

L_∞ spaces and derived loop spaces

Ryan Grady and Owen Gwilliam

ABSTRACT. We develop further the approach to derived differential geometry introduced in Costello’s work on the Witten genus (arXiv, 2011). In particular, we introduce several new examples of L_∞ spaces, discuss vector bundles and shifted symplectic structures on L_∞ spaces, and examine in some detail the example of derived loop spaces. This paper is background for a forthcoming paper in which we define a quantum field theory on a derived stack, building upon Costello’s definition of an effective field theory (AMS Monographs, 2011).

CONTENTS

1. Introduction and overview	231
2. The category of test objects	235
3. Derived stacks	241
4. L_∞ spaces	243
5. Geometry with L_∞ spaces	252
6. The derived loop space	254
Appendix A. Homotopy limits and cosimplicial simplicial sets	259
Appendix B. Brief overview of the Maurer–Cartan functor	263
Appendix C. Proof of Theorem 4.8	264
References	269

1. Introduction and overview

In his work constructing the Witten genus with a two-dimensional sigma model [Cos11], Costello introduces a framework for derived differential geometry. His approach is functorial in nature: he defines a *derived stack* as a functor from a category of test objects to the category of simplicial sets, satisfying some conditions familiar from geometry. His test objects are a

Received April 24, 2014.

2010 *Mathematics Subject Classification*. 14D23; 13D10, 18G55, 17B55.

Key words and phrases. derived geometry, deformation theory, higher Lie theory.

The first author was partially supported by the National Science Foundation under Award DMS-1309118.

The second author was supported as a postdoctoral fellow by the National Science Foundation under Award DMS-1204826.

modest enlargement of smooth manifolds to include nilpotent and derived-nilpotent directions, which is why this approach is a version of derived differential geometry. The conditions characterizing his derived stacks are mild.¹ Moreover, he points out a nice subset of derived stacks that have a kind of “ringed space” description, which he calls L_∞ spaces, that exploit the intimate relationship between dg Lie algebras and deformation theory. The appeal of L_∞ spaces is that they allow one to do explicit computations very efficiently, particularly computations that appear in (perturbative) quantum field theory (QFT).

Our goal here is to verify some simple properties of Costello’s formalism and to point out some appealing features. Much of the paper simply elaborates on Costello’s work in [Cos11]. In a companion paper [GG], we articulate how these definitions allow one to extend the reach of his formalism for perturbative quantum field theory to encompass less-perturbative QFT, developing explicitly some ideas that are implicit in Costello’s work.

1.1. An overview of the paper. We begin by developing the definition of a derived stack, in Costello’s sense. We then introduce a special class of derived stacks, the L_∞ spaces, which can be thought of as families of formal moduli problems parametrized by smooth manifolds. Thereafter, we introduce some simple examples of L_∞ spaces and a modicum of geometry on L_∞ spaces (notably vector bundles and shifted symplectic forms). Finally, we discuss various kinds of derived loop spaces that are relevant to our work on a 1-dimensional sigma model [GG14].

1.2. The motivation from physics. Before delving into the text itself, we remark on the motivations behind this approach. The immediate impetus arose from Costello’s goal of encoding a nonlinear sigma model into his formalism for quantum field theory [Cos11b]. The reader interested just in derived geometry is welcome to skip this short discussion. As we have not yet defined any of the principal objects of our formalism, we will speak loosely in geometric language.

Let M be the smooth manifold on which the fields of our field theory will live. In his book, Costello defined families of field theories over *nilpotent dg manifolds*. More explicitly, for \mathcal{N} a nilpotent dg manifold, suppose we have an M -fiber bundle $M \hookrightarrow P \rightarrow \mathcal{N}$ and a relative vector bundle $V \rightarrow P$. Costello’s formalism of renormalization works in this relative situation, where for each point x in \mathcal{N} , we have a field theory on $V_x \rightarrow M_x$ whose underlying fields are the smooth sections of the fiberwise vector bundle.

¹It would be interesting to work out the analogs of Deligne–Mumford and Artin stacks in this setting, as Costello’s conditions simply pick out homotopy sheaves and hence encompass functors that are not very geometric. It would also be quite useful and interesting to compare this approach to the approaches of Carchedi–Roytenberg [CarR13], Borisov–Noel [BorN], Lurie [Lura], Spivak [Spi10], Schreiber [Sch], and Joyce [Joy].

Thus Costello needed to rephrase a nonlinear sigma model — a field theory whose fields are $\text{Maps}(M, X)$, the space of smooth maps from M into a manifold X — within such a framework, where fields are sections of a vector bundles. There is a standard idea from physics that suggests how to do this, and L_∞ spaces allow one to formulate it as a mathematical procedure.

We would like to compute the path integral

$$\int_{\phi \in \text{Maps}(M, X)} e^{-S(\phi)/\hbar} \mathcal{D}\phi,$$

or — more accurately — provide an asymptotic expansion for this integral in the regime where \hbar is infinitesimally small. For very small \hbar , the measure should be concentrated in a small tubular neighborhood Tub of Sol , the subspace cut out by the solutions to the equations of motion. Thus, the integral can be well-approximated by pulling back the measure to the neighborhood Tub . We then identify this tubular neighborhood with the normal bundle N to Sol and compute the approximate integral in two steps. First, we use perturbative methods to compute the fiberwise integral and obtain a measure on Sol . Second, we integrate over Sol itself. This integral over Sol breaks up into a sum over the connected components of Sol .

Among the connected components of Sol , the lowest-energy solutions for a sigma model are given typically by the constant maps, and this component of the space of solutions looks like a copy of the target manifold X itself. This component usually provides the dominant term in the sum over components. Hence, our path integral should be well-approximated by integrating over a tubular neighborhood T just of the constant maps $X \hookrightarrow \text{Maps}(M, X)$. Note that a really small perturbation of such a constant map $\underline{x} : M \rightarrow x \in X$ can be viewed as a map from M to a small ball in X around x .

In physics, one often applies this heuristic idea as follows. One views T as a vector bundle over X whose fiber at $x \in X$ is $\Gamma(M, \underline{x}^*T_x X)/T_x X$. (This vector space takes the quotient of all the smooth sections of the pullback tangent bundle — which are the ways of wiggling the constant map x — by the subspace of sections that just move to a nearby constant map.) Formal derived geometry provides a nice mathematical language for describing the formal neighborhood \mathcal{T} of the constant maps inside $\text{Maps}(M, X)$. Indeed, Costello showed that \mathcal{T} is an L_∞ space. He then showed that his perturbative formalism interacted cleanly with L_∞ spaces to realize the two-step process of integration. In the language of [Cos11], the quantum BV theory he produces out of \mathcal{T} provides a *projective volume form*.

In our companion paper, we explain in more depth both the heuristic picture of path integral quantization of the nonlinear sigma model and the precise realization of that idea using Costello’s pair of formalisms (for QFT and derived geometry).

1.3. A comment on our imagined audience. Our primary purpose for this formalism is the construction of quantum field theories, and we imagine that some of our audience has a working knowledge of homological algebra and category theory at the level of Weibel [Wei94] and some familiarity with the language of L_∞ algebras, due to their role in deformation quantization *à la* Kontsevich [Kon03]. We have thus sought to explain constructions or concepts, like homotopy limits, that we use that fall outside those references. Those readers comfortable with homotopical algebra are encouraged to skip over such discussions.

1.4. Acknowledgements. First and foremost, we thank Kevin Costello for many helpful conversations and for introducing us to these ideas in the context of field theory. Our understanding on derived geometry, however limited, is rooted in discussions with John Francis, David Nadler, Anatoly Preygel, and Nick Rozenblyum. The influence of the ideas of Dennis Gaitsgory, Jacob Lurie, Bertrand Toën, and Gabriele Vezzosi should also be clear: we thank them for their inspiring ideas, lectures, and texts. We have also benefited from conversations with many people on this topic, including David Ayala, David Carchedi, Lee Cohn, Vasily Dolgushev, Si Li, Thel Seraphim, Yuan Shen, Jim Stasheff, Mathieu Stienon, Stephan Stolz, Peter Teichner, Ping Xu, and Brian Williams.

1.5. Notation.

- For M a smooth manifold, let Ω_M^* denote the sheaf of differential forms on M as a sheaf of commutative dg algebras. Let \mathbb{C}_M denote the locally constant sheaf assigning \mathbb{C} to any connected open.
- For \mathcal{C} a category, we denote the set of morphisms from x to y by $\mathcal{C}(x, y)$.
- For A a cochain complex, A^\sharp denotes the underlying graded vector space. If A is a cochain complex whose degree k space is A^k , then $A[1]$ is the cochain complex where $A[1]^k = A^{k+1}$.
- For A a dg module over a dg algebra R , the *dual* of A , denoted A^\vee , means the graded dual, whose k th component $(A^\vee)^k$ is the set of degree $-k$ elements of $\text{Hom}_{R^\sharp}(A^\sharp, R^\sharp)$. The differential is determined by requiring that the evaluation pairing $\text{ev} : A^\vee \otimes_{R^\sharp} A \rightarrow R$ be a map of R -modules.
- Let Sets denote the category whose objects are sets and whose morphisms are functions between sets.
- Let Δ denote the (finite) *ordinal category*, in which an object is a totally ordered finite set and a morphism is a nondecreasing function. We will usually restrict attention to the skeletal subcategory with objects

$$[n] := \{0 < 1 < \dots < n\}$$

and morphisms $f : [m] \rightarrow [n]$ such that $f(i) \leq f(j)$ for $i \leq j$.

- Let $s\text{Sets}$ denote the category of simplicial sets, namely the category of functors $\text{Fun}(\Delta^{\text{op}}, \text{Sets})$. A simplicial set X will often be written as X_\bullet , and $X_n := X([n])$ denotes the “set of n -simplices of X .”
- We denote by $\Delta[m]$ the simplicial set $\Delta(-, [m])$. Under the Yoneda lemma, this is the functor represented by the object $[m]$.²
- Let $c\text{Sets}$ denote the category *cosimplicial* sets $\text{Fun}(\Delta, \text{Sets})$. A cosimplicial set Y will often be written as Y^\bullet , and $Y^n := Y([n])$ denotes the “set of n -cosimplices of Y .”
- Let $cs\text{Sets}$ denote the category of cosimplicial simplicial sets

$$\text{Fun}(\Delta \times \Delta^{\text{op}}, \text{Sets}).$$

We will often denote a cosimplicial simplicial set Z by Z_\bullet^\bullet , and Z_\bullet^n is the simplicial set of n -cosimplices and Z_n^\bullet is the cosimplicial set of n -simplices.

- By Δ^n we mean the standard n -simplex in \mathbb{R}^n .
- To indicate the end of an example or remark, we use the symbol \diamond , just as we use \square to indicate the end of a proof.

2. The category of test objects

Let Man denote the category of smooth, finite-dimensional manifolds (without boundary) and smooth morphisms. From here on, *manifold* will mean smooth and finite-dimensional.

We “enlarge” the category Man by allowing a certain class of structure sheaves following ideas and constructions of Costello [Cos11b], [Cos11].

Definition 2.1. On a smooth manifold M , let

$$\text{Sym}(T_M^*[-1]) = \bigoplus_{n=0}^{\dim M} \bigwedge^n T_M^*[-n]$$

denote the \mathbb{Z} -graded vector bundle whose smooth sections are the differential forms on M (with their usual cohomological degree). Note that $\text{Sym}(T_M^*[-1])$ is a graded-commutative algebra in the category of vector bundles on M by (fiberwise) wedge product. This product induces the wedge product on its smooth sections, namely the wedge product of differential forms.

Definition 2.2. A *nilpotent dg manifold* \mathcal{M} is a triple $(M, \mathcal{O}_\mathcal{M}, \mathcal{I}_\mathcal{M})$ consisting of the following data and conditions.

- (1) A manifold M .
- (2) A \mathbb{Z} -graded vector bundle $A \rightarrow M$ of total finite rank equipped with:
 - a) The structure of a module over $\text{Sym}(T_M^*[-1])$.

²Note that we use the Greek letter Δ to denote the category and the triangle symbol \triangle to denote a simplex.

- b) A fiberwise multiplication map $m : A \otimes A \rightarrow A$ making A into a unital graded-commutative algebra in vector bundles on M ,³ so that (A, m) is an algebra over $\text{Sym}(T_M^*[-1])$.
- c) There is a short exact sequence of vector bundles

$$I \hookrightarrow A \rightarrow A/I \cong \mathbb{C} \times M$$

(i.e., the quotient is the trivial rank 1 bundle) that respects the multiplication m and an associated chain of vector bundles

$$0 = I^{n+1} \subset I^n \subset I^{n-1} \subset \dots \subset I$$

compatible with multiplication (i.e., $I^k \cdot I^\ell \subset I^{k+\ell}$). That is, I forms a nilpotent ideal in A .

Let $\mathcal{O}_{\mathcal{M}}$ denote the sheaf of smooth sections of the algebra bundle A . We equip $\mathcal{O}_{\mathcal{M}}$ with a derivation of cohomological degree 1 so that it is a sheaf of unital commutative dg Ω_M^* -algebras.

- (3) A map $q : \mathcal{O}_{\mathcal{M}} \rightarrow C_M^\infty$ of sheaves of Ω_M^* -algebras whose kernel is the sheaf $\mathcal{I}_{\mathcal{M}}$ of smooth sections of I . This sheaf $\mathcal{I}_{\mathcal{M}}$ forms a nilpotent dg ideal.⁴
- (4) We require the cohomology of $\mathcal{O}_{\mathcal{M}}(U)$ to be concentrated in nonpositive degrees for sufficiently small open sets $U \subset M$.

Note that the conditions on the sheaf $\mathcal{O}_{\mathcal{M}}$ imply that its global cohomology $\mathcal{O}_{\mathcal{M}}(M)$ lives in finitely many cohomological degrees. We call $\mathcal{O}_{\mathcal{M}}$ the *structure sheaf* of the nilpotent dg manifold \mathcal{M} .

Definition 2.3. Let \mathcal{N} be a nilpotent dg manifold with underlying manifold N and graded vector bundle B , and let \mathcal{M} be a nilpotent dg manifold with underlying manifold M and graded vector bundle A . A *map of nilpotent dg manifolds* $F : \mathcal{N} \rightarrow \mathcal{M}$ is a pair (f, ϕ) where f is a smooth map from N to M and ϕ is a map of graded vector bundles from the pullback bundle $f^{-1}A$ to B such that there is a commuting diagram of commutative dg $f^{-1}\Omega_M$ -algebra sheaves (on N)

$$\begin{array}{ccc} f^{-1}\mathcal{O}_{\mathcal{M}} & \xrightarrow{\phi} & \mathcal{O}_{\mathcal{N}} \\ \downarrow & & \downarrow \\ f^{-1}C_M^\infty & \longrightarrow & C_N^\infty. \end{array}$$

(In other words, the map of bundles is compatible with the graded algebra structure on the bundles, the differentials on the sheaf of sections, and the filtrations by the nilpotent dg ideal sheaves.)

There is one last construction that we will need.

³That is, m is a map of vector bundles from the Whitney tensor product $A \otimes A$ to A .

⁴We will generally suppress the map q from notation as it is nearly always obvious from context.

Definition 2.4. Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ and $\mathcal{N} = (N, \mathcal{O}_{\mathcal{N}})$ be nilpotent dg manifolds. The *product* $\mathcal{M} \times \mathcal{N}$ is the nilpotent dg manifold

$$(M \times N, \mathcal{O}_{\mathcal{M} \times \mathcal{N}}, \mathcal{I}_{\mathcal{M} \times \mathcal{N}}),$$

where

$$\mathcal{O}_{\mathcal{M} \times \mathcal{N}} := \varprojlim_{i,j} (\pi_M^{-1} \mathcal{O}_{\mathcal{M}} / \mathcal{I}_{\mathcal{M}}^i) \otimes_{\Omega_{M \times N}^*} (\pi_N^{-1} \mathcal{O}_{\mathcal{N}} / \mathcal{I}_{\mathcal{N}}^j)$$

equipped with the nilpotent ideal

$$\mathcal{I}_{\mathcal{M} \times \mathcal{N}} = (\pi_M^{-1} \mathcal{I}_{\mathcal{M}} \otimes_{\Omega_{M \times N}^*} \pi_N^{-1} \mathcal{O}_{\mathcal{N}}) \oplus (\pi_M^{-1} \mathcal{O}_{\mathcal{M}} \otimes_{\Omega_{M \times N}^*} \pi_N^{-1} \mathcal{I}_{\mathcal{N}}).$$

This definition arises by requiring the natural compatibility with the filtrations.

Remark 2.5. Our structure sheaves are always given as sections of graded vector bundles, equipped with extra structure, although it might seem more natural to work immediately with sheaves. This vector bundle condition appears in large part to ensure compatibility with the quantum field theory constructions in [Cos11b]. In our future work, [GG], we use the present results to construct effective field theories and quantization in families, where the parameterizing spaces are exactly the nilpotent dg manifolds we just defined.

2.1. Relation to smooth manifolds, formal geometry, complex geometry, and foliations. To give a sense of this new category dgMan of nilpotent dg manifolds, we now describe some important examples.

