the second term, one writes

$$(1+\mathcal{D}^2)^{-1/2} = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(1+\lambda+\mathcal{D}^2)^{-1}\, d\lambda,$$

then takes the commutator with a and multiplies by $\mathcal{D}$ yielding

$$\mathcal{D}[(1+\mathcal{D}^2)^{-1/2}, a] = \frac{1}{\pi}\mathcal{D}\int_0^\infty \lambda^{-1/2}[(1+\lambda+\mathcal{D}^2)^{-1}, a]\, d\lambda. \tag{2.2}$$

A careful analysis of the naive equality

$$\begin{aligned}\mathcal{D}[(1+\mathcal{D}^2)^{-1/2}, a] = \\ \frac{-1}{\pi}\int_0^\infty \lambda^{-1/2}\Big(&\mathcal{D}^2(1+\lambda+\mathcal{D}^2)^{-1}[\mathcal{D}, a](1+\lambda+\mathcal{D}^2)^{-1} \\ &+ \mathcal{D}(1+\lambda+\mathcal{D}^2)^{-1}[\mathcal{D}, a]\mathcal{D}(1+\lambda+\mathcal{D}^2)^{-1}\Big)\, d\lambda\end{aligned} \tag{2.3}$$

appears in [7, Lemmas 2.3 and 2.4]. There, and in the intervening remarks, it is proved that this equality is valid when a preserves the domain of $\mathcal{D}$. A similar analysis, employing the Cauchy integral formula, appears in [2, Proposition 1.1]. The remainder of the proof is to show that the right hand side of Equation (2.3) is a norm convergent integral with compact integrand, thus showing that the left hand side is compact.

The proof of [7, Lemma 2.3] makes it clear that the equality (2.3) requires careful domain considerations, and that (2.3) does not hold simply for algebraic reasons.

Thus we see that the Baaj–Julg approach to proving compactness of

$$[\mathcal{D}(1+\mathcal{D}^2)^{-1/2}, a]$$

using Equations (2.1) and (2.2) requires the assumption that a preserves the domain of $\mathcal{D}$. As a slight generalisation, it is asserted in [7] that the Baaj–Julg proof can be pushed through provided a maps a core for $\mathcal{D}$ into the domain of $\mathcal{D}$. We amplify on this in the next proposition, which generalises [8, Lemma 2.1].

Proposition 2.1. *Let* $\mathcal{D} : \operatorname{dom}\mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ *be a closed operator, let* $X \subset \operatorname{dom}\mathcal{D}$ *be a core for* $\mathcal{D}$, *and let* $a \in \mathcal{B}(\mathcal{H})$ *satisfy:*

(1) $a \cdot X \subset \operatorname{dom}\mathcal{D}$.
(2) $[\mathcal{D}, a] : X \to \mathcal{H}$ *is bounded on* X *and so extends to an operator in* $\mathcal{B}(\mathcal{H})$.

Then $a \cdot \operatorname{dom}\mathcal{D} \subset \operatorname{dom}\mathcal{D}$ *so that* $[\mathcal{D}, a] : \operatorname{dom}\mathcal{D} \to \mathcal{H}$ *is well-defined. If moreover there is an* $\mathcal{H}$*-norm dense subspace* $Y \subset \operatorname{dom}\mathcal{D}^*$ *such that* $a^* \cdot Y \subset \operatorname{dom}\mathcal{D}^*$, *then* $[\mathcal{D}, a] : \operatorname{dom}\mathcal{D} \to \mathcal{H}$ *extends to an operator in* $\mathcal{B}(\mathcal{H})$.

Proof. Since X is a core for $\mathcal{D}$, it is dense in $\operatorname{dom}\mathcal{D}$ in the graph norm. Let $x \in \operatorname{dom}\mathcal{D}$, and choose a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $x_n \to x$ in the graph norm, which is equivalent to $x_n \to x$ and $\mathcal{D}x_n \to \mathcal{D}x$ in the usual norm. Since $a \in \mathcal{B}(\mathcal{H})$, $ax_n \to ax$, and $\{\mathcal{D}ax_n\}_{n=1}^{\infty}$ is Cauchy in the usual norm since

$$\begin{aligned}\|\mathcal{D}ax_n - \mathcal{D}ax_m\| &= \|a\mathcal{D}x_n - a\mathcal{D}x_m + [\mathcal{D}, a]x_n - [\mathcal{D}, a]x_m\| \\ &\leq \|a\|\|\mathcal{D}x_n - \mathcal{D}x_m\| + \|[\mathcal{D}, a]\|\|x_n - x_m\| \to 0.\end{aligned}$$

Hence $\{ax_n\}_{n=1}^{\infty}$ is Cauchy in the graph norm, and since $\mathcal{D}$ is closed, there is some $y \in \operatorname{dom}\mathcal{D}$ such that $ax_n \to y$ in the graph norm. This implies that $ax_n \to y$ in the usual norm, and since $ax_n \to ax$ in the usual norm we see that $y = ax$. Hence $ax \in \operatorname{dom}\mathcal{D}$.

Now suppose that $Y \subset \operatorname{dom}\mathcal{D}^*$, $a^* \cdot Y \subset \operatorname{dom}\mathcal{D}^*$. To show that $[\mathcal{D}, a] : \operatorname{dom}\mathcal{D} \to \mathcal{H}$ is bounded, it is enough to show that $[\mathcal{D}, a]$ is closeable, since then $\overline{[\mathcal{D}, a]} \supset \overline{[\mathcal{D}, a]|_X}$ which is everywhere defined and bounded. Let $\xi \in \operatorname{dom}\mathcal{D}$ and $\eta \in Y$. Then

$$\begin{aligned}\langle [\mathcal{D}, a]\xi, \eta\rangle = \langle a\xi, \mathcal{D}^*\eta\rangle - \langle \mathcal{D}\xi, a^*\eta\rangle &= \langle \xi, a^*\mathcal{D}^*\eta\rangle - \langle \xi, \mathcal{D}^*a^*\eta\rangle \\ &= \langle \xi, -[\mathcal{D}^*, a^*]\eta\rangle .\end{aligned}$$

Hence $\operatorname{dom}([\mathcal{D}, a])^* \supset Y$. Since $[\mathcal{D}, a]$ is closeable if and only if $([\mathcal{D}, a])^*$ is densely defined, if Y is dense in $\mathcal{H}$ then $[\mathcal{D}, a]$ is closeable and thus extends to an operator in $\mathcal{B}(\mathcal{H})$. □

Corollary 2.2. *Condition* (1) *of Definition* 1.1 *is equivalent to:*

(i) *For all* $a \in \mathcal{A}$ *there exists a core* X *for* $\mathcal{D}$ *such that* $a \cdot X \subset \operatorname{dom}\mathcal{D}$, *and such that* $[\mathcal{D}, a] : X \to \mathcal{H}$ *is bounded on* X.

To simplify some later computations with bounded transforms

$$F = \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}$$

of unbounded self-adjoint operators, we include the following elementary lemma.

Lemma 2.3. *Let $\mathcal{D}$ be an unbounded self-adjoint operator on the Hilbert space $\mathcal{H}$, and suppose that $(1+\mathcal{D}^2)^{-1/2}$ is compact. Then with*

$$F = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}, \quad P_+ = \chi_{[0,\infty)}(\mathcal{D}), \quad P_- = 1 - P_+,$$

and $A \subset \mathcal{B}(\mathcal{H})$ a C^*-algebra, the operator $[F, a]$ is compact for all $a \in A$ if and only if $P_+ a P_-$ is compact for all $a \in A$.

