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Idempotents in βS that are only products trivially

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Abstract. All results mentioned in this abstract assume Martin's Axiom. (Some of them are known to not be derivable in ZFC.) It is known that if S is the free semigroup on countably many generators, then there exists an idempotent $p \in \beta S$ such that if $q, r \in \beta S$ and $qr = p$, then $q = r = p$. We show that the same conclusion holds for the semigroups $(\mathbb{N}, \cdot)$ and $(\mathcal{F}, \cup)$ where $\mathcal{F}$ is the set of finite nonempty subsets of $\mathbb{N}$. Such a strong conclusion is not possible if S is the free group on countably many generators or is the free semigroup on finitely many (but more than one) generators, since then any idempotent can be written as a product involving elements of S. But we show that in these cases we can produce p such that if $q, r \in \beta S$ and $qr = p$, then either $q = r = p$ or q and r satisfy one of the trivial exceptions that must exist. Finally, we show that for the free semigroup on countably many generators, the conclusion can be derived from a set theoretical assumption that is at least potentially weaker than what had previously been required.

Contents

1. Introduction

Given a discrete semigroup $(S, \cdot)$, we take the points of the Stone–Čech compactification, βS, of S to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. The operation on S has a natural extension to βS making $(\beta S, \cdot)$ a right topological semigroup, meaning that for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is

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continuous. The only thing we will need to know about the operation on βS in this paper is that if $p, q \in \beta S$ and $A \subseteq S$, then $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. Much more information, including an elementary introduction, can be found in [8].

Let $\langle x_t \rangle_{t=1}^{\infty}$ be a sequence in a semigroup $(S, \cdot)$. Then

$$FP(\langle x_t \rangle_{t=1}^{\infty}) = \left\{ \prod_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N}) \right\}$$

where $\mathbb{N}$ is the set of positive integers. For any set X, $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X and $\prod_{t \in F} x_t$ is the product in increasing order of indices. If the operation is denoted by $+$, we write

$$FS(\langle x_t \rangle_{t=1}^{\infty}) = \left\{ \sum_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N}) \right\}.$$

Given sequences $\langle x_t \rangle_{t=1}^{\infty}$ and $\langle y_t \rangle_{t=1}^{\infty}$ in S we say that $\langle y_t \rangle_{t=1}^{\infty}$ is a *product subsystem* of $\langle x_t \rangle_{t=1}^{\infty}$ if and only if there is a sequence $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $y_n = \prod_{t \in H_n} x_t$ and $\max H_n < \min H_{n+1}$. (For an additive semigroup, *sum subsystem* is defined analogously.)

An ultrafilter p on S is said to be *strongly productive* provided that, given any $A \in p$ there is a sequence $\langle x_t \rangle_{t=1}^{\infty}$ such that $FP(\langle x_t \rangle_{t=1}^{\infty}) \subseteq A$ and $FP(\langle x_t \rangle_{t=1}^{\infty}) \in p$. (The analogue in the additive situation is *strongly summable*.) See the introduction to [7] for the history behind the invention of strongly summable (or productive) ultrafilters.

It follows from [6, Theorem 2.3] that if $(S, +)$ is a countable, commutative, and cancellative semigroup, then any strongly summable ultrafilter on S is an idempotent in βS. Given any discrete semigroup S and an idempotent $p \in \beta S$, there is a largest subgroup $H(p)$ of βS with p as its identity. Often $H(p)$ is quite large. In fact, if S is an infinite cancellative semigroup with cardinality κ, then by [8, Corollary 7.39] βS contains a copy of the free group on 2^{2^κ} generators. It was shown in [5, Theorem 3.1] that if p is any strongly summable ultrafilter on $\mathbb{N}$, then any invertible element with respect to p is a member of $\mathbb{Z} + p$ and in particular, $H(p)$ is as small as possible; that is $H(p) = \mathbb{Z} + p$. And the question was asked in [5] whether a strongly summable ultrafilter p on $\mathbb{N}$ could be written as a sum of two elements, neither of which was a member of $\mathbb{Z} + p$. This question was answered in the negative in [9, Theorem 4]. (See the introduction to [7] for an explanation of why the negative answer follows.)

It was shown in [6, Theorem 4.5] that if $(G, +)$ is a countable group which can be embedded in the circle group $\mathbb{T}$, p is a *sparse* strongly summable ultrafilter on G, and $q, r \in G^* = \beta G \setminus G$ such that $q + r = p$, then p is an idempotent, $q \in G + p$, and $r \in G + p$.

Definition 1.1. Let $(S, +)$ be a semigroup and let $p \in \beta S$. Then p is a *sparse strongly summable ultrafilter* if and only if for every $A \in p$, there exist a sequence $\langle x_t\rangle_{t=1}^{\infty}$ and a subsequence $\langle y_t\rangle_{t=1}^{\infty}$ of $\langle x_t\rangle_{t=1}^{\infty}$ such that $FS(\langle x_t\rangle_{t=1}^{\infty}) \subseteq A$, $FS(\langle y_t\rangle_{t=1}^{\infty}) \in p$, and $\{x_n : n \in \mathbb{N}\} \setminus \{y_n : n \in \mathbb{N}\}$ is infinite.

In [7, Theorem 4.2] it was shown that if S is a countable subsemigroup of $\mathbb{T}$ and p is a nonprincipal strongly summable ultrafilter on S, then p is sparse, and thus as a consequence of [6, Theorem 4.5], if G is the group generated by S and $q, r \in G^*$ with $q + r = p$, then q and r are in $G + p$. It was recently shown in [3, Theorem 2.1] that all nonprincipal strongly summable ultrafilters on $\bigoplus_{n<\omega} \mathbb{Z}_2$ are sparse.

All of the results cited so far in this introduction deal with commutative semigroups. It was shown in [11, Theorem 3.10] that, assuming Martin's Axiom, if S is the free semigroup on countably many generators, then there is an idempotent $p \in \beta S$ such that, if $q, r \in \beta S$ and $q \cdot r = p$, then $q = r = p$. That idempotent is a strongly productive ultrafilter on S. In fact it satisfied the following stronger requirement.

Definition 1.2. Let S be the free semigroup on the generators $\langle a_t\rangle_{t=1}^{\infty}$ and let $p \in \beta S$. Then p is a *very strongly productive* ultrafilter on S if and only if for every $A \in p$ there is a product subsystem $\langle x_t\rangle_{t=1}^{\infty}$ of $\langle a_t\rangle_{t=1}^{\infty}$ such that $FP(\langle x_t\rangle_{t=1}^{\infty}) \subseteq A$ and $FP(\langle x_t\rangle_{t=1}^{\infty}) \in p$.

Very strongly productive ultrafilters correspond to *ordered union* ultrafilters introduced in [1]. Given a sequence $\langle A_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$,

$$FU(\langle A_n\rangle_{n=1}^{\infty}) = \left\{ \bigcup_{t\in F} A_t : F \in \mathcal{P}_f(\mathbb{N}) \right\}.$$

Definition 1.3. Let Θ be an ultrafilter on $\mathcal{P}_f(\mathbb{N})$.

(a) Θ is a *union ultrafilter* if and only if for each $\mathcal{A} \in \Theta$ there exists a sequence $\langle A_n\rangle_{n=1}^{\infty}$ of pairwise disjoint elements of $\mathcal{P}_f(\mathbb{N})$ such that $FU(\langle A_n\rangle_{n=1}^{\infty}) \subseteq \mathcal{A}$ and $FU(\langle A_n\rangle_{n=1}^{\infty}) \in \Theta$.

(b) Θ is an *ordered union ultrafilter* if and only if for each $\mathcal{A} \in \Theta$ there exists a sequence $\langle A_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $\max A_n < \min A_{n+1}$, $FU(\langle A_n\rangle_{n=1}^{\infty}) \subseteq \mathcal{A}$, and $FU(\langle A_n\rangle_{n=1}^{\infty}) \in \Theta$.

It was shown in [1, Theorem 2.4] that the Continuum Hypothesis implies the existence of ordered union ultrafilters, it was shown in [4, Theorem 4.1] that Martin's axiom implies the existence of union ultrafilters, and it was shown in [2, Theorem 3] that the existence of union ultrafilters cannot be established in ZFC.

If S is the free semigroup on a finite alphabet A with at least two members, then there is no idempotent $p \in \beta S$ such that, if $q, r \in \beta S$ and $q \cdot r = p$, then $q = r = p$. The reason is that for $p \in S^* = \beta S \setminus S$, $\bigcup_{a\in A} aS \in p$ so some $aS \in p$. Then $a^{-1}p = \{B \subseteq S : aB \in p\} \in S^*$ and thus

$$(pa) \cdot (a^{-1}p) = p \cdot p = p.$$

In Section 2 we show that the existence of ordered union ultrafilters implies the existence of an idempotent p in βS and distinct elements $b, c \in A$ such that if $q, r \in \beta S$, $q \cdot r = p$, and it is not the case that $q = r = p$, then some one of the following trivial cases must hold, and in particular $H(p) = \{p\}$.

(1) There is some $n \in \mathbb{N}$ such that $q = b^n$ and $r = b^{-n}p$;
(2) there is some $n \in \mathbb{N}$ such that $q = pb^n$ and $r = b^{-n}p$;
(3) there is some $n \in \mathbb{N}$ such that $q = pc^{-n}$ and $r = c^n$; or
(4) there is some $n \in \mathbb{N}$ such that $q = pc^{-n}$ and $r = c^n p$.

Similarly, if G is the free group on countably many generators, then there is no idempotent $p \in \beta G$ such that, if $q, r \in \beta S$ and $q \cdot r = p$, then $q = r = p$. The reason is that given any $w \in G$, one may let $q = pw$ and $r = w^{-1}p$. We show in Section 3 that Martin's axiom implies the existence of a *sparse* ordered union ultrafilter, and thus of a sparse very strongly productive ultrafilter. It is also shown that if p is a sparse very strongly productive ultrafilter, then the only way to write p nontrivially as a product in βG is as $(pw)(w^{-1}p)$, $w(w^{-1}p)$, or $(pw)w^{-1}$ for some $w \in G$.

In Section 4 we show that the existence of a union ultrafilter implies the existence of an idempotent $p \in (\beta\mathbb{N}, \cdot)$ such that if $q, r \in \beta\mathbb{N}\setminus\{1\}$ and $qr = p$, then $q = r = p$. We also show in this section that Martin's Axiom implies that there is an idempotent $p \in \beta S$, where S is the free semigroup on the generators $\langle a_t \rangle_{t=1}^{\infty}$, which is not very strongly productive, in fact not even strongly productive, but still has the property that it can only be written trivially as a product.

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2. The free semigroup on a finite alphabet

Throughout this section we shall let D be a finite alphabet with at least two members and will fix distinct elements b and c of D. We will let S be the free semigroup, with identity ι, on the alphabet D. We write $[\mathbb{N}]^{<\omega}$ for the set of finite subsets of $\mathbb{N}$. Thus $[\mathbb{N}]^{<\omega} = \mathcal{P}_f(\mathbb{N}) \cup \{\emptyset\}$. The following notions are based on the similar definitions in [11]. We agree that $\prod_{t\in\emptyset} x_t = \iota$, $\max\emptyset = 0$, and $\min\emptyset = \infty$.

We shall denote by T the subsemigroup of S generated by $\langle b^t c^t \rangle_{t=1}^{\infty}$. Then T is a copy of the free semigroup on countably many generators. Recall from [1] that the Continuum Hypothesis implies that ordered union ultrafilters exist, and by [11, Theorem 3.3] the existence of ordered union ultrafilters implies the existence of very strongly productive ultrafilters.

Lemma 2.1. *Let p be a very strongly productive ultrafilter on T. For each $A \in p$, there is a product subsystem $\langle x_t \rangle_{t=1}^{\infty}$ of $\langle b^t c^t \rangle_{t=1}^{\infty}$ such that $FP(\langle x_t \rangle_{t=1}^{\infty}) \subseteq A$ and for all $m \in \mathbb{N}$, $FP(\langle x_t \rangle_{t=m}^{\infty}) \in p$.*

Proof. For $i \in \{1,2\}$, let $C_i = \{3^n(3k+i) : n, k \in \omega\}$. (Note that C_i is the set of elements of $\mathbb{N}$ whose rightmost nonzero ternary digit is i.) For $i \in \{1,2\}$, let $D_i = \{x \in S \setminus \{\iota\} : \ell(x) \in C_i\}$, where $\ell(x)$ is the length of the word x. Pick $i \in \{1,2\}$ such that $D_i \in p$. Define $f : S \setminus \{\iota\} \to \omega$ by $f(x) = n$ where 3^n divides $\ell(x)$ and 3^{n+1} does not divide $\ell(x)$. (Thus $f(x)$ is the number of rightmost 0's in the ternary expansion of $\ell(x)$.) If $u, v \in D_i$ and $f(u) = f(v)$, then $uv \notin D_i$. Consequently, if $\{u, v, uv\} \subseteq D_i$, then $f(uv) = \min\{f(u), f(v)\}$.

Let $A \in p$ and pick a product subsystem $\langle x_t\rangle_{t=1}^{\infty}$ of $\langle b^t c^t\rangle_{t=1}^{\infty}$ such that $FP(\langle x_t\rangle_{t=1}^{\infty}) \subseteq A \cap D_i$ and $FP(\langle x_t\rangle_{t=1}^{\infty}) \in p$. Let $m \in \mathbb{N}$ and suppose that $FP(\langle x_t\rangle_{t=m}^{\infty}) \notin p$. Then $m > 1$. Since

$$\begin{aligned} FP(\langle x_t\rangle_{t=1}^{\infty}) &= FP(\langle x_t\rangle_{t=m}^{\infty}) \cup FP(\langle x_t\rangle_{t=1}^{m-1}) \\ &\quad \cup \bigcup \{u \cdot FP(\langle x_t\rangle_{t=m}^{\infty}) : u \in FP(\langle x_t\rangle_{t=1}^{m-1})\} \end{aligned}$$

and $FP(\langle x_t\rangle_{t=1}^{m-1}) \notin p$, because p is nonprincipal, there must be some $u \in FP(\langle x_t\rangle_{t=1}^{m-1})$ such that $u \cdot FP(\langle x_t\rangle_{t=m}^{\infty}) \in p$.