Example 2.6. For M a smooth manifold, the nilpotent dg manifold (M, Ω_M^*) is known as the *de Rham space* of M and is denoted by M_{dR} . By definition, every other nilpotent dg manifold \mathcal{M} with M as the underlying smooth manifold possesses a distinguished map $\mathcal{M} \rightarrow M_{dR}$. \diamond

Remark 2.7. There are several ways to think about the role of M_{dR} . Observe that the simplest natural sheaf of algebras to put on M is \mathbb{C}_M , the locally constant sheaf, but this sheaf is not soft and not well-suited to the techniques of differential geometry. The de Rham complex is then a pleasant replacement for \mathbb{C}_M , since it locally recovers \mathbb{C}_M as its cohomology. Note that the structure sheaves of nilpotent dg manifolds are algebras over the de Rham complex, and thus well-behaved replacements for algebras over the constant sheaf.

Another aspect is more categorical in nature. If we were to define a category of \mathcal{O} -modules over M_{dR} (although we will not develop such a formalism here), it should be equivalent to the category of D -modules on M . Heuristically, this relationship is clearest when one thinks about vector bundles on M_{dR} : these are essentially vector bundles with flat connection on the smooth manifold M . The role of M_{dR} is to provide the natural space over which live all “things with a flat connection over M ” (or more accurately, things with a system of differential equations over M). Indeed, we view the work

of Block and collaborators [Blo10], [BenbB13], [BloS14] as an approach to this question. (Works of Kapranov [Kap91] and of Simpson–Teleanu [SiT] are also highly relevant here.)

Finally, in Lemma 4.11, we indicate how this notion of de Rham space connects with the de Rham space in algebraic geometry. \diamond

Example 2.8. Let M be a smooth manifold. Then (M, C_M^∞) is a nilpotent dg manifold where C_M^∞ is viewed as an Ω_M^* module by the quotient map $\Omega_M^* \rightarrow \Omega_M^0$ whose kernel is the differential ideal generated by the 1-forms. We denote it by M_{sm} . \diamond

It is straightforward to see the following, which explains why dgMan is an enlargement of the natural test objects for smooth geometry.

Proposition 2.9. *The inclusion functor $i : \text{Man} \rightarrow \text{dgMan}$ where*

$$M \mapsto M_{sm} = (M, C_M^\infty)$$

is fully faithful.

In dgMan , the de Rham space of a smooth manifold M represents the set of constant maps. More precisely we have the following.

Proposition 2.10. *Let N and M be smooth manifolds. The set of maps $\text{dgMan}(N_{dR}, M_{sm})$ is in bijection with the underlying set of M (viewed as the constant maps from N to M).*

Proof. A map of nilpotent dg manifolds $N_{dR} \rightarrow (M, C_M^\infty)$ consists of a smooth map $f : N \rightarrow M$ and a map of sheaves of dg $f^{-1}\Omega_M$ algebras $\phi : f^{-1}C_M^\infty \rightarrow \Omega_N^*$ such that

$$\begin{array}{ccc} f^{-1}C_M^\infty & \xrightarrow{\phi} & \Omega_N^* \\ \text{Id} \downarrow & & \downarrow \\ f^{-1}C_M^\infty & \longrightarrow & C_N^\infty \end{array}$$

commutes. In particular, the map ϕ commutes with differentials. The differential on the structure sheaf (M, C_M^∞) is trivial, and hence we see that the pullback of a smooth function on M must be a constant function. \square

Remark 2.11. By definition, every nilpotent dg manifold $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ “lives between” M_{sm} and M_{dR} in the sense that we have canonical maps

$$M_{sm} \rightarrow \mathcal{M} \rightarrow M_{dR},$$

where the underlying map of manifolds is the identity and the algebra maps are provided by definition. \diamond

Example 2.12. Let R be an artinian algebra over \mathbb{C} , such as the *dual numbers* $\mathbb{C}[\epsilon]/(\epsilon^2)$. Then (pt, R) is a nilpotent dg manifold. \diamond

Let $\text{Art}_\mathbb{C}$ denote the category of artinian algebras over \mathbb{C} . The opposite category $\text{Art}_\mathbb{C}^{\text{op}}$ encodes “fat points.” It’s easy to see that as the underlying manifold is just a point we have a full and faithful embedding.

Proposition 2.13. *The Spec functor*

$$\begin{aligned} \text{Art}_\mathbb{C}^{\text{op}} &\rightarrow \text{dgMan} \\ A &\mapsto \text{Spec } A = (\text{pt}, A) \end{aligned}$$

is fully faithful.

We can also include with dg artinian algebras concentrated in nonpositive degrees, such as the *shifted dual numbers* $\mathbb{C}[\epsilon]/(\epsilon^2)$ where ϵ has negative cohomological degree. Thus, dgMan includes the basic ingredient of *derived deformation theory*.

This proposition also indicates one way that formal deformation theory will relate to dgMan , since we can fatten any manifold this way. For any smooth manifold M , we can “thicken” M by a “Spec R -bundle” to obtain an interesting nilpotent dg manifold. (That is, let the structure sheaf be sections of an R -bundle over M .)

Example 2.14. Given a complex structure on a smooth manifold M , there is a nilpotent dg manifold encoding the complex manifold, namely $(M, \Omega_M^{0,*})$, where $\Omega_M^{0,*}$ is the Dolbeault complex for this complex structure. We view $\Omega_M^{0,*}$ as the quotient of Ω_M^* by the differential ideal generated by the $(1, *)$ -forms. \diamond

Proposition 2.15. *The inclusion functor is a fully faithful embedding from the category of complex manifolds into dgMan .*

Example 2.16. Let \mathcal{F} be a (regular) foliation of M . Equivalently, let $\rho : T_{\mathcal{F}} \hookrightarrow T_M$ be a subbundle of the tangent bundle that is *integrable*: the Lie bracket of any two sections of $T_{\mathcal{F}}$ is always a section of $T_{\mathcal{F}}$. Thus, \mathcal{F} provides a Lie algebroid $T_{\mathcal{F}}$, and a standard construction for the theory of Lie algebroids then provides a nilpotent dg manifold, as follows. (See, for example, [Rin63], [Mac05], [Meh09], [AC12].)

The *Chevalley–Eilenberg cochain complex* of $T_{\mathcal{F}}$ is a sheaf of commutative dg algebras determined by the Lie algebroid. We denote it $C^*T_{\mathcal{F}}$. The underlying sheaf of graded algebras $C^\sharp T_{\mathcal{F}}$ is given by the smooth sections of the bundle $\text{Sym}(T_{\mathcal{F}}^*[-1])$, and multiplication is the pointwise wedge product. Hence it is a graded algebra over C^∞ , but we equip it with a differential that is not C^∞ -linear.⁵ The differential d is determined by the following conditions. First, for any function f , viewed as an element of $C^0T_{\mathcal{F}}(M)$, we have that

$$d(f)(X) = \rho(X)(f)$$

⁵The differential is determined by the Lie bracket, which is not C^∞ -linear.

for every $X \in \Gamma(M, T_{\mathcal{F}})$. Second for any $\alpha \in C^1 T_{\mathcal{F}}(M)$,

$$d(\alpha)(X \wedge Y) = \rho(X)(\alpha(Y)) - \rho(Y)(\alpha(X)) - \alpha([X, Y]),$$

for all $X, Y \in \Gamma(M, T_{\mathcal{F}})$. Third, we require that $d^2 = 0$ and

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\alpha} \alpha \wedge (d\beta)$$

for all elements α and β .⁶ Observe that for U an open on which the foliation decomposes as $U \cong \mathbb{R}^p \times \mathbb{R}^{\dim M - p}$, where the leaves are codimension p , then

$$H^k(C^* T_{\mathcal{F}}(U)) = \begin{cases} C^\infty(\mathbb{R}^p), & k = 0, \\ 0, & \text{else.} \end{cases}$$

(This is a direct consequence of the usual Poincaré lemma.) Note that $C^* T_{\mathcal{F}}$ has a nilpotent dg ideal given by $C^{\geq 1} T_{\mathcal{F}}$. Lastly, note that $C^* T_{\mathcal{F}}$ is a dg Ω_M^* -algebra — indeed a quotient algebra — via the algebra map determined by the dual to the anchor map $\rho^* : T_M^* \rightarrow T_{\mathcal{F}}^*$.

Let $M_{\mathcal{F}}$ denote the nilpotent dg manifold $(M, C^* T_{\mathcal{F}})$. It provides a dg manifold describing the “derived leaf space” of the foliation, as the Lie algebroid cohomology is precisely the derived functor for taking invariants of functions along leaves.

This construction encompasses several earlier examples: when $T_{\mathcal{F}} = 0$, we recover M_{sm} ; when $T_{\mathcal{F}} = T_M$, we recover M_{dR} ; and when M is a complex manifold, $M_{\bar{\partial}}$ is associated to the foliation given by $T_M^{0,1}$. \diamond

Proposition 2.17. *The functor from the category of regular foliations to dgMan sending \mathcal{F} to $M_{\mathcal{F}}$ is a fully faithful embedding.*

2.2. A notion of weak equivalence. Costello introduces an interesting notion of weak equivalence between nilpotent dg manifolds. His notion relies on the existence of a natural filtration on the structure sheaf $\mathcal{O}_{\mathcal{M}}$ of a nilpotent dg manifold \mathcal{M} . In particular, let $\mathcal{I}_{\mathcal{M}} = \ker q$ denote the sheaf of nilpotent dg ideals in $\mathcal{O}_{\mathcal{M}}$. We have the filtration

$$F^k \mathcal{O}_{\mathcal{M}} = \mathcal{I}_{\mathcal{M}}^k.$$

Let $\text{Gr } \mathcal{O}_{\mathcal{M}}$ denote the associated graded dg algebra.

Definition 2.18. A map $F : \mathcal{M} \rightarrow \mathcal{N}$ in dgMan is a *weak equivalence* if:

- (1) The smooth map $f : M \rightarrow N$ is a diffeomorphism.
- (2) The map of commutative dg algebras $\text{Gr } \phi : f^{-1} \text{Gr } \mathcal{O}_{\mathcal{N}} \rightarrow \text{Gr } \mathcal{O}_{\mathcal{M}}$ is a quasi-isomorphism.

This definition provides a well-behaved notion of weak equivalence because:

- Every isomorphism of nilpotent dg manifolds (i.e., a diffeomorphism with a strict isomorphism of structure sheaves) is a weak equivalence.

⁶This construction is a systematic generalization of the de Rham complex: when $T_{\mathcal{F}} = T_M$, the Chevalley–Eilenberg complex is precisely $\Omega^*(M)$.

- The notion satisfies the 2-out-of-3 property because diffeomorphism and quasi-isomorphism do.

Nonetheless, this definition might look a little strange because of the role played by the associated graded algebra. Observe, however, that it implies that ϕ is a quasi-isomorphism: ϕ preserves the natural filtration on the structure sheaves and hence induces a map of spectral sequences that is a quasi-isomorphism on the first page. Thus the definition is *stronger* than requiring that ϕ is a quasi-isomorphism, which might be the first definition that comes to mind. This stronger condition depends crucially on the filtration.

As further motivation for the definition, we note that the role of nilpotent dg manifolds here is supposed to parallel the role of artinian algebras in formal deformation theory. Arguments in deformation theory often proceed by *artinian induction*: every local artinian \mathbb{C} -algebra (A, \mathfrak{m}) possesses a canonical tower of quotients

$$A \rightarrow A/\mathfrak{m}^n \rightarrow \dots \rightarrow A/\mathfrak{m}^2 \rightarrow A/\mathfrak{m} \cong \mathbb{C},$$

and at each step we extend an artinian algebra by a square-zero ideal so it suffices to prove some property holds for such small extensions

$$I \hookrightarrow B \rightarrow A,$$

where I is the kernel of a ring map and $I^2 = 0$ inside B . The filtration on $\mathcal{O}_{\mathcal{M}}$ is our substitute for this canonical tower of quotients, and we will prove our main theorem by using a version of artinian induction.

3. Derived stacks

Our notion of “derived stack” or “derived space” will be, as usual, a kind of sheaf of simplicial sets on the site of “test objects” (cf. [TV05]). Thus we need to equip dgMan with a site structure.

Recall that the category Man has a site structure where a covering is simply an open cover in the usual sense. A *covering* of $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ in dgMan is a collection $\{\mathcal{U}_i = (U_i, \mathcal{O}_i)\}$ of nilpotent dg manifolds with maps $\{F_i : \mathcal{U}_i \rightarrow \mathcal{M}\}$ such that the collection $\{U_i\}$ forms a cover of M in Man and the maps of structure sheaves $\phi_i : f_i^{-1}\mathcal{O}_{\mathcal{M}} \rightarrow \mathcal{O}_i$ are isomorphisms.

Definition 3.1. A *derived stack* is a functor $\mathbf{X} : \text{dgMan}^{\text{op}} \rightarrow \text{sSets}$ satisfying:

- (1) \mathbf{X} sends weak equivalences of nilpotent dg manifolds to weak equivalences of simplicial sets.
- (2) \mathbf{X} satisfies Čech descent (see below for a reminder on what this means).

Note that a derived stack is merely a homotopical kind of sheaf, and thus the definition does not capture any particularly *geometric* properties. For instance, we do not require \mathbf{X} to locally resemble a ringed space or orbifold.

Our focus in this paper is on a class of examples, the L_∞ spaces, that *do* have a very geometric flavor. (It would be interesting to work out the analogs of orbifold or Artin stack in this context.)

Definition 3.2. Let $\mathbf{X}, \mathbf{Y} : \text{dgMan}^{\text{op}} \rightarrow \text{sSets}$ be derived stacks. A *map of stacks* $\alpha : \mathbf{X} \rightarrow \mathbf{Y}$ is just a natural transformation between the functors. A *weak equivalence of stacks* is a map of stacks α such that

$$\alpha(\mathcal{M}) : \mathbf{X}(\mathcal{M}) \rightarrow \mathbf{Y}(\mathcal{M})$$

is a weak equivalence for every nilpotent dg manifold \mathcal{M} .

Let dSt denote the category of derived stacks. Hence, we will denote the morphisms from a derived stack \mathbf{X} to \mathbf{Y} by $dSt(\mathbf{X}, \mathbf{Y})$.

3.1. Čech descent and homotopy sheaves. We recall the usual notion of a sheaf before giving the souped-up version we need.

Let X be a topological space and let Opens_X denote the poset category whose objects are opens in X and whose morphisms are inclusions of opens. A *presheaf* on X with values in the category \mathcal{C} is a functor $\mathcal{F} : \text{Opens}_X^{\text{op}} \rightarrow \mathcal{C}$. A *sheaf* is a presheaf such that for any open U and any cover $\mathfrak{V} = \{V_i\}$ of U , we have

$$\mathcal{F}(U) \cong \text{eq} \left(\prod_i \mathcal{F}(V_i) \rightrightarrows \prod_{i,j} \mathcal{F}(V_i \cap V_j) \right),$$

where eq denotes “equalizer” and the two arrows are the two natural restriction maps for \mathcal{F} .

In our setting, the value category \mathcal{C} is the category sSets , and we want to view two simplicial sets as the same if they are weakly equivalent. Thus, wherever we would ordinarily compute (co)limits, we should work with *homotopy* (co)limits instead. Moreover, we don’t merely want to require agreement on overlaps; we want to coherently agree on overlaps-of-overlaps and so on.

These desires indicate how we should refine the notion of sheaf.

Definition 3.3. Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ be a nilpotent dg manifold. Let $\mathfrak{V} = \{V_i, : i \in I\}$ be a cover for the underlying smooth manifold M . We also use \mathfrak{V} to denote, abusively, the nilpotent dg manifold $\coprod_{i \in I} (V_i, \mathcal{O}_{\mathcal{M}}|_{V_i})$, given by disjoint union over the opens in the cover. Let $\check{\mathcal{C}}\mathfrak{V}_\bullet$ denote the simplicial nilpotent dg manifold whose n -simplices are

$$\check{\mathcal{C}}\mathfrak{V}_n := \mathfrak{V} \times_M \cdots \times_M \mathfrak{V},$$

where the fiber product is taken $n + 1$ times, and the simplicial maps are the usual ones. We call $\check{\mathcal{C}}\mathfrak{V}_\bullet$ the *Čech nerve* of the cover \mathfrak{V} .

A homotopy sheaf will be a simplicial presheaf that satisfies homotopical descent for every such cover.

Definition 3.4. A simplicial presheaf \mathcal{F} on a nilpotent dg manifold \mathcal{M} is a *homotopy sheaf* if for every open U of M and every cover \mathfrak{U} of U , we have

$$\mathcal{F}(U) \xrightarrow{\simeq} \operatorname{holim}_{\mathcal{C}\mathfrak{U}} \mathcal{F},$$

where \simeq denotes weak equivalence of simplicial sets and holim denotes the homotopy limit.

We provide a concise introduction to homotopy limits, with a focus on the case of interest, in Appendix A.