Proof. The phase of $\mathcal{D}$ is

$$\mathrm{Ph}(\mathcal{D}) = P_+ - P_-,$$

and is a compact perturbation of $F = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$, so for $a \in A$, the commutator $[F, a]$ is compact if and only if $[\mathrm{Ph}(\mathcal{D}), a]$ is compact. Since $P_+ + P_- = 1$, we see that

$$[\mathrm{Ph}(\mathcal{D}), a] = (P_+ + P_-)[\mathrm{Ph}(\mathcal{D}), a](P_+ + P_-) = 2P_+ a P_- - 2P_- a P_+,$$

so that $[\mathrm{Ph}(\mathcal{D}), a]$ is compact if and only if $P_+ a P_- - P_- a P_+$ is compact. If $P_+ a P_- - P_- a P_+$ is compact, then so are

$$P_+(P_+ a P_- - P_- a P_+) = P_+ a P_- \quad \text{and} \quad -P_-(P_+ a P_- - P_- a P_+) = P_- a P_+,$$

so $[F, a]$ is compact if and only if $P_+ a P_-$ and $P_- a P_+$ are compact. Since $(P_+ a P_-)^* = P_- a^* P_+$, we have $[F, a]$ is compact for all $a \in A$ if and only if $P_+ a P_-$ is compact for all $a \in A$. □

3. The counterexamples

In this section we produce counterexamples to statements appearing in [3, Theorems 1.2, 1.3, 6.2]. The first and fourth of our counterexamples below also show that the definition of spectral triple using condition (1′) in place of condition (1) does not guarantee that we obtain a Fredholm module.

3.1. Finite deficiency indices: the unit interval. Initially, the authors of [3] confine their attention to symmetric operators with equal and finite deficiency indices, [3, Definition 1.1, Theorem 1.2]. We begin with our counterexample to their claims that a Fredholm module is obtained from any self-adjoint extension of such an operator (which must also satisfy spectral-triple-like conditions). Our extension will also satisfy the definition of spectral triple using condition (1′). In particular, [4, Proposition 17.11.3] and [3, Theorem 1.2] are false.

The basic properties of the following example are worked out in [11]. Let $\mathcal{H} = L^2([0,1])$ and let $AC([0,1])$ be the absolutely continuous functions. Set

$$\mathrm{dom}\,\mathcal{D} = \{f \in AC([0,1]) :\, f' \in L^2([0,1]),\ f(0) = f(1) = 0\}, \quad \mathcal{D} = \frac{1}{i}\frac{d}{dx},$$

so that $\mathcal{D}$ is a closed symmetric operator with adjoint

$$\operatorname{dom}\mathcal{D}^* = \{f \in AC([0,1]) : f' \in L^2([0,1])\}, \quad \mathcal{D}^* = \frac{1}{i}\frac{d}{dx}.$$

The deficiency indices of $\mathcal{D}$ are both 1. The operator $\mathcal{D}^*\mathcal{D}$ has normalised eigenvectors

$$\mathcal{D}^*\mathcal{D}\left(\sqrt{2}\sin(\pi n x)\right) = \pi^2 n^2 \sqrt{2}\sin(\pi n x), \quad n \in \mathbb{Z},$$

which are known to be complete for $L^2([0,1])$. Since $n^2\pi^2 \to \infty$ as $|n| \to \infty$, it follows that

$$(1 + \mathcal{D}^*\mathcal{D})^{-1/2} \in \mathcal{K}(\mathcal{H}).$$

It is clear that $C^\infty([0,1])$ preserves both $\operatorname{dom}\mathcal{D}$ and $\operatorname{dom}\mathcal{D}^*$, and that $[\mathcal{D}^*, a]$ is bounded for all $a \in C^\infty([0,1])$. In particular, the data

$$(C^\infty([0,1]), L^2([0,1]), \mathcal{D})$$

satisfy [3, Definition 1.1]. Let $\mathcal{D}_0$ be the self-adjoint extension defined by

$$\operatorname{dom}\mathcal{D}_0 = \{f \in AC([0,1]) : f' \in L^2([0,1]), f(0) = f(1)\}.$$

The eigenvectors of $\mathcal{D}_0$ are

$$\mathcal{D}_0 e^{2\pi i n x} = 2\pi n\, e^{2\pi i n x}, \quad n \in \mathbb{Z},$$

which by Fourier theory form a complete basis for $\mathcal{H}$. Hence the nonnegative spectral projection P_+ associated to $\mathcal{D}_0$ is the projection onto

$$\overline{\operatorname{span}}\{e^{2\pi i n x} : n \geq 0\}.$$

Since $\mathcal{D}_0$ has compact resolvent and is self-adjoint, any failure to obtain a Fredholm module (and so K-homology class) must arise from some function $f \in C([0,1])$ having noncompact commutator with $F := \mathcal{D}_0(1 + \mathcal{D}_0^2)^{-1/2}$. Indeed this is the case, and to see this let x be the identity function on $[0,1]$, which generates $C([0,1])$ along with the constant functions. Lemma 2.3 shows that to prove that $[F, x]$ is not compact, it suffices to prove that $P_+ x P_-$ is not compact. We observe that the noncompactness of $P_+ x P_-$ can also be deduced from [13, Theorem 1.(iv)]. This is described in detail in the appendix to [10], in which it is shown that the compactness of $P_+ x P_-$ and $P_- x P_+$ is equivalent to the vanishing mean oscillation (VMO) of x viewed as an L^∞ function on the circle [10, Theorem A.3]. Since x is not VMO, $P_+ x P_-$ is not compact. The calculations below have the virtue of making this explicit, and also indicate how we will deal with further counterexamples in higher dimensions where we do not have the VMO characterisation at our disposal.

Elementary Fourier theory shows that for $\sum_{n\in\mathbb{Z}} f_n e^{2\pi i n x} \in L^2([0,1])$

$$x \cdot \sum_{n\in\mathbb{Z}} f_n\, e^{2\pi i n x} = \sum_{n,l\in\mathbb{Z}} f_n \left(\frac{1 - \delta_{\ell n}}{2\pi i (n - \ell)} + \frac{1}{2}\delta_{\ell n}\right) e^{2\pi i \ell x}.$$

With P_+ the nonnegative spectral projection associated to D_0 and $P_- = 1 - P_+$, we find that

$$P_+ x P_- \cdot \sum_{n\in\mathbb{Z}} f_n\, e^{2\pi i n x} = \frac{-1}{2\pi i} \sum_{n\geq 1,\, \ell\geq 0} \frac{f_{-n}}{n+\ell}\, e^{2\pi i \ell x}.$$

Then for $m \in \mathbb{N}$ we define the sequence of vectors

$$\xi_m = \sum_{n=1}^{\infty} \frac{\sqrt{m}}{n+m}\, e^{-2\pi i n x}.$$

Lemma 3.1. *The sequence* $\{\xi_m\}_{m=1}^{\infty}$ *is bounded.*

Proof. We have

$$\|\xi_m\|^2 = m \sum_{n=1}^{\infty} \frac{1}{(m+n)^2} = m\psi^{(1)}(m+1),$$

where $\psi^{(k)}(x) = (d^{k+1}/dx^{k+1})(\log(\Gamma))(x)$ is the polygamma function of order k. As $m \to \infty$, $(m+1)\psi^{(1)}(m+1) \to 1$, so

$$\lim_{m\to\infty} \|\xi_m\|^2 = \lim_{m\to\infty} m \cdot \frac{1}{m+1} = 1. \qquad \square$$

With $\zeta_m = P_+ x P_- \xi_m$ and $\psi^{(0)}(x) = (d/dx)(\log(\Gamma))(x)$ the digamma function, we find that

$$\begin{aligned}
\|\zeta_m\|^2 &= \frac{m}{4\pi^2} \sum_{\ell=0}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{(n+m)(n+\ell)} \right)^2 \\
&\geq \frac{m}{4\pi^2} \sum_{\ell=0}^{m-1} \left(\sum_{n=1}^{\infty} \frac{1}{(n+m)(n+\ell)} \right)^2 \\
&= \frac{m}{4\pi^2} \sum_{\ell=0}^{m-1} \left(\frac{\psi^{(0)}(m+1) - \psi^{(0)}(\ell+1)}{m-\ell} \right)^2 \\
&= \frac{m}{4\pi^2} \sum_{\ell=0}^{m-1} \frac{1}{(m-\ell)^2} \left(\sum_{k=0}^{m-\ell-1} \frac{1}{\ell+k+1} \right)^2 \\
&\geq \frac{m}{4\pi^2} \sum_{\ell=0}^{m-1} \frac{1}{(m-\ell)^2} \left(\frac{m-\ell}{\ell+(m-\ell-1)+1} \right)^2 \\
&= \frac{m}{4\pi^2} \sum_{\ell=0}^{m-1} \frac{1}{m^2} = \frac{1}{4\pi^2}.
\end{aligned} \tag{3.1}$$

Lemma 3.2. *If* $\{\zeta_m\}_{m=1}^{\infty}$ *has a norm convergent subsequence* $\{\zeta_{m_j}\}_{j=1}^{\infty}$*, then* $\zeta_{m_j} \to 0$.