We claim that for all $x \in u \cdot FP(\langle x_t\rangle_{t=m}^{\infty})$, $f(x) \le f(u)$. To see this, let $x \in u \cdot FP(\langle x_t\rangle_{t=m}^{\infty})$ and pick $v \in FP(\langle x_t\rangle_{t=m}^{\infty})$ such that $x = uv$. Then $\{u, v, uv\} \subseteq FP(\langle x_t\rangle_{t=1}^{\infty}) \subseteq D_i$ so $f(x) = \min\{f(u), f(v)\}$.

Choose a sequence $\langle y_t\rangle_{t=1}^{\infty}$ such that $FP(\langle y_t\rangle_{t=1}^{\infty}) \subseteq u \cdot FP(\langle x_t\rangle_{t=m}^{\infty})$ and $FP(\langle y_t\rangle_{t=1}^{\infty}) \in p$. Then for all $k \in \mathbb{N}$, $f(y_k) \le f(u)$ so pick $k < t$ such that $f(y_k) = f(y_t)$. Then $y_k y_t \notin D_i$, a contradiction. □

We pause to note that every very strongly productive ultrafilter is an idempotent.

Lemma 2.2. *Let p be a very strongly productive ultrafilter on T. Then p is an idempotent.*

Proof. Let $A \in p$. We need to show that $\{y \in S : y^{-1}A \in p\} \in p$. Pick $\langle x_t\rangle_{t=1}^{\infty}$ as guaranteed by Lemma 2.1 for A. It suffices to show that

$$FP(\langle x_t\rangle_{t=1}^{\infty}) \subseteq \{y \in S : y^{-1}A \in p\},$$

so let $y \in FP(\langle x_t\rangle_{t=1}^{\infty})$ and pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $y = \prod_{t\in F} x_t$. Let $m = \max F + 1$. Then $FP(\langle x_t\rangle_{t=m}^{\infty}) \in p$ and $FP(\langle x_t\rangle_{t=m}^{\infty}) \subseteq y^{-1}A$. □

Definition 2.3. Let $\langle x_t\rangle_{t=1}^{\infty}$ be a sequence in S and let $k \in \mathbb{N}$.

(a) $$\begin{aligned} R(\langle x_t\rangle_{t=k}^{\infty}) = \Bigg\{ \Bigg(\prod_{t\in F} x_t\Bigg) u : u \in S \setminus \{\iota\}\,,\ & F \in [\mathbb{N}]^{<\omega}, \text{ and} \\ & (\exists s \in \mathbb{N})(\exists v \in S \setminus \{\iota\})(k \le \min F\,,\ \max F < s\,,\ k \le s \\ & \text{and } uv = x_s) \Bigg\}. \end{aligned}$$

(b) $$L(\langle x_t\rangle_{t=k}^{\infty}) = \Bigg\{ v\left(\prod_{t\in F} x_t\right) : v\in S\setminus\{\iota\}\,,\ F\in[\mathbb{N}]^{<\omega}, \text{ and } (\exists s\in\mathbb{N})(\exists u\in S\setminus\{\iota\})(k\le s<\min F\,,\ \text{ and } uv=x_s)\Bigg\}.$$

Note that, with $F=\emptyset$ in the definition, we have that

$$\{u\in S\setminus\{\iota\} : (\exists s\in\mathbb{N})(\exists v\in S\setminus\{\iota\})(k\le s \text{ and } uv=x_s)\}\subseteq R(\langle x_t\rangle_{t=k}^{\infty}),$$
$$\{v\in S\setminus\{\iota\} : (\exists s\in\mathbb{N})(\exists u\in S\setminus\{\iota\})(k\le s \text{ and } uv=x_s)\}\subseteq L(\langle x_t\rangle_{t=k}^{\infty}).$$

Lemma 2.4. *Let $\langle x_t\rangle_{t=1}^{\infty}$ be a sequence in $S\setminus\{\iota\}$ and let $y,z\in S\setminus\{\iota\}$ such that $yz\in FP(\langle x_t\rangle_{t=1}^{\infty})$. If either $y\notin FP(\langle x_t\rangle_{t=1}^{\infty})$ or $z\notin FP(\langle x_t\rangle_{t=1}^{\infty})$, then $y\in R(\langle x_t\rangle_{t=1}^{\infty})$ and $z\in L(\langle x_t\rangle_{t=1}^{\infty})$.*

Proof. Assume that either $y\notin FP(\langle x_t\rangle_{t=1}^{\infty})$ or $z\notin FP(\langle x_t\rangle_{t=1}^{\infty})$. Pick $H\in\mathcal{P}_f(\mathbb{N})$ such that $yz=\prod_{t\in H}x_t$ and write $H=\{n_1,n_2,\ldots,n_s\}$ where $n_1<n_2<\ldots<n_s$. Then $\ell(yz)=\sum_{i=1}^{s}\ell(x_{n_i})$.

Case 1. $\ell(y)\le\ell(x_{n_1})$. If $\ell(y)=\ell(x_{n_1})$, then $y=x_{n_1}$ and either $s=1$ in which case $z=\iota$ or $s>1$ in which case $z=\prod_{i=2}^{s}x_{n_i}$. Thus $\ell(y)<\ell(x_{n_1})$. Pick $v\in S\setminus\{\iota\}$ such that $x_{n_1}=yv$. If $s=1$, then $z=v$ and if $s>1$, then $z=v(\prod_{i=2}^{s}x_{n_i})$. Therefore $y\in R(\langle x_t\rangle_{t=1}^{\infty})$ and $z\in L(\langle x_t\rangle_{t=1}^{\infty})$.

Note that if $s=1$, then Case 1 applies.

Case 2. $s>1$ and $\ell(y)\ge\sum_{i=1}^{s-1}\ell(x_{n_i})$. If $\ell(y)=\sum_{i=1}^{s-1}\ell(x_{n_i})$, then $y\in FP(\langle x_t\rangle_{t=1}^{\infty})$ and $z\in FP(\langle x_t\rangle_{t=1}^{\infty})$. If $\ell(y)=\sum_{i=1}^{s}\ell(x_{n_i})$, then $z=\iota$. So we must have that $\sum_{i=1}^{s-1}\ell(x_{n_i})<\ell(y)<\sum_{i=1}^{s}\ell(x_{n_i})$. Pick $u\in S\setminus\{\iota\}$ such that $y=(\prod_{i=1}^{s-1}x_{n_i})u$. Then $x_{n_s}=uz$ so $y\in R(\langle x_t\rangle_{t=1}^{\infty})$ and $z\in L(\langle x_t\rangle_{t=1}^{\infty})$.

Case 3. $s>1$ and $\ell(x_{n_1})<\ell(y)<\sum_{i=1}^{s-1}\ell(x_{n_i})$. Then $s>2$. Pick $j\in\{1,2,\ldots,s-2\}$ such that

$$\sum_{i=1}^{j}\ell(x_{n_i})<\ell(y)\le\sum_{i=1}^{j+1}\ell(x_{n_i}).$$

If $\ell(y)=\sum_{i=1}^{j+1}\ell(x_{n_i})$, then $y=\prod_{i=1}^{j+1}x_{n_i}$ and $z=\prod_{i=j+2}^{s}x_{n_i}$, so

$$\sum_{i=1}^{j}\ell(x_{n_i})<\ell(y)<\sum_{i=1}^{j+1}\ell(x_{n_i}).$$

Pick $u,v\in S\setminus\{\iota\}$ such that $y=(\prod_{i=1}^{j}x_{n_i})u$ and $yv=\prod_{i=1}^{j+1}x_{n_i}$. Then $uv=x_{n_{j+1}}$ and $z=v(\prod_{i=j+2}^{s}x_{n_i})$ so $y\in R(\langle x_t\rangle_{t=1}^{\infty})$ and $z\in L(\langle x_t\rangle_{t=1}^{\infty})$. □

Lemma 2.5. *Let p be a very strongly productive ultrafilter on T. Assume that $q,r\in\beta S\setminus\{\iota\}$, $qr=p$, and it is not the case that $q=r=p$. Let $A\in p$. Then there is a product subsystem $\langle x_t\rangle_{t=1}^{\infty}$ of $\langle b^tc^t\rangle_{t=1}^{\infty}$ such that $FP(\langle x_t\rangle_{t=1}^{\infty})\subseteq A$ and for each $k\in\mathbb{N}$, $FP(\langle x_t\rangle_{t=k}^{\infty})\in p$, $R(\langle x_t\rangle_{t=k}^{\infty})\in q$, and $L(\langle x_t\rangle_{t=k}^{\infty})\in r$.*

Proof. Assume first that $q \neq p$ and pick $B \in q \setminus p$ such that $\iota \notin B$. By Lemma 2.1, pick a product subsystem $\langle x_t \rangle_{t=1}^{\infty}$ of $\langle b^t c^t \rangle_{t=1}^{\infty}$ such that $FP(\langle x_t \rangle_{t=1}^{\infty}) \subseteq A \setminus B$ and for all $k \in \mathbb{N}$, $FP(\langle x_t \rangle_{t=k}^{\infty}) \in p$. Let $k \in \mathbb{N}$. Then $\{y \in S : y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty}) \in r\} \in q$.

Suppose that $R(\langle x_t \rangle_{t=k}^{\infty}) \notin q$ and pick $y \in B \setminus R(\langle x_t \rangle_{t=k}^{\infty})$ such that $y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty}) \in r$. Pick $v \in y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty})$. Then $yv = \prod_{t \in H} x_t$ for some $H \in \mathcal{P}_f(\mathbb{N})$ with $\min H \geq k$. Since $y \in B$, $y \notin FP(\langle x_t \rangle_{t=k}^{\infty})$ so by Lemma 2.4, $y \in R(\langle x_t \rangle_{t=k}^{\infty})$, a contradiction.

Now suppose that $L(\langle x_t \rangle_{t=k}^{\infty}) \notin r$. Pick $y \in B$ with $y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty}) \in r$. Pick $z \in y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty}) \setminus L(\langle x_t \rangle_{t=k}^{\infty})$. Then $yz \in FP(\langle x_t \rangle_{t=k}^{\infty})$ and $y \notin FP(\langle x_t \rangle_{t=k}^{\infty})$ so by Lemma 2.4, $z \in L(\langle x_t \rangle_{t=k}^{\infty})$, a contradiction.

Now assume that $r \neq p$ and pick $B \in r \setminus p$ such that $\iota \notin B$. By Lemma 2.1, pick a product subsystem $\langle x_t \rangle_{t=1}^{\infty}$ of $\langle b^t c^t \rangle_{t=1}^{\infty}$ such that $FP(\langle x_t \rangle_{t=1}^{\infty}) \subseteq A \setminus B$ and for all $k \in \mathbb{N}$, $FP(\langle x_t \rangle_{t=k}^{\infty}) \in p$. Let $k \in \mathbb{N}$. Then

$$\{y \in S \setminus \{\iota\} : y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty}) \in r\} \in q.$$

Suppose that $L(\langle x_t \rangle_{t=k}^{\infty}) \notin r$. Pick $y \in S \setminus \{\iota\}$ with $y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty}) \in r$ and pick $z \in B \cap y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty}) \setminus L(\langle x_t \rangle_{t=k}^{\infty})$. Then $yz \in FP(\langle x_t \rangle_{t=k}^{\infty})$ and $z \notin FP(\langle x_t \rangle_{t=k}^{\infty})$ so we can again apply Lemma 2.4.

Finally suppose $R(\langle x_t \rangle_{t=k}^{\infty}) \notin q$. Pick $y \in S \setminus (R(\langle x_t \rangle_{t=k}^{\infty}) \cup \{\iota\})$ with $y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty}) \in r$; pick $z \in B \cap y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty})$. Then $yz \in FP(\langle x_t \rangle_{t=k}^{\infty})$ and $z \notin FP(\langle x_t \rangle_{t=k}^{\infty})$ so we can again apply Lemma 2.4. □

Lemma 2.6. *Let $p \in \beta S$ with $FP(\langle b^t c^t \rangle_{t=1}^{\infty}) \in p$. Assume that $q, r \in \beta S \setminus \{\iota\}$ and $qr = p$. Then there is some $n \in \mathbb{N}$ such that $Sb^n \notin q$.*

Proof. Suppose that for all $n \in \mathbb{N}$, $Sb^n \in q$. Let

$$A = \{x \in S : x^{-1}FP(\langle b^t c^t \rangle_{t=1}^{\infty}) \in r\}.$$

Then $A \in q$ so pick $w \in Sb \cap A$. Then there is some $n \in \mathbb{N}$ such that either $w = b^n$ or $w = uab^n$ for some $u \in S$ and some $a \in D \setminus \{b\}$. Pick $z \in Sb^{n+1} \cap A$. Then there is some $m > n$ such that either $z = b^m$ or $c = vdb^m$ for some $v \in S$ and some $d \in D \setminus \{b\}$.

Pick

$$y \in w^{-1}FP(\langle b^t c^t \rangle_{t=1}^{\infty}) \cap z^{-1}FP(\langle b^t c^t \rangle_{t=1}^{\infty}).$$

Since $wy \in FP(\langle b^t c^t \rangle_{t=1}^{\infty})$ there is some $l \geq n$ such that $y = b^{l-n}c^l$ or y begins $b^{l-n}c^l b$. Since $zy \in FP(\langle b^t c^t \rangle_{t=1}^{\infty})$ there is some $s \geq m$ such that $y = b^{s-m}c^s$ or y begins $b^{s-m}c^s b$. This is impossible, since $m > n$. □

Note that if $s \in \mathbb{N}$ and $b^s c^s$ occurs in some $z \in S$, then so does $b^t c^t$ for all $t \in \{1, 2, \ldots, s\}$. We omit the routine proof of the following lemma which allows us to conclude more from the occurrence of $bc^s b$.