3.2. The road not (yet) taken. So far we have given the basic skeleton of an approach to geometry, but much remains to be fleshed out. We wish here to point out a few possibilities that we find particularly interesting and then to explain some choices made in [Cos11].

First, we only spell out categories with weak equivalences here, both of nilpotent dg manifolds and of derived stacks. Many constructions would undoubtedly work more easily if one carefully constructed simplicially-enriched or quasi-categories of these objects. (We expect that for doing serious work in this setting, there might be more geometrically natural ways of constructing these ∞ -categories than taking the Dwyer–Kan localization.)

Second, it would be useful to construct categories of \mathcal{O}_X -modules, quasi-coherent sheaves, and so on over these spaces. In general, there are many techniques and examples in derived algebraic geometry whose analogues would be very useful in our setting. For example, for applications to field theory, it would be nice to have stacks like the moduli of Riemann surfaces or the moduli of holomorphic G -bundles on a complex manifold.

Finally, we note that nilpotent dg manifolds appear already in Costello’s approach to quantum field theory, where he shows that renormalization and Feynman diagram computations behave well in families over nilpotent dg manifolds (see Section 13 of Chapter 2 of [Cos11b]). There, he relies crucially on the fact that any constant or linear terms in the action functional are a multiple of the nilpotent ideal. It would be interesting to see if one could modify that analysis to work over dg manifolds whose structure sheaves are cohomologically artinian (and not already artinian at the cochain level), as these test objects are possibly more natural from the derived geometry perspective.

4. L_∞ spaces

In deformation theory, there is a governing, heuristic principle: every deformation functor is given by a dg Lie algebra.⁷ In other words, we can describe the “formal neighborhood of a point in some space” using Lie-theoretic

⁷This idea has a long history, which we will not trace here. See Hinich’s paper [Hin01] for one treatment of this idea that is quite close to what we do here. In [Lurb], Lurie proves a theorem that makes this principle precise and connects it with global derived geometry. Hennion [Hen] extends Lurie’s result to a relative context, working over a base derived Artin stack.

constructions, rather than commutative algebra constructions. (One recovers the functions on the formal neighborhood by taking the Chevalley–Eilenberg cochain complex that computes the cohomology of the dg Lie algebra.) Often, this perspective is incredibly helpful, partly because the manipulations on the Lie side may be simpler.

Our primary interest is in families of deformation problems parametrized by smooth manifolds, so we might hope we get a nice kind of derived stack by equipping a smooth manifold with a sheaf of dg Lie algebras. We make this idea precise via the notion of an L_∞ space.

4.1. Curved L_∞ algebras. It is convenient to enlarge the Lie-theoretic side to make it more flexible. We will work with curved L_∞ algebras rather than dg Lie algebras.

Definition 4.1. Let A be a commutative dg algebra with a nilpotent dg ideal I . A *curved L_∞ algebra over A* consists of:

- (1) A locally free, \mathbb{Z} -graded A^\sharp -module V .
- (2) A linear map of cohomological degree 1

$$d : \mathrm{Sym}(V[1]) \rightarrow \mathrm{Sym}(V[1])$$

satisfying:

- (i) $d^2 = 0$.
- (ii) $(\mathrm{Sym}(V[1]), d)$ is a cocommutative dg coalgebra over A (i.e., d is a coderivation).
- (iii) Modulo I , the coderivation d vanishes on the constants (i.e., on Sym^0).

The notation $\mathrm{Sym}(V[1])$ indicates the graded vector space known as the symmetric algebra over the graded algebra A^\sharp underlying the dg algebra A . We only remember its natural coalgebra structure in this setting. To reduce notation, we use $C_*(V)$ to denote the cocommutative dg coalgebra $(\mathrm{Sym}(V[1]), d)$. We use this notation because we call it the *Chevalley–Eilenberg homology complex* of V , as we are extending the usual notions of Lie algebra homology.

Recall that we obtain a sequence of maps

$$\ell_n : (\Lambda^n V)[n-2] \rightarrow V,$$

the n -fold bracket on V , from the composition

$$\mathrm{Sym}^n(V[1]) \hookrightarrow C_*(V) \xrightarrow{d} C_*(V) \xrightarrow{\pi} \mathrm{Sym}^1(V[1]) = V[1],$$

by shifting by 1. Thinking of V equipped with these brackets is why we use the terminology L_∞ algebra; it is often easier to work with the Chevalley–Eilenberg homology complex, which assembles all the brackets into a single map.

There is also a natural Chevalley–Eilenberg *cohomology* complex $C^*(V)$. It is $(\widehat{\text{Sym}}(V^\vee[-1]), d)$, where the notation $\widehat{\text{Sym}}(V^\vee[-1])$ indicates the completed symmetric algebra over the graded algebra A^\sharp underlying the dg algebra A . The differential d is the “dual” differential to that on $C_*(V)$. In particular, it makes $C^*(V)$ into a commutative dg algebra, so d is a derivation.

We usually think of a curved L_∞ algebra \mathfrak{g} over A as describing a derived space $B\mathfrak{g}$ over $\text{Spec } A$. The algebra of functions of $B\mathfrak{g}$ is precisely its Chevalley–Eilenberg cohomology complex $C^*(\mathfrak{g})$. Thanks to the natural pairing between the cohomology and homology complexes, we view $C_*(\mathfrak{g})$ as the coalgebra of distributions on $B\mathfrak{g}$.

Definition 4.2. A map of curved L_∞ algebras $\phi : V \rightarrow W$ is a map of co-commutative dg coalgebras $\phi_* : C_*(V) \rightarrow C_*(W)$ respecting the cofiltration by I . A map is a *weak equivalence* if ϕ_* is a quasi-isomorphism.

4.1.1. Commentary on curving. A curious aspect of this definition is the curving, since the uncurved case is discussed far more often. Indeed, “flat” L_∞ algebras (i.e., with zero curving) are usually understood as describing pointed formal moduli problems (see, for example, Lurie’s ICM talk [Lur10] for a recent discussion). If \mathfrak{g} is the flat L_∞ algebra over a commutative dg algebra R , the moduli problem $B\mathfrak{g}$ has a marked point. On the commutative algebra side, this appears as the fact that $C^*\mathfrak{g}$ is augmented: there is a distinguished map $C^*\mathfrak{g} \rightarrow R$. The derived space $B\mathfrak{g} = \text{Spec } C^*\mathfrak{g}$ is thus pointed by the augmentation map $\text{Spec } R \rightarrow \text{Spec } C^*\mathfrak{g}$. A curved L_∞ algebra \mathfrak{g} then corresponds to a nonpointed formal moduli spaces, because $C^*\mathfrak{g}$ is not augmented. We now elaborate on this idea.

Let R be a commutative dg algebra with nilpotent ideal I and let S denote R/I . Given a curved L_∞ algebra $\tilde{\mathfrak{g}}$ over R , let \mathfrak{g} denote the reduction modulo I , which is a flat L_∞ algebra over S . Let $B\tilde{\mathfrak{g}}$ denote the space associated to the algebra $C^*_R\tilde{\mathfrak{g}}$, which is a semi-free algebra over R , and let $B\mathfrak{g}$ denote the space associated to $C^*_S\mathfrak{g}$, which is a semi-free algebra over S . The space $B\tilde{\mathfrak{g}}$ encodes a fattening of the pointed space $B\mathfrak{g}$, where we cannot extend the map $p : \text{Spec } S \rightarrow \text{Spec } C^*_R\tilde{\mathfrak{g}}$ to an R -point $\tilde{p} : \text{Spec } R \rightarrow \text{Spec } C^*_R\tilde{\mathfrak{g}}$.

$$\begin{array}{ccc}
 B\mathfrak{g} & \longrightarrow & B\tilde{\mathfrak{g}} \\
 \downarrow & \nearrow p & \downarrow \\
 \text{Spec } S & \longrightarrow & \text{Spec } R
 \end{array}$$

The curving is the obstruction to such an extension.

Remark 4.3. Here is a different way of concocting such a situation. Consider a map of commutative dg algebras $f : A \rightarrow B$, which we view as a map of “derived spaces” $\text{Spec } B \rightarrow \text{Spec } A$. This map makes B an A -algebra and so we can find a semi-free resolution $\text{Sym}_A(M)$ of B as an A -algebra. This replacement $\text{Sym}_A(M)$ expresses B as a kind of L_∞ algebra over A ,

namely $\mathfrak{g}_B = M^\vee[-1]$.⁸ Note that if f factors through a quotient A/I of A , then \mathfrak{g}_B will be curved. (The differential for the semi-free resolution will produce I as the image of the differential's Taylor component mapping to $\mathrm{Sym}_A^0(M) = A$.) This curving appears because $\mathrm{Spec} B$ really only lives over the subscheme $\mathrm{Spec} A/I \subset \mathrm{Spec} A$, and extending it over the rest of $\mathrm{Spec} A$ is obstructed. \diamond

This kind of situation appears in the category of nilpotent dg manifolds. For any $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$, we see that \mathcal{M} “lives between” the smooth manifold M_{sm} and its de Rham space M_{dR} because we have algebra maps

$$\Omega_M^* \rightarrow \mathcal{O}_{\mathcal{M}} \xrightarrow{q} C_M^\infty$$

by definition. These maps induce maps of nilpotent dg manifolds

$$M_{sm} \rightarrow \mathcal{M} \rightarrow M_{dR},$$

where the underlying map of manifolds is simply the identity. We will see that we can often find a “replacement” of \mathcal{M} as a kind of L_∞ algebra over M_{dR} .

4.2. The Maurer–Cartan equation. For an element α of degree 1 in a curved L_∞ algebra \mathfrak{g} , let the *Maurer–Cartan element* be

$$mc(\alpha) := \sum_{n=0}^{\infty} \frac{1}{n!} \ell_n(\alpha^{\otimes n}).$$

The *Maurer–Cartan equation* is then $mc(\alpha) = 0$. There are many useful interpretations of this equation (and we discuss some in Appendix B).⁹

Here we will emphasize that a map of commutative dg algebras $a : C^*(\mathfrak{g}) \rightarrow A$ is determined by a cochain map $a : \mathrm{Sym}^1(\mathfrak{g}^\vee[-1]) = \mathfrak{g}^\vee[-1] \rightarrow A$, since a map of algebras is determined by where the generators go. Consider the element $\alpha \in \mathfrak{g}^{\vee\vee}[1] \otimes A$ that is dual to $a \in \mathrm{Hom}(\mathfrak{g}^\vee[-1], A)$. Then the condition of a being a cochain map is precisely the Maurer–Cartan equation on α (under the finiteness condition that $\mathfrak{g}^{\vee\vee} \cong \mathfrak{g}$). In sum, if we view \mathfrak{g} as encoding some kind of space $B\mathfrak{g}$, the Maurer–Cartan equation lets us understand its A -points.

We now construct a simplicial set of solutions to the Maurer–Cartan equation. (As explained in Appendix B, Getzler’s paper [Get09] is a wonderful reference for this Maurer–Cartan functor and much more.)

⁸This is not strictly true, because we are not working with the completed symmetric algebra, but we’re simply providing motivation here.

⁹Something that might rightly bother the reader is that the equation involves an infinite sum, which will not be well-defined in most cases. We will only work with *nilpotent* elements α , so that the sum is actually finite. Indeed, we use \mathfrak{g} and mc to construct a functor on artinian algebras: tensoring \mathfrak{g} with the maximal ideal of an artinian algebra gives us a nilpotent L_∞ algebra on which mc is well-behaved.

Definition 4.4. Let \mathfrak{g} be a curved L_∞ algebra. The *Maurer–Cartan space* $\mathrm{MC}_\bullet(\mathfrak{g})$ is the simplicial set whose n -simplices are solutions to the Maurer–Cartan equation in the curved L_∞ algebra $\mathfrak{g} \otimes \Omega^*(\Delta^n)$.

This space $\mathrm{MC}_\bullet(\mathfrak{g})$ has several nice properties when \mathfrak{g} is nilpotent: for instance, it is a Kan complex. See Appendix B for more discussion and references.

4.3. L_∞ spaces. We now describe the version of “families of curved L_∞ algebras parametrized by a smooth manifold” appropriate to our context.

Definition 4.5. Let X be a smooth manifold.

- (1) A *curved L_∞ algebra over Ω_X^** consists of a \mathbb{Z} -graded topological¹⁰ vector bundle $\pi : V \rightarrow X$ and the structure of a curved L_∞ algebra structure on its sheaf of smooth sections, denoted \mathfrak{g} , where the base algebra is Ω_X^* with nilpotent ideal $\mathcal{I} = \Omega_X^{\geq 1}$.
- (2) An *L_∞ space* is a pair (X, \mathfrak{g}) , where \mathfrak{g} is a curved L_∞ algebra over Ω_X^* .

We now explain how every L_∞ space defines a derived stack. Let $B\mathfrak{g} := (X, \mathfrak{g})$ denote an L_∞ space. Given a smooth map $f : Y \rightarrow X$, we obtain a curved L_∞ algebra over Ω^*Y by

$$f^*\mathfrak{g} := f^{-1}\mathfrak{g} \otimes_{f^{-1}\Omega_X^*} \Omega_Y^*,$$

where $f^{-1}\mathfrak{g}$ denotes sheaf of smooth sections of the pullback vector bundle $f^{-1}V$.

Definition 4.6. For $B\mathfrak{g} = (X, \mathfrak{g})$ an L_∞ space, its *functor of points* is the functor

$$\mathbf{B}\mathfrak{g} : \mathrm{dgMan}^{\mathrm{op}} \rightarrow s\mathrm{Sets}$$

for which an n -simplex of $\mathbf{B}\mathfrak{g}(\mathcal{M})$ is a pair (f, α) : a smooth map $f : M \rightarrow X$ and a solution α to the Maurer–Cartan equation in $f^*\mathfrak{g} \otimes_{\Omega_M^*} \mathcal{I}_{\mathcal{M}} \otimes_{\mathbb{R}} \Omega^*(\Delta^n)$.

The basic idea of the definition is hopefully clear, but we want to remark upon several choices made in this definition. First, note that we use the nilpotent ideal $\mathcal{I}_{\mathcal{M}}$, not the whole algebra $\mathcal{O}_{\mathcal{M}}$. This restriction ensures that we have a nilpotent curved L_∞ algebra, and hence a Kan complex. It also encodes the idea that we are deforming in the nilpotent direction away from an underlying map. Second, note that we are *not* allowing the smooth

¹⁰That is, the fibers are topological vector spaces and the gluing maps are continuous linear maps. In examples, the fibers will be nuclear Fréchet spaces, typically smooth sections of a vector bundle on some manifold. For instance, consider a smooth fiber bundle $p : T \rightarrow X$ where the fiber is diffeomorphic to some fixed manifold M and consider a relative vector bundle E on T , so that we have a vector bundle on each fiber of T over X . Then the vector bundle V might be relative sections of this relative vector bundle: to a point $x \in X$, we associate the vector space of smooth sections of the vector bundle $E_x \rightarrow p^{-1}(x) \cong M$.

map to vary over the n -simplices. In other words, $\mathbf{Bg}(\mathcal{M})$ is the disjoint union

$$\bigsqcup_{f \in C^\infty(M, X)} \mathrm{MC}_\bullet(f^* \mathfrak{g} \otimes_{\Omega_M^*} \mathcal{I}_\mathcal{M}),$$

where the union is over the *set* of smooth maps $C^\infty(M, X)$.

Remark 4.7. If we view $C^* \mathfrak{g}$ as the *structure sheaf* of the L_∞ space, then a vertex of $\mathbf{Bg}(\mathcal{M})$ is a map of the underlying manifolds $f : M \rightarrow X$ and a map of commutative dg algebras $f^{-1} C^* \mathfrak{g} \rightarrow \mathcal{O}_\mathcal{M}$. In other words, it is a map of dg ringed spaces. \diamond

Theorem 4.8. *The functor \mathbf{Bg} associated to an L_∞ space $B\mathfrak{g}$ defines a derived stack.*

As the proof of this theorem is lengthy and somewhat technical, we banish it to Appendix C to maintain our narrative flow.

Remark 4.9. A few interpretive comments are in order.

First, one can view an L_∞ space (X, \mathfrak{g}) as a (nonnilpotent) dg manifold whose structure sheaf $C^* \mathfrak{g}$ is a cofibrant commutative dg algebra over Ω_X^* . These are particularly well-behaved class of manifolds equipped with sheaves of commutative algebras, although we do not develop that formalism here. Note that every nilpotent dg manifold (M, \mathcal{A}) has a “replacement” by an L_∞ space, precisely by taking a semi-free resolution of its structure sheaf \mathcal{A} over Ω_M^* . See remark 4.13 for more discussion of this point.

Second, as previously noted, there is a well-known correspondence between L_∞ algebras and formal moduli spaces. An L_∞ space is a relative version of this idea: we have a family of formal moduli spaces parametrized by a smooth manifold. For a recent and enlightening treatment of an analogous idea in derived algebraic geometry, see [Hen], which also gives a clear explanation of how such relative formal stacks compare to derived Artin stacks. \diamond

4.4. Examples.

4.4.1. The functor of points evaluated on a smooth manifold. Let $M_{sm} = (M, C_M^\infty)$ be a smooth manifold viewed as a nilpotent dg manifold. Note that the nilpotent ideal $\mathcal{I}_M = 0$ here. Then we have the following simple observation.