Proof. We show that $\lim_{m\to\infty}\langle\zeta_m \,|\, e^{2\pi ipx}\rangle = 0$ for all $p\in\mathbb{Z}$, which shows that if $\zeta_{m_j}\to\zeta$, then $\zeta=0$. We have

$$\langle\zeta_m \,|\, e^{2\pi ipx}\rangle = \begin{cases}\sum_{n=1}^{\infty}\frac{-\sqrt{m}}{2\pi i(m+n)(n+p)} & p\geq 0\\ 0 & \text{otherwise.}\end{cases}$$

Thus we can ignore the case $p<0$. Computing further gives

$$\langle\zeta_m \,|\, e^{2\pi ipx}\rangle = \begin{cases}\frac{-\sqrt{m}}{2\pi i}\left(\frac{\psi^{(0)}(m+1)-\psi^{(0)}(p+1)}{m-p}\right) & p\geq 0,\ p\neq m\\ \frac{-\sqrt{m}}{2\pi i}\psi^{(1)}(m+1) & p=m.\end{cases}$$

Since $\psi^{(0)}(m+1)\sim\log(m+1)$ as $m\to\infty$, we see that in all cases $\langle\zeta_m \,|\, e^{2\pi ipx}\rangle\to 0$ as $m\to\infty$. □

Corollary 3.3. *The sequence* $\{\zeta_m\}_{m=1}^{\infty}$ *has no norm convergent subsequences.*

Proof. If ζ_m had a convergent subsequence $\{\zeta_{m_j}\}_{j=1}^{\infty}$, then $\zeta_{m_j}\to 0$ by Lemma 3.2. But by Equation (3.1), $\|\zeta_{m_j}\|\not\to 0$, which is a contradiction. □

Corollary 3.4. *The operator* P_+xP_- *is not compact.*

Proof. By Lemma 3.1, $\{\xi_m\}_{m=1}^{\infty}$ is bounded, but $\{P_+xP_-\xi_m\}_{m=1}^{\infty}$ contains no convergent subsequence. Hence P_+xP_- is not compact. □

In summary we have shown the following:

Proposition 3.5. *The self-adjoint extension* $\mathcal{D}_0$ *of the closed symmetric operator* $\mathcal{D}$ *has compact resolvent, and for all* $a\in C^\infty([0,1])$*, the commutators* $[\mathcal{D}_0,a]$ *are defined on* $\operatorname{dom}\mathcal{D}$*, and are bounded on this dense subset. The bounded transform* $F:=\mathcal{D}_0(1+\mathcal{D}_0^2)^{-\frac{1}{2}}$ *has the property that the commutator* $[F,x]$ *is not a compact operator. Therefore* $(C([0,1]),L^2([0,1]),F)$ *does not define a Fredholm module.*

3.2. Infinite deficiency indices: the unit disc. The next three subsections produce counterexamples to three statements appearing in [3, Theorems 1.3, 6.2]. These theorems rely on both the finite deficiency index case, and the extended definition in [3, Definition 6.3], which allows for symmetric operators having infinite (and equal) deficiency indices. The third of the counterexamples below again shows that the definition of spectral triple using condition (1′) in place of condition (1) does not guarantee that we obtain a Fredholm module.

The counterexamples below will be described using a single basic example. For this we let $\mathbb{D}$ be the closed unit disc in $\mathbb{R}^2$, and take the Hilbert space $L^2(\mathbb{D},\mathbb{C}^2)$ with the measure

$$C(\mathbb{D})\ni f\mapsto\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 f(r,\theta)\,r\,dr\,d\theta.$$

Write $\mathring{\mathbb{D}} := \mathbb{D} \setminus \partial\mathbb{D}$ for the interior of $\mathbb{D}$. We will use the Dirac operator on $\mathring{\mathbb{D}}$ for our example. This is a densely defined symmetric operator on $L^2(\mathbb{D}, \mathbb{C}^2)$, which is given in local polar coordinates by

$$\mathcal{D}_c := \begin{pmatrix} 0 & e^{-i\theta}(-\partial_r + ir^{-1}\partial_\theta) \\ e^{i\theta}(\partial_r + ir^{-1}\partial_\theta) & 0 \end{pmatrix} : C_c^\infty(\mathring{\mathbb{D}}, \mathbb{C}^2) \to C_c^\infty(\mathring{\mathbb{D}}, \mathbb{C}^2).$$

Let $\mathcal{D}$ be the closure of $\mathcal{D}_c$, and observe that its domain is given by

$$\operatorname{dom}\mathcal{D} = \{f \in L^2(\mathbb{D}, \mathbb{C}^2) : \exists f_n \in C_c^\infty(\mathring{\mathbb{D}}, \mathbb{C}^2),\ f_n \to f,\ \mathcal{D}_c f_n \to g \in L^2(\mathbb{D}, \mathbb{C}^2)\}.$$

This is also referred to as the *minimal domain* (or minimal extension) of the Dirac operator.

The *maximal domain* (or maximal extension) of the Dirac operator is the domain of its adjoint $\mathcal{D}^*$. This extension can be described using distributions. The symmetric operator $\mathcal{D}_c$ induces a dual operator

$$\mathcal{D}_c^\dagger : C_c^\infty(\mathring{\mathbb{D}}, \mathbb{C}^2)^\dagger \to C_c^\infty(\mathring{\mathbb{D}}, \mathbb{C}^2)^\dagger,$$

on the space of distributions $C_c^\infty(\mathring{\mathbb{D}}, \mathbb{C}^2)^\dagger$, uniquely determined by the formula

$$\langle \mathcal{D}_c^\dagger \phi, f\rangle := \langle \phi, \mathcal{D}_c f\rangle, \qquad \phi \in C_c^\infty(\mathring{\mathbb{D}}, \mathbb{C}^2)^\dagger,\ f \in C_c^\infty(\mathring{\mathbb{D}}, \mathbb{C}^2).$$

A similar formula embeds $L^2(\mathbb{D}, \mathbb{C}^2)$ into the space of distributions. Using these identifications, the domain of $\mathcal{D}^*$ is given by

$$\operatorname{dom}\mathcal{D}^* = \{f \in L^2(\mathbb{D}, \mathbb{C}^2) : \mathcal{D}_c^\dagger f \in L^2(\mathbb{D}, \mathbb{C}^2)\}.$$

The domain of $\mathcal{D}^*$ coincides with the first Sobolev space $H^1(\mathbb{D}, \mathbb{C}^2)$, [6, Proposition 20.7]. With this characterisation it is straightforward to check that for any smooth bounded function a on the disc, $a : \operatorname{dom}\mathcal{D} \to \operatorname{dom}\mathcal{D}$ and $a : \operatorname{dom}\mathcal{D}^* \to \operatorname{dom}\mathcal{D}^*$, and $[\mathcal{D}^*, a]$ is bounded on both $\operatorname{dom}\mathcal{D}$ and $\operatorname{dom}\mathcal{D}^*$.