Lemma 2.7. *Let $\langle x_t \rangle_{t=1}^{\infty}$ be a product subsystem of $\langle b^t c^t \rangle_{t=1}^{\infty}$ and for each $n \in \mathbb{N}$, let $H_n \in \mathcal{P}_f(\mathbb{N})$ such that $x_n = \prod_{t \in H_n} b^t c^t$. Let $s, k \in \mathbb{N}$, let $z \in L(\langle x_t \rangle_{t=k}^{\infty})$, and assume that either z ends with bc^s or $bc^s b$ occurs in z. Then $s \in H_n$ for some $n \geq k$.*

Lemma 2.8. *Let p be a very strongly productive ultrafilter on T. Assume that $q, r \in S^*$, $qr = p$, and it is not the case that $q = r = p$. If $Sb \in q$, then there is some $n \in \mathbb{N}$ such that $q = pb^n$.*

Proof. Suppose not. By Lemma 2.6 we may choose the largest $l \in \mathbb{N}$ such that $Sb^l \in q$. Then $Sb^l = \{b^l\} \cup \bigcup_{d\in D} Sdb^l$, $q \notin S$, and $Sb^{l+1} \notin q$ so there is some $d \in D \setminus \{b\}$ such that $Sdb^l \in q$. Since $Sc \in p$, we have that $p \neq q$. Pick $A \in q$ such that $\overline{A} \cap \{p, pb, pb^2, \ldots, pb^l\} = \emptyset$. Let

$$B = S \setminus (A \cup Ab^{-1} \cup Ab^{-2} \cup \ldots \cup Ab^{-l}).$$

Then $B \in p$ so pick by Lemma 2.5 a product subsystem $\langle x_t \rangle_{t=1}^{\infty}$ of $\langle b^t c^t \rangle_{t=1}^{\infty}$ such that for each $k \in \mathbb{N}$,

$$FP(\langle x_t \rangle_{t=k}^{\infty}) \in p, \quad R(\langle x_t \rangle_{t=k}^{\infty}) \in q, \quad \text{and} \quad L(\langle x_t \rangle_{t=k}^{\infty}) \in r.$$

For each $n \in \mathbb{N}$, pick $H_n \in \mathcal{P}_f(\mathbb{N})$ such that $x_n = \prod_{t\in H_n} b^t c^t$. Since $\langle x_t \rangle_{t=1}^{\infty}$ is a product subsystem of $\langle b^t c^t \rangle_{t=1}^{\infty}$ and $R(\langle x_t \rangle_{t=1}^{\infty}) \in q$, We must have $d = c$ and thus $Scb^l \in q$. Since $FP(\langle x_t \rangle_{t=l+1}^{\infty}) \in p$,

$$\{w \in S : w^{-1} FP(\langle x_t \rangle_{t=l+1}^{\infty}) \in r\} \in q.$$

Pick $w \in R(\langle x_t \rangle_{t=l+1}^{\infty}) \cap A \cap Scb^l$ such that $w^{-1} FP(\langle x_t \rangle_{t=l+1}^{\infty}) \in r$.

There are some $F \in [\mathbb{N}]^{<\omega}$ and $j \in \mathbb{N}$ with $\min F \geq l+1$, $\max F < j$ (and, if $F = \emptyset$, then $j \geq l+1$), and $v \in S$ such that $w = (\prod_{t\in F} x_t) \cdot u$ and $u \cdot v = x_j$. Since $w \in Scb^l$, we must have that u ends in cb^l. (If the length of u were at most l, then we would have $u = b^t$ for some $t \in \{1, 2, \ldots, l\}$ and thus that $\prod_{s\in F} x_s = wb^{-t} \in Ab^{-t}$, a contradiction.)

Since $uv = x_j = \prod_{i\in H_j} b^i c^i$ and u ends in cb^l, there exist $L \in \mathcal{P}_f(\mathbb{N})$, $s \in \mathbb{N}$, and (possibly empty) $M \in [\mathbb{N}]^{<\omega}$ such that $\max L < s < \min M$, $H_j = L \cup \{s\} \cup M$, $u = (\prod_{i\in L} b^i c^i) \cdot b^l$, and $v = b^{s-l} c^l \cdot \prod_{i\in M} b^i c^i$. (Note that $j > l$ so $s > l$.)

Since $L(\langle x_t \rangle_{t=j+1}^{\infty}) \in r$, pick $z \in w^{-1} FP(\langle x_t \rangle_{t=l+1}^{\infty}) \cap L(\langle x_t \rangle_{t=j+1}^{\infty})$. Then $wz \in FP(\langle x_t \rangle_{t=l+1}^{\infty})$ and $w = (\prod_{t\in F} x_t) \cdot (\prod_{i\in L} b^i c^i) \cdot b^l$. Also

$$wz = \prod_{t\in K} x_t = \prod_{t\in K} \prod_{i\in H_t} b^i c^i$$

for some $K \in \mathcal{P}_f(\mathbb{N})$ with $\min K > l$. Since $L \neq \emptyset$, pick $i \in L$. Then $b^i c^i b$ occurs in w and $i \in H_j$ so $j \in K$. Also

$$\begin{aligned} x_j &= \left(\prod_{i\in L} b^i c^i\right) \cdot b^l b^{s-l} c^s \cdot \prod_{i\in M} b^i c^i \\ &= w \cdot b^{s-l} c^s \cdot \prod_{i\in M} b^i c^i \end{aligned}$$

so z begins $b^{s-l} c^s$. So either z ends as $b^{s-l} c^s$ (if $M = \emptyset$) or $bc^s b$ occurs in z. In either case, by Lemma 2.7, $s \in H_n$ for some $n \geq j+1$. But $s \in H_j$, a contradiction. $\square$

Theorem 2.9. *Let p be a very strongly productive ultrafilter on T. Assume that $q, r \in S^*$, $qr = p$, and it is not the case that $q = r = p$. If $Sb \in q$, then there is some $n \in \mathbb{N}$ such that $q = pb^n$ and $r = b^{-n}p$.*

Proof. By Lemma 2.8, pick $n \in \mathbb{N}$ with $q = pb^n$. Suppose $r \neq b^{-n}p$. Then $p \neq b^n r$ so pick $A \in p$ such that $A \notin b^n r$. Pick a product subsystem $\langle x_t \rangle_{t=1}^{\infty}$ of $\langle b^t c^t \rangle_{t=1}^{\infty}$ with $FP(\langle x_t \rangle_{t=1}^{\infty}) \in p$ and $FP(\langle x_t \rangle_{t=1}^{\infty}) \subseteq A$. Then

$$\{w \in S : w^{-1}FP(\langle x_t \rangle_{t=1}^{\infty}) \in r\} \cap FP(\langle x_t \rangle_{t=1}^{\infty})b^n \in q$$

so pick $w \in FP(\langle x_t \rangle_{t=1}^{\infty})b^n$ with $w^{-1}FP(\langle x_t \rangle_{t=1}^{\infty}) \in r$. Since $b^{-n}(S \setminus A) \in r$, pick $y \in w^{-1}FP(\langle x_t \rangle_{t=1}^{\infty}) \cap b^{-n}(S \setminus A)$. Pick F and H in $\mathcal{P}_f(\mathbb{N})$ such that $w = (\prod_{t \in F} x_t) \cdot b^n$ and $wy = \prod_{t \in H} x_t$. Then $\prod_{t \in H} x_t = (\prod_{t \in F} x_t) \cdot b^n \cdot y$ so F is an initial segment of H and $\prod_{t \in H \setminus F} x_t = b^n \cdot y$ and thus

$$y \in b^{-n}FP(\langle x_t \rangle_{t=1}^{\infty}) \subseteq b^{-n}A,$$

a contradiction. $\square$

By a very similar sequence of lemmas, one can prove the following theorem.

Theorem 2.10. *Let p be a very strongly productive ultrafilter on T. Assume that $q, r \in S^*$, $qr = p$, and it is not the case that $q = r = p$. If $cS \in r$, then there is some $n \in \mathbb{N}$ such that $q = pc^{-n}$ and $r = c^n p$.*

Theorem 2.11. *Let p be a very strongly productive ultrafilter on T. Assume that $q, r \in \beta S$, $qr = p$, and it is not the case that $q = r = p$. Then either $Sb \in q$ or $cS \in r$. If $q \in S$ then there is some $n \in \mathbb{N}$ such that $q = b^n$. If $r \in S$, then there is some $n \in \mathbb{N}$ such that $r = c^n$.*

Proof. Suppose first that $q \in S$ and let n be the length of q. Pick by Lemma 2.1 a product subsystem $\langle x_t \rangle_{t=1}^{\infty}$ of $\langle b^t c^t \rangle_{t=1}^{\infty}$ such that $FP(\langle x_t \rangle_{t=n}^{\infty}) \in p$. In particular $FP(\langle b^t c^t \rangle_{t=n}^{\infty}) \in p = qr$ so $q^{-1}FP(\langle b^t c^t \rangle_{t=n}^{\infty}) \in r$. Pick $w \in q^{-1}FP(\langle b^t c^t \rangle_{t=n}^{\infty})$. Then $qw \in FP(\langle b^t c^t \rangle_{t=n}^{\infty})$ and thus the leftmost n letters of qw are all equal to b.

The proof for the case $r \in S$ is very similar. (At the appropriate point in the argument, pick w such that $r \in w^{-1}FP(\langle b^t c^t \rangle_{t=n}^{\infty})$. Then the rightmost n letters of wr are all equal to c.)

Now assume that q and r are in S^* and suppose that $Sb \notin q$ and $cS \notin r$. Pick some $a \in D \setminus \{b\}$ and some $d \in D \setminus \{c\}$ such that $Sa \in q$ and $dS \in r$.

By Lemma 2.5 pick a product subsystem $\langle x_t \rangle_{t=1}^{\infty}$ of $\langle b^t c^t \rangle_{t=1}^{\infty}$ such that for each $k \in \mathbb{N}$, $FP(\langle x_t \rangle_{t=k}^{\infty}) \in p$, $R(\langle x_t \rangle_{t=k}^{\infty}) \in q$, and $L(\langle x_t \rangle_{t=k}^{\infty}) \in r$. For each $n \in \mathbb{N}$, pick $H_n \in \mathcal{P}_f(\mathbb{N})$ such that $x_n = \prod_{t \in H_n} b^t c^t$. Pick $w \in Sa \cap R(\langle x_t \rangle_{t=1}^{\infty})$ such that $w^{-1}FP(\langle x_t \rangle_{t=1}^{\infty}) \in r$. Pick $F \in [\mathbb{N}]^{<\omega}$, $j \in \mathbb{N}$, and $u, v \in S \setminus \{\iota\}$ such that $\max F < j$, $w = (\prod_{t \in F} x_t) \cdot u$, and

$$uv = x_j = \prod_{t \in H_j} b^t c^t.$$

Since $a \neq b$ and the rightmost letter of w is the rightmost letter of u, we have $a = c$. Pick $s \in H_j$ such that

$$\sum\{2t : t \in H_j \text{ and } t < s\} < \ell(u) \leq \sum\{2t : t \in H_j \text{ and } t \leq s\},$$

where $\ell(u)$ is the length of u. Then the rightmost letter of u occurs in $b^s c^s$. We have $K_1, K_2 \in [\mathbb{N}]^{<\omega}$ and s such that $K_1 \cup \{s\} \cup K_2 = H_j$, $\max K_1 < s < \min K_2$, $u = (\prod_{t \in K_1} b^t c^t) \cdot b^s c^i$, and $v = c^{s-i} \cdot \prod_{t \in K_2} b^t c^t$ for some $i \in \{1, 2, \ldots, s\}$.

Pick $y \in w^{-1} FP(\langle x_t \rangle_{t=1}^{\infty}) \cap dS \cap L(\langle x_t \rangle_{t=j+1}^{\infty})$. Since $wy \in FP(\langle x_t \rangle_{t=1}^{\infty})$, the leftmost letter of y is b or c, and $d \neq c$, we have that $d = b$. Pick h and z in $S \setminus \{\iota\}$, $N \in [\mathbb{N}]^{<\omega}$, and $k < \min N$ with $k \geq j + 1$ such that $y = z \cdot \prod_{t \in N} x_t$ and $hz = x_k = \prod_{t \in H_k} b^t c^t$. Pick $f \in H_k$ such that

$$\sum\{2t : t \in H_k \text{ and } t < f\} < \ell(z) \leq \sum\{2t : t \in H_k \text{ and } t \leq f\}.$$

Since the leftmost letter of z is the leftmost letter of y which is b, we have $M_1, M_2 \in [\mathbb{N}]^{<\omega}$ and g such that $M_1 \cup \{g\} \cup M_2 = H_k$, $\max M_1 < g < \min M_2$, $h = (\prod_{t \in M_1} b^t c^t) \cdot b^{g-\alpha}$, and $z = b^\alpha c^g \cdot \prod_{t \in M_2} b^t c^t$ for some $\alpha \in \{1, 2, \ldots, g\}$.