Lemma 4.10. *For any L_∞ space $B\mathfrak{g} = (X, \mathfrak{g})$, $\mathbf{Bg}(M)$ is the discrete simplicial set given by the set $C^\infty(M, X)$ of smooth maps from M to X .*

Proof. For any smooth map $f : M \rightarrow X$, we see that $f^* \mathfrak{g} \otimes_{\Omega_M^*} \mathcal{I}_M = 0$. Hence there is exactly one solution to the Maurer–Cartan equation: zero. \square

4.4.2. The de Rham space X_{dR} . Let X be a smooth manifold. Consider the zero vector bundle, equipped with the trivial Ω_X^* structure. This L_∞ space $(X, 0)$ has associated structure sheaf $C^*\mathfrak{g} = \Omega_X^*$: we are recovering the de Rham complex itself. Thus, this L_∞ space provides a derived stack associated to the de Rham space X_{dR} . Abusively, we will also denote this derived stack by X_{dR} .

Lemma 4.11. *For $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ a nilpotent dg manifold, $X_{dR}(\mathcal{M})$ is the discrete simplicial set of smooth maps $\text{Maps}(M, X)$.*

In other words, $X_{dR}(\mathcal{M}) = X_{dR}(M_{sm})$ for any nilpotent dg manifold: the derived stack only cares about the underlying smooth manifold and not the structure sheaf. Note that this behavior agrees with the definition of the de Rham stack in algebraic geometry.¹¹

Proof. For $f : M \rightarrow X$ a smooth map, we note that $f^{-1}0 = 0$, so that we have the trivial L_∞ algebra on M , no matter the structure sheaf on \mathcal{M} , and so there is only the zero solution to the Maurer–Cartan equation. \square

4.4.3. An L_∞ space encoding a smooth manifold. Finally, we turn to our main example of an L_∞ space: the one that encodes the smooth geometry of a manifold X . More precisely, we have the following existence lemma from [GG14]. We include the proof here to illustrate how ∞ -jet bundles allow us to find a “Koszul dual” L_∞ space to an actual manifold.

Recall that the ∞ -jet bundle J for the trivial bundle is a pro-vector bundle, whose fiber at a point $x \in X$ encodes the “Taylor series around x ” for smooth functions. The bundle J comes equipped with a canonical flat connection, whose horizontal sections are exactly the smooth functions on X . We denote the sheaf of smooth sections of J by \mathcal{J} . The de Rham complex for \mathcal{J} , whose differential is given by the canonical flat connection, is denoted $dR(\mathcal{J})$, to lessen the profusion of Ω throughout this paper.

Lemma 4.12. *Let X be a smooth manifold. There is a curved L_∞ algebra \mathfrak{g}_X over Ω_X , with nilpotent ideal $\Omega_X^{>0}$, such that:*

- (a) $\mathfrak{g}_X \cong \Omega_X^\sharp(T_X[-1])$ as an Ω_X^\sharp module.
- (b) $dR(\mathcal{J}) \cong C^*(\mathfrak{g}_X)$ as commutative Ω_X algebras.
- (c) The map sending a smooth function to its ∞ -jet

$$C_X^\infty \hookrightarrow dR(\mathcal{J}) \cong C^*(\mathfrak{g}_X)$$

is a quasi-isomorphism of Ω_X -algebras.

Proof. We need to show that we can equip $\widehat{\text{Sym}}_{C_X^\infty}(T_X^\vee) \otimes_{C_X^\infty} \Omega_X$ with a degree 1 derivation d such that $d^2 = 0$ (this is the curved L_∞ structure) and

¹¹In algebraic geometry, the de Rham stack X_{dR} of a stack X is given by the sheafification of the functor $X_{dR}(R) = X(R/Nil(R))$, where $Nil(R)$ denotes the nilradical. In short, the de Rham stack does not “see” nilpotent directions, only the underlying reduced scheme. For further discussion, see [GaR].

such that this Chevalley–Eilenberg complex is quasi-isomorphic to C_X^∞ as an Ω_X module. In this process we will see property (b) explicitly.

We start by working with D_X modules and then use the de Rham functor to translate our constructions to Ω_X modules. Consider the sheaf \mathcal{J} of infinite jets of smooth functions. Observe that there is a natural descending filtration on \mathcal{J} by “order of vanishing.” To see this explicitly, note that the fiber of \mathcal{J} at a point x is isomorphic (after picking local coordinates (x_1, \dots, x_n) to $\mathbb{C}[[x_1, \dots, x_n]]$, and we can filter this vector space by powers of the ideal $\mathfrak{m} = (x_1, \dots, x_n)$. We define $F^k \mathcal{J}$ to be those sections of \mathcal{J} which live in \mathfrak{m}^k for every point. This filtration is not preserved by the flat connection, but the connection does send a section in $F^k \mathcal{J}$ to a section of $F^{k-1} \mathcal{J} \otimes_{C_X^\infty} \Omega_X^1$.

Observe that $F^1 \mathcal{J} / F^2 \mathcal{J} \cong \Omega_X^1$, because the first-order jets of a function encode its exterior derivative. Moreover, $F^k \mathcal{J} / F^{k+1} \mathcal{J} \cong \text{Sym}^k(\Omega_x^1)$ for similar reasons. Pick a splitting of the map $F^1 \mathcal{J} \rightarrow \Omega_X^1$ as C_X^∞ modules; we denote the splitting by σ . (Note that there is a contractible space of such splittings, see the following subsection.) By the universal property of the symmetric algebra, we get a map of nonunital C_X^∞ algebras that is, in fact, an isomorphism

$$\text{Sym}_{C_X^\infty}^{>0}(\Omega_X^1) \xrightarrow{\cong} F^1 \mathcal{J}.$$

Now both $\widehat{\text{Sym}}_{C_X^\infty}(\Omega_x^1)$ and \mathcal{J} are augmented C_X^∞ algebras with augmentations

$$p : \widehat{\text{Sym}}_{C_X^\infty}(\Omega_X^1) \rightarrow \text{Sym}^0 = C_X^\infty \quad \text{and} \quad q : \mathcal{J} \rightarrow \mathcal{J} / F^1 \mathcal{J} \cong C_X^\infty.$$

Further, $\text{Sym}_{C_X^\infty}^{>0}(\Omega_X^1) = \ker p$ and $F^1 \mathcal{J} = \ker q$, so we obtain an isomorphism of C_X^∞ algebras

$$\widehat{\text{Sym}}_{C_X^\infty}(\Omega_X^1) \xrightarrow{\cong \sigma} \mathcal{J}$$

by extending the previous isomorphism by the identity on $\text{Sym}_{C_X^\infty}^0$ and $\mathcal{J} / F^1 \mathcal{J}$. The preceding discussion is just one instance of the equivalence of categories between commutative nonunital A algebras and commutative augmented A algebras for A any commutative algebra.

We then equip $\widehat{\text{Sym}}(\Omega_X^1)$ with the flat connection for \mathcal{J} , via the isomorphism, thus making it into a D_X algebra. Applying the de Rham functor dR , we get an isomorphism of Ω_X algebras

$$\widehat{\text{Sym}}_{C_X^\infty}(\Omega_X^1) \otimes_{C_X^\infty} \Omega_X \xrightarrow{\cong \sigma} \mathcal{J} \otimes_{C_X^\infty} \Omega_X.$$

Recall that the symmetric algebra is compatible with base change, that is

$$\begin{aligned} \widehat{\text{Sym}}_{C_X^\infty}(\Omega_X^1) \otimes_{C_X^\infty} \Omega_X^\sharp &= \widehat{\text{Sym}}_{\Omega_X^\sharp}(\Omega_X^1 \otimes_{C_X^\infty} \Omega_X^\sharp) \\ &\cong \widehat{\text{Sym}}_{\Omega_X^\sharp} \left((T_X[-1] \otimes_{C_X^\infty} \Omega_X^\sharp)^\vee[-1] \right), \end{aligned}$$

where we dualize over Ω_X^\sharp . Via the de Rham functor we have constructed a derivation on this completed symmetric algebra defining the L_∞ structure over Ω_X .

Property (c) follows from a standard argument; see [CFT02] for an explicit contracting homotopy. \square

Note that we pick a splitting σ in the proof but that the space of splittings is contractible, and that all the associated L_∞ algebras are strictly isomorphic. We thus make no fuss over the choice of σ and denote the resulting L_∞ space (X, \mathfrak{g}_X) by $B\mathfrak{g}_X$.

Remark 4.13. We view $B\mathfrak{g}_X$ as a natural *derived enhancement* of the smooth manifold X for the following reason. From the functor of points perspective, any sheaf of sets on the site Man of smooth manifolds

$$\mathcal{M} : \text{Man}^{\text{op}} \rightarrow \text{Sets}$$

is a kind of “generalized smooth manifold.” The representable functor $\underline{X} = \text{Man}(-, X)$ is such a sheaf. Similarly, a homotopy sheaf of simplicial sets \mathcal{M} on Man is then a smooth *stack*.¹² A *derived enhancement* of a smooth stack \mathcal{M} is a derived stack

$$\widetilde{\mathcal{M}} : \text{dgMan}^{\text{op}} \rightarrow \text{sSets}$$

such that the restriction to the subsite $\text{Man} \subset \text{dgMan}$ agrees with \mathcal{M} .¹³ By Lemma 4.10, (the derived stack of) any L_∞ space (X, \mathfrak{g}) is a derived enhancement of \underline{X} , since the restriction to Man does not care about \mathfrak{g} . But Lemma 4.12 above shows that $B\mathfrak{g}_X$ essentially provides a “cofibrant replacement” for the smooth manifold X : we have replaced the structure sheaf C_X^∞ with a semi-free resolution over Ω_X . In this sense, it is the most natural derived enhancement. \diamond

4.4.4. An L_∞ space encoding a complex manifold. The construction of the L_∞ space $B\mathfrak{g}_X$ is inspired by Costello’s work in the holomorphic setting. If Y is a complex manifold, then there exists an L_∞ space $Y_{\bar{\partial}} = (Y, \mathfrak{g}_{Y_{\bar{\partial}}})$ with the following properties.

Proposition 4.14 (Lemma 3.1.1 of [Cos10]). *Let Y be a complex manifold. The L_∞ space $Y_{\bar{\partial}}$ is well defined up to contractible choice. Further:*

- (a) *As an Ω_Y^\sharp -module, $\mathfrak{g}_{Y_{\bar{\partial}}}$ is isomorphic to $\Omega_Y^\sharp(T_Y^{1,0}[-1])$.*
- (b) *The derived stack $\mathbf{B}\mathfrak{g}_{Y_{\bar{\partial}}}$ represents the moduli problem of holomorphic maps into Y : for any complex manifold X (viewed as a nilpotent dg manifold), $\mathbf{B}\mathfrak{g}_{Y_{\bar{\partial}}}(X)$ is the discrete simplicial set of holomorphic maps from X to Y .*

¹²A generalized smooth manifold \mathcal{M} defines a smooth stack by taking the discrete simplicial set $\mathcal{M}(Y)$ on every manifold Y . The argument parallels Lemma C.6.

¹³This perspective is standard in the setting of derived geometry. See for instance [SchuTV] or [Toë09].

5. Geometry with L_∞ spaces

Especially important for us will be that we can thus define shifted symplectic structures on L_∞ spaces, which play a crucial role in the classical Batalin–Vilkovisky formalism.

5.1. Vector bundles on L_∞ spaces. The notion of L_∞ space is sufficiently geometric to admit notions of vector bundles, in particular, (co)tangent bundles.

Definition 5.1. Let $B\mathfrak{g} := (X, \mathfrak{g})$ be an L_∞ space. A *vector bundle* on $B\mathfrak{g}$ is a \mathbb{Z} -graded topological vector bundle $\pi : V \rightarrow X$ for which its sheaf of smooth sections \mathcal{V} over X has the structure of an Ω_X^\sharp -module and for which $\mathfrak{g} \oplus \mathcal{V}$ has the structure of a curved L_∞ algebra over Ω_X^* such that:

- (1) The inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathcal{V}$ and the projection $\mathfrak{g} \oplus \mathcal{V} \rightarrow \mathfrak{g}$ are maps of L_∞ algebras.
- (2) The Taylor coefficients ℓ_n of the L_∞ structure vanish on tensors containing two or more sections of \mathcal{V} .

The *sheaf of sections of \mathcal{V} over $B\mathfrak{g}$* is given by the sheaf on X of dg $C^*(\mathfrak{g})$ -modules $C^*(\mathfrak{g}, \mathcal{V}[1])$, the Chevalley–Eilenberg complex for an \mathfrak{g} -module. The *total space* for the vector bundle \mathcal{V} over $B\mathfrak{g}$ is the L_∞ space $(X, \mathfrak{g} \oplus \mathcal{V})$.

In particular, for X a point, we recover the usual notion of a representation of \mathfrak{g} . Note that we have merely picked out a class of well-behaved sheaves of \mathfrak{g} -modules.

We now pick out two important examples. Recall that for a semi-free commutative dg algebra $\mathcal{A} = (\widehat{\text{Sym}}(V), d)$, the *derivations* are the module $\text{Der } \mathcal{A} := (\widehat{\text{Sym}}(V) \otimes V^\vee, d)$. We view the derivations as the vector fields — the sections of the tangent bundle $T_{\mathcal{A}}$ — on the space corresponding to \mathcal{A} . By this correspondence, we obtain the following.

Definition 5.2. The *tangent bundle* $T_{B\mathfrak{g}}$ is given by $\mathfrak{g}[1]$ equipped with the (shifted) adjoint action of \mathfrak{g} . Likewise, the *cotangent bundle* $T_{B\mathfrak{g}}^*$ is given by $\mathfrak{g}^\vee[-1]$ equipped with the (shifted) coadjoint action of \mathfrak{g} .

There are also *shifted* (co)tangent bundles. For instance, we let $T[k]B\mathfrak{g}$ denote the L_∞ space $(X, \mathfrak{g} \oplus \mathfrak{g}[k+1])$, which is the total space of the vector bundle $T_{B\mathfrak{g}}[k]$.

Sections of $T_{B\mathfrak{g}}^*$ over $B\mathfrak{g}$ are the Kähler differentials $\Omega_{B\mathfrak{g}}^1$ of $\mathcal{O}_{B\mathfrak{g}} = C^*\mathfrak{g}$. They are $\widehat{\text{Sym}}(\mathfrak{g}^\vee[-1]) \otimes_k (\mathfrak{g}^\vee[-1])$ equipped with the differential

$$d_{\Omega^1} : f \otimes x \mapsto d_{\mathfrak{g}}f \otimes x + (-1)^{|f|} f \cdot d_{dR}(d_{\mathfrak{g}}x),$$

where $f \in \mathcal{O}_{B\mathfrak{g}}$ and $x \in \mathfrak{g}^\vee[-1]$. Here $d_{\mathfrak{g}}$ denotes the differential on $C^*\mathfrak{g}$ and $d_{dR} : \mathcal{O}_{B\mathfrak{g}} \rightarrow \Omega_{B\mathfrak{g}}^1$ denotes the universal derivation

$$d_{dR} : x \mapsto 1 \otimes x$$

for $x \in \mathfrak{g}^\vee[-1]$. Note that $d_\Omega \circ d_{dR} = d_{dR} \circ \partial$. From hereon, we will denote $1 \otimes x$ by dx and $f \otimes x$ by $f dx$.

There is a natural analog of the de Rham complex $DR_{B\mathfrak{g}}$ where $\Omega_{B\mathfrak{g}}^k := C^*(\mathfrak{g}, \wedge^k \mathfrak{g}[-k])$ and

$$DR_{B\mathfrak{g}} := \bigoplus_{k \geq 0} \Omega_{B\mathfrak{g}}^k[-k]$$

with total differential d_{DR} given by adding the “internal” differential of Ω^k as a \mathfrak{g} -module to the “exterior derivative” d_{dR} (the universal derivation described in the preceding paragraph).

5.2. Shifted symplectic structures. We now formulate the analog of a symplectic form ω on an L_∞ space $B\mathfrak{g} = (X, \mathfrak{g})$. Recall that on a smooth manifold M , a 2-form $\omega \in \Omega^2(M)$ is *symplectic* if ω is closed and provides an isomorphism of vector bundles $\underline{\omega} : T_M \rightarrow T_M^*$. We must provide derived versions of these two conditions. (For a more sophisticated and thorough treatment of derived symplectic geometry, see [PTVV13]. Our definitions are essentially L_∞ space versions of theirs.)¹⁴

As described above, we have an analog of the de Rham complex. A *closed 2-form* is then a cocycle in the truncated de Rham complex

$$\Omega_{cl}^2 := \left(\bigoplus_{k \geq 2} \Omega_{B\mathfrak{g}}^k[-k+2], d_{DR} \right),$$

shifted to put Ω^2 “in degree 0.” Observe that there is a natural map

$$i : \Omega_{cl}^2 \rightarrow \Omega_{B\mathfrak{g}}^2$$

by forgetting components living in $\Omega^{>2}$. Hence being closed is *data*, not a *property*: given a 2-form $\omega \in \Omega_{B\mathfrak{g}}^2$, one must find a lift to some cocycle $\tilde{\omega} \in \Omega_{cl}^2$ to have a closed 2-form.