Lemma 3.6. *The operator* $(1 + \mathcal{D}^*\mathcal{D})^{-1/2}$ *is compact.*

Proof. The eigenvectors of $\mathcal{D}^*\mathcal{D}$ are

$$\left\{ \begin{pmatrix} J_n(r\alpha_{n,k})e^{in\theta} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ J_n(r\alpha_{n,k})e^{in\theta} \end{pmatrix} : n \in \mathbb{Z}, k = 1, 2, \dots \right\},$$

where $\alpha_{n,k}$ denotes the k-th positive root of the Bessel function J_n. These eigenvectors are complete for $L^2(\mathbb{D}, \mathbb{C}^2)$ by arguments similar to those in Section 3.5: namely $\{e^{in\theta} : n \in \mathbb{Z}\}$ is complete for S^1, and

$$\{J_n(r\alpha_{n,k}) : k \geq 1\}$$

is complete for $L^2([0,1], r\,dr)$ for all $n \in \mathbb{Z}$, [5].

We note that

$$\mathcal{D}^*\mathcal{D} \begin{pmatrix} J_n(r\alpha_{n,k})e^{in\theta} \\ 0 \end{pmatrix} = \alpha_{n,k}^2 \begin{pmatrix} J_n(r\alpha_{n,k})e^{in\theta} \\ 0 \end{pmatrix},$$

$$\mathcal{D}^*\mathcal{D}\begin{pmatrix}0\\ J_n(r\alpha_{n,k})e^{in\theta}\end{pmatrix} = \alpha_{n,k}^2\begin{pmatrix}0\\ J_n(r\alpha_{n,k})e^{in\theta}\end{pmatrix},$$

so the eigenvalues of $\mathcal{D}^*\mathcal{D}$ are $\{\alpha_{n,k}^2\}_{n=0,k=1}^{\infty}$. Each of these eigenvalues has multiplicity 4. Since $\alpha_{n,k}\to\infty$ as $n,k\to\infty$, it follows that $(1+\mathcal{D}^*\mathcal{D})^{-1/2}$ is compact. □

Since $(1+\mathcal{D}^*\mathcal{D})^{-1/2}$ is compact, the data $(C^\infty(\mathbb{D}), L^2(\mathbb{D},\mathbb{C}^2), \mathcal{D})$ satisfies the definition of symmetric unbounded Fredholm module in [3, Definition 6.3]. The closed symmetric operator $\mathcal{D}$ has infinite deficiency indices, since one may check directly that

$$\ker(\mathcal{D}^*\mp i)\supset\overline{\operatorname{span}}\left\{\begin{pmatrix}\pm ie^{in\theta}I_n(r)\\ e^{i(n+1)\theta}I_{n+1}(r)\end{pmatrix}, \begin{pmatrix}\pm ie^{-i(n+1)\theta}I_{n+1}(r)\\ e^{-in\theta}I_n(r)\end{pmatrix} : n\in\mathbb{N}\right\},$$

where the I_n are modified Bessel functions of the first kind. Thus $\mathcal{D}$ has self-adjoint extensions. It is a well known general fact that any closed symmetric extension $\mathcal{D}_{ext}$ of $\mathcal{D}$ must satisfy $\operatorname{dom}\mathcal{D}\subset\operatorname{dom}\mathcal{D}_{ext}\subset\operatorname{dom}\mathcal{D}^*$, [12].

3.3. An example with noncompact resolvent. The arguments in the proofs of [3, Theorems 1.2 and 6.2] purport to show that all self-adjoint extensions of an operator such as $\mathcal{D}$ above give rise to a Fredholm module (for $C^\infty(\mathbb{D})$ in this example). As in the finite deficiency index case, this fails, but it can fail in more ways.

The issue of (relatively) compact resolvent is addressed on [3, page 198]. The assertions about extensions used there are false[2], and we now show how to obtain an extension with noncompact resolvent. Write

$$\mathcal{D}=\begin{pmatrix}0 & \mathcal{D}_-\\ \mathcal{D}_+ & 0\end{pmatrix}.$$

Then define a self-adjoint extension of $\mathcal{D}$ by

$$\mathcal{D}_{ext}:=\begin{pmatrix}0 & \mathcal{D}_+^*\\ \mathcal{D}_+ & 0\end{pmatrix},$$

where $\mathcal{D}_+=(\mathcal{D}_+)_{\min}$ is the minimal extension, and $((\mathcal{D}_+)_{\min})^*=(\mathcal{D}_-)_{\max}$ is the maximal extension of $\mathcal{D}_-$, [6, Proposition 20.7]. As in Equation (3.2) in the next section, it is easily checked that

$$\ker(\mathcal{D}_-)_{\max}=\overline{\operatorname{span}}\{r^ne^{-in\theta} : n=0,1,\dots\},$$

thus $\mathcal{D}_{ext}$ has infinite dimensional kernel and so the resolvent is not compact. As the constant function $1\in C^\infty(\mathbb{D})$ acts as the identity on the Hilbert space, this shows that we fail to obtain a spectral triple for $C^\infty(\mathbb{D})$. Since this also means that

$$1-F_{\mathcal{D}_{ext}}^2=1-\mathcal{D}_{ext}^2(1+\mathcal{D}_{ext}^2)^{-1}=(1+\mathcal{D}_{ext}^2)^{-1}$$

is not compact, we do not obtain a Fredholm module for $C(\mathbb{D})$.

[2]There are no nontrivial self-adjoint extensions of a self-adjoint operator.

3.4. The dependence of K-homology classes on the choice of extension. Next we show that the claim in [3, Theorem 6.2] that the K-homology class of a symmetric operator with equal deficiency indices is independent of the self-adjoint extension is false. This example also shows that [3, Theorem 1.3] is false.

To define our self-adjoint extensions, we use boundary conditions. The trace theorem, [6, Theorem 11.4], gives the continuity of $f \mapsto f|_{\partial\mathbb{D}}$ as a map $\operatorname{dom}\mathcal{D}^* \to H^{1/2}(\partial\mathbb{D}, \mathbb{C}^2) \subset L^2(S^1, \mathbb{C}^2)$. Thus we can use the boundary values to specify domains of extensions of $\mathcal{D}$ inside $\operatorname{dom}\mathcal{D}^*$.

We consider APS-type extensions arising from the projections

$$P_N : L^2(S^1) \to L^2(S^1),$$

$N \in \mathbb{Z}$, defined by

$$P_N\left(\sum_{k\in\mathbb{Z}} c_k e^{ik\theta}\right) = \sum_{k\geq N} c_k e^{ik\theta}, \qquad \sum_{k\in\mathbb{Z}} c_k e^{ik\theta} \in L^2(S^1).$$

We use P_N to define self-adjoint extensions by setting

$$\operatorname{dom}\mathcal{D}_{P_N} := \left\{ \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \in \operatorname{dom}\mathcal{D}^* : \, P_N(\xi_1|_{\partial\mathbb{D}}) = 0, \; (1 - P_{N+1})(\xi_2|_{\partial\mathbb{D}}) = 0 \right\}$$
$$\mathcal{D}_{P_N}\xi := \mathcal{D}^*\xi, \qquad \text{for } \xi \in \operatorname{dom}\mathcal{D}_{P_N}.$$

