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Ideal Whitehead graphs in $Out(F_r)$. I. Some unachieved graphs

Catherine Pfaff

Abstract. Masur and Smillie, 1993, proved precisely which singularity index lists arise from pseudo-Anosov mapping classes. In search of an analogous theorem for outer automorphisms of free groups, Handel and Mosher, 2011, ask: Is each connected, simplicial, $(2r-1)$-vertex graph the ideal Whitehead graph of a fully irreducible $\phi \in Out(F_r)$? We answer this question in the negative by exhibiting, for each r, examples of connected (2r-1)-vertex graphs that are not the ideal Whitehead graph of any fully irreducible $\phi \in Out(F_r)$. In the course of our proof we also develop machinery used in Pfaff, 2012, to fully answer the question in the rank-three case.

Contents

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1. Introduction

For a compact surface S, the *mapping class group* $\mathcal{MCG}(S)$ is the group of isotopy classes of homeomorphisms $h\colon S \to S$. A generic (see, for example, [Mah11]) mapping class is *pseudo-Anosov*, i.e., has a representative leaving invariant a pair of transverse measured singular minimal foliations. The foliation has an associated singularity index list. Masur and Smillie determined precisely which singularity index lists, permitted by the Poincare–Hopf index formula, arise from pseudo-Anosov homeomorphisms [MS93]. The search for an analogous theorem in the setting of an outer automorphism group of a free group is still open.

We let $Out(F_r)$ denote the outer automorphism group of the free group of rank r. Analogous to pseudo-Anosov mapping classes are *fully irreducible* outer automorphisms, i.e., those such that no power leaves invariant the conjugacy class of a proper free factor. In fact, some fully irreducible outer automorphisms, called *geometrics*, are induced by pseudo-Anosovs. It is noteworthy that the index lists of geometrics are fully understood through the Masur–Smillie index theorem.

Related to pseudo-Anosov index lists are three fully irreducible invariants, namely indices, index lists, and ideal Whitehead graphs. Each is invariant under taking powers and is in fact an invariant of the conjugacy class of a fully irreducible within $Out(F_r)$. The ideal Whitehead graph is the finest of the three invariants and is the one we focus on in this paper, in [Pfa13a], and in [Pfa13b]. We focus on index lists in [Pfa13c].

Singularity indices for fully irreducible outer automorphisms were first introduced by Gaboriau, Jaeger, Levitt, and Lustig in [GJLL98]. In [GJLL98] they additionally proved an $Out(F_r)$-analogue to the Poincare–Hopf index equality. Using a "rotationless index," where the sign is switched for consistency with the surface case and the "rotationless" power is taken, the [GJLL98] index sum inequality becomes $0 \geq i(\phi) \geq 1-r$, where $\phi \in Out(F_r)$ is any fully irreducible. It is proved by Bestvina and Feighn in [BF94] that the equality $i(\phi) = 1 - r$ holds precisely in the cases of geometric and "parageometric" fully irreducibles. We will focus on the third category of fully irreducibles, "ageometrics," and hence on the strict inequality.

Having an inequality, instead of just an equality, adds a rich layer of complexity to the search for an analogue to the Masur–Smillie theorem. Toward this goal, Handel and Mosher asked in [HM11]:

Question 1.1. *Which index types, satisfying $0 \geq i(\phi) > 1-r$, are achieved by nongeometric fully irreducible $\phi \in Out(F_r)$?*

Beyond the existence of an inequality, instead of just an equality, "ideal Whitehead graphs" (see [HM11] or Definition 2.1 below) give yet another layer of complexity for fully irreducible outer automorphisms.

The ideal Whitehead graph for a pseudo-Anosov mapping class is just a disjoint union of circles, with each circle corresponding to an ideal polygon formed by the lifted lamination leaves bounding a principle region, as in Nielsen theory [N86]. In contrast, what we show in [Pfa13a] and [Pfa13b] is that the ideal Whitehead graph $\mathcal{IW}(\phi)$ for a fully irreducible $\phi \in Out(F_r)$ can even be the complete graph in each rank. It hence gives a finer outer automorphism invariant than just the index list. Indeed, each connected component C_i of $\mathcal{IW}(\phi)$ contributes the index $1 - \frac{k_i}{2}$ to the list, where C_i has k_i vertices.

The deeper information the ideal Whitehead graph records regards the dynamical behavior of the attracting lamination for a fully irreducible under the action of the fully irreducible on its attracting tree. In [LL03], Levitt and Lustig proved that, as with a pseudo-Anosov acting on Teichmüller space, each fully irreducible $\phi \in Out(F_r)$ acts with North-South dynamics on the natural compactification of outer space $\overline{CV_r}$. For a fully irreducible ϕ, the lamination is "almost equal" to the zero lamination of the repelling tree T_ϕ^- [KL11]. It is also exactly equal to the support of the attracting current (as defined in [Mar95], see also [Uya13]) for ϕ [KL11].

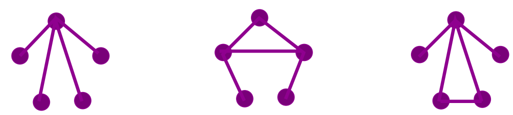
While related, since the ideal Whitehead graph can give significantly more information than simply an index list, the deeper, more appropriate question we focus on is:

Question 1.2. *Which isomorphism types of graphs occur as the ideal Whitehead graph $\mathcal{IW}(\phi)$ of a fully irreducible outer automorphism ϕ?*

[Pfa13b] will give a complete answer to Question 1.2 in rank 3 for the single-element index list $(-\frac{3}{2})$. In Theorem 9.1 of this paper we provide examples in each rank of connected (2r-1)-vertex graphs that are not the ideal Whitehead graph $\mathcal{IW}(\phi)$ for any fully irreducible $\phi \in Out(F_r)$, i.e., that are *unachieved* in rank r:

Theorem A (Theorem 9.1)**.** *For each $r \geq 3$, let $\mathcal{G}_r$ be the graph consisting of $2r-2$ edges adjoined at a single vertex.*

(I) *For no fully irreducible $\phi \in Out(F_r)$ is $\mathcal{IW}(\phi) \cong \mathcal{G}_r$.*
(II) *The following connected graphs are not the ideal Whitehead graph $\mathcal{IW}(\phi)$ for any fully irreducible $\phi \in Out(F_3)$:*



For a fully irreducible $\phi \in Out(F_r)$ to have the index list $(\frac{3}{2}-r)$, ϕ must be ageometric with a connected, (2r-1)-vertex ideal Whitehead graph $\mathcal{IW}(\phi)$. We chose to focus on the single-element index list $(\frac{3}{2}-r)$ because it is the closest to that achieved by geometrics, without being achieved by a geometric. We denote the set of connected $(2r-1)$-vertex, simplicial graphs by $\mathcal{PI}_{(r;(\frac{3}{2}-r))}$.

1.1. Elements of the proof. One often studies outer automorphisms via topological representatives. Let R_r be the r-petaled rose, with its fundamental group identified with F_r. For a finite graph Γ with only valence-three or greater vertices, a homotopy equivalence $R_r \to \Gamma$ is called a *marking.* Such a graph Γ, together with its marking $R_r \to \Gamma$, is called a *marked graph.* Each $\phi \in Out(F_r)$ can be represented by a homotopy equivalence $g\colon \Gamma \to \Gamma$ of a marked graph ($\phi = g_*\colon \pi_1(\Gamma) \to \pi_1(\Gamma)$). Thurston defined such a homotopy equivalence to be a *train track map* when g^k is locally injective on edge interiors for each $k > 0$. When g induces $\phi \in Out(F_r)$ and sends vertices to vertices, one says g is a *train track* (tt) *representative* for ϕ [BH92].

To prove Theorem 9.1(I), we give a necessary *Birecurrency Condition* (Proposition 3.7) on "lamination train track structures." For a train track representative $g\colon \Gamma \to \Gamma$ on a marked rose, we define a *lamination train track* (ltt) *structure* $G(g)$ obtainable from Γ by replacing the vertex v with the "local Whitehead graph" $\mathcal{LW}(g;v)$. The local Whitehead graph encodes how lamination leaves enter and exit v. In our circumstance, $\mathcal{IW}(\phi)$ will be a subgraph of $\mathcal{LW}(g;v)$, hence of $G(g)$. We additionally define "higher lamination train track structures" $G^k(g)$ giving even further information.

The lamination train track structures are given a smooth structure so that leaves of the expanding lamination are realized as locally smoothly embedded lines. It is called *birecurrent* if it has a locally smoothly embedded line containing each edge infinitely many times, in each end, i.e., as any assigned parameter $r \in \mathbb{R}$ satisfies $r \to \infty$ and as $r \to -\infty$.

Proposition 3.7 (Birecurrency Condition)**.** *Let $\phi \in Out(F_r)$ be a fully irreducible outer automorphism. Then the ltt structures $G^k(g)$ for each train track representative $g\colon \Gamma \to \Gamma$ of ϕ are birecurrent.*

Combinatorial proofs (not included here) of Theorem 9.1(I) exist. However, we include a proof using the Birecurrency Condition to highlight what we have observed to be a significant obstacle to achievability, namely the birecurrency of ltt structures. The Birecurrency Condition is also used in our proof of Theorem 9.1(II). We use it in [Pfa13a], where we prove the achievability of the complete graph in each rank. Finally, the condition is used in [Pfa13b] to prove precisely which of the twenty-one connected, simplicial, five-vertex graphs are $\mathcal{IW}(\phi)$ for fully irreducible $\phi \in Out(F_3)$.

In Proposition 4.3 we show that each ϕ such that

$$\mathcal{IW}(\phi) \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$$

has a power ϕ^R with a rotationless representative whose Stallings fold decomposition (see Subsection 4.2) consists entirely of proper full folds of roses (see Subsection 4.3). The representatives of Proposition 4.3 are called "ideally decomposable." We define in Section 8 automata, ideal "decomposition ($\mathcal{ID}$) diagrams" with ltt structures as nodes. Every ideally decomposed representative is realized by a loop in an $\mathcal{ID}$ diagram. To prove Theorem 9.1(II) we show ideally decomposed representatives cannot exist by showing that the $\mathcal{ID}$ diagrams do not have the correct kind of loops.