Pick $L \in \mathcal{P}_f(\mathbb{N})$ such that $wy = \prod_{t \in L} x_t$. Then

$$\prod_{t \in L} x_t = \left(\prod_{t \in F} x_t\right) \cdot \left(\prod_{t \in K_1} b^t c^t\right) \cdot b^s c^i b^\alpha c^g \cdot \left(\prod_{t \in M_2} b^t c^t\right) \cdot \left(\prod_{t \in N} x_t\right)$$

so

$$\prod_{t \in L \setminus (F \cup N)} x_t = \left(\prod_{t \in K_1} b^t c^t\right) \cdot b^s c^i b^\alpha c^g \cdot \left(\prod_{t \in M_2} b^t c^t\right).$$

Since $K_1 \subseteq H_j$, $s \in H_j$, $g \in H_k$, $M_2 \subseteq H_k$, and $j < k$, we must have $L \setminus (F \cup N) = \{j, k\}$, $i = s$, $\alpha = g$, $x_j = (\prod_{t \in K_1} b^t c^t) \cdot b^s c^s$, and $x_k = b^g c^g \cdot (\prod_{t \in M_2} b^t c^t)$. But then $H_j = K_1 \cup \{s\}$ so $K_2 = \emptyset$ and, since $i = s$, $v = \iota$, a contradiction. □

Corollary 2.12. *Let p be a very strongly productive ultrafilter on T. Assume that $q, r \in \beta S$, $qr = p$, and it is not the case that $q = r = p$. Then one of the following statements holds.*

1. *There is some $n \in \mathbb{N}$ such that $q = b^n$ and $r = b^{-n} p$;*
2. *there is some $n \in \mathbb{N}$ such that $q = pb^n$ and $r = b^{-n} p$;*
3. *there is some $n \in \mathbb{N}$ such that $q = pc^{-n}$ and $r = c^n$; or*
4. *there is some $n \in \mathbb{N}$ such that $q = pc^{-n}$ and $r = c^n p$.*

Proof. If q and r are in S^*, the conclusion follows from Theorems 2.9, 2.10, and 2.11. If q and r were both in S, then qr would be in S.

Assume that $q \in S$ and $r \in S^*$. Then by Theorem 2.11, pick $n \in \mathbb{N}$ such that $q = b^n$. Then $b^n r = p$ so, computing in βG, where G is the free group on the alphabet D, we have that $r = b^{-n} p$. Similarly, if $r \in S$, then there is some $n \in \mathbb{N}$ such that $r = c^n$ and $q = pc^{-n}$. □

3. The free group on a countable alphabet

Throughout this section we will let S and G be respectively the free semigroup with identity and the free group on the generators $\langle a_t\rangle_{t=1}^{\infty}$. We will let $T = \bigcap_{m=1}^{\infty} \overline{FP(\langle a_t\rangle_{t=m}^{\infty})}$. We will show that, assuming Martin's Axiom, there is an idempotent $p \in S^*$ with the property that if $q, r \in \beta G$ and $qr = p$, then there is some $w \in G$ such that (1) $q = w$ and $r = w^{-1}p$, (2) $q = pw$ and $r = w^{-1}p$, or (3) $q = pw$ and $r = w^{-1}$.

Members of G are the members of the free semigroup with identity on the alphabet $\{a_n : n \in \mathbb{N}\} \cup \{a_n^{-1} : n \in \mathbb{N}\}$ which do not have adjacent occurrences of a_n and a_n^{-1} for any n. We denote concatenation by $\frown$. Thus, for example, if $u = a_2 a_3^{-1} a_2^{-1}$ and $v = a_2 a_4$, then $uv = a_2 a_3^{-1} \frown a_4$.

Definition 3.1. Let $w \in G \setminus \{\iota\}$.

(a) $A_w = \{x \in G : x \text{ begins with } w\}$.
(b) $B_w = \{x \in G : x \text{ ends with } w^{-1}\}$.

When we write "let l be a letter", we mean that

$$l \in \{a_n : n \in \mathbb{N}\} \cup \{a_n^{-1} : n \in \mathbb{N}\}.$$

Lemma 3.2. *Let $q, r \in G^*$ and assume that $qr \in T$. Let l be a letter. If $A_l \in r$, then $B_l \in q$.*

Proof. Assume first that $l = a_s^{-1}$ for some $s \in \mathbb{N}$ and suppose that $B_l \notin q$. Pick $x \in G \setminus B_l$ such that $x^{-1}FP(\langle a_t\rangle_{t=1}^{\infty}) \in r$. Pick $y \in x^{-1}FP(\langle a_t\rangle_{t=1}^{\infty}) \cap A_l$. Since x does not end in a_s, a_s^{-1} occurs in xy, a contradiction.

Now assume that $l = a_s$ for some $s \in \mathbb{N}$ and suppose that $B_l \notin q$. Pick $x \in G \setminus B_l$ such that $x^{-1}FP(\langle a_t\rangle_{t=s+1}^{\infty}) \in r$. Pick $y \in x^{-1}FP(\langle a_t\rangle_{t=s+1}^{\infty}) \cap A_l$. Then a_s occurs in xy, a contradiction. $\square$

Lemma 3.3. *Let $q, r \in G^*$ and assume that $qr \in T$. If either $S \notin q$ or $S \notin r$, then there is a letter l such that $A_l \in r$.*

Proof. Assume first that $S \notin q$. Pick $x \in G \setminus S$ with $x^{-1}FP(\langle a_t\rangle_{t=1}^{\infty}) \in r$. Pick $u \in G$, $v \in S$, and $t \in \mathbb{N}$ such that $x = u \frown a_t^{-1} \frown v$. Assume first that $v = \iota$. We claim $A_{a_t} \in r$. Suppose instead that $A_{a_t} \notin r$ and pick $y \in x^{-1}FP(\langle a_i\rangle_{i=1}^{\infty}) \setminus A_{a_t}$. Then a_t^{-1} occurs in xy, a contradiction. Now assume that $v \in S$ and let a_s be the rightmost letter of v. Then as above we see that $A_{a_s^{-1}} \in r$.

The case that $S \in q$ and $S \notin r$ is handled in a similar fashion. $\square$

Lemma 3.4. *Let $q, r \in G^*$ and assume that $qr \in T$. If either $S \notin q$ or $S \notin r$, then there is a letter l such that $A_l \in r$ and $B_l \in q$.*

Proof. Lemmas 3.2 and 3.3. $\square$

Lemma 3.5. *Let $k \in \mathbb{N}$, let $r \in G^*$, let $w = l_1 l_2 \cdots l_k$ where each l_i is a letter, and assume that $A_w \in r$. Then $A_{l_k^{-1}} \notin w^{-1}r$.*

Proof. We proceed by induction on k. For $k = 1$, let l be a letter and suppose that $A_l \in r$ and $A_{l^{-1}} \in l^{-1}r$. Then $lA_{l^{-1}} \in r$. Pick $x \in A_l \cap lA_{l^{-1}}$. Since $x \in lA_{l^{-1}}$ we have $x = l(l^{-1}\frown w)$ where w does not begin with l so $x = w \notin A_l$, a contradiction.

Now assume that $k > 1$ and the lemma is valid for $k - 1$. Suppose that $A_{l_k^{-1}} \in w^{-1}r$ Let $w' = l_2l_3 \cdots l_k$ and $r' = l_1^{-1}r$. We claim that $A_{w'} \in r'$ and $A_{l_k^{-1}} \in (w')^{-1}r'$, contradicting the induction hypothesis.

Now $A_w \in r$ so $l_1^{-1}A_w \in r'$. We claim that $l_1^{-1}A_w \subseteq A_{w'}$ so let $x \in l_1^{-1}A_w$. Then $l_1x = l_1l_2 \cdots l_k\frown u$ for some $u \in G$ so $l_1x \in A_{l_1}$. If $l_1x = l_1\frown x$, then $x = l_2l_3 \cdots l_k\frown u \in A_{w'}$ as desired. So suppose $l_1x \neq l_1\frown x$. Then $x = l_1^{-1}\frown v$ for some $v \in G \setminus A_{l_1}$ and thus $l_1x = v \notin A_{l_1}$, a contradiction.

Finally, $(w')^{-1}r' = (w')^{-1}l_1^{-1}r = (l_1w')^{-1}r = w^{-1}r$ so $A_{l_k^{-1}} \in (w')^{-1}r'$ as claimed. □

Lemma 3.6. *Let $q, r \in G^*$ and assume that $qr \in T$ and either $S \notin q$ or $S \notin r$. Then one of the following must hold:*

(1) *There is some $w \in G$ such that $w^{-1}r \in \beta S$ and $qw \in \beta S$.*
(2) *There exists a sequence $\langle l_t\rangle_{t=1}^{\infty}$ of letters such that $l_{t+1} \neq l_t^{-1}$ for each t and for each k, if $w_k = l_1l_2 \cdots l_k$, then $A_{w_k} \in r$ and $B_{w_k} \in q$.*

Proof. Assume that (1) fails. By Lemma 3.4 we have some letter l_1 such that $A_{l_1} \in r$ and $B_{l_1} \in q$. Let $k \in \mathbb{N}$ and assume that $l_1, l_2, \ldots, l_k$ have been chosen. Let $w_k = l_1l_2 \cdots l_k$. Then $A_{w_k} \in r$ and $B_{w_k} \in q$. Let $r' = w_k^{-1}r$ and $q' = qw_k$. Since (1) fails, either $S \notin r'$ or $S \notin q'$ so by Lemma 3.4, pick a letter l_{k+1} such that $A_{l_{k+1}} \in r'$ and $B_{l_{k+1}} \in q'$. By Lemma 3.5, $l_{k+1} \neq l_k^{-1}$. We claim that $A_{w_{k+1}} \in r$ and $B_{w_{k+1}} \in q$. Since $A_{l_{k+1}} \in r' = w_k^{-1}r$ and $B_{l_{k+1}} \in q' = qw_k$ we have that $w_kA_{l_{k+1}} \in r$ and $B_{l_{k+1}}w_k^{-1} \in q$. Since $l_{k+1} \neq l_k^{-1}$ we have immediately that $w_kA_{l_{k+1}} \subseteq A_{w_{k+1}}$ and $B_{l_{k+1}}w_k^{-1} \subseteq B_{w_{k+1}}$. □

We find it hard to believe that case (2) of the following theorem could hold, but we cannot prove that it does not.

Theorem 3.7. *Let p be a very strongly productive ultrafilter on S, let $q, r \in G^*$, and assume that $qr = p$ and either $S \notin q$ or $S \notin r$. Then one of the following must hold:*

(1) *There is some $w \in G$ such that $r = wp$ and $q = pw^{-1}$.*
(2) *There exists a sequence $\langle l_t\rangle_{t=1}^{\infty}$ of letters such that:*
    (a) *$l_{t+1} \neq l_t^{-1}$ for each t and for each k, if $w_k = l_1l_2 \cdots l_k$, then $A_{w_k} \in r$ and $B_{w_k} \in q$.*
    (b) *There exists $k \in \mathbb{N}$ such that $\langle l_t\rangle_{t=k}^{\infty}$ is a subsequence of $\langle a_t\rangle_{t=1}^{\infty}$.*

Proof. We have that either conclusion (1) or conclusion (2) of Lemma 3.6 holds. Assume first that conclusion (1) of Lemma 3.6 holds. By [11, Theorem 3.10] $w^{-1}r = qw = p$.

Now assume that conclusion (2) of Lemma 3.6 holds. Let $C = FP(\langle a_t\rangle_{t=1}^{\infty})$ and pick $x \in G$ such that $x^{-1}C \in r$. Let $k = \ell(x)+1$ and let $m > k$ be given.

Let $w_m = l_1 l_2 \cdots l_m$ and pick $y \in A_{w_m} \cap x^{-1}C$. Then $y = w_m {}^\frown v$ for some $v \in G \setminus A_{l_m^{-1}}$. In the computation of xy at most $k-1$ letters of w_m cancel so there exist $u \in G$ and $s \in \{1, 2, \ldots, k\}$ such that $xy = u^\frown l_s l_{s+1} \cdots l_m {}^\frown v$. Also $xy = \prod_{t\in F} a_t$ for some $F \in \mathcal{P}_f(\mathbb{N})$. Thus we have that for each $i \in \{0, 1, \ldots, m-s\}$, $l_{s+i} = a_{t_i}$ for some $t_0 < t_1 < \ldots < t_{m-s}$. □

Lemma 3.8. *Let p be a very strongly productive ultrafilter on S, let $q, r \in G^*$ such that $qr = p$, and assume that $\langle l_t\rangle_{t=1}^\infty$ and k are as in conclusion (2) of Theorem 3.7. Then $FP(\langle l_t\rangle_{t=k}^\infty) \in p$.*

Proof. Suppose instead $D = FP(\langle a_t\rangle_{t=1}^\infty) \setminus FP(\langle l_t\rangle_{t=k}^\infty) \in p$. Pick an increasing sequence $\langle \gamma(t)\rangle_{t=k}^\infty$ in $\mathbb{N}$ such that for each $t \geq k$, $l_t = a_{\gamma(t)}$. Pick a product subsystem $\langle x_t\rangle_{t=1}^\infty$ of $\langle a_t\rangle_{t=1}^\infty$ such that $E = FP(\langle x_t\rangle_{t=1}^\infty) \subseteq D$ and $E \in p$. For each $n \in \mathbb{N}$, pick $H_n \in \mathcal{P}_f(\mathbb{N})$ such that $x_n = \prod_{t\in H_n} a_t$. Pick $z \in B_{w_k}$ such that $z^{-1}E \in r$. Pick $\alpha \geq k$ and $u \in G$ such that $z = u^\frown l_\alpha^{-1} l_{\alpha-1}^{-1} \cdots l_1^{-1}$ and u does not end with $l_{\alpha+1}^{-1}$. (Note that $u = \iota$ is possible.)

Pick the first $\delta \in \mathbb{N}$ such that $\gamma(\alpha+1) \leq \max H_\delta$. Pick the largest $\nu \in \mathbb{N}$ such that $\gamma(\nu) \leq \max H_\delta$. Pick the first $\tau \in \mathbb{N}$ such that $\gamma(\nu+1) \leq \max H_\tau$. Pick the largest $\eta \in \mathbb{N}$ such that $\gamma(\eta) \leq \max H_\tau$. Pick $m \in \mathbb{N}$ such that $\gamma(m) > \max H_\tau$. Then $\alpha + 1 \leq \nu < \eta < m$.