As usual, every element of the \mathfrak{g} -module $\mathcal{V}^\vee \otimes \mathcal{W}$ determines an element of the module $\text{Hom}_{\mathfrak{g}}(\mathcal{V}, \mathcal{W})$. Given a 2-form ω of cohomological degree n , we thus obtain a bundle morphism $\underline{\omega} : T_{B\mathfrak{g}} \rightarrow T_{B\mathfrak{g}}^*$ of cohomological degree $-n$. We say ω is *nondegenerate* if the morphism $\underline{\omega}$ is a quasi-isomorphism from $T_{B\mathfrak{g}}$ to $T_{B\mathfrak{g}}^*[-n]$.

Definition 5.3. An *n -shifted symplectic L_∞ space* is an L_∞ space $B\mathfrak{g} = (X, \mathfrak{g})$ equipped with a closed 2-form $\omega \in \Omega_{cl}^2$ of cohomological degree $-n$ such that $i(\omega)$ is nondegenerate.

The shifted cotangent bundle $T^*[n]B\mathfrak{g}$ is equipped with a natural n -shifted symplectic structure by antisymmetrizing the (shifted) evaluation pairing.

¹⁴As in the case of the Lie-theoretic perspective on deformation theory, there is a long history to these ideas. Our approach grows directly out of the work of Schwarz and Kontsevich and their various collaborators (see, for instance, [Schw93], [MS06], and [ASZK97]).

6. The derived loop space

We now discuss an interesting collection of examples — derived loop spaces — whose analogs in derived algebraic geometry provide new perspectives on important constructions in representation theory and homological algebra (see, for instance, [BenzN12] and [TV09]). In our work [GG14], their geometry describes the classical field theory that we quantize.

Recall the usual free loop space LX , where

$$LX = \text{Man}(S^1, X) = \{\text{smooth maps } f : S^1 \rightarrow X\}.$$

There are various ways to view LX as a topological space or even an ∞ -dimensional smooth manifold. For us, the functor of points approach to geometry suggests that we view LX as the functor

$$L_{sm}X : \text{Man}^{\text{op}} \rightarrow \text{Sets}$$

$$M \mapsto \text{Man}(S^1 \times M, X)$$

where the right hand side is simply the *set* of smooth maps $S^1 \times M \rightarrow X$. Note that this functor is, in fact, a sheaf on the site Man .

When we move to the derived setting, we can ask for a derived enhancement of $L_{sm}X$, but we can also ask about other kinds of loop spaces using other circles. In particular, two other “circles” besides S^1 are:

- (1) the de Rham circle $S_{dR}^1 = (S^1, \Omega_{S^1}^*)$,
- (2) the “Betti circle” $S_B^1 = (\text{pt}, \mathbb{R}[\epsilon])$ with ϵ of degree 1 (i.e., the functions are the cohomology ring of the circle).

The de Rham circle is a nilpotent dg manifold but the Betti circle is not, strictly speaking, because the algebra does not sit in nonpositive degrees. Nonetheless, it is a reasonable ringed space to consider. In essence, S_{dR}^1 knows about the smooth topology of S^1 and S_B^1 knows about the real homotopy type of S^1 .

We now describe derived stacks that are natural loop spaces for these various circles. Note that we will always be discussing some version of a loop space for a manifold X , but our constructions work for an arbitrary L_∞ space by replacing $B\mathfrak{g}_X$ by an arbitrary $B\mathfrak{g}$.

6.1. Enhancing $L_{sm}X$. Lemma 4.12 gives us a natural derived enhancement of X itself: use the L_∞ space $B\mathfrak{g}_X$. We piggyback on that result to enhance $L_{sm}X$.

Definition 6.1. Let $\mathcal{L}_{sm}X$ denote the functor

$$\mathcal{L}_{sm}X : \text{dgMan}^{\text{op}} \rightarrow \text{sSets}$$

$$\mathcal{M} \mapsto \mathbf{B}\mathfrak{g}_X(S^1 \times \mathcal{M}).$$

This functor is, in fact, a derived stack because $\mathbf{B}\mathfrak{g}_X$ is.

6.2. The Betti loop space $\mathcal{L}_B X$. Let X be a smooth manifold and let $B\mathfrak{g}_X = (X, \mathfrak{g}_X)$ be the associated L_∞ space. Let $\mathcal{L}_B X$ denote the L_∞ space $(X, \mathbb{R}[\epsilon] \otimes \mathfrak{g}_X)$. We call it the *Betti loop space* of X .

Remark 6.2. If S_B^1 were a nilpotent dg manifold, the functor $\mathbf{B}\mathcal{L}_B X$ would agree with $\mathbf{B}\mathfrak{g}_X(S_B^1 \times -)$. Hence this definition is motivated by the same logic as $L_{sm} X$ or $\mathcal{L}_{sm} X$. \diamond

The Betti loop space has another description: it is just the L_∞ space $T[-1]B\mathfrak{g}_X$.¹⁵ Explicitly, this L_∞ space is

$$(X, \mathfrak{g}_X \oplus \mathfrak{g}_X[-1]),$$

where $\mathfrak{g}_X[-1]$ denotes $\Omega_X^*(T_X^*[-2])$ viewed as an \mathfrak{g}_X -module by the shifted adjoint action.

Observe that there is a natural quasi-isomorphism

$$\bigoplus_{k \geq 0} \Omega_X^k[k] \xrightarrow{\simeq} C^*(\mathfrak{g}_X \oplus \mathfrak{g}_X[-1]),$$

extending the quasi-isomorphism from Lemma 4.12. This equivalence says that “the functions on the Betti loop space are the (regraded) de Rham forms of X ,” that is, $\mathcal{O}(\mathcal{L}_B X) \simeq \bigoplus_{k \geq 0} \Omega_X^k[k]$. This description meshes nicely with the Hochschild–Kostant–Rosenberg theorem and its interpretation in derived geometry: the Betti loop space encodes the derived self-intersection of X as the diagonal in $X \times X$, and hence its algebra of functions should be the Hochschild homology of functions on X . (See [TV11] and [BenzN12] for a lot more on this topic.)

Remark 6.3. The Betti loop space, and the de Rham loop space defined below, only become interesting when evaluated on interesting nilpotent dg manifolds. When restricted to the subcategory Man of ordinary manifolds, these functors are isomorphic to the underlying smooth manifold X . \diamond

6.3. The de Rham loop space $\mathcal{L}_{dR} X$ and its completion along constant maps. Following the pattern so far, we define the *de Rham loop space* of X as follows.

Definition 6.4. Let $\mathcal{L}_{dR} X$ denote the functor

$$\begin{aligned} \mathcal{L}_{dR} X : \text{dgMan}^{\text{op}} &\rightarrow s\text{Sets} \\ \mathcal{M} &\mapsto \mathbf{B}\mathfrak{g}_X(S_{dR}^1 \times \mathcal{M}). \end{aligned}$$

This functor is again a derived stack.

Our main interest in [GG14], however, lies with a simpler space: we focus on maps out of $S_{dR}^1 \times \mathcal{M}$ where the underlying smooth map $S^1 \times M \rightarrow X$ is constant along S^1 . Thanks to a result of Costello, this subfunctor is, in fact,

¹⁵This statement is the manifestation, in our formalism, of the fact that the derived loop space of a scheme X is $T[-1]X$. See [BenzN12] for some discussion of this fact.

given by an L_∞ space. We start with his general result before discussing its use in our context.

Let $B\mathfrak{g} = (X, \mathfrak{g})$ be an L_∞ space and let $\mathcal{N} = (N, \mathcal{O}_\mathcal{N})$ be a nilpotent dg manifold. We define a new simplicial presheaf $\mathbf{B}\mathfrak{g}^\mathcal{N}$ on the site dgMan by

$$\mathbf{B}\mathfrak{g}^\mathcal{N} : \text{dgMan}^{\text{op}} \rightarrow \text{sSets}$$

$$\mathcal{M} \mapsto \mathbf{B}\mathfrak{g}(\mathcal{N} \times \mathcal{M}).$$

Consider the sub-simplicial presheaf $\widehat{\mathbf{B}\mathfrak{g}^\mathcal{N}} \subset \mathbf{B}\mathfrak{g}^\mathcal{N}$ given by Maurer–Cartan solutions where the underlying map of manifolds $N \times M \rightarrow X$ is independent of N . That is, for \mathcal{M} a nilpotent dg manifold

$$\widehat{\mathbf{B}\mathfrak{g}^\mathcal{N}}(\mathcal{M}) \subset \mathbf{B}\mathfrak{g}^\mathcal{N}(\mathcal{M})$$

consists of Maurer–Cartan elements (f, α) such that the underlying smooth map f factors through the projection onto M :

$$\begin{array}{ccc} N \times M & \xrightarrow{f} & X \\ \pi_M \downarrow & \nearrow & \\ M & & \end{array} .$$

Costello shows that this subfunctor is itself an L_∞ space, under certain conditions.

Every nilpotent dg manifold $\mathcal{N} = (N, \mathcal{O}_\mathcal{N})$ has a natural filtration

$$F^k \mathcal{O}_\mathcal{N} = \mathcal{I}_\mathcal{N}^k$$

by powers of its nilpotent ideal $\mathcal{I}_\mathcal{N}$. Let $\text{Gr}^k \mathcal{O}_\mathcal{N} = F^k \mathcal{O}_\mathcal{N} / F^{k+1} \mathcal{O}_\mathcal{N}$ denote the k th component of the associated graded sheaf of algebras.

Proposition 6.5 (Proposition 5.0.1 of [Cos10]). *Let $\mathcal{N} = (N, \mathcal{O}_\mathcal{N})$ be a nilpotent dg manifold such that the cohomology of $\text{Gr}^k \mathcal{O}_\mathcal{N}$ is concentrated in strictly positive degrees for $k \geq 1$. Further, let $B\mathfrak{g} = (X, \mathfrak{g})$ be an L_∞ space such that the cohomology of the sheaf of L_∞ algebras*

$$\mathfrak{g}^{\text{red}} := \mathfrak{g} / \Omega_X^{>0}$$

is concentrated in strictly positive degrees. Then, the simplicial presheaf $\widehat{\mathbf{B}\mathfrak{g}^\mathcal{N}}$ is weakly equivalent to the functor of points for the L_∞ space $\mathfrak{h} := (X, \mathfrak{g} \otimes_{\mathbb{R}} \mathcal{O}_\mathcal{N}(N))$, where $\mathcal{O}_\mathcal{N}(N)$ denotes the global sections of the structure sheaf $\mathcal{O}_\mathcal{N}$. In other words, we have a natural transformation of functors

$$\mathbf{B}\mathfrak{h} \xrightarrow{\simeq} \widehat{\mathbf{B}\mathfrak{g}^\mathcal{N}}$$

that is an objectwise weak equivalence.

Let us apply this result to the case of interest, where the L_∞ space is $B\mathfrak{g}_X = (X, \mathfrak{g}_X)$ and the nilpotent dg manifold is $S_{dR}^1 = (S^1, \Omega_{S^1})$. Then $\mathcal{L}_{dR} X$ is precisely the derived space $B\mathfrak{g}_X^{S_{dR}^1}$. Moreover, our spaces satisfy the

hypotheses of the proposition, since the sheaves $\mathrm{Gr}^k(\Omega_{S^1}^*) = \Omega_{S^1}^k$ are soft (and hence have vanishing higher cohomologies) and since $\mathfrak{g}_X^{\mathrm{red}}$ is already concentrated in positive degrees. Hence, we have the following.

Definition/Lemma 6.6. *The L_∞ space*

$$\widehat{\mathcal{L}_{dR}X} := (X, \Omega^*(S^1) \otimes \mathfrak{g}_X)$$

encodes the derived stack $B\widehat{\mathfrak{g}_X^{S^1_{dR}}}$.

Finally, we note the following, which makes this space particularly easy to understand.

Lemma 6.7. *The L_∞ spaces $\mathcal{L}_B X$ and $\widehat{\mathcal{L}_{dR}X}$ determine weakly equivalent derived stacks. In fact, any volume form ω on S^1 induces a natural transformation $\underline{\omega} : \mathcal{L}_B X \Rightarrow \widehat{\mathcal{L}_{dR}X}$ that is a homotopy equivalence on every nilpotent dg manifold.*

Proof. The first statement directly follows from the second. Note that a choice of ω gives a quasi-isomorphism of commutative dg algebras

$$\begin{aligned} \phi_\omega : \mathbb{R}[\epsilon] &\rightarrow \Omega^*(S^1) \\ a + b\epsilon &\mapsto a + b\omega. \end{aligned}$$

Hence we obtain a map of L_∞ spaces

$$(id_X, \phi_\omega) : \mathcal{L}_B X = (X, \mathfrak{g}_X \otimes \mathbb{R}[\epsilon]) \rightarrow (X, \mathfrak{g}_X \otimes \Omega^*(S^1)) = \widehat{\mathcal{L}_{dR}X}$$

that is a quasi-isomorphism of sheaves of curved L_∞ algebras. This map induces the desired natural transformation between their Maurer–Cartan functors. In particular, given a nilpotent dg manifold $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$, for each smooth map $f : M \rightarrow X$, we obtain a quasi-isomorphism of nilpotent curved L_∞ algebras

$$f^* \phi_\omega : f^*(\mathfrak{g}_X \otimes \mathbb{R}[\epsilon]) \otimes_{\Omega_M^*} \mathcal{S}_\mathcal{M} \rightarrow f^*(\mathfrak{g}_X \otimes \Omega^*(S^1)) \otimes_{\Omega_M^*} \mathcal{S}_\mathcal{M}$$

and hence a homotopy equivalence of Maurer–Cartan spaces (see Appendix B). \square

6.4. Symplectic structures and the AKSZ construction. We now explain how to equip $\widehat{\mathcal{L}_{dR}X}$ with a (-1) -symplectic structure when X is 0-symplectic. Our approach is a version of the AKSZ construction of Alexandrov, Kontsevich, Schwarz, and Zaboronsky [ASZK97], who introduced a general method for constructing shifted symplectic structures on mapping spaces. Their motivation was to produce “classical Batalin–Vilkovisky theories,” where the BV (Batalin–Vilkovisky) formalism is a homological approach to field theory. The language of L_∞ spaces is well-suited to applying the AKSZ approach, as we will demonstrate in the case of the derived loop space $\widehat{\mathcal{L}_{dR}X}$. Indeed, in [GG14], we start with this space and then quantize it using the BV formalism, as developed in [Cos11b].

Before equipping the derived loop space with a (-1) -symplectic structure, we provide a quick gloss of the AKSZ construction as motivation. The basic idea is simple. Let the d -dimensional source dg manifold Σ of a sigma model come equipped with a volume form dvol and let the target dg manifold X come equipped with a k -symplectic structure ω . Then the mapping space (*aka* fields) $\mathcal{F} := \text{Maps}(\Sigma, X)$ obtains a $k-d$ -symplectic structure as follows. For a fixed map $\phi : \Sigma \rightarrow X$, the tangent space $T_\phi\mathcal{F} = \Gamma(\Sigma, \phi^*T_X)$ has a natural pairing

$$\langle \zeta, \zeta' \rangle_\phi := \int_{s \in \Sigma} \phi^*\omega(s)(\zeta(s), \zeta'(s)) \text{dvol},$$

with $\zeta, \zeta' \in T_\phi\mathcal{F}$; and $\langle -, - \rangle_\phi$ has cohomological degree $n-d$ by construction and is skew-symmetric thanks to the skew-symmetry of ω . In many situations, it will be closed and nondegenerate, as well.

This construction can be realized in our context. Suppose X is a 0-symplectic manifold (i.e., a symplectic manifold in the standard sense) with symplectic form $\omega \in \Omega^2(X)$. Suppose we fix a 1-form $\nu \in \Omega^1(S^1)$ that is not exact, i.e., $\int_{S^1} \nu \neq 0$.

By Corollary 11.3 of [GG14], we know that the ∞ -jet $J(\omega)$ is a cocycle in both $\Omega^2(B\mathfrak{g}_X)$ and $\Omega_{cl}^2(B\mathfrak{g}_X)$. (This is a special feature of this situation.) Consider the pairing

$$\begin{aligned} \Omega_{\omega, \nu} : [\mathfrak{g}_X \otimes \Omega^*(S^1)]^{\otimes 2} &\rightarrow \Omega^*(X) \\ (Z \otimes \alpha) \otimes (Z' \otimes \alpha') &\mapsto \int_{\theta \in S^1} J(\omega)(Z \otimes \alpha(\theta), Z' \otimes \alpha'(\theta)) \wedge \nu(\theta), \end{aligned}$$

which is a direct application of the AKSZ approach. Note that it is a cocycle in $\Omega^2(\mathfrak{g}_X \otimes \Omega^*(S^1))$ by construction.