We again use the ideally decomposed representatives and $\mathcal{ID}$ diagrams in [Pfa13a] and [Pfa13b] to construct ideally decomposed representatives with particular ideal Whitehead graphs.

To determine the edges of the $\mathcal{ID}$ diagrams, we prove in Section 5 a list of "Admissible Map ($\mathcal{AM}$) properties" held by ideal decompositions. In Section 7 we use the $\mathcal{AM}$ properties to determine the two geometric "moves" one applies to ltt structures in defining edges of the $\mathcal{ID}$ diagrams. The geometric moves turn out to have useful properties expanded upon in [Pfa13a] and [Pfa13b].

1.2. Relations of our work to $\mathbb{R}$-trees and various index invariants. There are several results on related questions. For example, [JL09] gives examples of automorphisms with the maximal number of fixed points on ∂F_r, as dictated by a related inequality in [GJLL98]. Our work instead focuses on an $Out(F_r)$-version of the Masur–Smillie theorem. Hence, in this paper, [Pfa13a], [Pfa13b], and [Pfa13c] we restrict attention to fully irreducibles and the [GJLL98] index inequality.

While neither ideal Whitehead graphs nor index list realization have undergone deep analysis as of yet, index invariants of free group outer automorphisms have overall been quite extensively investigated. Before discussing the question we do answer, for context, we explain the relationship between the main object of this paper, namely the ideal Whitehead graph, and the various index invariants. At this point, there are three types of index invariants in the literature. (To keep discussion relevant, we restrict to discussing fully irreducible outer automorphisms.)

First, in [GJLL98] the index of an automorphism was defined in terms of the attracting fixed points of the homomorphism $\partial\phi\colon \partial F_r \to \partial F_r$ induced by the automorphism ϕ. Recall from [BH92] that a fully irreducible outer automorphism $\phi \in Out(F_r)$ can be represented by a train track representative, $g\colon \Gamma \to \Gamma$. For a representative without periodic Nielsen paths (nontrivial paths $\rho \subset \Gamma$ such that $g^k(\rho) \simeq \rho$ rel endpoints for some $k > 0$), there is a natural bijection between each class of attracting fixed points at infinity and the set of gates at a properly chosen vertex v of the representative. Via this bijection one can read off the index of ϕ directly from the ideal Whitehead graph of the vertex, in a similar manner as explained above.

Second, the index $ind_{geo}(T)$ of [GL95], for a fully irreducible ϕ, arises from the sum of branching indices in the attracting $\mathbb{R}$-tree T^+_ϕ that represents the attracting fixed point of the action of ϕ on ∂CV_r. This results again from a natural bijection between the gates at some vertex v and any branch point in T contained in the F_r-orbit of branch points corresponding to v. Thus, the index of ϕ is actually equal to the geometric index of T^+_ϕ, as established by Gaboriau–Levitt [GL95] for more general $\mathbb{R}$-trees.

Finally, much more recently, Coulbois and Hilion [CH] introduced yet another index for a certain class of $\mathbb{R}$-trees. This invariant $ind_Q(T)$ relies on the dual lamination and is apriori more difficult to compute. However, Coulbois–Hilion showed that, in the special case where T represents one of the two fixed points of ϕ acting on ∂CV_r, replacing ϕ with its rotationless power (as defined in [FH11]), one has the following fact [CH12]:

$$2ind(\phi) = ind_{geo}(T^+_\phi) = ind_Q(T^-_\phi).$$

As a consequence, the index of ϕ is an invariant of the repelling fixed point T^-_ϕ as the Q-index of the "backward limit tree" T^-_ϕ.

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2. Preliminary definitions and notation

We continue with the introduction's notation. *Further, we assume throughout this document that all representatives g of $\phi \in Out(F_r)$ are train track* (tt) *maps.*

We let $\mathcal{FI}_r$ denoted the subset of $Out(F_r)$ consisting of all fully irreducible elements.

2.1. Directions and turns. In general we use the definitions from [BH92] and [BFH00] when discussing train track maps. We give further definitions and notation here. $g : \Gamma \to \Gamma$ will represent some $\phi \in Out(F_r)$.

$E^+(\Gamma) = \{E_1, \dots, E_n\}$ will be the edge set of Γ with some prescribed orientation. For $E \in E^+(\Gamma)$, $\overline{E}$ will be E oppositely oriented.

$$E(\Gamma) := \{E_1, \overline{E_1}, \dots, E_n, \overline{E_n}\} = \{e_1, e_1, \dots, e_{2n-1}, e_{2n}\}.$$

If the indexing $\{E_1, \dots, E_n\}$ of the edges, and thus the indexing

$$\{e_1, e_1, \dots, e_{2n-1}, e_{2n}\}$$

is prescribed, we call Γ an *edge-indexed* graph. Edge-indexed graphs differing by an index-preserving homeomorphism will be considered equivalent.

$V(\Gamma)$ will denote the vertex set of Γ (V, when Γ is clear) and $\mathcal{D}(\Gamma)$ will denote $\underset{v \in \mathcal{V}(\Gamma)}{\cup} \mathcal{D}(v)$, where $\mathcal{D}(v)$ is the set of directions (germs of initial edge segments) at v.

For each $e \in E(\Gamma)$, $D_0(e)$ will denote the initial direction of e and $D_0\gamma := D_0(e_1)$ for each path $\gamma = e_1 \dots e_k$ in Γ. Dg will denote the direction map induced by g. We call $d \in \mathcal{D}(\Gamma)$ *periodic* if $Dg^k(d) = d$ for some $k > 0$ and *fixed* if $k = 1$. $Per(x)$ will consist of the periodic directions at an $x \in \Gamma$ and $Fix(x)$ of those fixed. $Fix(g)$ will denote the fixed point set for g.

$T(v)$ will denote the set of turns (unordered pairs of directions) at a vertex $v \in V(\Gamma)$ and $D^t g$ the induced map of turns. For a path $\gamma = e_1 e_2 \dots e_{k-1} e_k$ in Γ, we say γ *contains (or takes)* the turn $\{\overline{e_i}, e_{i+1}\}$ for each $1 \le i < k$. Sometimes we abusively write $\{\overline{e_i}, e_j\}$ for $\{D_0(\overline{e_i}), D_0(e_j)\}$. Recall that a turn is called *illegal* for g if $Dg^k(d_i) = Dg^k(d_j)$ for some k (d_i and d_j are in the same *gate*).

2.2. The attracting lamination Λ_ϕ for a fully irreducible outer automorphism. The attracting lamination Λ_ϕ for a $\phi \in \mathcal{FI}_r$ was defined in [BFH97]. While an outer automorphism invariant, it can be defined in terms of any train track representative $g \colon \Gamma \to \Gamma$ of ϕ. For a tt representative $g \colon \Gamma \to \Gamma$ of ϕ, a leaf of the realization $\Lambda_\phi(\Gamma)$ of Λ_ϕ is a bi-infinite unparameterized reduced edge-path γ in Γ such that, for each finite subpath β of γ there exists an $e \in E(\Gamma)$ and integer $n \ge 1$ such that β is a subpath of $g^n(e)$. $\Lambda_\phi(\Gamma)$ is the collection of all such leaves. Note that $\Lambda_\phi(\Gamma)$ is unique.

Leaves in a lamination are known to be "quasiperiodic" (hence birecurrent) in the following sense. Using the Perron–Frobenius eigenvector for the transition matrix, lengths can be assigned to edges in Γ in such a way so that g stretches each edge by a factor of λ, where λ is the Perron–Frobenius eigenvalue. From this assignment of edge lengths, one obtains a length on any path in Γ. A line γ in Γ is then *quasiperiodic* if for each $L > 0$ there exists an $L' > L$ such that each line segment γ' in γ of length at most L appears as a subpath of each segment of length L'.

It is well-known (see [BFH97, pg. 6]) that $\Lambda(g)$ contains *periodic* leaves obtained by iterating a neighborhood of a g-periodic point in the interior of each edge of Γ (possibly taking a power of g). Since g is irreducible, such a leaf will contain each edge $e \subset \Gamma$, hence each $g^n(e)$.

2.3. Periodic Nielsen paths, ageometrics, principal points, and rotationless powers. Recall [BF94] that a *periodic Nielsen path (pNp)* is a nontrivial path ρ between $x, y \in Fix(g)$ such that, for some k, $g^k(\rho) \simeq \rho$ rel endpoints. In later sections we use [BF94, Theorem 3.2] that a $\phi \in \mathcal{FI}_r$ is ageometric if and only if some ϕ^k has a representative with no pNps (closed or otherwise). $\mathcal{AF}_r$ will denote the subset of $\mathcal{FI}_r$ consisting precisely of its ageometric elements.

As in [HM11], we call a periodic point $v \in \Gamma$ *principal* that either has at least three periodic directions or is an endpoint of a periodic Nielsen path.

A tt representative is called *rotationless* if every principal point is fixed and every periodic direction at each principal point is fixed. In [FH11, Proposition 3.24] it is shown that one can define a fully irreducible outer automorphism to be *rotationless* if and only if one (hence all) of its tt representatives are rotationless.

2.4. Local Whitehead graphs, local stable Whitehead graphs, and ideal Whitehead graphs. We explain several different ideal Whitehead graph definitions and how they relate. These definitions can be found in [HM11], though it is not their original source, and versions here are specialized. Their equivalence reveals how, while definable using a single train track representative, the ideal Whitehead graph is an outer automorphism (conjugacy class) invariant. Even further explanations of the definitions and their invariance can be found in [Pfa12].

For this subsection $g : \Gamma \to \Gamma$ will be a pNp-free train track representative of some $\phi \in Out(F_r)$.

Definition 2.1. The *local Whitehead graph* $\mathcal{LW}(g; v)$ for g at a vertex v has:

(1) a vertex for each direction $d \in \mathcal{D}(v)$ and
(2) edges connecting vertices for $d_1, d_2 \in \mathcal{D}(v)$ where $\{d_1, d_2\}$ is taken by some $g^k(e)$, with $e \in E(\Gamma)$.

The *local Stable Whitehead graph* $\mathcal{SW}(g; v)$ is the subgraph obtained by restricting precisely to vertices with labels in $Per(v)$. For a rose Γ with vertex v, we denote the single local stable Whitehead graph $\mathcal{SW}(g; v)$ by $\mathcal{SW}(g)$ and the single local Whitehead graph $\mathcal{LW}(g; v)$ by $\mathcal{LW}(g)$.