The self-adjoint extensions above do define Fredholm modules, and so K-homology classes, for the algebra of functions constant on the boundary, since these functions preserve the domain, but each $\mathcal{D}_{P_N}$ defines a different class. This is easy, and not new: see [2, Appendix A], since the index (that is the pairing of the K-homology class with the constant function 1) is easily computed to be

$$\operatorname{Index}((\mathcal{D}_{P_N})_+) = N.$$

The reason is that

$$\ker(\mathcal{D}^*) = \overline{\operatorname{span}}\left\{ \begin{pmatrix} r^n e^{in\theta} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ r^n e^{-in\theta}\end{pmatrix} : n = 0, 1, 2, \ldots \right\}, \tag{3.2}$$

and so

$$\ker((\mathcal{D}_{P_N})_+) = \begin{cases} \{0\} & N \leq 0 \\ \overline{\operatorname{span}}\{r^n e^{in\theta} : 0 \leq n < N\} & N > 0, \end{cases}$$

whilst

$$\ker((\mathcal{D}_{P_N})_-) = \begin{cases} \{0\} & N > -1 \\ \overline{\operatorname{span}}\{r^n e^{-in\theta} : 0 \leq n \leq -N-1\} & N \leq -1. \end{cases}$$

3.5. Another noncompact commutator. In Section 3.1 we showed that the weakened definition of spectral triple does not suffice to guarantee that we obtain a Fredholm module. The example there also showed that [3, Theorem 1.2] is false. Now we show that the problem of noncompact commutators persists in the infinite deficiency index case. This shows that [3, Theorem 6.2] can not be repaired by requiring that the self-adjoint extensions employed have compact resolvents.

In this section, $\mathcal{D}_P$ shall denote the self-adjoint extension $\mathcal{D}_{P_0}$. As $\mathcal{D}_P$ is an extension of $\mathcal{D}$, we find that $[\mathcal{D}_P, a]$ is defined and bounded on the domain of $\mathcal{D}$, for all $a \in C^\infty(\mathbb{D})$. As in Section 3.1, we need to compute commutators with the phase of $\mathcal{D}_P$.

For $k \geq 1$, let $\alpha_{n,k}$ denote the k^{th} positive zero of the Bessel function J_n. Then the eigenvectors of $\mathcal{D}_P^2$ are

$$\left\{\begin{pmatrix} J_n(r\alpha_{n-1,k})e^{-in\theta} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ J_n(r\alpha_{n-1,k})e^{in\theta} \end{pmatrix}\right\}_{n,k=1}^{\infty}, \tag{3.3}$$
$$\left\{\begin{pmatrix} J_n(r\alpha_{n,k})e^{in\theta} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ J_n(r\alpha_{n,k})e^{-in\theta} \end{pmatrix}\right\}_{n=0,k=1}^{\infty}.$$

Lemma 3.7. *The eigenvectors* (3.3) *of* $\mathcal{D}_P^2$ *span* $L^2(\mathbb{D}, \mathbb{C}^2)$. *The corresponding set of eigenvalues is* $\{\alpha_{n,k}^2\}_{n=0,k=1}^{\infty}$, *and hence the resolvent of* $\mathcal{D}_P$ *is compact.*

Proof. With the measure $rdrd\theta$, we can take $\mathbb{D} = [0,1] \times S^1/\sim$, where $\sim$ is the identification $(0,z) \sim (0,1)$ for $z \in S^1$. It is well known that $\{e^{in\theta}\}_{n=-\infty}^{\infty}$ is complete for $L^2(S^1)$, so it is enough to show that:

(a) $\{r \mapsto J_n(r\alpha_{n-1,k})\}_{k=1}^{\infty}$ spans $L^2([0,1], r\,dr)$ for all $n = 1, 2, \ldots$.
(b) $\{r \mapsto J_n(r\alpha_{n,k})\}_{k=1}^{\infty}$ spans $L^2([0,1], r\,dr)$ for all $n = 0, 1, 2, \ldots$.

Statement (a) is true by [5, Theorem 6], and (b) is true by [5, Theorem 2].[3] Hence the eigenfunctions above are the entire set of eigenfunctions, and the set of eigenvalues is $\{\alpha_{n,k}^2\}_{n=0,k=1}^{\infty}$. Each of these eigenvalues has multiplicity 4. In particular $\mathcal{D}_P$ has no kernel, and since $\alpha_{n,k} \to \infty$ as $n, k \to \infty$, $(1 + \mathcal{D}_P^2)^{-1/2}$ is compact. □

To facilitate our computations we now describe an orthonormal eigenbasis for $\mathcal{D}_P$.

Proposition 3.8. *The vectors*

$$|1, n, k, \pm\rangle = \frac{1}{J_n(\alpha_{n-1,k})} \begin{pmatrix} J_n(r\alpha_{n-1,k})e^{-in\theta} \\ \pm J_{n-1}(r\alpha_{n-1,k})e^{-i(n-1)\theta} \end{pmatrix},$$
$$|2, n, k, \pm\rangle = \frac{1}{J_n(\alpha_{n-1,k})} \begin{pmatrix} J_{n-1}(r\alpha_{n-1,k})e^{i(n-1)\theta} \\ \mp J_n(r\alpha_{n-1,k})e^{in\theta} \end{pmatrix},$$

[3]In [5], Boas and Pollard take the usual measure on $[0,1]$ instead of $r\,dr$ and a slightly different set of functions, but it is easy to see that the two approaches are equivalent.

$n, k = 1, 2, \ldots$. form a normalised complete set of eigenvectors for $\mathcal{D}_P$. The corresponding set of eigenvalues is given by

$$\mathcal{D}_P \,|j,n,k,\pm\rangle = \pm\alpha_{n-1,k}\,|j,n,k,\pm\rangle\,.$$

Proof. From Lemma 3.7 it is straightforward to show that the eigenvectors and eigenvalues of $\mathcal{D}_P$ are

$$\mathcal{D}_P \begin{pmatrix} J_n(r\alpha_{n-1,k})e^{-in\theta} \\ \pm J_{n-1}(r\alpha_{n-1,k})e^{-i(n-1)\theta} \end{pmatrix} = \pm\alpha_{n-1,k} \begin{pmatrix} J_n(r\alpha_{n-1,k})e^{-in\theta} \\ \pm J_{n-1}(r\alpha_{n-1,k})e^{-i(n-1)\theta} \end{pmatrix},$$
$$\mathcal{D}_P \begin{pmatrix} J_{n-1}(r\alpha_{n-1,k})e^{i(n-1)\theta} \\ \mp J_n(r\alpha_{n-1,k})e^{in\theta} \end{pmatrix} = \pm\alpha_{n-1,k} \begin{pmatrix} J_{n-1}(r\alpha_{n-1,k})e^{i(n-1)\theta} \\ \mp J_n(r\alpha_{n-1,k})e^{in\theta} \end{pmatrix},$$

for $n, k = 1, 2, \ldots$. Note that these eigenvectors are complete for $L^2(\mathbb{D}, \mathbb{C}^2)$ since we can recover our spanning set (3.3) from linear combinations of these.

To normalise these eigenvectors, we use the following standard integrals which can be found in [14]:

$$\begin{aligned}
&\left\langle \begin{pmatrix} J_n(r\alpha_{n-1,k})e^{-in\theta} \\ \pm J_{n-1}(r\alpha_{n-1,k})e^{-i(n-1)\theta} \end{pmatrix}, \begin{pmatrix} J_n(r\alpha_{n-1,k})e^{-in\theta} \\ \pm J_{n-1}(r\alpha_{n-1,k})e^{-i(n-1)\theta} \end{pmatrix} \right\rangle \\
&= \frac{1}{2\pi}\int_0^{2\pi}\!\!\int_0^1 (J_n^2(r\alpha_{n-1,k}) + J_{n-1}^2(r\alpha_{n-1,k}))r\,dr\,d\theta \\
&= \frac{1}{2}\left(J_n^2(\alpha_{n-1,k}) + J_n^2(\alpha_{n-1,k})\right) = J_n^2(\alpha_{n-1,k}),
\end{aligned}$$

and similarly

$$\left\langle \begin{pmatrix} J_{n-1}(r\alpha_{n-1,k})e^{i(n-1)\theta} \\ \pm J_n(r\alpha_{n-1,k})e^{in\theta} \end{pmatrix}, \begin{pmatrix} J_{n-1}(r\alpha_{n-1,k})e^{i(n-1)\theta} \\ \pm J_n(r\alpha_{n-1,k})e^{in\theta} \end{pmatrix} \right\rangle = J_n^2(\alpha_{n-1,k}). \ \square$$

Our purpose is to find a function $a \in C(\mathbb{D})$ for which the commutator $[F, a]$ is not compact, where $F = \mathcal{D}_P(1+\mathcal{D}_P^2)^{-1/2}$ is the bounded transform. Let P_+ be the nonnegative spectral projection associated to $\mathcal{D}_P$, and let $P_- = 1-P_+$. By Lemma 2.3, we need only show that there is some $a \in C(\mathbb{D})$ for which the operator P_+aP_- is not compact.