Pick $y \in z^{-1}E \cap A_{w_m}$. Then $y = l_1 l_2 \cdots l_m {}^\frown v$ for some $v \in G$ which does not begin with l_m^{-1}. Then

$$(*) \qquad zy = u^\frown l_{\alpha+1} l_{\alpha+2} \cdots l_m {}^\frown v.$$

Since $zy \in E$, pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $zy = \prod_{n\in F} x_n$. Pick n_1 and n_2 in F such that $\gamma(\alpha+1) \in H_{n_1}$ and $\gamma(\nu+1) \in H_{n_2}$. Then $\gamma(\alpha+1) \leq \max H_{n_1}$ and $\gamma(\alpha+1) \geq \min H_{n_1} > \max H_{n_1-1}$ so $n_1 = \delta$. Similarly, $n_2 = \tau$. Let $K = \{n \in F : n < \delta\}$ and $L = \{n \in F : n > \tau\}$. Then

$$(**) \qquad zy = \prod_{n\in K} x_n \cdot \prod_{t\in H_\delta} a_t \cdot \prod_{t\in H_\tau} a_t \cdot \prod_{n\in L} x_n.$$

(Recall that we take $\prod_{n\in\emptyset} x_n = \iota$.)

Comparing ($*$) and ($**$) we see that

$$u^\frown l_{\alpha+1} l_{\alpha+2} \cdots l_\nu = \prod_{n\in K} x_n \cdot \prod_{t\in H_\delta} a_t$$

so that $\prod_{t\in H_\tau} a_t = l_{\nu+1} l_{\nu+2} \cdots l_\eta$ and thus $x_\tau = \prod_{t\in H_\tau} a_\tau \in FP(\langle l_t\rangle_{t=k}^\infty)$, a contradiction. □

Definition 3.9. Let $p \in \beta S$. Then p is *sparse* if and only if for each $A \in p$ there exist $\langle x_t\rangle_{t=1}^\infty$ in S and an infinite set $D \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus D$ is infinite, $FP(\langle x_t\rangle_{t=1}^\infty) \subseteq A$, and $FP(\langle x_n\rangle_{n\in D}) \in p$.

We will conclude this section with a proof that Martin's Axiom implies that sparse very strongly productive ultrafilters on S exist.

Theorem 3.10. *Let p be a sparse very strongly productive ultrafilter on S and let $q, r \in G^*$ such that $qr = p$. Then there exists $w \in G$ such that $r = wp$ and $q = pw^{-1}$.*

Proof. Suppose not. Then we may pick $\langle l_t \rangle_{t=1}^{\infty}$ and k as guaranteed by conclusion (2) of Theorem 3.7. By Lemma 3.8, $FP(\langle l_t \rangle_{t=k}^{\infty}) \in p$. Pick infinite $D \subseteq \mathbb{N}$ and $\langle x_t \rangle_{t=1}^{\infty}$ in S such that $\mathbb{N} \setminus D$ is infinite, $FP(\langle x_t \rangle_{t=1}^{\infty}) \subseteq FP(\langle l_t \rangle_{t=k}^{\infty})$, and $E = FP(\langle x_n \rangle_{n \in D}) \in p$. For each $n \in \mathbb{N}$ pick $H_n \in \mathcal{P}_f(\mathbb{N})$ such that $x_n = \prod_{t \in H_n} l_t$. Note that for each n, $\max H_n < \min H_{n+1}$ because $x_n x_{n+1} = \prod_{t \in H_n} l_t \cdot \prod_{t \in H_{n+1}} l_t$ and $x_n x_{n+1} \in FP(\langle l_t \rangle_{t=k}^{\infty})$.

Pick $z \in B_{w_k}$ such that $z^{-1}E \in r$. Pick $\alpha \geq k$ and $u \in G$ such that $z = u^\frown l_\alpha^{-1} l_{\alpha-1}^{-1} \cdots l_1^{-1}$ and u does not end with $l_{\alpha+1}^{-1}$.

Pick the first $\delta \in \mathbb{N}$ such that $\alpha + 1 \leq \max H_\delta$ and let $\nu = \max H_\delta$. Pick the first $\tau > \delta$ such that $\tau \notin D$ and let $m = \max H_\tau$. Pick $y \in z^{-1}E \cap A_{w_m}$. Then $zy = u^\frown l_{\alpha+1} l_{\alpha+2} \cdots l_m{}^\frown v$ where $v \in G$ and v does not begin with l_m^{-1}. Since $zy \in E$, pick $F \in \mathcal{P}_f(D)$ such that $zy = \prod_{n \in F} x_n$. Since $l_{\alpha+1}$ occurs in zy, we may pick $n \in F$ such that $\alpha + 1 \in H_n$. Then $\alpha + 1 \leq \max H_n$ and $\alpha + 1 \geq \min H_n > \max H_{n-1}$ so $\delta = n$. Let $K = \{n \in F : n < \delta\}$. Then $\prod_{n \in K} x_n \cdot \prod_{t \in H_\delta} l_t = u^\frown l_{\alpha+1} l_{\alpha+2} \cdots l_\nu$. Now $\tau > \delta$ and $m = \max H_\tau$ so $H_\tau \subseteq \nu+1, \nu+2, \ldots, m$. Pick $s \in H_\tau$. Since $\tau \notin D$, l_s does not occur in $\prod_{n \in F} x_n = zy$, a contradiction. $\square$

Corollary 3.11. *Let p be a sparse very strongly productive ultrafilter on S and let $q, r \in \beta G$ such that $qr = p$. Then there exists $w \in G$ such that:*

(1) *$r = wp$ and $q = pw^{-1}$;*
(2) *$r = w$ and $q = pw^{-1}$; or*
(3) *$r = wp$ and $q = w^{-1}$.*

Proof. If $q, r \in G^*$, then conclusion (1) holds by Theorem 3.10. If $r \in G$, let $w = r$. Then since $wq = p$, $q = w^{-1}p$. If $q \in G$, let $w = q^{-1}$. $\square$

Except for a question asked at the end, the rest of this section consists of a proof that Martin's Axiom implies the existence of a sparse very strongly productive ultrafilter on S (and thus that Martin's Axiom implies the existence of idempotents in βS that can only be written trivially as products of elements of βG). See [10, pages 53-61] or [8, Chapter 12] for an introduction to Martin's Axiom.

We actually produce a sparse ordered union ultrafilter on the semigroup $(\mathcal{F}, \cup)$, where $\mathcal{F} = \mathcal{P}_f(\mathbb{N})$.

Definition 3.12. Let Θ be an ultrafilter on $\mathcal{F}$. Then Θ is *sparse* if and only if for each $\mathcal{A} \in \Theta$, there exist a sequence $\langle X_n \rangle_{n=1}^{\infty}$ of members of $\mathcal{F}$ such that $\max X_n < \min X_{n+1}$ for each n and an infinite subset D of $\mathbb{N}$ such that $FU(\langle X_n \rangle_{n=1}^{\infty}) \subseteq \mathcal{A}$, $\mathbb{N} \setminus D$ is infinite, and $FU(\langle X_n \rangle_{n \in D}) \in \Theta$.

Definition 3.13.

(a) $\mathcal{I} = \{\langle X_n \rangle_{n=1}^{\infty} :$ for each $n \in \mathbb{N}$, $X_n \in \mathcal{F}$ and $\max X_n < \min X_{n+1}\}$.

(b) For $m,k\in\mathbb{N}$, $\mathcal{B}_{m,k}=FU(\langle\{2^k n\}\rangle_{n=m+1}^{\infty})$.

Note that if $(m_1,k_1),(m_2,k_2)\in\mathbb{N}\times\mathbb{N}$, $m_1\le m_2$, and $k_1\le k_2$, then $\mathcal{B}_{m_2,k_2}\subseteq\mathcal{B}_{m_1,k_1}$.

Definition 3.14. (Π,f) is a *sparse ordered union pair* if and only if the following hold:

(1) Π is a nonempty set of infinite subsets of $\mathcal{F}$.
(2) $f:\mathcal{P}_f(\Pi)\to\mathcal{I}$.
(3) For all $\Delta\in\mathcal{P}_f(\Pi)$, if $f(\Delta)=\langle X_n\rangle_{n=1}^{\infty}$, then:
  (a) $FU(\langle X_n\rangle_{n=1}^{\infty})\subseteq\bigcap\Delta$.
  (b) For all $m\in\mathbb{N}$, $FU(\langle X_{2n}\rangle_{n=m}^{\infty})\in\Pi$.

Lemma 3.15. *Let $\Pi=\{\mathcal{B}_{m,k}:(m,k)\in\mathbb{N}\times\mathbb{N}\}$. For $F\in\mathcal{P}_f(\mathbb{N}\times\mathbb{N})$, let*

$$\mu(F)=\max\big\{m:(\exists k)\big((m,k)\in F\big)\big\},$$
$$\kappa(F)=\max\big\{k:(\exists m)\big((m,k)\in F\big)\big\}.$$

Define $f:\mathcal{P}_f(\Pi)\to\mathcal{I}$ as follows. Given $\Delta\in\mathcal{P}_f(\Pi)$, let F be the subset of $\mathbb{N}\times\mathbb{N}$ such that $\Delta=\{\mathcal{B}_{m,k}:(m,k)\in F\}$ and let

$$f(\Delta)=\langle\{2^{\kappa(F)}(2\mu(F)+n)\}\rangle_{n=1}^{\infty}.$$

Then (Π,f) is a sparse ordered union pair.

Proof. Conditions (1) and (2) of the definition are immediate. For (3), let $\Delta\in\mathcal{P}_f(\Pi)$ be given and let F be the subset of $\mathbb{N}\times\mathbb{N}$ such that $\Delta=\{\mathcal{B}_{m,k}:(m,k)\in F\}$. For $n\in\mathbb{N}$, let $X_n=\{2^{\kappa(F)}(2\mu(F)+n)\}$. Then $FU(\langle X_n\rangle_{n=1}^{\infty})=FU(\langle\{2^{\kappa(F)}n\}\rangle_{n=2\mu(F)+1}^{\infty})=\mathcal{B}_{2\mu(F),\kappa(F)}$. For $(m,k)\in F$, $\mathcal{B}_{2\mu(F),\kappa(F)}\subseteq\mathcal{B}_{m,k}$ so $FU(\langle X_n\rangle_{n=1}^{\infty})\subseteq\mathcal{B}_{m,k}$ as required for (3a).

Also

$$\begin{aligned}FU(\langle X_{2n}\rangle_{n=1}^{\infty})&=FU(\langle\{2^{\kappa(F)}(2\mu(F)+2n)\}\rangle_{n=1}^{\infty})\\&=FU(\langle\{2^{\kappa(F)+1}(\mu(F)+n)\}\rangle_{n=1}^{\infty})\\&=\mathcal{B}_{\mu(F),\kappa(F)+1}.\end{aligned}$$ □

We now introduce the partially ordered set with which we will apply Martin's Axiom.

Given $X\in\mathcal{F}$ and $\mathcal{G}\subseteq\mathcal{F}$, by $-X+\mathcal{G}$ we mean $\{Y\in\mathcal{F}:X\cup Y\in\mathcal{G}\}$.

Definition 3.16. Let Π be a nonempty set of infinite subsets of $\mathcal{F}$. Define

$$\begin{aligned}Q(\Pi)=\{(\mathcal{G},\Delta):\ &\mathcal{G}\in\mathcal{P}_f(\mathcal{F}),\ \Delta\in\mathcal{P}_f(\Pi)\text{ and whenever }X\text{ and }Y\text{ are}\\&\text{distinct elements of }\mathcal{G}\text{, either }\max X<\min Y\\&\text{or }\max Y<\min X\}.\end{aligned}$$

We define a partial ordering on $Q(\Pi)$ as follows. for $(\mathcal{G},\Delta),(\mathcal{G}'\Delta')\in Q(\Pi)$, we set $(\mathcal{G}',\Delta')\le(\mathcal{G},\Delta)$ if the following conditions hold:

(a) $\mathcal{G}\subseteq\mathcal{G}'$.

(b) $\Delta \subseteq \Delta'$.
(c) $(\forall Y \in \mathcal{G}' \setminus \mathcal{G})(\forall X \in \mathcal{G})(\max X < \min Y)$.
(d) $\mathcal{G}' \setminus \mathcal{G} \subseteq \bigcap \Delta$.
(e) There exists $g : \mathcal{G}' \setminus \mathcal{G} \to \Delta'$ such that:
  (i) $(\forall X \in \mathcal{G}' \setminus \mathcal{G})\big(g(X) \subseteq \bigcap \Delta \cap (-X + \bigcap \Delta)\big)$.
  (ii) $(\forall X, Y \in \mathcal{G}' \setminus \mathcal{G})\big(\max X < \min Y \Rightarrow Y \in g(X)$ and
$$g(Y) \subseteq g(X) \cap (-Y + g(X))\big).$$

Note that for applications of Martin's Axiom, partial orders need not be antisymmetric. However, the relation on $Q(\Pi)$ is trivially antisymmetric.

Lemma 3.17. *Let Π be a nonempty set of infinite subsets of $\mathcal{F}$. Then $Q(\Pi)$ is a nonempty partially ordered set.*

Proof. Pick $\mathcal{A} \in \Pi$ and pick $F \in \mathcal{F}$. Then $(\{F\}, \{\mathcal{A}\}) \in Q(\Pi)$ so $Q(\Pi) \neq \emptyset$. Trivially $\leq$ is reflexive. (For (e), $\emptyset : \emptyset \to \Delta = \Delta'$ is as required.)

To verify transitivity, let $(\mathcal{G}, \Delta), (\mathcal{G}', \Delta'), (\mathcal{G}'', \Delta'') \in Q(\Pi)$ with
$$(\mathcal{G}'', \Delta'') \leq (\mathcal{G}', \Delta') \leq (\mathcal{G}, \Delta).$$
Trivially $\mathcal{G} \subseteq \mathcal{G}''$ and $\Delta \subseteq \Delta''$. To verify (c), let $Y \in \mathcal{G}'' \setminus \mathcal{G}$ and let $X \in \mathcal{G}$. If $Y \in \mathcal{G}'$, then $\max X < \min Y$ since $Y \in \mathcal{G}' \setminus \mathcal{G}$. If $Y \notin \mathcal{G}'$, then $\max X < \min Y$ since $X \in \mathcal{G}'$.