Lemma 6.8. *The 2-form $\Omega_{\omega, \nu}$ is a (-1) -symplectic form on $\widehat{\mathcal{L}_{dR}X}$.*

Proof. Checking that it is a closed 2-form is a direct computation. It remains to show that the pairing is nondegenerate. Observe that $J(\omega)$ is a nondegenerate pairing on $T_{B\mathfrak{g}_X}$, thanks to Corollary 11.3 of [GG14]. (Indeed, the horizontal sections of jets of vector fields on X is precisely the sheaf of smooth vector fields on X .) As ν is cohomological nontrivial on S^1 , we see that $\Omega_{\omega, \nu}$ is nontrivial at the level of cohomology as well. \square

Similar arguments should work for many L_∞ spaces of this form (i.e., an L_∞ space that arises as completion of a mapping stack along some set of maps). Thus, this formalism is a natural place to deploy the AKSZ construction.

Appendix A. Homotopy limits and cosimplicial simplicial sets

In a category with a notion of weak equivalence (such as topological spaces with weak homotopy equivalence or chain complexes with quasi-isomorphism), the *homotopy* limit of a diagram is better behaved than the ordinary limit with regards to questions about objects up to weak equivalence. There is an extensive literature that motivates, defines, and constructs homotopy limits in a variety of contexts, and we recommend [Dug] and [Shu] as nice places to start reading. (For a more succinct discussion close to the style of our overview, see [Dou14].)

In our context, we can take advantage of the existence of several explicit, well-behaved formulas for the homotopy limit of a diagram in simplicial sets.¹⁶ A standard source is the foundational work of Bousfield and Kan [BouK72] (a recent, thorough source is [Hir03]). Of course, it is not always obvious how the different formulas are related. We give a quick discussion of the ideas behind the formulas we use.

To motivate all this formalism, we remind the reader of its appearance in this paper. Let \mathcal{F} be a simplicial presheaf and $\mathfrak{V} = \{V_i : i \in I\}$ a cover of U . Applying the presheaf \mathcal{F} levelwise to the Čech nerve $\check{C}\mathfrak{V}_\bullet$, we obtain a cosimplicial simplicial set $\check{C}^\bullet(\mathfrak{V}; \mathcal{F})$ given by

$$\prod_{i_0} \mathcal{F}(V_{i_0 \in I}) \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \end{array} \prod_{i_0, i_1 \in I} \mathcal{F}(V_{i_0 i_1}) \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xrightarrow{d^2} \end{array} \prod_{i_0, i_1, i_2 \in I} \mathcal{F}(V_{i_0 i_1 i_2}) \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xrightarrow{d^2} \\ \xrightarrow{d^3} \end{array} \cdots,$$

called the *Čech cosimplicial diagram* associated to \mathfrak{V} with coefficients in \mathcal{F} . The standard notation is

$$\check{C}^p(\mathfrak{V}; \mathcal{F}) := \prod_{i_0 \cdots i_p} \mathcal{F}(V_{i_0 \cdots i_p}),$$

where $V_{i_0 \cdots i_p} = V_{i_0} \cap \cdots \cap V_{i_p}$. We want to compute

$$\text{holim}_\Delta \check{C}^\bullet(\mathfrak{V}; \mathcal{F}),$$

so we need to define a homotopy limit over a cosimplicial diagram.

A.1. A quick overview of the big picture. Let I denote a small category, and let $\text{Fun}(I, \mathcal{C})$ denote the category of functors from this index category I to \mathcal{C} . We call such a functor X an *I-diagram* in \mathcal{C} and so call $\text{Fun}(I, \mathcal{C})$ a *diagram category*. Recall that the limit of a diagram X arises from an adjunction

$$\mathcal{C} : \mathcal{C} \rightleftarrows \text{Fun}(I, \mathcal{C}) : \lim_I$$

¹⁶We emphasize that a homotopy limit should be defined by a homotopic version of the usual universal property for a limit. These formulas provide an explicit means for constructing an object that provides such a homotopy limit.

where for $x \in \mathcal{C}$, Cx denotes the diagram

$$i \mapsto x \text{ and } (i \rightarrow i') \mapsto (x \xrightarrow{1_x} x),$$

aka the constant diagram with value x . It is straightforward to unravel this definition to the definition by the terminal cone.

Now suppose \mathcal{C} is a category with a notion of weak equivalence. Let $\text{Ho}(\mathcal{C})$ denote the homotopy category given by localizing at these weak equivalences. We equip $\text{Fun}(I, \mathcal{C})$ with an associated notion of weak equivalence: a map of diagrams $f : X \rightarrow Y$ is a weakly equivalence if the map on each object $f(i) : X(i) \rightarrow Y(i)$ is a weak equivalence. Let $\text{Ho}(\text{Fun}(I, \mathcal{C}))$ denote the associated homotopy category.

Observe that the constant functor C preserves weak equivalences and hence induces a functor

$$\text{Ho}(C) : \text{Ho}(\mathcal{C}) \rightarrow \text{Ho}(\text{Fun}(I, \mathcal{C})).$$

We denote the right adjoint of $\text{Ho}(C)$ — if it exists — by $\text{Ho} - \lim_I$.¹⁷ *Note that $\text{Ho} - \lim_I$ does not arise from \lim_I — it is not “ $\text{Ho}(\lim_I)$ ” — except in very special situations, because \lim_I need not preserve weak equivalences.*

Not only do we want to construct $\text{Ho} - \lim_I$, but we want to go a step further and produce a functor $\text{holim}_I : \text{Fun}(I, \mathcal{C}) \rightarrow \mathcal{C}$ that induces $\text{Ho} - \lim_I$ at the level of homotopy categories.

Model categories are a well-established approach (among others) to accomplishing such goals.¹⁸ One useful feature of model categories is that they provide another tool for constructing functors between homotopy categories: given a Quillen adjunction

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R$$

between model categories, one obtains an adjunction

$$\mathbb{L}L : \text{Ho}(\mathcal{C}) \rightleftarrows \text{Ho} \mathcal{D} : \mathbb{R}R$$

between the homotopy categories. We now use this tool to find our desired functor holim_I , following Bousfield and Kan.

We are only interested here in the case where \mathcal{C} is the category $s\text{Sets}$ of simplicial sets, with the standard *aka* Quillen model structure. Bousfield and Kan introduced a model structure on the diagram category $\text{Fun}(I, s\text{Sets})$, now known as the *projective* model structure, where weak equivalences and fibrations are both objectwise.

The trick to making holim_I is to find a functor $C' : s\text{Sets} \rightarrow \text{Fun}(I, s\text{Sets})$ such that:

- (1) There is a Quillen adjunction $C' : s\text{Sets} \rightleftarrows \text{Fun}(I, s\text{Sets}) : F$.
- (2) C' is *not* C but $\mathbb{L}C' = \text{Ho}(C)$, so they agree at the level of homotopy categories.

¹⁷We use this ugly notation to avoid conflicting with commonly-used notations.

¹⁸Our discussion only assumes a basic familiarity with model categories. For more extensive discussion, there are many lovely expositions, such as [DwS95] or [Hov99].

We then know that $\mathbb{R}F$ is $\text{Ho} - \lim_I$, so we can view the right adjoint F as the desired holim_I . In particular, on an objectwise-fibrant diagram X , we know that $\text{holim}_I X$ is a simplicial set with the correct homotopy type.

We now construct C' . For any $i \in I$, we denote its over-category by $I_{/i}$. Let $N(I_{/i})$ denote the nerve of $I_{/i}$, which is a contractible simplicial set. We define a functor

$$C' = I_{/-} \times - : s\text{Sets} \rightarrow \text{Fun}(I, s\text{Sets})$$

$$x \mapsto (i \mapsto x \times N(I_{/i})).$$

By construction, the diagram $C'(x)$ assigns to every $i \in I$ a simplicial set $C'(x)(i) = x \times N(I_{/i})$ that is weakly equivalent to x . Indeed, there is a canonical natural transformation $\eta : C \Rightarrow C'$ because $N(I_{/i})$ has a canonical basepoint $*$ coming from the terminal object $i \rightarrow i$ in $I_{/i}$ and so $\eta(x)$ is the canonical map $x \hookrightarrow x \times * \subset x \times N(I_{/i})$. Then η is an objectwise weak equivalence.

Recall that $s\text{Sets}$ is enriched over itself, and there is an inner hom that we denote Maps , so

$$s\text{Sets}(x \times y, z) \cong s\text{Sets}(x, \text{Maps}(y, z)).$$

It is immediate from this adjunction that the set of n -simplices of $\text{Maps}(y, z)$ is $s\text{Sets}(\Delta[n] \times y, z)$. Piggybacking on this construction, we obtain a right adjoint to the functor C' :

$$\text{Maps}(I_{/-}, -) : \text{Fun}(I, s\text{Sets}) \rightarrow s\text{Sets}$$

$$X \mapsto ([n] \mapsto \text{Nat}(I_{/-} \times C' \Delta[n], X)),$$

where $\text{Nat}(A, B)$ denotes the set of natural transformations between the diagrams A and B .

Proposition A.1. *The adjunction*

$$C' = I_{/-} \times - : s\text{Sets} \rightleftarrows \text{Fun}(I, s\text{Sets}) : \text{Maps}(I_{/-}, -)$$

is a Quillen adjunction, using the projective model structure on the right hand side.

We thus define the *homotopy limit* holim_I to be $\text{Maps}(I_{/-}, -)$. Its right derived functor is $\text{Ho} - \lim_I$.

Our main interest is when $I = \Delta$, so that $\text{Fun}(I, s\text{Sets})$ is precisely $cs\text{Sets}$, the cosimplicial simplicial sets.

Definition A.2. The *homotopy limit* of a cosimplicial diagram of simplicial sets $X \in cs\text{Sets}$ is

$$\text{holim}_\Delta X = \text{Maps}(\Delta_{/-}, X).$$

We emphasize that this is not the only definition used in the literature, but any construction/definition should provide a weakly equivalent simplicial set (so long as we agree on the notion of weak equivalence of diagrams).

A.2. An alternative approach: fat totalization. When $I = \Delta$, one can give a different functor, known as *fat totalization* and denoted $\mathbb{T}ot$, whose right derived functor $\mathbb{R}\mathbb{T}ot$ is $\text{Ho} - \lim_{\Delta}$. In other words, we will recall a different approach to the homotopy limit which has its foundations in the work of Segal [Seg74]. Costello uses this version of holim in his proof, which is why we include this brief discussion.

Let Δ^{inj} denote the subcategory of Δ with the same objects but whose morphisms are just the injections. The superscript inj is to indicate that it is generated by the face inclusions $[n-1] \hookrightarrow [n]$. Let $\iota : \Delta^{\text{inj}} \rightarrow \Delta$ denote the inclusion functor. We immediately obtain a Quillen adjunction

$$\iota_! : \text{Fun}(\Delta^{\text{inj}}, s\text{Sets}) \rightleftarrows \text{Fun}(\Delta, s\text{Sets}) : \iota^*$$

where $\iota_!$ is the left Kan extension along ι . Note that it is a Quillen adjunction for the projective model structures because the forgetful functor ι clearly preserves objectwise fibrations and weak equivalences.

By our work above, we already have a Quillen adjunction

$$\Delta^{\text{inj}}_{/-} : s\text{Sets} \rightleftarrows \text{Fun}(\Delta^{\text{inj}}, s\text{Sets}) : \text{Maps}(\Delta^{\text{inj}}_{/-}, -).$$

Let $\mathbb{T}ot$ be the composition $\text{Maps}(\Delta^{\text{inj}}_{/-}, -) \circ \iota^*$. Composing these Quillen adjunctions, we obtain a Quillen adjunction

$$\iota_! \circ \Delta^{\text{inj}}_{/-} : s\text{Sets} \rightleftarrows \text{Fun}(\Delta, s\text{Sets}) : \mathbb{T}ot.$$

One can check that $\iota_! \circ \Delta^{\text{inj}}_{/-}$ is weakly equivalent to C , and hence $\mathbb{T}ot$ is weakly equivalent to holim_{Δ} .

This construction is called *fat totalization* to contrast it with totalization. The *totalization* of $X_{\bullet} \in cs\text{Sets}$ is the simplicial set given by

$$\text{Tot } X_{\bullet} := \text{Maps}(\Delta_{\bullet}, X_{\bullet}).$$

where Δ_{\bullet} is the *cosimplicial standard simplex* whose simplicial set of n -cosimplices Δ_{\bullet}^n is the standard n -simplex $\Delta[n]$. Totalization is thus the dual notion to geometric realization.

A.3. Yet another definition and its use in the linear setting. There is another model category structure that is often used on cosimplicial simplicial sets, known as the Reedy model structure (Bousfield and Kan also work with this structure). For $X_{\bullet} \in cs\text{Sets}$ that is (co)fibrant in this model structure, we say X_{\bullet} is *Reedy-(co)fibrant*. The cosimplicial standard simplex Δ_{\bullet} is a cofibrant replacement for the constant functor C in $cs\text{Sets}$, so we use it to define yet another version the homotopy limit, which we denote by holim_{Δ}^R .

Proposition A.3 (XI.4.5, [BouK72]). *For X_{\bullet} a Reedy-fibrant cosimplicial simplicial set, the natural map $\Delta_{/-} \rightarrow \Delta_{\bullet}$ induces a weak equivalence*

$$\text{Tot } X_{\bullet} \rightarrow \text{holim}_{\Delta}^R X_{\bullet}.$$

Moreover, Bousfield and Kan prove that every cosimplicial simplicial group is Reedy-fibrant (see X.4.9, [BouK72]). Hence, we can compute homotopy limits using totalization, which is sometimes simpler.

These results are particularly helpful in connecting the Čech complex for a sheaf of chain complexes to Čech descent using holim^R . We use this result to prove our main theorem about L_∞ spaces, so we explain what we need.

There is a conormalization functor $N^* : cAb \rightarrow Ch^+$ that turns a cosimplicial abelian group into a nonnegatively-graded cochain complex of abelian groups. It is the cosimplicial twin of the normalization functor $N_* : sAb \rightarrow Ch_+$ appearing in the Dold–Kan correspondence between simplicial abelian groups and nonnegatively-graded chain complexes. Let $DK : Ch_+ \rightarrow sAb$ denote the adjoint functor appearing in the Dold–Kan correspondence.

Given a cosimplicial simplicial abelian group A_\bullet , let TA denote the total chain complex of the double complex $N^*N_*A_\bullet$ (we use the product of groups in making the total complex). Let $\text{Tot } A$ denote the simplicial abelian group given by totalization as a cosimplicial simplicial set. Note that this computes $\text{holim}_\Delta^R A$, as A is fibrant.

Proposition A.4 (Lemma 2.2 [Bou89]). *There is a natural quasi-isomorphism $\phi : N_* \text{Tot } A \rightarrow TA$. Equivalently, under the Dold–Kan correspondence, there is a natural weak equivalence $\phi' : \text{Tot } A \rightarrow DK(TA)$.*

Given a simplicial presheaf \mathcal{F} such that $\mathcal{F}(U)$ is a simplicial abelian group for every open U , we thus see that $\text{holim}_{\mathcal{C}\mathcal{O}}^R \mathcal{F}$ is equivalent to the usual Čech complex for \mathcal{F} .

Appendix B. Brief overview of the Maurer–Cartan functor

We collect here some definitions and theorems about the Maurer–Cartan space MC_\bullet that we use in understanding the derived stack $\mathbf{B}\mathfrak{g}$ from an L_∞ space $B\mathfrak{g} := (X, \mathfrak{g})$. The elegant paper of Getzler [Get09] is the primary reference. Look there for the proofs and many further insights.

Definition B.1. The *lower central series* of a L_∞ algebra \mathfrak{g} over a commutative dg algebra A is the decreasing filtration $F^k \mathfrak{g}$ with $F^1 \mathfrak{g} = \mathfrak{g}$ and

$$F^k \mathfrak{g} = \sum_{i_1 + \dots + i_n = k} \ell_k(F^{i_1} \mathfrak{g}, \dots, F^{i_n} \mathfrak{g})$$

for $k > 1$. (This expression means “take the span of elements produced by applying the bracket ℓ_k to elements from the appropriate level of the filtrations.”)

A L_∞ algebra \mathfrak{g} is *nilpotent* if there is some positive integer N such that $F^N \mathfrak{g} = 0$. In other words, any sufficiently long sequence of brackets vanishes.

For \mathfrak{g} a nilpotent L_∞ algebra, the Maurer–Cartan equation $mc(\alpha) = 0$ is well-posed, since the infinite sum — which is *a priori* ill-defined — is actually a finite sum.

Let \mathfrak{g} denote a nilpotent L_∞ algebra. Getzler proves several properties that are useful for us:

- (1) $\mathrm{MC}_\bullet(\mathfrak{g})$ is a Kan complex.
- (2) For $f : \mathfrak{g} \rightarrow \mathfrak{h}$ a levelwise surjective map of nilpotent L_∞ algebras, the induced map $\mathrm{MC}_\bullet(f) : \mathrm{MC}_\bullet(\mathfrak{g}) \rightarrow \mathrm{MC}_\bullet(\mathfrak{h})$ is a fibration of simplicial sets.
- (3) For \mathfrak{g} an abelian L_∞ algebra (i.e., merely a cochain complex), $\mathrm{MC}_\bullet(\mathfrak{g})$ is homotopy equivalent to $\mathrm{DK}_\bullet(\tau_{\leq 0}(\mathfrak{g}[1]))$, where $\tau_{\leq 0}(\mathfrak{g}[1])$ denotes the brutal truncation where we drop all positive-degree elements of a cochain complex.