In terms of the eigenbasis of $\mathcal{D}_P$, for any $a \in C(\mathbb{D})$ we can write

(3.4)
$$P_+aP_- = \sum_{i,j=1,2}\sum_{n,m,k,\ell=1}^{\infty} |i,n,k,+\rangle\,\langle i,n,k,+|\,a\,|j,m,\ell,-\rangle\,\langle j,m,\ell,-|\,.$$

Now we fix $a = re^{-i\theta}$. The function $re^{-i\theta}$ generates $C(\mathbb{D})$ (along with the constant function 1), and fails to preserve the domain of $\mathcal{D}_P$; for instance $re^{-i\theta}\cdot|2,1,k,\pm\rangle \notin \mathrm{dom}(\mathcal{D}_P)$. To show that $P_+re^{-i\theta}P_-$ is not compact, we will construct a bounded sequence of vectors ξ_n, with the property that $P_+re^{-i\theta}P_-$ maps ξ_n to a sequence with no convergent subsequences. In order to find the sequence ξ_n, we first derive an explicit formula for $P_+re^{-i\theta}P_-$.

Lemma 3.9. *The operator $P_+ re^{-i\theta} P_-$ can be expressed as*

$$\begin{aligned} &P_+ re^{-i\theta} P_- \\ &= \sum_{m,k,\ell=1}^{\infty} \frac{2\alpha_{m,k}}{(\alpha_{m,k}-\alpha_{m-1,\ell})(\alpha_{m,k}+\alpha_{m-1,\ell})^2} \left|1,m+1,k,+\right\rangle \left\langle 1,m,\ell,-\right| \\ &\quad + \sum_{n,k,\ell=1}^{\infty} \frac{2\alpha_{n,\ell}}{(\alpha_{n-1,k}-\alpha_{n,\ell})(\alpha_{n,\ell}+\alpha_{n-1,k})^2} \left|2,n,k,+\right\rangle \left\langle 2,n+1,\ell,-\right| \\ &\quad + \sum_{k\neq\ell} \frac{1}{\alpha_{0,k}+\alpha_{0,\ell}} \left|1,1,k,+\right\rangle \left\langle 2,1,\ell,-\right| + \sum_{k=1}^{\infty} \frac{1}{\alpha_{0,k}} \left|1,1,k,+\right\rangle \left\langle 2,1,k,-\right|. \end{aligned}$$

Proof. In view of Equation (3.4), we first compute the operators

$$\left\langle i,n,k,+\right| re^{-i\theta} \left|j,m,\ell,-\right\rangle$$

for $i,j=1,2$. Using integration by parts and standard recursion relations and identities for the Bessel functions and their derivatives, [14], we find:

(1) Case $i=j=1$:

$$\begin{aligned} &\left\langle 1,n,k,+\right| re^{-i\theta} \left|1,m,\ell,-\right\rangle \\ &= \frac{1}{2\pi J_n(\alpha_{n-1,k})J_m(\alpha_{m-1,\ell})} \int_0^{2\pi}\int_0^1 r^2 e^{i(n-m-1)\theta}\big(J_n(r\alpha_{n-1,k})J_m(r\alpha_{m-1,\ell}) \\ &\qquad - J_{n-1}(r\alpha_{n-1,k})J_{m-1}(r\alpha_{m-1,\ell})\big)\,dr\,d\theta \\ &= \frac{\delta_{n,m+1}}{J_{m+1}(\alpha_{m,k})J_m(\alpha_{m-1,\ell})} \int_0^1 r^2 J_{m+1}(r\alpha_{m,k})J_m(r\alpha_{m-1,\ell}) \\ &\qquad - r^2 J_m(r\alpha_{m,k})J_{m-1}(r\alpha_{m-1,\ell})\,dr \\ &= \frac{2\alpha_{m,k}\delta_{n,m+1}}{(\alpha_{m,k}-\alpha_{m-1,\ell})(\alpha_{m,k}+\alpha_{m-1,\ell})^2}; \end{aligned}$$

(2) Case $i=1, j=2$:

$$\begin{aligned} &\left\langle 1,n,k,+\right| re^{-i\theta} \left|2,m,\ell,-\right\rangle \\ &= \frac{1}{2\pi J_n(\alpha_{n-1,k})J_m(\alpha_{m-1,\ell})} \int_0^{2\pi}\int_0^1 r^2 e^{i(m+n-2)\theta}\big(J_n(r\alpha_{n-1,k})J_{m-1}(r\alpha_{m-1,\ell}) \\ &\qquad + J_{n-1}(r\alpha_{n-1,k})J_m(r\alpha_{m-1,\ell})\big)\,dr\,d\theta \\ &= \begin{cases} \frac{1}{J_1(\alpha_{0,k})J_1(\alpha_{0,\ell})} \int_0^1 r^2 J_1(r\alpha_{0,k})J_0(r\alpha_{0,\ell}) & \\ \qquad + r^2 J_0(r\alpha_{0,k})J_1(r\alpha_{0,\ell})\,dr & n=m=1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{\alpha_{0,k}+\alpha_{0,\ell}} & n=m=1 \text{ and } k\neq\ell \\ \frac{1}{\alpha_{0,k}} & n=m=1 \text{ and } k=\ell \\ 0 & \text{otherwise;} \end{cases} \end{aligned}$$

(3) Case $i = 2, j = 1$:

$$\begin{aligned}
&\langle 2, n, k, + |\, re^{-i\theta}\, |1, m, \ell, -\rangle \\
&= \frac{1}{2\pi J_n(\alpha_{n-1,k}) J_m(\alpha_{m-1,\ell})} \int_0^{2\pi}\!\!\int_0^1 r^2 e^{-i(n+m)\theta} \big(J_{n-1}(r\alpha_{n-1,k}) J_m(r\alpha_{m-1,\ell}) \\
&\qquad + J_n(r\alpha_{n-1,k}) J_{m-1}(r\alpha_{m-1,k\ell}) \big)\, dr\, d\theta \\
&= 0;
\end{aligned}$$

(4) Case $i = j = 2$:

$$\begin{aligned}
&\langle 2, n, k, + |\, re^{-i\theta}\, |2, m, \ell, -\rangle \\
&= \frac{1}{2\pi J_n(\alpha_{n-1,k}) J_m(\alpha_{m-1,\ell})} \\
&\quad \cdot \int_0^{2\pi}\!\!\int_0^1 r^2 e^{i(m-n-1)\theta} \big(J_{n-1}(r\alpha_{n-1,k}) J_{m-1}(r\alpha_{m-1,\ell}) \\
&\qquad - J_n(r\alpha_{n-1,k}) J_m(r\alpha_{m-1,\ell}) \big)\, dr\, d\theta \\
&= \frac{\delta_{m,n+1}}{J_n(\alpha_{n-1,k}) J_{n+1}(\alpha_{n,\ell})} \int_0^1 r^2 J_{n-1}(r\alpha_{n-1,k}) J_n(r\alpha_{n,\ell}) \\
&\qquad - r^2 J_n(r\alpha_{n-1,k}) J_{n+1}(r\alpha_{n,\ell})\, dr \\
&= \frac{2\alpha_{n,\ell}\delta_{m,n+1}}{(\alpha_{n-1,k} - \alpha_{n,\ell})(\alpha_{n,\ell} + \alpha_{n-1,k})^2}.
\end{aligned}$$