To verify (d), let $X \in \mathcal{G}'' \setminus \mathcal{G}$. If $X \in \mathcal{G}'$, then $X \in \bigcap \Delta$ since $X \in \mathcal{G}' \setminus \mathcal{G}$. If $X \notin \mathcal{G}'$, then $X \in \mathcal{G}'' \setminus \mathcal{G}'$ so $X \in \bigcap \Delta' \subseteq \bigcap \Delta$.

To verify (e), let $g_1 : \mathcal{G}' \setminus \mathcal{G} \to \Delta'$ and $g_2 : \mathcal{G}'' \setminus \mathcal{G}' \to \Delta''$ be as guaranteed by the facts that $(\mathcal{G}', \Delta') \leq (\mathcal{G}, \Delta)$ and $(\mathcal{G}'', \Delta'') \leq (\mathcal{G}', \Delta')$. Let $g = g_1 \cup g_2$. Then $g : \mathcal{G}'' \setminus \mathcal{G} \to \Delta'$. To verify (ei), let $X \in \mathcal{G}'' \setminus \mathcal{G}$. If $X \in \mathcal{G}'$, then $g(X) = g_1(X) \subseteq \bigcap \Delta \cap (-X + \bigcap \Delta)$. If $X \notin \mathcal{G}'$, then
$$g(X) = g_2(X) \subseteq \bigcap \Delta' \cap \left(-X + \bigcap \Delta'\right) \subseteq \bigcap \Delta \cap \left(-X + \bigcap \Delta\right).$$

To verify (eii), let $X, Y \in \mathcal{G}'' \setminus \mathcal{G}$ with $\max X < \min Y$. If $\{X, Y\} \subseteq \mathcal{G}'' \setminus \mathcal{G}'$ or $\{X, Y\} \subseteq \mathcal{G}' \setminus \mathcal{G}$, the conclusion is immediate. By (c) the only other possibility is that $X \in \mathcal{G}' \setminus \mathcal{G}$ and $Y \in \mathcal{G}'' \setminus \mathcal{G}'$. Then $g(X) = g_1(X) \in \Delta'$ so $\bigcap \Delta' \subseteq g(X)$ and thus
$$g(Y) \subseteq \bigcap \Delta' \cap \left(-Y + \bigcap \Delta'\right) \subseteq g(X) \cap \big(-Y + g(X)\big). \qquad \square$$

Definition 3.18. Let Π be a nonempty set of infinite subsets of $\mathcal{F}$, let $\mathcal{V} \in \Pi$, and let $n \in \mathbb{N}$.

(1) $D(\mathcal{V}) = \{(\mathcal{G}, \Delta) \in Q(\Pi) : \mathcal{V} \in \Delta\}$.
(2) $E(n) = \{(\mathcal{G}, \Delta) \in Q(\Pi) : (\exists F \in \mathcal{G})(n < \min F)\}$.

Recall that in applications of Martin's Axiom, "dense" means "cofinal downward".

Lemma 3.19. *Let Π be a nonempty set of infinite subsets of $\mathcal{F}$ and let $\mathcal{V} \in \Pi$. Then $D(\mathcal{V})$ is dense in $Q(\Pi)$.*

Proof. If $(\mathcal{G},\Delta)\in Q(\Pi)$, then

$$(\mathcal{G},\Delta\cup\{\mathcal{V}\})\in Q(\Pi)\quad\text{and}\quad(\mathcal{G},\Delta\cup\{\mathcal{V}\})\leq(\mathcal{G},\Delta). \qquad \square$$

Lemma 3.20. *Let Π be a nonempty set of infinite subsets of $\mathcal{F}$ and let $n\in\mathbb{N}$. If there is some f such that (Π,f) is a sparse ordered union pair, then $E(n)$ is dense in $Q(\Pi)$.*

Proof. Pick f such that (Π,f) is a sparse ordered union pair. Let $(\mathcal{G},\Delta)\in Q(\Pi)$ and let $f(\Delta)=\langle X_t\rangle_{t=1}^{\infty}$. Pick $t\in\mathbb{N}$ such that $\min X_{2t}>n$ and $\min X_{2t}>\max\bigcup\mathcal{G}$. Let $\mathcal{B}=FU(\langle X_{2m}\rangle_{m=t+1}^{\infty})$. Then $\mathcal{B}\in\Pi$ and

$$(\mathcal{G}\cup\{X_{2t}\},\Delta\cup\{\mathcal{B}\})\in Q(\Pi)\cap E(n)\,.$$

We claim that $(\mathcal{G}\cup\{X_{2t}\},\Delta\cup\{\mathcal{B}\})\leq(\mathcal{G},\Delta)$. Requirements (a), (b), and (c) are immediate. Since $X_{2t}\subseteq FU(\langle X_j\rangle_{j=1}^{\infty})\subseteq\bigcap\Delta$, we have that (d) holds. To verify (e), define $g(X_{2t})=\mathcal{B}$. Then $\mathcal{B}\subseteq FU(\langle X_j\rangle_{j=1}^{\infty})\subseteq\bigcap\Delta$. To see that $\mathcal{B}\subseteq(-X_{2t}\cap\bigcap\Delta)$ let $Y\in\mathcal{B}$. Then $X_{2t}\cup Y\subseteq FU(\langle X_j\rangle_{j=1}^{\infty})\subseteq\bigcap\Delta$ so (ei) holds. And (eii) is vacuous. $\square$

Lemma 3.21. *Let Π be a nonempty set of infinite subsets of $\mathcal{F}$.*

(1) *If $(\mathcal{G},\Delta)$ and $(\mathcal{G}',\Delta')$ are incompatible, then $\mathcal{G}\neq\mathcal{G}'$. Consequently, $Q(\Pi)$ is a c.c.c. partial order.*
(2) *If $(\mathcal{G}',\Delta')\leq(\mathcal{G},\Delta)$, then $FU(\mathcal{G}'\setminus\mathcal{G})\subseteq\bigcap\Delta$.*

Proof. (1) If $\mathcal{G}=\mathcal{G}'$, then $(\mathcal{G},\Delta\cup\Delta')\leq(\mathcal{G},\Delta)$ and $(\mathcal{G},\Delta\cup\Delta')\leq(\mathcal{G}',\Delta')$.

(2) If $\mathcal{G}'\setminus\mathcal{G}=\{X\}$, then $FU(\mathcal{G}'\setminus\mathcal{G})=\{X\}\subseteq\bigcap\Delta$ by requirement (d) of Definition 3.16. Now assume that $n>1$ and $\mathcal{G}'\setminus\mathcal{G}=\{X_1,X_2,\ldots,X_n\}$ where, for each $t\in\{1,2,\ldots,n-1\}$, $\max X_t<\min X_{t+1}$. Pick $g:\mathcal{G}'\setminus\mathcal{G}\to\Delta'$ as guaranteed by (e) of Definition 3.16. We show by induction on $|T|$ that if $\emptyset\neq T\subseteq\{2,3,\ldots,n\}$ and $\min T=t$, then $\bigcup_{i\in T}X_i\in g(X_{t-1})$. Assume first that $|T|=1$. Then $X_{t-1},X_t\in\mathcal{G}'\setminus\mathcal{G}$ so by (eii), $X_t\in g(X_{t-1})$. Now assume that $|T|>1$, let $U=T\setminus\{t\}$ and let $u=\min U$. Then $\bigcup_{i\in U}X_i\in g(X_{u-1})$. If $u-1=t$, this says that $\bigcup_{i\in U}X_i\in g(X_t)$. If $u-1>t$, then $\max X_t<\min X_{u-1}$ so by (eii), $g(X_{u-1})\subseteq X_t$. Thus in either case $\bigcup_{i\in U}X_i\in g(X_t)$. Thus by (eii), $\bigcup_{i\in U}X_i\in -X_t+g(X_{t-1})$ so $\bigcup_{i\in T}X_i\in g(X_{t-1})$ as required.

Now let $L\subseteq\{1,2,\ldots,n\}$ with $\min L=l$. Assume first that $l>1$. Then $\bigcup_{i\in L}X_i\in g(X_{l-1})\subseteq g(X_1)\subseteq\bigcap\Delta$. Now assume that $l=1$. If $L=\{1\}$ we have by (d) that $X_l\in\bigcap\Delta$, so assume that $|L|>1$. Let $T=L\setminus\{1\}$ and let $t=\min T$. Then $\bigcup_{i\in T}X_i\in g(X_{t-1})\subseteq g(X_1)\subseteq -X_1+\bigcap\Delta$ by (ei) so $\bigcup_{i\in L}X_i\subseteq\bigcap\Delta$. $\square$

Lemma 3.22. *Let $\omega\leq\kappa<\mathfrak{c}$ and assume $MA(\kappa)$. Let (Π,f) be a sparse ordered union pair with $|\Pi|=\kappa$ and let $\mathcal{C}\subseteq\mathcal{F}$. There is a sparse ordered union pair (Ψ,g) such that:*

(1) $\Pi\subseteq\Psi$.
(2) $f\subseteq g$.

(3) $\mathcal{C} \in \Psi$ *or* $\mathcal{F} \setminus \mathcal{C} \in \Psi$.
(4) $|\Psi| = \kappa$.

Proof. By Lemmas 3.17 and 3.21(1), $Q(\Pi)$ is a c.c.c. partial order. By Lemmas 3.19 and 3.20, $\{D(\mathcal{V}) : \mathcal{V} \in \Pi\} \cup \{E(n) : n \in \mathbb{N}\}$ is a set of κ dense subsets of $Q(\Pi)$. Pick by $MA(\kappa)$ a filter G in $Q(\Pi)$ such that $G \cap D(\mathcal{V}) \neq \emptyset$ for each $\mathcal{V} \in \Pi$ and $G \cap E(n) \neq \emptyset$ for each $n \in \mathbb{N}$.

Since $G \cap E(n) \neq \emptyset$ for each $n \in \mathbb{N}$ we may choose a sequence $\langle F_t \rangle_{t=1}^{\infty}$ in $\mathcal{F}$ such that for each $t \in \mathbb{N}$, $\max F_t < \min F_{t+1}$ and there is some $(\mathcal{G}, \Delta) \in G$ such that $F_t \in \mathcal{G}$.

Pick by [8, Corollary 5.17] $\mathcal{D} \in \{\mathcal{C}, \mathcal{F} \setminus \mathcal{C}\}$ and a union subsystem $\langle X_t \rangle_{t=1}^{\infty}$ of $\langle F_t \rangle_{t=1}^{\infty}$ such that $FU(\langle X_t \rangle_{t=1}^{\infty}) \subseteq \mathcal{D}$. Let $\Psi = \Pi \cup \{\mathcal{D}\} \cup \{FU(\langle X_{2^k t} \rangle_{t=m}^{\infty}) : k, m \in \mathbb{N}\}$. Then conclusions (1), (3), and (4) hold. We claim that it suffices to show that

$$(*) \qquad \big(\forall \Delta \in \mathcal{P}_f(\Psi) \setminus \mathcal{P}_f(\Pi)\big)(\exists k, m \in \mathbb{N}) \left(FU(\langle X_{2^k t} \rangle_{t=m}^{\infty}) \subseteq \bigcap \Delta \right).$$

Assume we have done this. For $\Delta \in \mathcal{P}_f(\Psi)$, if $\Delta \subseteq \Pi$, let $g(\Delta) = f(\Delta)$. Otherwise, pick k and m as guaranteed by $(*)$ and let $g(\Delta) = \langle X_{2^k t} \rangle_{t=m}^{\infty}$. Then conclusion (2) holds. We need to show that (Ψ, g) is a sparse ordered union pair. Requirements (1) and (2) of Definition 3.14 hold. To verify (3), let $\Delta \in \mathcal{P}_f(\Psi)$. If $\Delta \subseteq \Pi$, then $g(\Delta) = f(\Delta)$ so (3a) and (3b) hold. So assume that $\Delta \setminus \Pi \neq \emptyset$ and pick k and m as guaranteed by $(*)$. For $t \in \mathbb{N}$, let $Y_t = X_{2^k(2m+t)}$. Then

$$FU(\langle Y_t \rangle_{t=1}^{\infty}) = FU(\langle X_{2^k(2m+t)} \rangle_{t=1}^{\infty}) \subseteq FU(\langle X_{2^k t} \rangle_{t=m}^{\infty}) \subseteq \bigcap \Delta$$

and, for $l \in \mathbb{N}$,

$$\begin{aligned} FU(\langle Y_{2t} \rangle_{t=l}^{\infty}) = FU(\langle X_{2^k(2m+2t)} \rangle_{t=l}^{\infty}) &= FU(\langle X_{2^{k+1}(m+t)} \rangle_{t=l}^{\infty}) \\ &= FU(\langle X_{2^{k+1} n} \rangle_{n=m+l}^{\infty}) \in \Psi. \end{aligned}$$

So we set out to establish $(*)$. Let $\Delta \in \mathcal{P}_f(\Psi) \setminus \mathcal{P}_f(\Pi)$. We may assume that $\Delta \cap \Pi \neq \emptyset$. We have that $\Delta \setminus \Pi \subseteq \{\mathcal{D}\} \cup \{FU(\langle X_{2^k t} \rangle_{t=m}^{\infty}) : k, m \in \mathbb{N}\}$ so pick $k, u \in \mathbb{N}$ such that $FU(\langle X_{2^k t} \rangle_{t=u}^{\infty}) \subseteq \bigcap(\Delta \setminus \Pi)$. For each $\mathcal{V} \in \Delta \cap \Pi$, pick $(\mathcal{G}_{\mathcal{V}}, \Delta_{\mathcal{V}}) \in G \cap D(\mathcal{V})$. Pick $(\mathcal{G}', \Delta') \in G$ such that $(\mathcal{G}', \Delta') \leq (\mathcal{G}_{\mathcal{V}}, \Delta_{\mathcal{V}})$ for each $\mathcal{V} \in \Delta \cap \Pi$.