For a pNp-free g, the *ideal Whitehead graph of ϕ*, $\mathcal{IW}(\phi)$, is isomorphic to

$$\bigsqcup_{\text{principal vertices } v \in \Gamma} \mathcal{SW}(g; v).$$

In particular, when Γ is a rose, $\mathcal{IW}(\phi) \cong \mathcal{SW}(g)$.

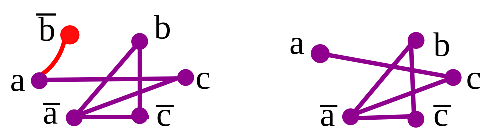
Example 2.2. Let $g : \Gamma \to \Gamma$, where Γ is a rose and g is the train track map such that the following describes the edge-path images of its edges:

$$g = \begin{cases} a \mapsto abacbaba\bar{c}abacbaba \\ b \mapsto ba\bar{c} \\ c \mapsto c\bar{a}\bar{b}\bar{a}\bar{b}\bar{a}\bar{b}\bar{c}\bar{a}\bar{b}\bar{a}c. \end{cases}$$

The vertices for $\mathcal{LW}(g)$ are $\{a, \bar{a}, b, \bar{b}, c, \bar{c}\}$ and the vertices of $\mathcal{SW}(g)$ are $\{a, \bar{a}, b, c, \bar{c}\}$: The periodic (actually fixed) directions for g are $\{a, \bar{a}, b, c, \bar{c}\}$. $\bar{b}$ is not periodic since $Dg(\bar{b}) = c$, which is a fixed direction, meaning that $Dg^k(\bar{b}) = c$ for all $k \geq 1$, and thus $Dg^k(\bar{b})$ does NOT equal $\bar{b}$ for any $k \geq 1$.

The turns taken by the $g^k(E)$, for $E \in E(\Gamma)$, are $\{a, \bar{b}\}$, $\{\bar{a}, \bar{c}\}$, $\{b, \bar{a}\}$, $\{b, \bar{c}\}$, $\{c, \bar{a}\}$, and $\{a, c\}$. Since $\{a, \bar{b}\}$ contains the nonperiodic direction $\bar{b}$, this turn does not give an edge in $\mathcal{SW}(g)$, though does give an edge in $\mathcal{LW}(g)$. All other turns listed give edges in both $\mathcal{SW}(g)$ and $\mathcal{LW}(g)$.

$\mathcal{LW}(g)$ and $\mathcal{SW}(g)$ respectively look like (reasons for colors become clear in Subsection 2.5):



To make clear that the ideal Whitehead graph is actually an outer automorphism invariant, as in [HM11] and [Pfa12], we relate the above definition to those relying solely on the attracting lamination. In what follows, ϕ will be a nongeometric fully irreducible.

Definition 2.3. A fixed point x is *repelling* for the action of g if it is an attracting fixed point for the action of g^{-1}, i.e., if there exists a neighborhood U of x such that, for each neighborhood $V \subset U$ of x, there exists an $N > 0$ such that $g^{-k}(y) \in V$ for all $y \in U$ and $k \geq N$.

Let $g : \Gamma \to \Gamma$ be a rotationless irreducible train track representative of $\phi \in Out(F_r)$ and let $\tilde{g} : \tilde{\Gamma} \to \tilde{\Gamma}$ be a *principal lift* of g, i.e., a lift to the universal cover such that the boundary extension $\hat{g}$ has at least three nonrepelling fixed points. $W(\tilde{g})$ is defined to be the graph where:

(1) Vertices correspond to nonrepelling fixed points of the boundary extension $\hat{g}$.
(2) Edges connect vertices P_1 and P_2 precisely when P_1 and P_2 are the ideal (boundary) endpoints of some leaf in the lift $\tilde{\Lambda}(\phi)$ of the attracting lamination to the universal cover $\tilde{\Gamma}$ of Γ.

We then define $W(g) = \sqcup W(\tilde{g})$, leaving out components with two or fewer vertices. One obtains the ideal Whitehead graph $\mathcal{IW}(g)$ from $W(g)$ by taking the quotient under conjugation by the deck transformation action on $\tilde{\Gamma}$.

Since the attracting lamination is an outer automorphism invariant (and, in particular, the properties of leaves having nonrepelling fixed point endpoints and sharing an endpoint are invariant), the definition we just gave does not rely on the choice of representative g for a given $\phi \in Out(F_r)$. Thus, once we establish equivalence between Definition 2.3 and Definition 2.1, it should be clear that the ideal Whitehead graph is an outer automorphism invariant.

[HM11, Corollary 3.2] (see below) relates Definition 2.1 with Definition 2.3. However, for [HM11, Corollary 3.2] to actually make sense, one needs the following definitions and identification from [HM11]. A *cut vertex* of a graph is a vertex separating a component of the graph into two components. $SW(\tilde{v};\tilde{\Gamma})$ denotes the lift of $SW(v;\Gamma)$ to the universal cover $\tilde{\Gamma}$ of Γ (having countably many disjoint copies of $SW(v;\Gamma)$, one for each lift of v).

Let $g : \Gamma \to \Gamma$ be an irreducible train track representative of an iterate of $\phi \in Out(F_r)$ such that:

(1) Each periodic vertex $v \in \Gamma$ is fixed.
(2) Each periodic direction at such a v is fixed.

Choose one of these fixed vertices v. Suppose $\tilde{v} \in \tilde{\Gamma}$ is a lift of v to the universal cover, $\tilde{g} : \tilde{\Gamma} \to \tilde{\Gamma}$ is a lift of g fixing $\tilde{v}$, and d is a direction at $\tilde{v}$ fixed by $D\tilde{g}$. Furthermore, let $\tilde{E}$ be the edge at $\tilde{v}$ whose initial direction is d. The *ray determined by* d (or *by* $\tilde{E}$) is defined as

$$\tilde{R} = \bigcup_{j=0}^{\infty} \tilde{g}^j(\tilde{E}).$$

This is a ray in $\tilde{\Gamma}$ converging to a nonrepelling fixed point for $\hat{g}$. Such a ray is called *singular* if the vertex $\tilde{v}$ it originates at is principal (i.e., v is principal). With these definitions:

(1) The vertices of $SW(\tilde{v};\tilde{\Gamma})$ correspond to singular rays $\tilde{R}$ based at $\tilde{v}$.
(2) Directions d_1 and d_2 represent endpoints of an edge in $SW(\tilde{v};\tilde{\Gamma})$ if and only if $\tilde{l} = \tilde{R}_1 \cup \tilde{R}_2$ is a singular leaf of $\tilde{\Lambda}$ realized in $\tilde{\Gamma}$, where $\tilde{R}_1$ and $\tilde{R}_2$ are the rays determined by d_1 and d_2 respectively.

Noticing that the ideal (boundary) endpoints of singular rays are precisely the nonrepelling fixed points at infinity for the action of $\tilde{g}$, combining this with what has already been said, as well as Proposition 2.4 ([HM11, Corollary 3.2]) and what follows, we have the correspondence proving ideal Whitehead graph invariance.

Proposition 2.4. *Let $\tilde{g}$ be a principal lift of g. Then:*

(1) *$W(\tilde{g})$ is connected.*
(2) *$W(\tilde{g}) = \bigcup_{\tilde{v} \in Fix(\tilde{g}) \in \Gamma} SW(\tilde{v})$.*
(3) *For $i \neq j$, $SW(\tilde{v}_i)$ and $SW(\tilde{v}_j)$ intersect in at most one vertex. If they do intersect at a vertex P, then P is a cut point of $W(\tilde{g})$, in fact P separates $SW(\tilde{v}_i)$ and $SW(\tilde{v}_j)$ in $W(\tilde{g})$.*

By [HM11, Lemma 3.1], in our case (where there are no pNps), there is in fact only one $\tilde{v} \in Fix(\tilde{g})$ and so the above corollary gives that $W(\tilde{g}) = SW(\tilde{v})$.

This concludes our justification of how an ideal Whitehead graph is an outer automorphism invariant. One can consult [HM11] for clarification of the relationship between ideal Whitehead graphs and $\mathbb{R}$-trees or for other ideal Whitehead graph characterizations.

Please note that the ideal Whitehead graphs, local Whitehead graphs, and stable Whitehead graphs used here (defined in [HM11]) differ from other Whitehead graphs in the literature. We clarify a difference. In general, Whitehead graphs record turns taken by immersions of 1-manifolds into graphs. In our case, the 1-manifold is a set of lines, the attracting lamination. In much of the literature the 1-manifolds are circuits representing conjugacy classes of free group elements. For example, for the Whitehead graphs of [CV86], edge images are viewed as cyclic words. This is not true for ours.

2.5. Lamination train track structures. We define here "lamination train track (ltt) structures." Bestvina, Feighn, and Handel discussed in their papers slightly different train track structures. However, those we define contain as smooth paths lamination leaf realizations. This makes them useful for deeming unachieved particular ideal Whitehead graphs and for constructing representatives (see [Pfa13a] and [Pfa13b]). "Higher ltt structures" will be defined in Section 3.

Again, $g : \Gamma \to \Gamma$ will be a pNp-free train track map on a marked rose with vertex v.

Definition 2.5. The *colored local Whitehead graph $\mathcal{CW}(g)$ at v*, is $\mathcal{LW}(g)$, but with the subgraph $\mathcal{SW}(g)$ colored purple and $\mathcal{LW}(g) - \mathcal{SW}(g)$ colored red (nonperiodic direction vertices are red).

Let $\Gamma_N = \Gamma - N(v)$ where $N(v)$ is a contractible neighborhood of v. For each $E_i \in E^+(\Gamma)$, add vertices d_i and $\overline{d_i}$ at the corresponding boundary points of the partial edge $E_i - (N(v) \cap E_i)$. A *lamination train track* (ltt) *structure* $G(g)$ for g is formed from $\Gamma_N \bigsqcup \mathcal{CW}(g)$ by identifying the vertex d_i in Γ_N with the vertex d_i in $\mathcal{CW}(g)$. Vertices for nonperiodic directions are red, edges of Γ_N black, and all periodic vertices purple.