These results extend to the nilpotent curved L_∞ algebras that we work with.

It might help to know what the Maurer–Cartan space $\mathrm{MC}_\bullet(\mathfrak{g})$ means when \mathfrak{g} is a nilpotent Lie algebra, in the classical sense, in order to interpret the general construction. The introduction to [Get09] gives a beautiful explanation, with connections to many topics. He shows, for instance, that for G a simply-connected, nilpotent Lie group and $\mathfrak{g} = \mathrm{Lie}(G)$ its associated nilpotent Lie algebra, there is a natural homotopy equivalence $N_\bullet G \rightarrow \mathrm{MC}_\bullet \mathfrak{g}$. (Getzler, in fact, goes further and finds a replacement for the Maurer–Cartan functor that “integrates” a nilpotent L_∞ algebra to its “group.” For a nilpotent Lie algebra, his functor recovers the nerve of the group on the nose.)

Appendix C. Proof of Theorem 4.8

We follow the structure of Costello’s argument in [Cos11], reworking and elaborating on several steps of the argument. Let $B\mathfrak{g} = (X, \mathfrak{g})$ be an L_∞ space. Recall that $\mathbf{B}\mathfrak{g}$ denotes the associated functor of points.

The essential idea is to exploit the nilpotent ideal $\mathcal{I}_\mathcal{N} = \ker q$ inside $\mathcal{O}_\mathcal{N}$ for any nilpotent dg manifold \mathcal{N} on which we evaluate $\mathbf{B}\mathfrak{g}$. Let n be the integer such that $\mathcal{I}_\mathcal{N}^{n+1} = 0$. Then we have a tower of commutative dg Ω_N^* -algebras

$$\mathcal{O} \rightarrow \mathcal{O}/\mathcal{I}^n \rightarrow \mathcal{O}/\mathcal{I}^{n-1} \rightarrow \cdots \rightarrow \mathcal{O}/\mathcal{I}^2 \rightarrow \mathcal{O}/\mathcal{I} \cong C_N^\infty$$

and an associated tower of nilpotent dg manifolds

$$N = \mathcal{N}_1 \rightarrow \mathcal{N}_2 \rightarrow \cdots \rightarrow \mathcal{N}_n \rightarrow \mathcal{N},$$

where $\mathcal{N}_k = (N, \mathcal{O}_\mathcal{N}/\mathcal{I}_\mathcal{N}^k)$. We also obtain a tower of simplicial sets

$$\mathbf{B}\mathfrak{g}(\mathcal{N}) \rightarrow \mathbf{B}\mathfrak{g}(\mathcal{N}_n) \rightarrow \cdots \rightarrow \mathbf{B}\mathfrak{g}(\mathcal{N}_2) \rightarrow \mathbf{B}\mathfrak{g}(\mathcal{N}_1) = \mathbf{B}\mathfrak{g}(N)$$

for any nilpotent dg manifold \mathcal{N} .

Our arguments will proceed by induction up this tower (this is a direct analog of artinian induction). For instance, we will show that weak equivalences go to weak equivalences stage by stage along the tower.

Remark C.1. There is an important feature of the functor $\mathbf{B}\mathfrak{g}$ that we wish to emphasize: the simplicial set $\mathbf{B}\mathfrak{g}(\mathcal{N})$ is a disjoint union of simplicial sets

over the set of the smooth maps $f : N \rightarrow X$. In the construction of the k -simplices $\mathbf{Bg}(\mathcal{N})_k$, we solve for solutions of the Maurer–Cartan equation in an algebra depending on $\Omega^*(\Delta^k)$ but where the underlying smooth map is independent of Δ^k . *From hereon in the proof of the theorem, we fix a map f and simply study solutions over that f .* We remind the reader of this assumption periodically, for clarity’s sake. \diamond

We start with the base case for the induction.

Lemma C.2. *For any smooth manifold N , the simplicial set $\mathbf{Bg}(N)$ is the discrete simplicial set of smooth maps $C^\infty(N, X)$. It is, in particular, a Kan complex.*

Proof. See Lemma 4.10 above. \square

We now show that each stage of the induction is well-behaved homotopically.

Lemma C.3. *The map*

$$q : \mathbf{Bg}(\mathcal{N}_{k+1}) \rightarrow \mathbf{Bg}(\mathcal{N}_k)$$

is a fibration.

Proof. Recall that q is a Kan fibration if for any map of an m -simplex $s : \Delta^m \rightarrow \mathbf{Bg}(\mathcal{N}_k)$ and any map of a m -horn $t : \Lambda_j^m \rightarrow \mathbf{Bg}(\mathcal{N}_{k+1})$ such that

$$\begin{array}{ccc} \Lambda_j^m & \xrightarrow{t} & \mathbf{Bg}(\mathcal{N}_{k+1}) \\ \iota \downarrow & & \downarrow q \\ \Delta^m & \xrightarrow{s} & \mathbf{Bg}(\mathcal{N}_k) \end{array}$$

is a commutative diagram, we can lift to a map $\tilde{s} : \Delta^m \rightarrow \mathbf{Bg}(\mathcal{N}_{k+1})$ such that $t = \tilde{s} \circ \iota$ and $s = q \circ \tilde{s}$.

As explained in remark C.1, we are free to fix the underlying smooth map $f : N \rightarrow X$ in all these constructions. Let $f : N \rightarrow X$ denote that fixed map from hereon.

Now suppose we have an m -simplex (f, α) of $\mathbf{Bg}(\mathcal{N}_k)$ and an m -horn of $\mathbf{Bg}(\mathcal{N}_{k+1})$ agreeing with (f, α) modulo \mathcal{I}^k . We want to extend to a full m -simplex.

As a first step, we consider the problem of lifting α simply as an element of *graded vector spaces*, ignoring the Maurer–Cartan equation. We have a short exact sequence

$$\begin{aligned} 0 \rightarrow (\mathcal{I}^k / \mathcal{I}^{k+1})^\#(N) \otimes \Omega^\#(\Delta^m) &\rightarrow (\mathcal{O} / \mathcal{I}^{k+1})^\#(N) \otimes \Omega^\#(\Delta^m) \\ &\xrightarrow{Q} (\mathcal{O} / \mathcal{I}^k)^\#(N) \otimes \Omega^\#(\Delta^m) \rightarrow 0 \end{aligned}$$

of graded vector spaces. We denote this sequence

$$0 \rightarrow K \rightarrow B \xrightarrow{Q} A \rightarrow 0$$

for simplicity. Note that K is a square-zero ideal of B and hence a very simple *nonunital* commutative dg algebra. We eventually want to study solutions to the Maurer–Cartan equation when we tensor $f^*\mathfrak{g}$ with each term in this sequence. Ignoring the differentials for now, observe that we get a sequence of graded vector spaces

$$0 \rightarrow f^*\mathfrak{g} \otimes K \rightarrow f^*\mathfrak{g} \otimes B \xrightarrow{\text{Id} \otimes Q} f^*\mathfrak{g} \otimes A \rightarrow 0.$$

Thus, given a solution α to the Maurer–Cartan equation in $f^*\mathfrak{g} \otimes A$, there exist lifts $\tilde{\alpha}$ in $f^*\mathfrak{g} \otimes B$ such that $Q \circ \tilde{\alpha} = \alpha$ because we can split Q as a map of vector spaces. These lifts form a torsor for $f^*\mathfrak{g} \otimes K$.

We now ask when such a lift $\tilde{\alpha}$ satisfies the Maurer–Cartan equation. We know α does, so the failure to satisfy the Maurer–Cartan equation lives in $f^*\mathfrak{g} \otimes K$. In other words, we have an obstruction living in the second cohomology group of $f^*\mathfrak{g} \otimes K$. By hypothesis, we know that $\tilde{\alpha}$ satisfies the Maurer–Cartan equation when restricted to the horn. As

$$\Omega^*(\Lambda_j^m) \rightarrow \Omega^*(\Delta^m)$$

is a quasi-isomorphism, we see that the obstruction must vanish. \square

This tower of fibrations gives us a procedure for checking a property by working our way up the tower and simply working with $\mathcal{G}^k/\mathcal{G}^{k+1}$ at each stage. As this is simply a cochain complex, rather than a nontrivial commutative dg algebra, the problem has become more tractable.

Proposition C.4. *Let $F : \mathcal{N} \rightarrow \mathcal{N}'$ be a weak equivalence of nilpotent dg manifolds. Then the map $F^* : \mathbf{Bg}(\mathcal{N}') \rightarrow \mathbf{Bg}(\mathcal{N})$ is a weak equivalence of simplicial sets.*

Proof. We have a diffeomorphism $f : N \rightarrow N'$ and a map of commutative dg algebras $\phi : f^{-1}\mathcal{O}_{N'} \rightarrow \mathcal{O}_N$ that is compatible with the filtration. Our plan is to show we have a weak equivalence

$$F_k^* : \mathbf{Bg}(N'_k) \rightarrow \mathbf{Bg}(N_k)$$

for each level k of the tower of nilpotent dg manifolds.

The base case is straightforward:

$$F_1^* = f^* : \mathbf{Bg}(N') \rightarrow \mathbf{Bg}(N)$$

is an isomorphism because f is a diffeomorphism.

Now suppose F_{k-1}^* is a weak equivalence. To show F_k^* is a weak equivalence, it suffices to show that the induced map on fibers is a weak equivalence by the long exact sequence in homotopy groups. Here we use remark C.1: it suffices to study the problem for each distinct smooth map $g : N' \rightarrow C$. This choice then fixes the smooth map $g \circ f : N \rightarrow X$. We will compare the fibers over g and $h = g \circ f$ but suppress them from discussion.

The fiber for N is then the Maurer–Cartan space for $h^*\mathfrak{g} \otimes_{\Omega_N^*} \mathcal{I}_N^{k-1} / \mathcal{I}_N^k$. This curved L_∞ algebra is, in fact, abelian because the $\mathcal{I}_N^{k-1} / \mathcal{I}_N^k$ is square-zero, so this simplicial set has another name: it arises from the cochain complex $h^*\mathfrak{g} \otimes_{\Omega_N^*} \mathcal{I}_N^{k-1} / \mathcal{I}_N^k[1]$ under the Dold–Kan correspondence.

By the hypothesis, we know that $\text{Gr } \phi$ is a quasi-isomorphism. This means that the cochain complex $\mathcal{I}_N^{k-1} / \mathcal{I}_N^k$ is quasi-isomorphic to $\mathcal{I}_{N'}^{k-1} / \mathcal{I}_{N'}^k$ for every k . Hence the fibers — as simplicial sets constructed by the Dold–Kan correspondence — are weakly equivalent. \square

It remains to show that $\mathbf{B}\mathfrak{g}$ satisfies Čech descent. This argument is the trickiest because it involves homotopy limits.

Proposition C.5. *The simplicial presheaf $\mathbf{B}\mathfrak{g}$ satisfies Čech descent. That is, for \mathfrak{V} any cover of N , the map*

$$\mathbf{B}\mathfrak{g}(N) \rightarrow \text{holim}_{\check{C}\mathfrak{V}} \mathbf{B}\mathfrak{g} = \text{holim } \check{C}^\bullet(\mathfrak{V}, \mathbf{B}\mathfrak{g})$$

is a weak equivalence.

This argument also proceeds by artinian induction. We fix useful notation. Let $\mathfrak{V} = \{V_i\}$ denote a cover of a smooth manifold N , and let $\mathfrak{V}_k = \{(V_i, \mathcal{O}_{N_k}|_{V_i})\}$ denote the associated cover of the nilpotent dg manifold \mathcal{N}_k .

The base case of the argument is the following standard fact.

Lemma C.6. *The natural map $\mathbf{B}\mathfrak{g}(N) \rightarrow \text{holim}_{\check{C}\mathfrak{V}} \mathbf{B}\mathfrak{g}$ is a weak equivalence of simplicial sets.*

Proof. By Lemma C.2, we know that all the simplicial sets here are discrete. In particular, there is the associated map of sets

$$C^\infty(N, X) \rightarrow \text{holim } \check{C}(\mathfrak{V}, C^\infty(-, X)).$$

Hence the homotopy limit agrees with the limit, and we know that $C^\infty(-, X)$ is a sheaf of sets on Man , so that map of sets is an isomorphism. \square

With the base case behind us, we tackle the induction step. Observe that we have a commuting square of simplicial sets

$$\begin{array}{ccc} \mathbf{B}\mathfrak{g}(\mathcal{N}_k) & \longrightarrow & \text{holim}_{\check{C}\mathfrak{V}_k} \mathbf{B}\mathfrak{g} \\ \downarrow & & \downarrow \\ \mathbf{B}\mathfrak{g}(\mathcal{N}_{k-1}) & \longrightarrow & \text{holim}_{\check{C}\mathfrak{V}_{k-1}} \mathbf{B}\mathfrak{g} \end{array}$$

where the left vertical map is a fibration and the bottom horizontal map is a weak equivalence. We will show that the right vertical map is a fibration and that the induced map on fibers is a weak equivalence. This then implies that the top horizontal map is a weak equivalence, finishing the induction step.

Lemma C.7. *The map*

$$\check{C}(\mathfrak{A}_k, \mathbf{B}\mathfrak{g}) \rightarrow \check{C}(\mathfrak{A}_{k-1}, \mathbf{B}\mathfrak{g})$$

is a fibration of cosimplicial simplicial sets. As $\text{holim}_\Delta : \text{csSets} \rightarrow \text{sSets}$ preserves fibrations,¹⁹ we see that

$$\text{holim}_{\check{C}\mathfrak{A}_k} \mathbf{B}\mathfrak{g} \rightarrow \text{holim}_{\check{C}\mathfrak{A}_{k-1}} \mathbf{B}\mathfrak{g}$$

is a fibration of simplicial sets.

Proof. We already showed in Lemma C.3 that $\mathbf{B}\mathfrak{g}$ is a fibration along any square-zero extension. Hence, we know the map of Čech diagrams is an objectwise fibration and hence a fibration in the projective model structure on csSets . Thus, the homotopy limit, as a right Quillen adjoint, gives us a fibration of simplicial sets. \square

It remains to show that the map of fibers is a weak equivalence. We know that the fiber of the map $\mathbf{B}\mathfrak{g}(\mathcal{N}_k) \rightarrow \mathbf{B}\mathfrak{g}(\mathcal{N}_{k-1})$ is simply

$$\text{MC}_\bullet(\mathfrak{g} \otimes \mathcal{I}^{k-1} / \mathcal{I}^k(N)),$$

the Maurer–Cartan space for the L_∞ algebra $\mathfrak{g} \otimes \mathcal{I}^{k-1} / \mathcal{I}^k(N)$. (We include the notation “ (N) ” to emphasize that we are taking global sections of this sheaf of L_∞ algebras.) On the other side of the square above, we know that holim preserves fibrations, so that the fiber of the map $\text{holim}_{\check{C}\mathfrak{A}_k} \mathbf{B}\mathfrak{g} \rightarrow \text{holim}_{\check{C}\mathfrak{A}_{k-1}} \mathbf{B}\mathfrak{g}$ is given by

$$\text{holim } \check{C}^\bullet(\mathfrak{A}, \text{MC}_\bullet(\mathfrak{g} \otimes \mathcal{I}^{k-1} / \mathcal{I}^k)),$$

namely the Čech cosimplicial diagram for the cover \mathfrak{A} with respect to the simplicial presheaf that assigns the Maurer–Cartan space for the sheaf of L_∞ algebras $\mathfrak{g} \otimes \mathcal{I}^{k-1} / \mathcal{I}^k$.

Lemma C.8. *The map between the fibers*

$$\text{MC}(\mathfrak{g} \otimes \mathcal{I}^{k-1} / \mathcal{I}^k)(N) \rightarrow \text{holim } \check{C}^\bullet(\mathfrak{A}, \text{MC}(\mathfrak{g} \otimes \mathcal{I}^{k-1} / \mathcal{I}^k))$$

is a weak equivalence.

This lemma concludes the proof of Proposition C.5.

Proof. Recall that for an abelian L_∞ algebra \mathfrak{h} , the Maurer–Cartan space $\text{MC}_\bullet(\mathfrak{h})$ is just the Dold–Kan space $\text{DK}_\bullet(\tau_{\leq 0}(\mathfrak{h}[1]))$. Note that the Dold–Kan space only depends on the truncation $\tau_{\leq 0}(\mathfrak{h}[1])$, where all components of \mathfrak{h} of degree greater than 1 are eliminated. In particular, we see that

$$\text{MC}_\bullet(\mathfrak{g} \otimes \mathcal{I}^{k-1} / \mathcal{I}^k(N)) = \text{DK}_\bullet(\tau_{\leq 0}(\mathfrak{g} \otimes \mathcal{I}^{k-1} / \mathcal{I}^k(N)[1])),$$

so that the fiber is determined by a cochain complex.