Let $s = \max(\bigcup \mathcal{G}) + 1$. We claim that $FU(\langle F_t \rangle_{t=s}^{\infty}) \subseteq \bigcap(\Delta \cap \Pi)$. This will complete the proof for then we let $m = \max\{s, u\}$. Since $\langle X_t \rangle_{t=1}^{\infty}$ is a union subsystem of $\langle F_t \rangle_{t=1}^{\infty}$ we have $FU(\langle X_{2^k t} \rangle_{t=m}^{\infty}) \subseteq FU(\langle F_t \rangle_{t=s}^{\infty}) \subseteq \bigcap(\Delta \cap \Pi)$ and $FU(\langle X_{2^k t} \rangle_{t=m}^{\infty}) \subseteq FU(\langle X_{2^k t} \rangle_{t=u}^{\infty}) \subseteq \bigcap(\Delta \setminus \Pi)$.

So let $H \in \mathcal{P}_f(\mathbb{N})$ with $\min H \geq s$ be given. For $t \in H$, pick $(\mathcal{G}_t, \Delta_t) \in G$ such that $F_t \in \mathcal{G}_t$. Pick $(\mathcal{G}'', \Delta'') \in G$ such that $(\mathcal{G}'', \Delta'') \leq (\mathcal{G}', \Delta')$ and $(\mathcal{G}'', \Delta'') \leq (\mathcal{G}_t, \Delta_t)$ for each $t \in H$. Then for each $t \in H$, $F_t \in \mathcal{G}''$ and, since $\min F_t \geq t > \max \bigcup \mathcal{G}'$, we have $F_t \notin \mathcal{G}'$. By Lemma 3.21(2), we have $\bigcup_{t \in H} F_t \in FU(\mathcal{G}'' \setminus \mathcal{G}') \subseteq \bigcap \Delta'$ and $\bigcap \Delta' \subseteq \bigcap(\Delta \cap \Pi)$ since $(\mathcal{G}', \Delta') \leq (\mathcal{G}_{\mathcal{V}}, \Delta_{\mathcal{V}})$ for each $\mathcal{V} \in \Delta \cap \Pi$. $\square$

Theorem 3.23. *Let (Π, f) be a sparse ordered union pair with $\omega \leq |\Pi| < \mathfrak{c}$ and assume Martin's Axiom. There is a sparse ordered union ultrafilter Θ with $\Pi \subseteq \Theta$.*

Proof. Well order $\mathcal{P}(\mathcal{F})$ as $\langle \mathcal{C}_\sigma \rangle_{\sigma<\mathfrak{c}}$ with $\mathcal{C}_0 \in \Pi$. Let $\sigma < \mathfrak{c}$ and assume that we have chosen $\langle \Psi_\delta \rangle_{\delta<\sigma}$ and $\langle g_\delta \rangle_{\delta<\sigma}$ such that for each $\delta < \sigma$:

(1) (Ψ_δ, g_δ) is a sparse ordered union pair.
(2) If $\tau < \delta$, then $\Psi_\delta \subseteq \Psi_\tau$ and $g_\delta \subseteq g_\tau$.
(3) $\mathcal{C}_\delta \in \Psi_\delta$ or $\mathcal{F} \setminus \mathcal{C}_\delta \in \Psi_\delta$.
(4) $|\Psi_\delta| \leq \max\{|\Pi|, |\delta|\}$.

These hypotheses hold at $\delta = 0$, (2) vacuously. Let $\Psi'_\sigma = \bigcup_{\delta<\sigma} \Psi_\delta$ and $g'_\sigma = \bigcup_{\delta<\sigma} g_\delta$. It is routine to verify that $(\Psi'_\sigma, g'_\sigma)$ is a sparse ordered union pair. Also $|\Psi'_\sigma| \leq \max\{|\Pi|, |\sigma|\}$. (If $\sigma \leq |\Pi|$, then $|\Psi'_\sigma| \leq \sum_{\delta<\sigma} |\Pi| = |\Pi|$. If $\sigma > |\Pi|$, then $|\Psi'_\sigma| \leq \sum_{\delta<\sigma} |\sigma| = |\sigma|$.)

Pick by Lemma 3.22 a sparse ordered union pair (Ψ_σ, g_σ) such that $\Psi'_\sigma \subseteq \Psi_\sigma$, $g'_\sigma \subseteq g_\sigma$, $|\Psi_\sigma| = |\Psi'_\sigma|$, and either $\mathcal{C}_\sigma \in \Psi_\sigma$ or $\mathcal{F} \setminus \mathcal{C}_\sigma \in \Psi_\sigma$. Hypotheses (1) through (4) all hold.

The construction being complete, let $\Theta = \bigcup_{\sigma<\mathfrak{c}} \Psi_\sigma$. If (Ξ, h) is a sparse ordered union pair, then by Definition 3.14(3a), Ξ has the finite intersection property. Therefore by induction hypotheses (1) and (3), we have that Θ is an ultrafilter on $\mathcal{F}$. To see that Θ is a sparse ordered union ultrafilter, let $\mathcal{A} \in \Theta$. Pick $\sigma < \mathfrak{c}$ such that $\mathcal{A} \in \Psi_\sigma$, let $\Delta = \{\mathcal{A}\}$, and let $\langle X_n \rangle_{n=1}^\infty = g_\sigma(\Delta)$. Then $FU(\langle X_n \rangle_{n=1}^\infty) \subseteq \mathcal{A}$ and $FU(\langle X_{2n} \rangle_{n=1}^\infty \in \Psi_\sigma \subseteq \Theta$. □

Corollary 3.24. *Assume Martin's Axiom. There exists a sparse ordered union ultrafilter on $\mathcal{F}$.*

Proof. Lemma 3.15 and Theorem 3.23. □

Recall that in this section we are taking S to be the free semigroup with identity on the generators $\langle a_t \rangle_{t=1}^\infty$.

Corollary 3.25. *Assume Martin's Axiom. There exists a sparse very strongly productive ultrafilter on S.*

Proof. By Corollary 3.24, pick a sparse ordered union ultrafilter Θ. Let $p = \{C \subseteq S : (\exists \mathcal{A} \in \Theta)(\{\prod_{n\in B} a_n : B \in \mathcal{A}\} \subseteq C)\}$. By [11, Theorem 3.3], p is a very strongly productive ultrafilter. To see that p is sparse, let $C \in p$. By Definition 3.9, we need to show that there are a sequence $\langle x_t \rangle_{t=1}^\infty$ in S and an infinite set $D \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus D$ is infinite, $FP(\langle x_t \rangle_{t=1}^\infty) \subseteq C$, and $FP(\langle x_t \rangle_{t\in D}) \in p$.

Pick $\mathcal{A} \in \Theta$ such that $\{\prod_{n\in B} a_n : B \in \mathcal{A}\} \subseteq C$. By Definition 3.12 we may pick a sequence $\langle X_n \rangle_{n=1}^\infty$ of members of $\mathcal{F}$ such that $\max X_n < \min X_{n+1}$ for each n and an infinite subset D of $\mathbb{N}$ such that $\mathbb{N} \setminus D$ is infinite, $FU(\langle X_n \rangle_{n=1}^\infty) \subseteq \mathcal{A}$, and $FU(\langle X_n \rangle_{n\in D}) \in \Theta$. For each $n \in \mathbb{N}$, let $x_n = \prod_{t\in X_n} a_t$. Since $\max X_n < \min X_{n+1}$ for each n, we have that if $H \in \mathcal{P}_f(\mathbb{N})$ and $K = \bigcup_{n\in H} X_n$, then $\prod_{n\in H} x_n = \prod_{t\in K} a_t$. Therefore $\langle x_n \rangle_{n=1}^\infty$ is as required. □

Recall from the introduction that there are many situations in which it is known that all strongly summable ultrafilters are sparse.

Question 3.26. *Let S be the free semigroup on countably many generators. Are all very strongly productive ultrafilters on S sparse?*

4. More idempotents which are products only trivially

Let S be the free semigroup on the generators $\langle a_t\rangle_{t=1}^\infty$ and let $\mathcal{F}=\mathcal{P}_f(\mathbb{N})$. Denote by $\uplus$ the operation on $\beta\mathcal{F}$ extending the operation $\cup$ on $\mathcal{F}$ making $(\beta\mathcal{F},\uplus)$ a right topological semigroup with $\mathcal{F}$ contained in its topological center. (Normally we use the same symbol to denote the extended operation. But in this case, if $\Theta,\Psi\in\beta\mathcal{F}$, then $\Theta\cup\Psi$ already means something.) We show in this section that Martin's Axiom implies that there is an idempotent p in βS which is not very strongly productive, in fact is not even strongly productive, and p can only be written trivially as a product. We also show that the existence of a union ultrafilter implies that there is an idempotent p in $(\beta\mathbb{N},\cdot)$ which can only be written trivially as a product and that there is an idempotent Θ in $(\beta\mathcal{F},\uplus)$ so that, if Ψ and Ξ are in $\beta\mathcal{F}$ and $\Psi\uplus\Xi=\Theta$, then $\Psi=\Xi=\Theta$.

We begin by showing in Theorem 4.2 that if p is a strongly productive ultrafilter on S and $FP(\langle a_t\rangle_{t=1}^\infty)\in p$, then in fact p is very strongly productive.

Lemma 4.1. *Let S be the free semigroup on the generators $\langle a_t\rangle_{t=1}^\infty$ and let $\langle x_t\rangle_{t=1}^\infty$ be a sequence in S. If $FP(\langle x_t\rangle_{t=1}^\infty)\subseteq FP(\langle a_t\rangle_{t=1}^\infty)$, then $\langle x_t\rangle_{t=1}^\infty$ is a product subsystem of $\langle a_t\rangle_{t=1}^\infty$.*

Proof. For each $n\in\mathbb{N}$ pick $H_n\in\mathcal{P}_f(\mathbb{N})$ such that $x_n=\prod_{t\in H_n}a_t$. We claim that for each n, $\max H_n<\min H_{n+1}$. Otherwise

$$x_n\cdot x_{n+1}=\prod_{t\in H_n}a_t\cdot\prod_{t\in H_{n+1}}a_t\notin FP(\langle a_t\rangle_{t=1}^\infty).\qquad\square$$

Theorem 4.2. *Let S be the free semigroup on the generators $\langle a_t\rangle_{t=1}^\infty$ and let p be a strongly productive ultrafilter on S such that $FP(\langle a_t\rangle_{t=1}^\infty)\in p$. Then p is a very strongly productive ultrafilter.*

Proof. Let $A\in p$. Pick a sequence $\langle x_t\rangle_{t=1}^\infty$ such that

$$FP(\langle x_t\rangle_{t=1}^\infty)\subseteq A\cap FP(\langle a_t\rangle_{t=1}^\infty)\text{ and }FP(\langle x_t\rangle_{t=1}^\infty)\in p\,.$$

By Lemma 4.1, $\langle x_t\rangle_{t=1}^\infty$ is a product subsystem of $\langle a_t\rangle_{t=1}^\infty$. $\square$

When we say that a sequence $\langle x_t\rangle_{t=1}^\infty$ satisfies uniqueness of finite products, we mean that whenever $F,H\in\mathcal{P}_f(\mathbb{N})$ and $\prod_{t\in F}x_t=\prod_{t\in H}x_t$, one must have that $F=H$.

The subsemigroup $\mathbb{H}$ of $(\beta\mathbb{N},+)$ is defined by $\mathbb{H}=\bigcap_{n=1}^\infty\overline{2^n\mathbb{N}}$. This semigroup contains all of the idempotents of $(\beta\mathbb{N},+)$ and much of the remaining known algebraic structure of $(\beta\mathbb{N},+)$. See [8, Section 6.1]. The proof of the following lemma is only a slight variation of the proof of [8, Theorem 6.27] so we omit it.

Lemma 4.3. *Let S be any semigroup, let $\langle x_t\rangle_{t=1}^\infty$ be a sequence in S satisfying uniqueness of finite products, and let $T = \bigcap_{n=1}^\infty \overline{FP(\langle x_t\rangle_{t=n}^\infty)}$. Define $\varphi : \mathbb{N} \to S$ by, for $H \in \mathcal{P}_f(\mathbb{N})$, $\varphi(\sum_{t\in H} 2^{t-1}) = \prod_{t\in H} x_t$ and let $\widetilde{\varphi} : \beta\mathbb{N} \to \beta S$ be the continuous extension of φ. The restriction of $\widetilde{\varphi}$ to $\mathbb{H}$ is an isomorphism and a homeomorphism onto T.*

Lemma 4.4. *Define $\psi : \mathcal{F} \to \mathbb{N}$ by, for $F \in \mathcal{F}$, $\psi(F) = \sum_{t\in F} 2^{t-1}$ and let $\widetilde{\psi} : \beta\mathcal{F} \to \beta\mathbb{N}$ be its continuous extension. If Θ is a union ultrafilter on $\mathcal{F}$, then $\widetilde{\psi}(\Theta)$ is a strongly summable ultrafilter on $\mathbb{N}$.*

Proof. This is the easy half of [2, Theorem 1]. $\square$

Lemma 4.5. *Let S be the free semigroup on the generators $\langle a_t\rangle_{t=1}^\infty$, define $\varphi : \mathbb{N} \to S$ and $\psi : \mathcal{F} \to \mathbb{N}$ by, for $F \in \mathcal{F}$, $\varphi(\sum_{t\in F} 2^{t-1}) = \prod_{t\in F} a_t$ and $\psi(F) = \sum_{t\in F} 2^{t-1}$. Let $\widetilde{\varphi} : \beta\mathbb{N} \to \beta S$ and $\widetilde{\psi} : \beta\mathcal{F} \to \beta\mathbb{N}$ be the continuous extensions of φ and ψ. Let $\Theta \in \beta\mathcal{F}$ and let $p = \widetilde{\varphi}\big(\widetilde{\psi}(\Theta)\big)$. If p is a very strongly productive ultrafilter, then Θ is an ordered union ultrafilter.*

Proof. Define $\tau : \mathcal{F} \to S$ by, for $F \in \mathcal{F}$, $\tau(F) = \prod_{t\in F} a_t$. Let $\widetilde{\tau} : \beta\mathcal{F} \to \beta S$ be its continuous extension. Then $\tau = \varphi \circ \psi$ so $p = \widetilde{\tau}(\Theta)$.