An ltt structure $G(g)$ is given a *smooth structure* via a partition of the edges at each vertex into two sets: $\mathcal{E}_b$ (containing the black edges of $G(g)$) and $\mathcal{E}_c$ (containing the colored edges of $G(g)$). A *smooth path* will mean a path alternating between colored and black edges.

An edge connecting a vertex pair $\{d_i, d_j\}$ will be denoted $[d_i, d_j]$, with interior (d_i, d_j). Additionally, $[e_i]$ will denote the black edge $[d_i, \overline{d_i}]$ for $e_i \in E(\Gamma)$.

For a smooth (possibly infinite) path γ in $G(g)$, the *path (or line) in Γ corresponding to γ* is

$$\ldots e_{-j} e_{-j+1} \ldots e_{-1} e_0 e_1 \ldots e_j \ldots,$$

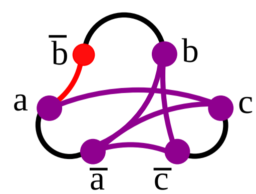
with

$$\gamma = \dots [d_{-j}, \overline{d_{-j}}][\overline{d_{-j}}, d_{-j+1}] \dots [d_0, \overline{d_0}][\overline{d_0}, d_1] \dots [d_j, \overline{d_j}] \dots,$$

where each $d_i = D_0(e_i)$, each $[d_i, \overline{d_i}]$ is the black edge $[e_i]$, and each $[d_i, \overline{d_{i+1}}]$ is a colored edge. We denote such a path

$$\gamma = [\dots, d_{-j}, \overline{d_{-j}}, d_{-j+1}, \dots, \overline{d_{-1}}, d_0, \overline{d_0}, d_1, \dots, d_j, \overline{d_j} \dots].$$

Example 2.6. Let g be as in Example 2.2. The vertex $\bar{b}$ in $G(g)$ is red. All others are purple. $G(g)$ has a purple edge for each edge in $\mathcal{SW}(g)$ and a single red edge for the turn $\{a, \bar{b}\}$ (represented by an edge in $\mathcal{LW}(g)$, but not in $\mathcal{SW}(g)$). $\mathcal{CW}(g)$ is $\mathcal{LW}(g)$ with the coloring of Example 2.2. And $G(g)$ is obtained from $\mathcal{CW}(g)$ by adding black edges connecting the vertex pairs $\{a, \bar{a}\}$, $\{b, \bar{b}\}$, and $\{c, \bar{c}\}$ (corresponding precisely to the edges a, b, and c of Γ).



One can check that each $g(e)$ is realized by a smooth path in $G(g)$.

Remark 2.7. If Γ had more than one vertex, one could define $G(g)$ by creating a colored graph $\mathcal{CW}(g; v)$ for each vertex, removing an open neighborhood of each vertex when forming Γ_N, and then continuing with the identifications as above in $\Gamma_N \bigsqcup (\cup \mathcal{CW}(g; v))$.

3. Birecurrency of ltt structures and higher Rauzy graphs

Proposition 3.7 of this section gives a necessary condition for ltt structures to belong to train track representatives of fully irreducible outer automorphisms. We in fact use it in Theorem 9.1(I) to show that certain ideal Whitehead graphs are not achieved.

We first establish several definitions we will use. In particular, we define higher ltt graphs and the Rauzy graphs of tiling theory inspiring them. While higher ltt structures are not used outside this section, it is the belief of the author that much about them can be profitably explored, a belief justifying their inclusion. Rauzy graphs have already been used to study $Out(F_r)$ in papers such as [Kap05] and [Kap06], where they are referred to as "initial graphs."

For this section we again fix a basis $X_1, \dots, X_r$ for F_r and let R_r denote the r-petaled rose endowed with a marking identifying its petals with the generators $X_1, \dots, X_r$ of F_r.

Rauzy graphs in general were introduced in [Rau82] and appear in a number of works. Sequences of Rauzy graphs for infinite words are studied, in particular, in [Sal10]. We use the definition of [Sal10] to define the notion in this setting.

Definition 3.1. Suppose $g\colon \Gamma \to \Gamma$ is a train track representative of an ageometric fully irreducible outer automorphism $\phi \in Out(F_r)$. The *order-k Rauzy graph* $\mathcal{R}_k(g)$ is the graph with:

- **Vertices:** a vertex for each length-k edge-path w appearing in any lamination leaf of $\Lambda(\Gamma)$.
- **Edges:** a directed edge connecting $u_1 \dots u_k$ to $u_2 \dots u_{k+1}$ for each length-$(k+1)$ edge-path $w = u_1 \dots u_{k+1}$ appearing in a lamination leaf.

The Rauzy graph definition is that consistent with tiling theory if Γ is the marked rose R_r and if we define a language whose alphabet consists of generators of F_r (and their inverses) and whose words are those realized as edge-paths that are subpaths of lamination leaves. However, to properly generalize our ltt structure definition to higher ltt structures, we must alter the Rauzy graph definitions. As in the ltt structures of Section 2.5, we want for vertices to correspond to "directions" (by which we mean here oriented words of length k), we want black edges connecting the vertices for the two directions (orientations) of a word appearing in a leaf of $\Lambda(\Gamma)$, and we want colored edges for generalized "taken turns."

In the spirit of the Rauzy graphs we define the level-k ltt structure $G^k(g)$ for a train track representative $g\colon \Gamma \to \Gamma$ of a fully irreducible $\phi \in Out(F_r)$:

Definition 3.2. Suppose $g\colon \Gamma \to \Gamma$ is a train track representative of an ageometric fully irreducible outer automorphism $\phi \in Out(F_r)$. The level-k ltt structure $G^k(g)$ is the train track graph satisfying:

- **Vertices:** For each length-k edge-path w_i in any leaf of $\Lambda(\Gamma)$, $G^k(g)$ will contain a vertex for w_i (and a vertex for w_i^{-1}).
- **Black Edges:** $G^k(g)$ will contain a black edge connecting each pair of vertices of the form w_i and w_i^{-1}.
- **Colored Edges:** $G^k(g)$ will contain a colored edge connecting the vertices $u = u_1 \dots u_k$ and $v = v_1 \dots v_k$ if $v_k = u_2^{-1}, v_{k-1} = u_3^{-1}, \dots, v_2 = u_k^{-1}$ and either uv_1^{-1} or vu_1^{-1} is an edge-path of a leaf of $\Lambda(\Gamma)$.

Remark 3.3. By considering both purple and red leaves as just colored, it follows from the definitions that $G(g) = G^1(g)$.

Definition 3.4. A *train track* (tt) *graph* is a finite graph G satisfying:

(tt1) G has no valence-1 vertices.
(tt2) Each edge of G has 2 distinct vertices (single edges are never loops).
(tt3) The edge set of G is partitioned into two subsets, $\mathcal{E}_b$ (the "black" edges) and $\mathcal{E}_c$ (the "colored" edges), such that each vertex is incident to at least one $E_b \in \mathcal{E}_b$ and at least one $E_c \in \mathcal{E}_c$.

We consider tt graphs *equivalent* that are isomorphic as graphs via an isomorphism preserving the edge partition. We call a path in a tt graph *smooth* that alternates between edges in $\mathcal{E}_b$ and edges in $\mathcal{E}_c$.

Example 3.5. The ltt structure $G(g)$ for a pNp-free representative g on R_r is a tt graph where $\mathcal{E}_b$ is the set of black edges of $G(g)$ and where $\mathcal{E}_c$ is the edge set of $\mathcal{C}(G(g))$. The $G^k(g)$ are also tt graphs.

Definition 3.6. A tt graph is *birecurrent* if it has a locally smoothly embedded line containing each edge infinitely many times in each end, i.e., as any assigned parameter $r \in \mathbb{R}$ satisfies $r \to \infty$ and $r \to -\infty$.

Proposition 3.7 (Birecurrency Condition)**.** *Let $\phi \in Out(F_r)$ be an fully irreducible outer automorphism. Then the ltt structures $G^k(g)$ for each train track representative $g\colon \Gamma \to \Gamma$ of ϕ are birecurrent.*

The key to this proof is showing that each lamination leaf gives a smooth, surjective, birecurrent line.

Lemma 3.8. *Let $g\colon \Gamma \to \Gamma$ be a train track representative of some $\phi \in \mathcal{FI}_r$. Then each $G^k(g)$ contains a smooth surjective path corresponding to the realization in Γ of each leaf of Λ_ϕ.*

Proof. Given a subpath $a_1 \cdots a_n$ of a lamination leaf and an integer k satisfying $n > k > 0$, we obtain a smooth path in $G^k(g)$ starting with the vertex $a_1 \cdots a_k$, traversing the colored edge to $(a_2 \cdots a_{k+1})^{-1}$, traversing the black edge to $a_2 \cdots a_{k+1}$, traversing the colored edge to $(a_3 \cdots a_{k+2})^{-1}$, $\dots$, traversing the colored edge from $a_{n-k} \cdots a_{n-1}$ to $(a_{n-k+1} \cdots a_n)^{-1}$, and traversing the black edge from $(a_{n-k+1} \cdots a_n)^{-1}$ to $a_{n-k+1} \cdots a_n$. Extending infinitely in both directions, one gets a smooth realization of the entire lamination leaf.

We now prove surjectivity. Given an edge $[u = u_1 \dots u_k, v = v_1 \dots v_k]$ in a $G^k(g)$, then either uv_1^{-1} or vu_1^{-1} is a subsegment of some leaf in $\Lambda(g)$, hence of some $g^n(e)$, hence of each periodic leaf γ in $\Lambda(g)$. By construction, the path induced by γ will traverse the black edge from u^{-1} to u, the colored edge from u to v, and the black edge from v to v^{-1}. Since every u is contained in a longer γ subsegment, this implies the surjectivity of the path induced by γ. $\square$

Proof of Proposition 3.7. The periodic lines in $G^k(g)$ induced by the lamination leaves are birecurrent (and quasiperiodic) by the quasiperiodicity of lamination leaves for fully irreducibles. Hence, by the above lemma, each $G^k(g)$ is birecurrent. $\square$

Remark 3.9.