The desired equation is now obtained by using these cases in combination with (3.4). □

For convenience we write

$$|\ell, -\rangle := |2, 1, \ell, -\rangle\,, \qquad |k, +\rangle := |1, 1, k, +\rangle\,,$$

and define the sequence

$$\xi_n := \sum_{\ell=1}^{\infty} \frac{\sqrt{n}}{n+\ell}\, |\ell, -\rangle\,, \quad n = 1, 2, \ldots.$$

Lemma 3.10. *The sequence $\{\xi_n\}_{n=1}^{\infty}$ is bounded.*

Proof. As in Lemma 3.1 we have

$$\|\xi_n\|^2 = n \sum_{\ell=1}^{\infty} \frac{1}{(n+\ell)^2} = n\psi^{(1)}(n+1),$$

where $\psi^{(m)}(x) = (d^{m+1}/dx^{m+1})(\log(\Gamma))(x)$ is the polygamma function of order m. As $n \to \infty$, $(n+1)\psi^{(1)}(n+1) \to 1$, so $\|\xi_n\|^2 \to 1$. □

To simplify the computations, we subtract the operator

$$K := \sum_{k=1}^{\infty} \frac{1}{2\alpha_{0,k}}\, |1, 1, k, +\rangle \langle 2, 1, k, -|$$

from $P_+re^{-i\theta}P_-$, since K is obviously compact, and define

$$\zeta_n := (P_+re^{-i\theta}P_- - K)\xi_n.$$

Our purpose is to show that ζ_n has no convergent subsequence. To this end we investigate its limiting behaviour.

Lemma 3.11.

$$\liminf_{n\to\infty} \|\zeta_n\| \geq \frac{1}{2\pi}.$$

Proof. We have

$$\zeta_n = \sum_{k,\ell=1}^{\infty} \frac{\sqrt{n}}{(n+\ell)(\alpha_{0,k}+\alpha_{0,\ell})} |k,+\rangle .$$

It is proved in [9, Lemma 1] that for all $\ell \geq 1$,

$$\pi(\ell - 1/4) < \alpha_{0,\ell} < \pi(\ell - 1/8), \tag{3.5}$$

yielding the inequality

$$\frac{\sqrt{n}}{(n+\ell)(\alpha_{0,k}+\alpha_{0,\ell})} > \frac{\sqrt{n}}{(n+\ell)(\alpha_{0,k}+\pi(\ell-1/8))}.$$

This allows us to estimate the coefficients of ζ_n via

$$\begin{aligned}
&\sum_{\ell=1}^{\infty} \frac{\sqrt{n}}{(n+\ell)(\alpha_{0,k}+\alpha_{0,\ell})}\\
&\geq \sum_{\ell=1}^{\infty} \frac{\sqrt{n}}{(n+\ell)(\alpha_{0,k}+\pi(\ell-1/8))}\\
&= \frac{\sqrt{n}}{\pi(n-\alpha_{0,k}/\pi+1/8)} \sum_{\ell=1}^{\infty}\left(\frac{1}{\ell+\alpha_{0,k}/\pi-1/8} - \frac{1}{n+\ell}\right)\\
&= \frac{\sqrt{n}}{\pi(n-\alpha_{0,k}/\pi+1/8)} \sum_{\ell=1}^{\infty}\left(\frac{1}{\ell+\alpha_{0,k}/\pi-1/8} - \frac{1}{\ell} + \frac{1}{\ell} - \frac{1}{\ell+n}\right)\\
&= \frac{\sqrt{n}}{\pi(n-\alpha_{0,k}/\pi+1/8)}\left(-\psi^{(0)}(\alpha_{0,k}/\pi+7/8) + \psi^{(0)}(n+1)\right)\\
&= \frac{\sqrt{n}}{\pi}\frac{\psi^{(0)}(n+1) - \psi^{(0)}(\alpha_{0,k}/\pi+7/8)}{n-\alpha_{0,k}/\pi+1/8}
\end{aligned}$$

which in turn allows us to bound $\|\zeta_n\|$ by

$$\begin{aligned}
\|\zeta_n\|^2 &\geq \frac{n}{\pi^2}\sum_{k=1}^{\infty}\left(\frac{\psi^{(0)}(n+1) - \psi^{(0)}(\alpha_{0,k}/\pi+7/8)}{n-\alpha_{0,k}/\pi+1/8}\right)^2 \\
&\geq \frac{n}{\pi^2}\sum_{k=1}^{n}\left(\frac{\psi^{(0)}(n+1) - \psi^{(0)}(\alpha_{0,k}/\pi+7/8)}{n-\alpha_{0,k}/\pi+1/8}\right)^2 .
\end{aligned} \tag{3.6}$$

Now, $\alpha_{0,k}/\pi \in (k-1/4, k-1/8)$ by Equation (3.5), and $\psi^{(0)}$ increases monotonically on $(0,\infty)$, so for $k \le n$ we have

$$0 \le \psi^{(0)}(n+1) - \psi^{(0)}(k+1) < \psi^{(0)}(n+1) - \psi^{(0)}(k+3/4) \\ < \psi^{(0)}(n+1) - \psi^{(0)}(\alpha_{0,k}/\pi + 7/8).$$

For $k \le n$,

$$\psi^{(0)}(n+1) - \psi^{(0)}(k+1) = \sum_{j=0}^{n-k-1} \frac{1}{k+j+1},$$

and so

$$0 \le \sum_{j=0}^{n-k-1} \frac{1}{k+j+1} < \psi^{(0)}(n+1) - \psi^{(0)}(\alpha_{0,k}/\pi + 7/8).$$

For $k \le n$ we also have

$$0 < n - \alpha_{0,k}/\pi + 1/8 < n - k + 3/8,$$

allowing us to obtain the estimate

$$\begin{aligned} (3.7) \qquad & \sum_{k=1}^{n} \left(\frac{\psi^{(0)}(n+1) - \psi^{(0)}(\alpha_{0,k}/\pi + 7/8)}{n - \alpha_{0,k}/\pi + 1/8} \right)^2 \\ & > \sum_{k=1}^{n} \frac{1}{(n-k+3/8)^2} \left(\sum_{j=0}^{n-k-1} \frac{1}{k+j+1} \right)^2 \\ & \ge \sum_{k=1}^{n} \frac{1}{(n-k+3/8)^2} \cdot \left(\frac{n-k}{k+(n-k-1)+1} \right)^2 \\ & = \sum_{k=1}^{n} \frac{(n-k)^2}{(n-k+3/8)^2} \frac{1}{n^2} \\ & \ge \sum_{k=1}^{n} \frac{(n-k)^2}{(n-k+1)^2} \frac{1}{n^2} \\ & = \frac{1}{n^2} \sum_{j=2}^{n} \frac{(j-1)^2}{j^2} \ge \frac{1}{n^2} \frac{n-1}{4}. \end{aligned}$$

Thus combining Equations (3.6) and (3.7) yields

$$(3.8) \qquad \|\zeta_n\|^2 \ge \frac{n}{\pi^2} \sum_{k=1}^{n} \left(\frac{\psi^{(0)}(n+1) - \psi^{(0)}(\alpha_{0,k}/\pi + 7/8)}{n - \alpha_{0,k}/\pi + 1/8} \right)^2 \ge \frac{n-1}{4n\pi^2}.$$

As $n \to \infty$,

$$\liminf_{n\to\infty} \|\zeta_n\|^2 \ge \frac{1}{4\pi^2}$$

□

Next we analyse the possible limits of convergent subsequences of ζ_n, should they exist.