To see that Θ is an ordered union ultrafilter, let $\mathcal{W} \in \Theta$. Then $\tau[\mathcal{W}] \in p$. Pick a product subsystem $\langle x_t\rangle_{t=1}^\infty$ of $\langle a_t\rangle_{t=1}^\infty$ such that $FP(\langle x_t\rangle_{t=1}^\infty) \subseteq \tau[\mathcal{W}]$ and $FP(\langle x_t\rangle_{t=1}^\infty) \in p$. Pick a sequence $\langle H_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that for each n, $x_n = \prod_{t\in H_n} a_t$ and $\max H_n < \min H_{n+1}$. For each n, pick $F_n \in \mathcal{W}$ such that $x_n = \tau(F_n)$. Then $\prod_{t\in F_n} a_t = x_n = \prod_{t\in H_n} a_t$, so $F_n = H_n$. We have $\tau^{-1}[FP(\langle x_t\rangle_{t=1}^\infty)] \in \Theta$, $\tau^{-1}[FP(\langle x_t\rangle_{t=1}^\infty)] \subseteq \tau^{-1}\big[\tau[\mathcal{W}]\big] = \mathcal{W}$, and $\tau^{-1}[FP(\langle x_t\rangle_{t=1}^\infty)] = FU(\langle F_n\rangle_{n=1}^\infty)$. $\square$

Lemma 4.6. *Let S be the free semigroup on the generators $\langle a_t\rangle_{t=1}^\infty$ and let $T = \bigcap_{n=1}^\infty \overline{FP(\langle a_t\rangle_{t=n}^\infty)}$. If $r, s \in \beta S$ and $rs \in T$, then $r \in T$ and $s \in T$.*

Proof. Let $n \in \mathbb{N}$. We will show that $FP(\langle a_t\rangle_{t=n}^\infty) \in r$ and $FP(\langle a_t\rangle_{t=n}^\infty) \in s$. Since $FP(\langle a_t\rangle_{t=n}^\infty) \in rs$, we have $B = \{x \in S : x^{-1}FP(\langle a_t\rangle_{t=n}^\infty) \in s\} \in r$. We claim $B \subseteq FP(\langle a_t\rangle_{t=n}^\infty)$, so let $x \in B$ and pick $y \in x^{-1}FP(\langle a_t\rangle_{t=n}^\infty)$. Then $xy = \prod_{t\in H} a_t$ for some $H \in \mathcal{P}_f(\mathbb{N})$ such that $\min H \geq n$. Therefore $x \in FP(\langle a_t\rangle_{t=n}^\infty)$ and $y \in FP(\langle a_t\rangle_{t=n}^\infty)$. Thus $B \subseteq FP(\langle a_t\rangle_{t=n}^\infty)$ and, since y was an arbitrary member of $x^{-1}FP(\langle a_t\rangle_{t=n}^\infty)$, $x^{-1}FP(\langle a_t\rangle_{t=n}^\infty) \subseteq FP(\langle a_t\rangle_{t=n}^\infty)$. $\square$

All previously known examples of elements of βS which could not be written nontrivially as a product were very strongly productive.

Theorem 4.7. *Let S be the free semigroup on the generators $\langle a_t\rangle_{t=1}^\infty$ and assume Martin's Axiom. There exists an idempotent $p \in \beta S$ such that:*

(1) *If $r, s \in \beta S$ and $rs = p$, then $r = s = p$.*

(2) *p is not strongly productive.*

Proof. By [2, Theorem 5] pick a union ultrafilter Θ on $\mathcal{F}$ such that Θ is not an ordered union ultrafilter. Let ψ, φ, $\widetilde{\psi}$, and $\widetilde{\varphi}$ be as in Lemma 4.5. Let $T = \bigcap_{n=1}^{\infty} \overline{FP(\langle a_t\rangle_{t=n}^{\infty})}$ and let $q = \widetilde{\psi}(\Theta)$. By Lemma 4.4, q is strongly summable.

Now $(\mathbb{N}, +)$ can be embedded in the circle group so by [7, Corollary 4.3], if $x, y \in \mathbb{N}^*$ and $x + y = q$, then $x, y \in \mathbb{Z} + q$. Consequently, if $x, y \in \mathbb{H}$ and $x + y = q$, then $x, y \in (\mathbb{Z} + q) \cap \mathbb{H} = \{q\}$.

Let $p = \widetilde{\varphi}(q)$. By Lemma 4.3, p is an idempotent and $p \in T$. Assume that $r, s \in \beta S$ and $rs = p$. By Lemma 4.6, $r \in T$ and $s \in T$ so by Lemma 4.3 pick $x, y \in \mathbb{H}$ such that $r = \widetilde{\varphi}(x)$ and $s = \widetilde{\varphi}(y)$. Then $\widetilde{\varphi}(x + y) = rs = p$ so $x + y = q$. Therefore $x = y = q$ and thus $r = s = p$.

Finally suppose that p is strongly productive. By Theorem 4.2 p is very strongly productive so by Lemma 4.5, Θ is an ordered union ultrafilter, a contradiction. □

The following corollary is an immediate consequence of the proof of Theorem 4.7.

Corollary 4.8. *Let φ, and $\widetilde{\varphi}$ be as in Lemma 4.5 and assume Martin's Axiom. There is a strongly summable ultrafilter q on $\mathbb{N}$ such that $\widetilde{\varphi}(q)$ is not strongly productive.*

We conclude the paper with some results which are consequences of the existence of union ultrafilters. This is certainly a weaker assumption than Martin's Axiom since it is known that the existence of union ultrafilters follows from the axiom known as $P(\mathfrak{c})$. (See the discussion in [2, Page 97].) It is not known whether this is a weaker assumption than the existence of ordered union ultrafilters.

Theorem 4.9. *Let S be any semigroup, let $\langle x_t\rangle_{t=1}^{\infty}$ be a sequence in S satisfying uniqueness of finite products, and let $T = \bigcap_{n=1}^{\infty} \overline{FP(\langle x_t\rangle_{t=n}^{\infty})}$. Define φ and $\widetilde{\varphi}$ as in Lemma 4.3. If whenever $r, s \in \beta S$ and $rs \in T$, one must have $r \in T$ and $s \in T$, then for any strongly summable ultrafilter q on $\mathbb{N}$, if $r, s \in \beta S$ and $\widetilde{\varphi}(q) = rs$, then $r = s = \widetilde{\varphi}(q)$.*

Proof. Pick $r, s \in \beta S$ such that $\widetilde{\varphi}(q) = rs$. Then $r, s \in T$ so by Lemma 4.3, pick $x, y \in \mathbb{H}$ such that $\widetilde{\varphi}(x) = r$ and $\widetilde{\varphi}(y) = s$. Then $x + y = q$ so by [7, Corollary 4.3], $x, y \in \mathbb{H} \cap (\mathbb{Z} + q) = \{q\}$. Thus $x = y = q$ so $r = s = \widetilde{\varphi}(q)$. □

Corollary 4.10. *Assume there exists a union ultrafilter on $\mathcal{F}$. There is an idempotent p in $(\beta\mathbb{N}, \cdot)$ such that if $r, s \in \beta\mathbb{N}$ and $rs = p$, then $r = s = p$.*

Proof. By Lemma 4.4, pick a strongly summable ultrafilter q on $\mathbb{N}$. Let $\langle x_t\rangle_{t=1}^{\infty}$ be a sequence of distinct primes and define $T = \bigcap_{n=1}^{\infty} \overline{FP(\langle x_t\rangle_{t=n}^{\infty})}$. By Theorem 4.9 it suffices to show that if $r, s \in \beta\mathbb{N}$ and $rs \in T$, then $r \in T$ and $s \in T$. This follows easily from the fact that if $n, y, z \in \mathbb{N}$ and

$yz \in FP(\langle x_t\rangle_{t=n}^{\infty})$, then all prime factors of y and of z are in $\{x_t : t \geq n\}$ and neither y nor z has a repeated prime factor. $\square$

Corollary 4.11. *Assume there exists a union ultrafilter on $\mathcal{F}$. There is an idempotent Θ in $\beta\mathcal{F}$ such that if $\Psi, \Xi \in \beta\mathcal{F}$ and $\Psi \uplus \Xi = \Theta$, then $\Psi = \Xi = \Theta$.*

Proof. By Lemma 4.4, pick a strongly summable ultrafilter q on $\mathbb{N}$. For each $n \in \mathbb{N}$, let $X_n = \{n\}$. Then, given $n \in \mathbb{N}$,

$$FU(\langle X_t\rangle_{t=n}^{\infty}) = \{H \in \mathcal{F} : \min H \geq n\}\,.$$

Note that $\langle X_n\rangle_{n=1}^{\infty}$ satisfies uniqueness of finite unions. Define $\varphi : \mathbb{N} \to \mathcal{F}$ by, for $H \in \mathcal{F}$, $\varphi(\sum_{t\in H} 2^{t-1}) = H$ and let $T = \bigcap_{n=1}^{\infty} \overline{FU(\langle X_t\rangle_{t=n}^{\infty})}$.

By Theorem 4.9 it suffices to show that if $\Psi, \Xi \in \beta\mathcal{F}$ and $\Psi \uplus \Xi \in T$, then $\Psi \in T$ and $\Xi \in T$. To this end, let $n \in \mathbb{N} \setminus \{1\}$. We need to show that $\{H \in \mathcal{F} : \min H \geq n\} \in \Psi$ and $\{H \in \mathcal{F} : \min H \geq n\} \in \Xi$. Now $\{H \in \mathcal{F} : \min H < n\}$ is an ideal of $(\mathcal{F}, \cup)$ so by [8, Corollary 4.18], $\overline{\{H \in \mathcal{F} : \min H < n\}}$ is an ideal of $(\beta\mathcal{F}, \uplus)$ so if either

$$\{H \in \mathcal{F} : \min H \geq n\} \notin \Psi \text{ or } \{H \in \mathcal{F} : \min H \geq n\} \notin \Xi\,,$$

we would have $\{H \in \mathcal{F} : \min H \geq n\} \notin \Psi \uplus \Xi$. $\square$

References

[1] Blass, Andreas. Ultrafilters related to Hindman's finite-unions theorem and its extensions. *Logic and Combinatorics*, 89–124. Contep. Math., 65, *Amer. Math. Soc., Providence, RI*, 1987. MR0891244 (88g:04002), Zbl 0634.03045, doi: 10.1090/conm/065/891244.

[2] Blass, Andreas; Hindman, Neil. On strongly summable ultrafilters and union ultrafilters. *Trans. Amer. Math. Soc.* **304** (1987), no. 1, 83–97. MR0906807 (88i:03080), Zbl 0643.03032, doi: 10.2307/2000705.

[3] Fernández Bretón, David J. Every strongly summable ultrafilter on $\bigoplus \mathbb{Z}_2$ is sparse. *New York J. Math.* **19** (2013), 117–129. MR3065919, Zbl 06220382, arXiv:1302.5676.

[4] Hindman, Neil. Summable ultrafilters and finite sums. *Logic and Combinatorics* 263–274. Contemp. Math. 65, *Amer. Math. Soc., Providence, RI*, (1987). MR0891252 (88h:03070), Zbl 0634.03046, doi: 10.1090/conm/065/891252.

[5] Hindman, Neil. Strongly summable ultrafilters on $\mathbb{N}$ and small maximal subgroups of $\beta\mathbb{N}$. *Semigroup Forum* **42** (1991), no. 1, 63–75. MR1075195 (92a:54025), Zbl 0716.22002, doi: 10.1007/BF02573407.

[6] Hindman, Neil; Protasov, Igor; Strauss, Dona. Strongly summable ultrafilters on abelian groups. *Mat. Stud.* **10** (1998), no, 2, 121–132. MR1687143 (2001d:22003), Zbl 0934.22005.

[7] Hindman, Neil; Steprāns, Juris; Strauss, Dona. Semigroups in which all strongly summable ultrafilters are sparse. *New York J. Math.* **18** (2012), 835–848. MR2991425, Zbl 1257.54029.

[8] Hindman, Neil; Strauss, Dona. Algebra in the Stone–Čech compactification. Theory and applications. Second edition. *Walter de Gruyter & Co., Berlin*, 2012. xviii+591 pp. ISBN: 978-3-11-025623-9. MR2893605, Zbl 1241.22001, doi: 10.1515/9783110258356.

[9] Krautzberger, Peter. On strongly summable ultrafilters. *New York J. Math.* **16** (2010), 629–649. MR2740593 (2012k:03135), Zbl 1234.03034, arXiv:1006.3816.

[10] Kunen, Kenneth. Set theory. An introduction to independence proofs. Studies in Logic and Foundations of Mathematics, 102. *North–Holland Publishing Co., Amsterdam–New York*, 1980. xvi+313 pp. ISBN: 0-444-85401-0. MR0597342 (82f:03001). Zbl 0443.03021, http://projecteuclid.org/euclid.bams/1183551426.

[11] Legette, Lakeshia. Maximal groups in βS can be trivial. *Topology Appl.* **156** (2009), no. 16, 2632–2641. MR2561215 (2010h:22003), Zbl 1181.22007, doi: 10.1016/j.topol.2009.04.022.

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