(1) For each k, each lamination leaf for g additionally gives an infinite path (line) in $\mathcal{R}_k(g)$, in fact a pair of oriented lines in $\mathcal{R}_k(g)$. The

pair of oriented lines in $\mathcal{R}_k(g)$ corresponding to the two orientations of any periodic leaf of the lamination will traverse every edge of $\mathcal{R}_k(g)$.

(2) The lines in $\mathcal{R}_k(g)$ induced by the lamination leaves are also birecurrent (and quasiperiodic) by the quasiperiodicity of lamination leaves for fully irreducibles.

4. Ideal decompositions

This section contains our proof of Proposition 4.3: if $\mathcal{G} \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$ is $\mathcal{IW}(\phi)$ for a $\phi \in \mathcal{AF}_r$, then ϕ has a rotationless power with a representative satisfying several nice properties, including that its Stallings fold decomposition consists entirely of proper full folds of roses. We call such a decomposition an *ideal decomposition*. Proving an ideal decomposition cannot exist will suffice to deem a $\mathcal{G}$ unachieved.

We remind the reader of definitions of folds and a Stallings fold decomposition before introducing ideal decompositions, as Stallings fold decompositions are central in our proof of Proposition 4.3.

4.1. Folds. Stallings introduced "folds" in [Sta83] and Bestvina and Handel use several versions in their train track algorithm of [BH92].

Let $g\colon \Gamma \to \Gamma$ be a homotopy equivalence of marked graphs. Suppose $g(e_1) = g(e_2)$ as edge paths, where $e_1, e_2 \in E(\Gamma)$ emanate from a common vertex $v \in V(\Gamma)$. One can obtain a graph Γ_1 by identifying e_1 and e_2 in such a way that $g\colon \Gamma \to \Gamma$ projects to $g_1\colon \Gamma_1 \to \Gamma_1$ under the quotient map induced by the identification of e_1 and e_2. g_1 is also a homotopy equivalence and one says g_1 and Γ_1 are obtained from g by an *elementary fold* of e_1 and e_2.

To generalize one requires $e_1' \subset e_1$ and $e_2' \subset e_2$ only be maximal, initial, nontrivial subsegments of edges emanating from a common vertex such that $g(e_1') = g(e_2')$ as edge paths and such that the terminal endpoints of e_1 and e_2 are in $g^{-1}(V(\Gamma))$. Possibly redefining Γ to have vertices at the endpoints of e_1' and e_2', one can fold e_1' and e_2' as e_1 and e_2 were folded above. We say $g_1\colon \Gamma_1 \to \Gamma_1$ is obtained by:

- a *partial fold* of e_1 and e_2 if both e_1' and e_2' are proper subedges;
- a *proper full fold* of e_1 and e_2 if only one of e_1' and e_2' is a proper subedge (the other a full edge);
- an *improper full fold* of e_1 and e_2 if e_1' and e_2' are both full edges.

4.2. Stallings fold decompositions. Stallings [Sta83] also showed a tight homotopy equivalence of graphs is a composition of elementary folds and a final homeomorphism. We call such a decomposition a *Stallings fold decomposition*.

A description of a Stallings fold decomposition can be found in [Sko89], where Skora described a Stallings fold decomposition for a $g\colon \Gamma \to \Gamma'$ as a sequence of folds performed continuously. Consider a lift $\tilde{g}\colon \tilde{\Gamma} \to \tilde{\Gamma}'$, where

here $\tilde{\Gamma}'$ is given the path metric. Foliate $\tilde{\Gamma}$ x $\tilde{\Gamma}'$ with the leaves $\tilde{\Gamma}$ x $\{x'\}$ for $x' \in \Gamma'$. Define $N_t(\tilde{g}) = \{(x, x') \in \tilde{\Gamma} \text{ x } \tilde{\Gamma}' \mid d(\tilde{g}(x), x') \leq t\}$. For each t, by restricting the foliation to N_t and collapsing all leaf components, one obtains a tree Γ_t. Quotienting by the F_r-action, one sees the sequence of folds performed on the graphs below over time.

Alternatively, at an illegal turn for $g\colon \Gamma \to \Gamma$, fold maximal initial segments having the same image in $\tilde{\Gamma}'$ to obtain a map $g^1\colon \Gamma_1 \to \Gamma'$ of the quotient graph Γ_1. Repeat for g^1. If some g^k has no illegal turn, it will be a homeomorphism and the fold sequence is complete. Using this description, we can assume only the final element of the decomposition is a homeomorphism. Thus, a Stallings fold decomposition of $g\colon \Gamma \to \Gamma$ can be written $\Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n$ where each g_k, with $1 \leq k \leq n-1$, is a fold and g_n is a homeomorphism.

4.3. Ideal decompositions. This subsection contains our proof of Proposition 4.3. We remark first that it follows from the rotationless and ideal Whitehead graph definitions given in [HM11] that: For $\phi \in \mathcal{AF}_r$ such that $\mathcal{IW}(\phi) \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$, ϕ is rotationless if and only if the vertices of $\mathcal{IW}(\phi)$ are fixed by the action of ϕ. Finally, we need the following lemmas.

Lemma 4.1. *Let $g\colon \Gamma \to \Gamma$ be a tt representative of $\phi \in Out(F_r)$ and*

$$\Gamma = \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n = \Gamma$$

a decomposition of g into homotopy equivalences of marked graphs. Then the composition

$$h\colon \Gamma_k \xrightarrow{g_{k+1}} \Gamma_{k+1} \xrightarrow{g_{k+2}} \cdots \xrightarrow{g_{k-1}} \Gamma_{k-1} \xrightarrow{g_k} \Gamma_k$$

is also a tt representative of ϕ. Further, if g is pNp-free, then h is pNp-free.

Proof. Let $\pi\colon R_r \to \Gamma$ mark Γ_1. Since g_1 is a homotopy equivalence, $g_1 \circ \pi$ gives a marking on Γ. So g and h differ by a change of marking and thus represent the same outer automorphism ϕ.

We show h is a tt map. For contradiction's sake suppose $h(e)$ contains an illegal turn $\{d_1, d_2\}$. Since each g_j is surjective, some $(g_k \circ \cdots \circ g_1)(e_i)$ would traverse e. So $(g_k \circ \cdots \circ g_1)(e_i)$ would contain $\{d_1, d_2\}$. And

$$g^2(e_i) = (g_n \circ \cdots \circ g_{k+1}) \circ h \circ (g_k \circ \cdots \circ g_1)(e_i)$$

would contain $\{D(g_n \circ \cdots \circ g_{k+1})(d_1), D(g_n \circ \cdots \circ g_{k+1})(d_2)\}$, which would either be illegal or degenerate (since $\{d_1, d_2\}$ is an illegal turn). This would contradict that g is a tt map. So h is a tt map.

Suppose that g is pNp-free and h had a pNp ρ and $h^p(\rho) \simeq \rho$ rel endpoints. Let $\rho_1 = g_n \circ \cdots \circ g_{k+1}(\rho)$. If ρ_1 were trivial,

$$h^p(\rho) = (g_k \circ \cdots \circ g_1 \circ g^{p-1})(g_n \circ \cdots \circ g_{k+1}(\rho)) = (g_k \circ \cdots \circ g_1 \circ g^{p-1})(\rho_1)$$

would be trivial, contradicting ρ being a pNp. So assume ρ_1 is not trivial.

$g^p(\rho_1) = g^p((g_k \circ \cdots \circ g_1)(\rho)) = (g_n \circ \cdots \circ g_{k+1}) \circ h^p(\rho)$. Now, $h^p(\rho) \simeq \rho$ rel endpoints and so $(g_n \circ \cdots \circ g_{k+1}) \circ h^p(\rho) \simeq (g_n \circ \cdots \circ g_{k+1})(\rho)$ rel endpoints. So $g^p(\rho_1) = g^p((g_k \circ \cdots \circ g_1)(\rho)) = (g_n \circ \cdots \circ g_{k+1}) \circ h^p(\rho)$ is homotopic to $(g_n \circ \cdots \circ g_{k+1})(\rho) = \rho_1$ rel endpoints. This makes ρ_1 a pNp for g, contradicting that g is pNp-free. Thus, h is pNp-free. $\square$

Lemma 4.2. *Let $g\colon \Gamma \to \Gamma$ be a tt map with $2r-1$ fixed directions and Stallings fold decomposition $\Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n$. Let g^i be such that $g = g^i \circ g_i \circ \cdots \circ g_1$. Let $d_{(1,1)}, \dots, d_{(1,2r-1)}$ be the fixed directions for Dg and let $d_{j,k} = D(g_j \circ \cdots \circ g_1)(d_{1,k})$ for each $1 \leq j \leq n$ and $1 \leq k \leq 2r-1$. Then $D(g^i)$ is injective on $\{d_{(i,1)}, \dots, d_{(i,2r-1)}\}$.*

Proof. Let $d_{(1,1)}, \dots, d_{(1,2r-1)}$ be the fixed directions for Dg. If $D(g^i)$ identified any of $d_{(i,1)}, \dots, d_{(i,2r-1)}$, then Dg would have fewer than 2r-1 directions in its image. $\square$

Proposition 4.3. *Let $\phi \in Out(F_r)$ be an ageometric, fully irreducible outer automorphism whose ideal Whitehead graph $\mathcal{IW}(\phi)$ is a connected, $(2r-1)$-vertex graph. Then there exists a train track representative g of a power $\psi = \phi^R$ of ϕ that is:*

(1) *on the rose,*
(2) *rotationless,*
(3) *pNp-free, and*
(4) *decomposable as a sequence of proper full folds of roses.*

In fact, it decomposes as $\Gamma = \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n = \Gamma$, where:

(I) *The index set $\{1, \dots, n\}$ is viewed as the set $\mathbf{Z}/n\mathbf{Z}$ with its natural cyclic ordering.*
(II) *Each Γ_k is an edge-indexed rose with an indexing*

$$\{e_{(k,1)}, e_{(k,2)}, \dots, e_{(k,2r-1)}, e_{(k,2r)}\}$$

where:

(a) *One can edge-index Γ with $E(\Gamma) = \{e_1, e_2, \dots, e_{2r-1}, e_{2r}\}$ such that, for each t with $1 \leq t \leq 2r$, $g(e_t) = e_{i_1} \dots e_{i_s}$ where*

$$(g_n \circ \cdots \circ g_1)(e_{0,t}) = e_{n,i_1} \dots e_{n,i_s}.$$

(b) *For some i_k, j_k with $e_{k,i_k} \neq (e_{k,j_k})^{\pm 1}$*

$$g_k(e_{k-1,t}) := \begin{cases} e_{k,t}e_{k,j_k} & \text{for } t = i_k \\ e_{k,t} & \text{for all } e_{k-1,t} \neq e^{\pm 1}_{k-1,j_k} \end{cases}$$

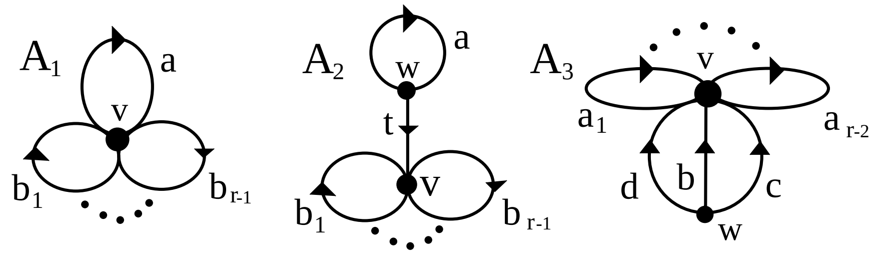
(the edge index permutation for the homeomorphism in the decomposition is trivial, so left out).