¹⁹It is the *right* adjoint in a Quillen adjunction

$$c : \text{sSets} \rightleftarrows \text{csSets} : \text{holim}$$

as discussed in Proposition A.1.

As $\mathfrak{g} \otimes \mathcal{I}^{k-1} / \mathcal{I}^k$ is a sheaf of abelian L_∞ algebras, we are simply applying DK_\bullet to the value on every open. Hence $\check{C}^\bullet(\mathfrak{V}, \mathrm{DK}_\bullet(\tau_{\leq 0}(\mathfrak{g} \otimes \mathcal{I}^{k-1} / \mathcal{I}^k[1])))$ is a cosimplicial simplicial abelian group. By Proposition A.4, we know that $\mathrm{holim} \check{C}^\bullet(\mathfrak{V}, \mathrm{MC}_\bullet(\mathfrak{g} \otimes \mathcal{I}^{k-1} / \mathcal{I}^k))$ is thus weakly equivalent to the Dold–Kan simplicial set of

$$T\check{C}(\mathfrak{V}, \tau_{\leq 0}(\mathfrak{g} \otimes \mathcal{I}^{k-1} / \mathcal{I}^k[1])),$$

the total complex formed from the Čech double complex for the sheaf $\tau_{\leq 0}(\mathfrak{g} \otimes \mathcal{I}^{k-1} / \mathcal{I}^k[1])$ on the cover \mathfrak{V} .

Thus, to prove the lemma, we verify instead that

$$\tau_{\leq 0}(\mathfrak{g} \otimes \mathcal{I}^{k-1} / \mathcal{I}^k)(N)[1] \rightarrow T\check{C}(\mathfrak{V}, \tau_{\leq 0}(\mathfrak{g} \otimes \mathcal{I}^{k-1} / \mathcal{I}^k[1]))$$

is a quasi-isomorphism, as this implies that the Dold–Kan simplicial sets are weakly equivalent.

This map is a quasi-isomorphism because $\mathfrak{g} \otimes \mathcal{I}^{k-1} / \mathcal{I}^k$ consists of smooth sections of a vector bundle on N , so the usual partition of unity arguments apply. \square

References

- [AC12] ABAD, CAMILO ARIAS; CRAINIC, MARIUS. Representations up to homotopy of Lie algebroids. *J. Reine Angew. Math.* **663** (2012), 91–126. [MR2889707](#), [Zbl 1238.58010](#), [arXiv:0901.0319](#), doi: [10.1515/CRELLE.2011.095](#).
- [ASZK97] ALEXANDROV, M.; SCHWARZ, A.; ZABORONSKY, O.; KONTSEVICH, M. The geometry of the master equation and topological quantum field theory. *Internat. J. Modern Phys. A* **12** (1997), no. 7, 1405–1429. [MR1432574](#) (98a:81235), [Zbl 1073.81655](#), [arXiv:hep-th/9502010](#), doi: [10.1142/S0217751X97001031](#).
- [BenbB13] BEN-BASSAT, OREN; BLOCK, JONATHAN. Milnor descent for cohesive dg-categories. *J. K-Theory* **12** (2013), no. 3, 433–459. [MR3165183](#), [Zbl 1296.18005](#), [arXiv:1201.6118](#), doi: [10.1017/is013007003jkt236](#).
- [BenzN12] BEN-ZVI, DAVID; NADLER, DAVID. Loop spaces and connections. *J. Topol.* **5** (2012), no. 2, 377–430. [MR2928082](#), [Zbl 1246.14027](#), [arXiv:1002.3636](#), doi: [10.1112/jtopol/jts007](#).
- [Blo10] BLOCK, JONATHAN. Duality and equivalence of module categories in noncommutative geometry. *A celebration of the mathematical legacy of Raoul Bott*, 311–339, CRM Proc. Lecture Notes, 50. *Amer. Math. Soc., Providence, RI*, 2010, pp. 311–339. [MR2648899](#) (2011k:19005), [Zbl 1201.58002](#).
- [BloS14] BLOCK, JONATHAN; SMITH, AARON M. The higher Riemann–Hilbert correspondence. *Adv. Math.* **252** (2014), 382–405. [MR3144234](#), [Zbl 1304.18036](#), doi: [10.1016/j.aim.2013.11.001](#).
- [BorN] BORISOV, DENNIS; NOEL, JUSTIN. Simplicial approach to derived differential manifolds. Preprint, 2011. [arXiv:1112.0033](#).
- [Bou89] BOUSFIELD, A. K. Homotopy spectral sequences and obstructions. *Israel J. Math.* **66** (1989), no. 1-3, 54–104. [MR1017155](#) (91a:55027), [Zbl 0677.55020](#), doi: [10.1007/BF02765886](#).
- [BouK72] BOUSFIELD, A. K.; KAN, D. M. Homotopy limits, completions and localizations. *Lecture Notes in Mathematics*, 304. *Springer-Verlag, Berlin-New York*, 1972. v+348 pp. [MR0365573](#) (51 #1825), [Zbl 0259.55004](#), doi: [10.1007/978-3-540-38117-4](#).

- [CarR13] CARCHEDI, DAVID; ROYTENBERG, DMITRY. On theories of superalgebras of differentiable functions. *Theory Appl. Categ.* **28** (2013), no. 30, 1022–1098. [MR3121621](#), [Zbl 1291.18007](#), [arXiv:1211.6134](#).
- [CFT02] CATTANEO, ALBERTO S.; FELDER, GIOVANNI; TOMASSINI, LORENZO. From local to global deformation quantization of Poisson manifolds. *Duke Math. J.* **115** (2002), no. 2, 329–352. [MR1944574](#) (2004a:53114), [Zbl 1037.53063](#), [arXiv:math/0012228](#), doi: [10.1215/S0012-7094-02-11524-5](#).
- [Cos10] COSTELLO, KEVIN. A geometric construction of the Witten genus, I. *Proceedings of the International Congress of Mathematicians. II*, 942–959, *Hindustan Book Agency, New Delhi*, 2010. [MR2827826](#), [Zbl 1242.58013](#), [arXiv:1006.5422](#), doi: [10.1142/9789814324359_0081](#).
- [Cos11] COSTELLO, KEVIN. A geometric construction of the Witten genus, II. Preprint, 2011. [arXiv:1112.0816](#).
- [Cos11b] COSTELLO, KEVIN. Renormalization and effective field theory. *Mathematical Surveys and Monographs*, 170. *American Mathematical Society, Providence, RI*, 2011. viii+251 pp. ISBN: 978-0-8218-5288-0. [MR2778558](#) (2012f:81177), [Zbl 1221.81004](#), doi: [10.1090/surv/170](#).
- [Dou14] DOUGLAS, CHRISTOPHER L. Sheaves in homotopy theory. Topological modular forms. Based on the Talbot workshop, North Conway, NH, USA, March 25–31, 2007. Douglas, Christopher L.; Francis, John; Henriques, André G.; Hill, Michael A.; Dirs. *Mathematical Surveys and Monographs*, 201. *American Mathematical Society, Providence, RI*, 2014. pp. xxxi + 318. [Zbl 1304.55002](#), doi: [10.1090/surv/201](#).
- [Dug] DUGGER, DANIEL. A primer on homotopy colimits. <http://math.uoregon.edu/~ddugger/hocolim.pdf>.
- [DwS95] DWYER, W. G.; SPALIŃSKI, J. Homotopy theories and model categories. *Handbook of algebraic topology*, 73–126, *North-Holland, Amsterdam*, 1995. [MR1361887](#) (96h:55014), [Zbl 0869.55018](#), doi: [10.1016/B978-044481779-2/50003-1](#).
- [GaR] GAITSGORY, DENNIS; ROZENBLYUM, NICK. Crystals and D-modules. *Pure Appl. Math. Q* **10** (2014), no. 1, 57–154. [MR3264953](#), [Zbl 06384336](#), [arXiv:1111.2087](#), doi: [10.4310/PAMQ.2014.v10.n1.a2](#).
- [Get09] GETZLER, EZRA. Lie theory for nilpotent L_∞ -algebras. *Ann. of Math. (2)* **170** (2009), no. 1, 271–301. [MR2521116](#) (2010g:17026), [Zbl 1246.17025](#), [arXiv:math/0404003](#), doi: [10.4007/annals.2009.170.271](#).
- [GG] GRADY, RYAN; GWILLIAM, OWEN. L_∞ spaces and the Batalin-Vilkovisky formalism for quantum field theory. In progress.
- [GG14] GWILLIAM, OWEN; GRADY, RYAN. One-dimensional Chern–Simons theory and the \hat{A} genus. *Algebr. Geom. Topol.* **14** (2014), no. 4, 419–497. [Zbl 1305.81114](#), [arXiv:1110.3533](#), doi: [10.2140/agt.2014.14.2299](#).
- [Hen] HENNION, BENJAMIN. Tangent Lie algebra of derived Artin stacks. Preprint, 2013. [arXiv:1312.3167](#).
- [Hin01] HINICH, VLADIMIR. DG coalgebras as formal stacks. *J. Pure Appl. Algebra* **162** (2001), no. 2-3, 209–250. [MR1843805](#) (2002f:14008), [Zbl 1020.18007](#), [arXiv:math/9812034](#), doi: [10.1016/S0022-4049\(00\)00121-3](#).
- [Hir03] HIRSCHHORN, PHILIP S. Model categories and their localizations. *Mathematical Surveys and Monographs*, 99. *American Mathematical Society, Providence, RI*, 2003. [MR1944041](#) (2003j:18018), [Zbl 1017.55001](#).
- [Hov99] HOVEY, MARK. Model categories. *Mathematical Surveys and Monographs*, 63. *American Mathematical Society, Providence, RI*, 1999. xii+209 pp. ISBN: 0-8218-1359-5. [MR1650134](#) (99h:55031), [Zbl 0909.55001](#).

- [Joy] JOYCE, DOMINIC. *D*-manifolds and *d*-orbifolds: a theory of derived differential geometry. Preprint, 2012. Available at <http://people.maths.ox.ac.uk/~joyce/dmanifolds.html>.
- [Kap91] KAPRANOV, M. M. On DG-modules over the de Rham complex and the vanishing cycles functor. *Algebraic geometry* (Chicago, IL, 1989), 57–86, Lecture Notes in Math., 1479. Springer, Berlin, 1991. MR1181207 (94e:14028), Zbl 0757.14010, doi: 10.1007/BFb0086264.
- [Kon03] KONTSEVICH, MAXIM. Deformation quantization of Poisson manifolds, *Lett. Math. Phys.* **66** (2003), no. 3, 157–216. MR2062626 (2005i:53122), Zbl 1058.53065, arXiv:q-alg/9709040, doi: 10.1023/B:MATH.0000027508.00421.bf,
- [Lura] LURIE, JACOB. Derived algebraic geometry V: structured spaces. Preprint, 2009. arXiv:0905.0459.
- [Lurb] LURIE, JACOB. Derived algebraic Geometry X: formal moduli P. Preprint, 2011. <http://www.math.harvard.edu/~lurie/papers/DAG-X.pdf>.
- [Lur10] LURIE, JACOB. Moduli problems for ring spectra. *Proceedings of the International Congress of Mathematicians. II*, 1099–1125, Hindustan Book Agency, New Delhi, 2010. MR2827833 (2012i:55008), Zbl 1244.55007, doi: 10.1142/9789814324359_0088.
- [Mac05] MACKENZIE, KIRILL C. H. General theory of Lie groupoids and Lie algebroids. London Mathematical Society Lecture Note Series, 213. Cambridge University Press, Cambridge, 2005. xxviii+501 pp. ISBN: 978-0-521-49928-3; 0-521-49928-3. MR2157566 (2006k:58035), Zbl 1078.58011, doi: 10.1112/blms/bdp115.
- [Meh09] MEHTA, RAJAN AMIT. *Q*-algebroids and their cohomology. *J. Symplectic Geom.* **7** (2009), no. 3, 263–293. MR2534186 (2011b:58040), Zbl 1215.22002, arXiv:math/0703234, doi: 10.4310/JSG.2009.v7.n3.a1.
- [MS06] MOVSHEV, M.; SCHWARZ, A. Algebraic structure of Yang–Mills theory. *The unity of mathematics*, 473–523, Progr. Math., 244. Birkhäuser Boston, Boston, MA, 2006. MR2181815 (2007a:81147), Zbl 1229.81281, arXiv:hep-th/0404183. doi: 10.1007/0-8176-4467-9_14.
- [PTVV13] PANTEV, TONY; TOËN, BERTRAND; VAQUIÉ, MICHEL; VEZZOSI, GABRIELE. Shifted symplectic structures. *Publ. Math. Inst. Hautes Études Sci.* **117** (2013), 271–328. MR3090262, Zbl 06185246, arXiv:1111.3209, doi: 10.1007/s10240-013-0054-1.
- [Rin63] RINEHART, GEORGE S. Differential forms on general commutative algebras. *Trans. Amer. Math. Soc.* **108** (1963), 195–222. MR0154906 (27 #4850), Zbl 0113.26204, doi: 10.2307/1993603.
- [Sch] SCHREIBER, URS. Differential cohomology in a cohesive infinity-topos. Preprint, 2013. arXiv:1310.7930.
- [SchuTV] SCHÜRG, TIMO; TOËN, BERTRAND; VEZZOSI, GABRIELE. Derived algebraic geometry, determinants of perfect complexes, and applications to obstruction theories for maps and complexes. *J. Reine Angew. Math.* **679** (2013). arXiv:1102.1150.
- [Schw93] SCHWARZ, ALBERT. Geometry of Batalin–Vilkovisky quantization. *Comm. Math. Phys.* **155** (1993), no. 2, 249–260. MR1230027 (95f:81095), Zbl 0786.58017, arXiv:hep-th/9205088, doi: 10.1007/BF02097392.
- [Seg74] SEGAL, GRAEME. Categories and cohomology theories. *Topology* **13** (1974), 293–312. MR0353298 (50 #5782), Zbl 0284.55016, doi: 10.1016/0040-9383(74)90022-6.
- [Shu] SHULMAN, MICHAEL. Homotopy limits and colimits and enriched homotopy theory. Preprint, 2006. arXiv:math/0610194.
- [SiT] SIMPSON, CARLOS; TELEMAN, CONSTANTIN. De Rham’s theorem for ∞ -stacks. Preprint. <http://math.berkeley.edu/~teleman/math/simpson.pdf>.

- [Spi10] SPIVAK, DAVID I. Derived smooth manifolds. *Duke Math. J.* **153** (2010), no. 1, 55–128. [MR2641940](#) (2012a:57043), [Zbl 05717812](#), [arXiv:0810.5174](#), doi: [10.1215/00127094-2010-021](#).
- [Toë09] TOËN, BERTRAND. Higher and derived stacks: a global overview. *Algebraic geometry—Seattle 2005. Part 1*, 435–487, Proc. Sympos. Pure Math., 80, Part 1. *Amer. Math. Soc., Providence, RI*, 2009. [MR2483943](#) (2010h:14021), [Zbl 1183.14001](#), [arXiv:math/0604504](#).
- [TV05] TOËN, BERTRAND; VEZZOSI, GABRIELE. Homotopical algebraic geometry. I. Topos theory. *Adv. Math.* **193** (2005), no. 2, 257–372. [MR2137288](#) (2007b:14038), [Zbl 1120.14012](#), [arXiv:math/0207028](#), doi: [10.1016/j.aim.2004.05.004](#).
- [TV09] TOËN, BERTRAND; VEZZOSI, GABRIELE. Chern character, loop spaces and derived algebraic geometry. *Algebraic topology*, 331–354, Abel Symp., 4. *Springer, Berlin*, 2009. [MR2597742](#) (2011d:14031), [Zbl 1177.14042](#), doi: [10.1007/978-3-642-01200-6_11](#).
- [TV11] TOËN, BERTRAND; VEZZOSI, GABRIELE. Algèbres simpliciales S^1 -équivariantes, théorie de de Rham et théorèmes HKR multiplicatifs. *Compos. Math.* **147** (2011), no. 6, 1979–2000. [MR2862069](#), [Zbl 1257.18014](#), doi: [10.1112/S0010437X11005501](#).
- [Wei94] WEIBEL, CHARLES A. An introduction to homological algebra, Cambridge Studies in Advanced Mathematics, 38. *Cambridge University Press, Cambridge*, 1994. xiv+450 pp. ISBN: 0-521-43500-5; 0-521-55987-1. [MR1269324](#) (95f:18001), [Zbl 0797.18001](#).

(Ryan Grady) DEPARTMENT OF MATHEMATICS AND STATISTICS, BOSTON UNIVERSITY,
111 CUMMINGTON MALL, BOSTON MA 02215, USA
regrady@math.bu.edu

(Owen Gwilliam) MAX PLANCK INSTITUT FÜR MATHEMATIK, VIVATSGASSE 7, 53111
BONN, GERMANY
gwilliam@mpim-bonn.mpg.de

This paper is available via <http://nyjm.albany.edu/j/2015/21-11.html>.