Lemma 3.12. *If $\{\zeta_n\}_{n=1}^\infty$ has a norm convergent subsequence $\{\zeta_{n_j}\}_{j=1}^\infty$, then $\zeta_{n_j} \to 0$.*

Proof. We show that $\lim_{n\to\infty} \langle \zeta_n \,|k,+\rangle = 0$ for all $k = 1, 2, \ldots$, which shows that if $\zeta_{n_j} \to \zeta$, then $\zeta = 0$. We have

$$\langle \zeta_n \,|\, k,+\rangle = \sum_{\ell=1}^\infty \frac{\sqrt{n}}{(n+\ell)(\alpha_{0,k}+\alpha_{0,\ell})}$$

Since $\alpha_{0,k} \in (\pi k - \pi/4, \pi k - \pi/8)$ by Equation (3.5), we have

$$\frac{1}{\alpha_{0,k}+\alpha_{0,\ell}} < \frac{1}{\pi(k+\ell-1/2)}.$$

Hence

$$\begin{aligned} 0 \le \langle \zeta_n | k,+\rangle &\le \frac{\sqrt{n}}{\pi} \sum_{\ell=1}^\infty \frac{1}{(n+\ell)(k+\ell-1/2)} \\ &= \frac{\sqrt{n}}{\pi(n-k+1/2)} \sum_{\ell=1}^\infty \left(\frac{1}{k+\ell-1/2} - \frac{1}{n+\ell}\right) \\ &= \frac{\sqrt{n}}{\pi(n-k+1/2)} \left(\psi^{(0)}(n+1) - \psi^{(0)}(k+1/2)\right). \end{aligned}$$

As $n \to \infty$, $\psi^{(0)}(n) \sim \ln(n)$, showing that

$$\begin{aligned} &\lim_{n\to\infty} \frac{\sqrt{n}}{\pi(n-k+1/2)} \left(\psi^{(0)}(n+1) - \psi^{(0)}(k+1/2)\right) \\ &= \lim_{n\to\infty} \left(\frac{\sqrt{n}(\ln(n+1) - \psi^{(0)}(k+1/2))}{\pi(n-k+1/2)}\right) \\ &= 0. \end{aligned}$$

Hence $\lim_{n\to\infty} \langle \zeta_n \,|\, k,+\rangle = 0$. □

Corollary 3.13. *The sequence $\{\zeta_n\}_{n=1}^\infty$ has no norm convergent subsequences.*

Proof. If ζ_n had a convergent subsequence $\{\zeta_{n_j}\}_{j=1}^\infty$, then $\zeta_{n_j} \to 0$ by Lemma 3.12. But by Lemma 3.11, $\|\zeta_{n_j}\| \not\to 0$, which is a contradiction. □

Corollary 3.14. *The operator $P_+ re^{-i\theta} P_-$ is not compact.*

Proof. By Lemma 3.10, $\{\xi_n\}_{n=1}^\infty$ is bounded, but $\{(P_+ re^{-i\theta} P_- - K)\xi_n\}_{n=1}^\infty$ contains no convergent subsequence. As $P_+ re^{-i\theta} P_-$ and $P_+ re^{-i\theta} P_- - K$ differ by a compact operator, $P_+ re^{-i\theta} P_-$ is not compact. □

In summary we have shown the following:

Proposition 3.15. *The self-adjoint extension $\mathcal{D}_P$ of the closed symmetric operator $\mathcal{D}$ has compact resolvent, and for all $a \in C^\infty(\mathbb{D})$, the commutators $[\mathcal{D}_P, a]$ are defined on $\operatorname{dom}\mathcal{D}$, and are bounded on this dense subset. The bounded transform $F := \mathcal{D}_P(1+\mathcal{D}_P^2)^{-\frac{1}{2}}$ has the property that the commutator $[F, re^{-i\theta}]$ is not a compact operator. Therefore $(C(\mathbb{D}), L^2(\mathbb{D}, \mathbb{C}^2), F)$ does not define a Fredholm module.*

References

[1] Baaj, Saad; Julg, Pierre. Théorie bivariante de Kasparov et opérateurs non bornés dans les C^*-modules hilbertiens. *C. R. Acad. Sci. Paris Sér. I Math.* **296** (1983), no. 21, 875–878. MR0715325 (84m:46091), Zbl 0551.46041.

[2] Baum, Paul; Douglas, Ronald G.; Taylor, Michael E. Cycles and relative cycles in analytic K-homology. *J. Differ. Geom.* **30** (1989), no. 3, 761–804. MR1021372 (91b:58244), Zbl 0697.58050.

[3] Bettaieb, Hela; Matthey, Michel; Valette, Alain. Unbounded symmetric operators in K-homology and the Baum–Connes conjecture. *J. Funct. Anal.* **229** (2005), no. 1, 184–237. MR2180079 (2006i:19006), Zbl 1086.19002, arXiv:math/0412364, doi: 10.1016/j.jfa.2005.01.013.

[4] Blackadar, Bruce. K-Theory for operator algebras. Second edition. Mathematical Sciences Research Institute Publications, 5. *Cambridge University Press, Cambridge*, 1998. xx+300 pp. ISBN: 0-521-63532-2. MR1656031 (99g:46104), Zbl 0913.46054.

[5] Boas, R. P., Jr.; Pollard, Harry. Complete sets of Bessel and Legendre functions. *Ann. of Math.* (2) **48** (1947), 366–384. MR0020660 (8,578b), Zbl 0029.35602.

[6] Booss-Bavnbek, Bernhelm; Wojciechowski, Krzysztof P. Elliptic boundary problems for Dirac operators. Mathematics: Theory & Applications. *Birkhäuser, Inc., Boston, MA*, 1993. xviii+307 pp. ISBN: 0-8176-3681-1. MR1233386 (94h:58168), Zbl 0797.58004.

[7] Carey, Alan; Phillips, John. Unbounded Fredholm modules and spectral flow. *Canad. J. Math.* **50** (1998), no. 4, 673–718. MR1638603 (2000g:58041), Zbl 0915.46063, doi: 10.4153/CJM-1998-038-x.

[8] Hilsum, Michel. Bordism invariance in KK-theory. *Math. Scand.* **107** (2010), no. 1, 73–89. MR2679393 (2011j:19010), Zbl 1198.19005.

[9] Lorch, Lee; Muldoon, Martin E. Monotonic sequences related to zeros of Bessel functions. *Numer. Algorithms* **49** (2008), no. 1–4, 221–233. MR2457101 (2009i:33008), Zbl 1171.33304, doi: 10.1007/s11075-008-9189-4.

[10] Power, S. C. Hankel operators on Hilbert space. Research Notes in Mathematics, 64. *Pitman (Advanced Publishing Program), Boston, Mass.-London*, 1982. vii+87 pp. ISBN: 0-273-08518-2. MR0666699 (84e:47037), Zbl 0489.47011.

[11] Reed, Michael; Simon, Barry. Methods of modern mathematical physics. I. Functional analysis. Second edition. *Academic Press, Inc., New York*, 1980. xv+400 pp. ISBN: 0-12-585050-6. MR0751959 (85e:46002), Zbl 0459.46001.

[12] Reed, Michael; Simon, Barry. Methods of modern mathematical physics. II. Fourier analysis, self-adjointness. *Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London*, 1975. xv+361 pp. MR0493420 (58 #12429b), Zbl 0308.47002.

[13] Sarason, Donald. Functions of vanishing mean oscillation. *Trans. Amer. Math. Soc.* **207** (1975), 391–405. MR0377518 (51 13690), Zbl 0319.42006, doi: 10.1090/S0002-9947-1975-0377518-3.

[14] Watson, G. N. A treatise on the theory of bessel functions. *Merchant Books,* 2008.

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