(c) *For each $e_t \in E(\Gamma)$ such that $t \neq j_n$, we have $Dg(d_t) = d_t$, where $d_t = D_0(e_t)$.*

Proof. Since $\phi \in \mathcal{AF}_r$, there exists a pNp-free tt representative g of a power of ϕ. Let $h = g^k : \Gamma \to \Gamma$ be rotationless. Then h is also a pNp-free tt representative of some ϕ^R and h (and all powers of h) satisfy (2)–(3). Since h has no pNps (meaning

$$\mathcal{IW}(\phi^R) \cong \bigsqcup_{\text{singularities } v \in \Gamma} \mathcal{SW}(h; v)$$

and, if Γ is the rose, $\mathcal{SW}(h) \cong \mathcal{IW}(\phi^R)$), since h fixes all its periodic directions, and since $\mathcal{IW}(\phi)$ (hence $\mathcal{IW}(\phi^R)$) is in $\mathcal{PI}_{(r;(\frac{3}{2}-r))}$, Γ must have a vertex with $2r-1$ fixed directions. Thus, Γ must be one of:



If $\Gamma = A_1$, h satisfies (3). We show, in this case, we also have the decomposition for (4). However, first we show Γ cannot be A_2 or A_3 by ruling out all possibilities for folds in h's Stallings decomposition.

If $\Gamma = A_2$, v has to be the vertex with 2r-1 fixed directions. h has an illegal turn unless it it is a homeomorphism, contradicting irreducibility. Note w could not be mapped to v in a way not forcing an illegal turn at w, as this would force either an illegal turn at v (if t were wrapped around some b_i) or we would have backtracking on t. Because all 2r-1 directions at v are fixed by h, if h had an illegal turn, it would have to occur at w (no two fixed directions can share a gate).

The turns at w are $\{a, \bar{a}\}$, $\{a, t\}$, and $\{\bar{a}, t\}$. By symmetry we only need to rule out illegal turns at $\{a, \bar{a}\}$ and $\{a, t\}$.

First, suppose $\{a, \bar{a}\}$ were illegal and the first fold in the Stallings decomposition. Fold $\{a, \bar{a}\}$ maximally to obtain $(A_2)_1$. Completely collapsing a would change the homotopy type of A_2. (See Figure 1.)

Let $h_1 : (A_2)_1 \to (A_2)_1$ be the induced map of [BH92]. Since the fold of $\{a, \bar{a}\}$ was maximal, $\{a_1, \overline{a_1}\}$ must be legal. Since h was a train track, $\{t_1, a_1\}$ and $\{t_1, \overline{a_1}\}$ would also be legal. But then h_1 would fix all directions at both vertices of Γ_1 (since it still would need to fix all directions at v). This would make h_1 a homeomorphism, again contradicting irreducibility. So $\{a, \bar{a}\}$ could not have been the first turn folded. We are left to rule out $\{a, t\}$.

Suppose the first turn folded in the Stallings decomposition were $\{a, t\}$. Fold $\{a, t\}$ maximally to obtain $(A_2)'_1$. Let $h'_1 \colon (A_2)'_1 \to (A_2)'_1$ be the induced map of [BH92]. Then one of the following holds:

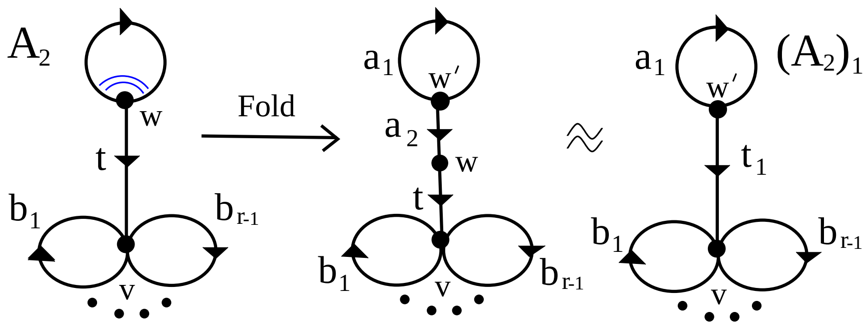


FIGURE 1. a_1 is the portion of a not folded, a_2 is the edge created by the fold, w' is the vertex created by the fold, and t_1 is $a_2 \cup t$ without the (now unnecessary) vertex w.

(A) All of t was folded with a full power of a.
(B) All of t was folded with a partial power of a.
(C) Part of t was folded with either a full or partial power of a.

If (A) or (B) held, $(A_2)'_1$ would be a rose and h'_1 would give a representative on the rose, returning us to the case of A_1. So we just need to analyze (C).

Consider first (C), i.e., suppose that part of t is folded with either a full or partial power of a:

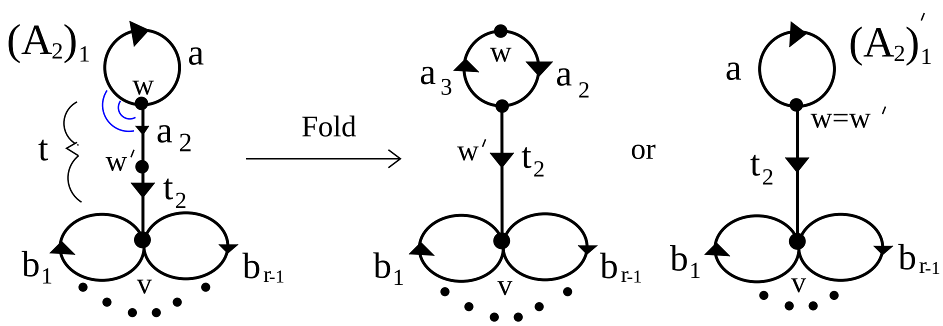


FIGURE 2. Of the two scenarios on the right, the leftmost is where the fold ends in the middle of a. a_2 is a possible portion of a folded with the portion of t, a_3 would be the portion of a not folded with t, and t_2 would be the portion of t not folded with a.

If $h = h^1 \circ g_1$, where g_1 is the single fold performed thus far, then h^1 could not identify any directions at w': identifying a_2 and t_2 would lead to h back-tracking on t; identifying t_2 and $\bar{a}$ would lead to h back-tracking on a; and h^1 could not identify t_2 and $\overline{a_3}$ because the fold was maximal. But then all directions of $(A_2)'_1$ would be fixed by h^1, making h^1 a homeomorphism and the decomposition complete. However, this would make h consist of the single fold g_1 and a homeomorphism, contradicting h's irreducibility. Thus, all cases where $\Gamma = A_2$ are either impossible or yield the representative on the rose for (1).

Now assume $\Gamma = A_3$. v must have $2r-1$ fixed directions. As with A_2, since h must fix all directions at v, if h had an illegal turn (which it still has to) it would be at w. Without losing generality assume $\{b, d\}$ is an illegal turn and that the first Stallings fold maximally folds $\{b, d\}$. Folding all of b and d would change the homotopy type. So assume (again without generality loss) either:

- all of b is folded with part of d, or
- only proper initial segments of b and d are folded with each other.

If all of b is folded with part of d, we get a pNp-free tt map on the rose. So suppose only proper initial segments of b and d are identified. Let $h_1\colon (A_3)_1 \to (A_3)_1$ be the [BH92] induced map.

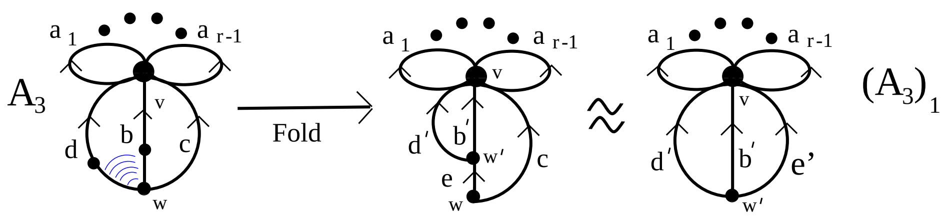


FIGURE 3. e was created by the fold and e' is $\bar{e} \cup c$ without the (now unnecessary) vertex w.

The new vertex w' has 3 distinct gates: $\{b', d'\}$ is legal since the fold was maximal and $\{b', \bar{e}\}$ and $\{d', \bar{e}\}$ must be legal or h would have back-tracked on b or d, respectively. This leaves that the entire decomposition is a single fold and a homeomorphism, again contradicting h's irreducibility.

We have ruled out A_3 and proved for (1) that we have a pNp-free representative on the rose of some $\psi = \phi^R$. We now prove (4).

Let h be the pNp-free tt representative of ϕ^R on the rose and

$$\Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n$$

the Stallings decomposition. Each g_i is either an elementary fold or locally injective (thus a homeomorphism). We can assume g_n is the only homeomorphism. Let $h^i = g_n \circ \cdots \circ g_{i+1}$. Since h has precisely $2r-1$ gates, h has precisely one illegal turn. We first determine what g_1 could be. g_1 cannot be a homeomorphism or $h = g_1$, making h reducible. So g_1 must maximally fold the illegal turn. Suppose the fold is a proper full fold. (If it is not, see the analysis below of cases of improper or partial folds.)

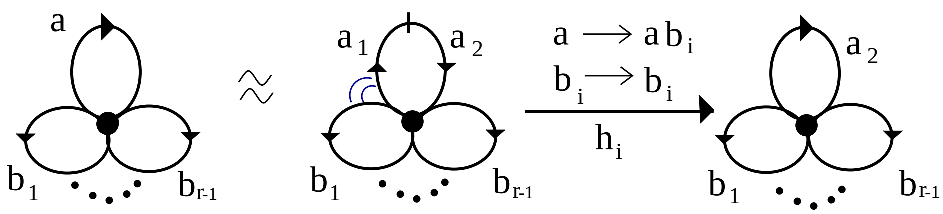
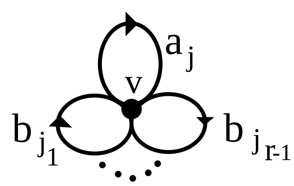


FIGURE 4. Proper full fold.

By Lemma 4.2, h^1 can only have one turn $\{d_1, d_2\}$ where $Dh^1(\{d_1, d_2\})$ is degenerate (we call such a turn an *order-1 illegal turn* for h^1). If it has no order-1 illegal turn, h^1 is a homeomorphism and the decomposition is determined. So suppose h^1 has an order-1 illegal turn (with more than one, h could not have 2r-1 distinct gates). The next Stallings fold must maximally fold this turn. With similar logic, we can continue as such until either h is obtained, in which case the desired decomposition is found, or until the next fold is not a proper full fold. The next fold cannot be an improper full fold or the homotopy type would change. Suppose after the last proper full fold we have:



Without losing generality, suppose the illegal turn is $\{a_j, \overline{a_j}\}$. Maximally folding $\{a_j, \overline{a_j}\}$ yields A_2, as above. This cannot be the final fold in the decomposition since A_1 is not homeomorphic to A_2. By Lemma 4.1, the illegal turn must be at w. The fold of Figure 3 cannot be performed, as our fold was maximal. If the fold of Figure 4 were performed, there would be backtracking on a.

Now suppose, without loss of generality, that the first Stallings fold that is not a proper full fold is a partial fold of b' and c', as in the following figure.

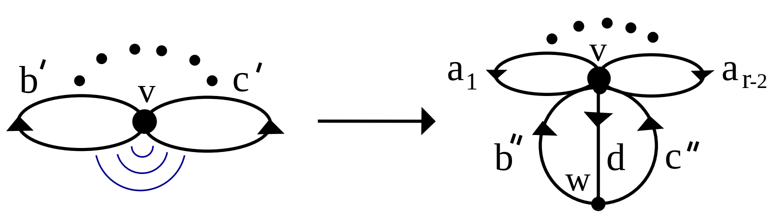


FIGURE 5. d is the edge created by folding initial segments of b' and c', b'' is the terminal segment of b' not folded, and c'' is the terminal segment of c' not folded.

As in the case of $\Gamma = A_3$ above, the next fold has to be at w or the next generator would be a homeomorphism, contradicting that the image of h is a rose, while A_3 is not a rose. Since the previous fold was maximal, the next fold cannot be of $\{b'', c''\}$. Also, $\{b'', \bar{d}\}$ and $\{c'', \bar{d}\}$ cannot be illegal turns or h would have had edge backtracking. Thus, h_i was not possible in the first place, meaning that all folds in the Stallings decomposition must be proper full folds between roses, proving (4).

Since all Stallings folds are proper full folds of roses, for each $1 \leq k \leq n-1$, one can index $\mathcal{E}_k = E(\Gamma_k)$

$$\{E_{(k,1)}, \overline{E_{(k,1)}}, E_{(k,2)}, \overline{E_{(k,2)}}, \dots, E_{(k,r)}, \overline{E_{(k,r)}}\} \\ = \{e_{(k,1)}, e_{(k,2)}, \dots, e_{(k,2r-1)}, e_{(k,2r)}\}$$

so that

(a) $g_k \colon e_{k-1,j_k} \mapsto e_{k,i_k} e_{k,j_k}$ where $e_{k-1,j_k} \in \mathcal{E}_{k-1}$, $e_{k,i_k}, e_{k,j_k} \in \mathcal{E}_k$ and
(b) $g_k(e_{k-1,i}) = e_{k,i}$ for all $e_{k-1,i} \neq e^{\pm 1}_{k-1,j_k}$.

Suppose we similarly index the directions $D_0(e_{k,i}) = d_{k,i}$.

Let $g_n = h'$ be the Stallings decomposition's homeomorphism and suppose its edge index permutation were nontrivial. Some power p of the permutation would be trivial. Replace h by h^p, rewriting h^p's decomposition as follows. Let σ be the permutation defined by $h'(e_{n-1,i}) = e_{n-1,\sigma(i)}$ for each i. For $n \leq k \leq 2n-p$, define g_k by $g_k : e_{k-1,\sigma^{-s+1}(j_t)} \mapsto e_{k,\sigma^{-s+1}(i_t)} e_{k,\sigma^{-s+1}(j_t)}$ where $k = sp+t$ and $0 \leq t \leq p$. Adjust the corresponding proper full folds accordingly. This decomposition still gives h^p, but now the homeomorphism's edge index permutation is trivial, making it unnecessary for the decomposition. $\square$

tt maps with a decomposition satisfying (I)–(II) of Proposition 4.3 will be called *ideally decomposable* ($\mathcal{ID}$) with an *ideal decomposition.* Note that (IIc) implies that g (hence ϕ) is rotationless, provided that g is pNp-free.

Standard Notation/Terminology 4.4 (Ideal decompositions)**.** We will consider the notation of the proposition standard for an ideal decomposition. Additionally:

(1) We denote e_{k-1,j_k} by e^{pu}_{k-1}, denote e_{k,j_k} by e^u_k, denote e_{k,i_k} by e^a_k, and denote $e_{k-1,i_{k-1}}$ by e^{pa}_{k-1}.
(2) $\mathcal{D}_k$ will denote the set of directions corresponding to $\mathcal{E}_k$.
(3) $f_k := g_k \circ \cdots \circ g_1 \circ g_n \circ \cdots \circ g_{k+1} : \Gamma_k \to \Gamma_k$.
(4)
$$g_{k,i} := \begin{cases} g_k \circ \cdots \circ g_i \colon \Gamma_{i-1} \to \Gamma_k & \text{if } k > i, \\ g_k \circ \cdots \circ g_1 \circ g_n \circ \cdots \circ g_i & \text{if } k < i. \end{cases}$$
(5) d^u_k will denote $D_0(e^u_k)$, sometimes called the *unachieved direction* for g_k, as it is not in $Im(Dg_k)$.
(6) d^a_k will denote $D_0(e^a_k)$, sometimes called the *twice-achieved direction* for g_k, as it is the image of both d^{pu}_{k-1} ($= D_0(e_{k-1,j_k})$) and d^{pa}_{k-1}

($= D_0(e_{k-1,i_k})$) under Dg_k. d^{pu}_{k-1} will sometimes be called the *pre-unachieved direction* for g_k and d^{pa}_{k-1} the *pre-twice-achieved direction* for g_k.

(7) G_k will denote the ltt structure $G(f_k)$

(8) $G_{k,l}$ will denote the subgraph of G_l containing
   - all black edges and vertices (given the same colors and labels as in G_l) and
   - all colored edges representing turns in $g_{k,l}(e)$ for some $e \in \mathcal{E}_{k-1}$.

(9) For any k, l, we have a direction map $Dg_{k,l}$ and an induced map of turns $Dg^t_{k,l}$. The *induced map of ltt structures* $Dg^T_{k,l} : G_{l-1} \mapsto G_k$ (which we show below exists) is such that
   - the vertex corresponding to a direction d is mapped to the vertex corresponding to $Dg_{k,l}(d)$,
   - the colored edge $[d_1, d_2]$ is mapped linearly as an extension of the vertex map to the edge $[Dg^t_{k,l}(\{d_1, d_2\})] = [Dg_{k,l}(d_1), Dg_{k,l}(d_2)]$, and
   - the interior of the black edge of G_{l-1} corresponding to the edge $E \in E(\Gamma_{l-1})$ is mapped to the interior of the smooth path in G_k corresponding to $g(E)$.

Example 4.5. We describe an induced map of rose-based ltt structures for $g_2 : x \mapsto xz$:

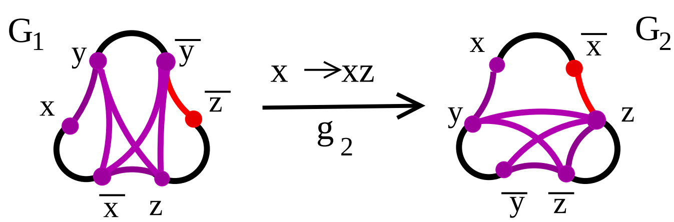


FIGURE 6. The induced map for $g_2 : x \mapsto xz$ sends vertex $\bar{x}$ of G_1 to vertex $\bar{z}$ of G_2 and every other vertex of G_1 to the identically labeled vertex of G_2. $[y]$ in G_1 maps to $[y]$ in G_2, $[z]$ in G_1 maps to $[z]$ in G_2, and $[x]$ in G_1 maps to $[x] \cup [\bar{x}, z] \cup [z]$ in G_2. The purple edge $[\bar{x}, y]$ in G_1 maps to the purple edge $[\bar{z}, y]$ in G_2, the purple edge $[\bar{x}, \bar{y}]$ in G_1 maps to the purple edge $[\bar{z}, \bar{y}]$ in G_2, $[\bar{x}, z]$ in G_1 maps to the purple edge $[\bar{z}, z]$ in G_2, and each other purple edge in G_1 is sent to the identically labeled purple edge in G_2. The red edge $[\bar{z}, \bar{y}]$ in G_1 maps to the purple edge $[\bar{z}, \bar{y}]$ in G_2.

. We return to Standard Notation/Terminology 4.4:

(10) $\mathcal{C}(G_k)$ will denote the subgraph of G_k, coming from $\mathcal{LW}(f_k)$ and containing all colored (red and purple) edges of G_k.

(11) Sometimes we use $\mathcal{PI}(G_k)$ to denote the purple subgraph of G_k coming from $\mathcal{SW}(f_k)$.
(12) $Dg^C_{k,l}$ will denote the restriction (which we show below exists) to $\mathcal{C}(G_{l-1})$ of $Dg^T_{k,l}$.
(13) If we additionally require $\phi \in \mathcal{AF}_r$ and $\mathcal{IW}(\phi) \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$, then we will say g is *potentially* $(r;(\frac{3}{2}-r))$ or *has* $(r;(\frac{3}{2}-r))$ *potential.* (By saying g has $(r;(\frac{3}{2}-r))$ potential, it will be implicit that, not only is $\phi \in \mathcal{AF}_r$, but ϕ is ideally decomposed, or at least $\mathcal{ID}$). In particular, we are assuming that ϕ is rotationless.

Remark 4.6. For typographical clarity, we sometimes put parantheses around subscripts. We refer to $E_{k,i}$ as E_i, and Γ_k as Γ, for all k when k is clear.

5. Admissible map properties

We prove that the ideal decomposition of a potentially $(r;(\frac{3}{2}-r))$ representative satisfies "Admissible Map Properties" listed in Proposition 5.1. In Section 7 we use the properties to show there are only two possible (fold/peel) relationship types between adjacent ltt structures in an ideal decomposition. Using this, in Section 8, we define the "ideal decomposition diagram" for $\mathcal{G} \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$.

The statement of Proposition 5.1 comes at the start of this section, while its proof comes after a sequence of technical lemmas used in the proof.

Unless otherwise stated, $g : \Gamma \to \Gamma$ will represent a rotationless $\phi \in Out(F_r)$ such that $\mathcal{IW}(\phi)$ is a connected $(2r-1)$-vertex graph (in other words, g will have $(r;(\frac{3}{2}-r))$ potential). Further, g will be ideally decomposed as:

$$\Gamma = \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n = \Gamma.$$

We use the "Standard 4.4 Notation".

Proposition 5.1. *Suppose $g : \Gamma \to \Gamma$ represents a rotationless $\phi \in Out(F_r)$ such that $\mathcal{IW}(\phi)$ is a connected $(2r-1)$-vertex graph (in other words, g has $(r;(\frac{3}{2}-r))$ potential) and is ideally decomposed as*

$$\Gamma = \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n = \Gamma.$$

Then g satisfies each of the following.

$\mathcal{AM}$I *Each G_j is birecurrent.*
$\mathcal{AM}$II *For each G_j, the illegal turn T_j for the generator g_{j+1} exiting G_j contains the unachieved direction d^u_j for the generator g_j entering G_j, i.e., either $d^u_j = d^{pa}_j$ or $d^u_j = d^{pu}_j$.*
$\mathcal{AM}$III *In each G_j, the vertex labeled d^u_j and edge $[t^R_j] = [d^u_j, \overline{d^a_j}]$ are both red.*
$\mathcal{AM}$IV *If $[d_{(j,i)}, d_{(j,l)}]$ is in $\mathcal{C}(G_j)$, then $D^C g_{m,j+1}([d_{(j,i)}, d_{(j,l)}])$ is a purple edge in G_m, for each $m \neq j$.*

$\mathcal{AM}$V *For each* j, $[t_j^R] = [d_j^u, \overline{d_j^a}]$ *is the unique edge containing* d_j^u.

$\mathcal{AM}$VI *Each* g_j *is defined by* $g_j : e_{j-1}^{pu} \mapsto e_j^a e_j^u$, *where*

$$D_0(e_j^u) = d_j^u, \quad D_0(\overline{e_j^a}) = \overline{d_j^a}, \quad e_j^u = e_{j,m}, \quad e_{j-1}^{pu} = e_{j-1,m}.$$

$\mathcal{AM}$VII $Dg_{l,j+1}$ *induces an isomorphism from* $\mathcal{SW}(f_j)$ *onto* $\mathcal{SW}(f_l)$ *for all* $j \neq l$.

$\mathcal{AM}$VIII *For each* $1 \leq j \leq r$:

(a) *There exists a* k *such that either* $e_k^u = E_{k,j}$ *or* $e_k^u = \overline{E_{k,j}}$.

(b) *There exists a* k *such that either* $e_k^a = E_{k,j}$ *or* $e_k^a = \overline{E_{k,j}}$.

The proof of Proposition 5.1 will come at the end of this section.

Definition 5.2. An edge path $\gamma = e_1 \dots e_k$ in Γ has *cancellation* if $\overline{e_i} = e_{i+1}$ for some $1 \leq i \leq k-1$. We say g has *no cancellation on edges* if for no $l > 0$ and edge $e \in E(\Gamma)$ does $g^l(e)$ have cancellation.

Lemma 5.3. *For this lemma we index the generators in the decomposition of all powers* g^p *of* g *so that*

$$g^p = g_{pn} \circ g_{pn-1} \circ \dots \circ g_{(p-1)n} \circ \dots \circ g_{(p-2)n} \circ \dots \circ g_{n+1} \circ g_n \circ \dots \circ g_1$$

($g_{mn+i} = g_i$, *but we want to use the indices to keep track of a generator's place in the decomposition of* g^p*). With this notation,* $g_{k,l}$ *will mean* $g_k \circ \dots \circ g_l$. *Then:*

(1) *For each* $e \in E(\Gamma_{l-1})$, *no* $g_{k,l}(e)$ *has cancellation.*
(2) *For each* $0 \leq l \leq k$ *and* $E_{l-1,i} \in E^+(\Gamma_{l-1})$, *the edge* $E_{k,i}$ *is in the path* $g_{k,l}(E_{l-1,i})$.
(3) *If* $e_k^u = e_{k,j}$, *then the turn* $\{\overline{d_k^a}, d_k^u\}$ *is in the edge path* $g_{k,l}(e_{l-1,j})$, *for all* $0 \leq l \leq k$.

Proof. Let s be minimal so that some $g_{s,t}(e_{t-1,j})$ has cancellation. Before continuing with our proof of (1), we first proceed by induction on $k-l$ to show that (2) holds for $k < s$. For the base case observe that $g_{l+1}(e_{l,j}) = e_{l+1,j}$ for all $e_{l+1,j} \neq (e_l^{pu})^{\pm 1}$. Thus, if $e_{l,j} \neq e_l^{pu}$ and $e_{l,j} \neq \overline{e_l^{pu}}$ then $g_{l+1}(e_{l,j})$ is precisely the path $e_{l+1,j}$ and so we are only left for the base case to consider when $e_{l,j} = (e_l^{pu})^{\pm 1}$. If $e_{l,j} = e_l^{pu}$, then $g_{l+1}(e_{l,j}) = e_{l+1}^a e_{l+1,j}$ and so the edge path $g_{l+1}(e_{l,j})$ contains $e_{l+1,j}$, as desired. If $e_{l,j} = \overline{e_l^{pu}}$, then $g_{l+1}(e_{l,j}) = e_{l+1,j}\overline{e_{l+1}^a}$ and so the edge path $g_{l+1}(e_{l,j})$ also contains $e_{l+1,j}$ in this case. Having considered all possibilities, the base case is proved.

For the inductive step, we assume $g_{k-1,l+1}(e_{l,j})$ contains $e_{k-1,j}$ and show $e_{k,j}$ is in the path $g_{k,l+1}(e_{l,j})$. Let

$$g_{k-1,l+1}(e_{l,j}) = e_{i_1} \dots e_{i_{q-1}} e_{k-1,j} e_{i_{q+1}} \dots e_{i_r}$$

for some edges $e_i \in \mathcal{E}_{k-1}$. As in the base case, for all $e_{k-1,j} \neq (e_k^u)^{\pm 1}$, $g_k(e_{k-1,j})$ is precisely the path $e_{k,j}$. Thus (since g_k is an automorphism and since there is no cancellation in $g_{j_1,j_2}(e_{j_1,j_2})$ for $1 \leq j_1 \leq j_2 \leq k$), $g_{k,l+1}(e_{l,j}) = \gamma_1 \dots \gamma_{q-1}(e_{k,j})\gamma_{q+1} \dots \gamma_m$ where each $\gamma_{i_j} = g_l(e_{i_j})$ and where no $\{\overline{\gamma_i}, \gamma_{i+1}\}$, $\{\overline{e_{k,j}}, \gamma_{q+1}\}$, or $\{\overline{\gamma_{q-1}}, e_{k,j}\}$ is an illegal turn. So each $e_{k,j}$ is in $g_{k,l+1}(e_{l,j})$. We are only left to consider for the inductive step the cases $e_{k-1,j} = e_k^{pu}$ and $e_{k-1,j} = \overline{e_k^{pu}}$.

If $e_{k-1,j} = e_k^{pu}$, then $g_k(e_{k-1,j}) = e_k^a e_{k,j}$, and so

$$g_{k,l+1}(e_{l,j}) = \gamma_1 \dots \gamma_{q-1} e_k^a e_{k,j} \gamma_{q+1} \dots \gamma_m$$

(where no $\{\overline{\gamma_i}, \gamma_{i+1}\}$, $\{\overline{e_{k,j}}, \gamma_{q+1}\}$, or $\{\overline{\gamma_{q-1}}, e_k^a\}$ is an illegal turn), which contains $e_{k,j}$, as desired. If instead $e_{k-1,j} = \overline{e_k^{pu}}$, then $g_k(e_{k-1,j}) = e_{k,j}\overline{e_k^a}$ and so $g_{k,l+1}(e_{l,j}) = \gamma_1 \dots \gamma_{q-1} e_{k,j} \overline{e_k^a} \gamma_{q+1} \dots \gamma_m$, which also contains $e_{k,j}$. Having considered all possibilities, the inductive step is now also proven and the proof is complete for (2) in the case of $k < s$.

We finish the proof of (1). s is still minimal. So $g_{s,t}(e_{t-1,j})$ has cancellation for some $e_{t-1,j} \in \mathcal{E}_j$. Suppose $g_{s,t}(e_{t-1,j})$ has cancellation. For $1 \le j \le m$, let $\alpha_j \in \mathcal{E}_{s-1}$ be such that $g_{s-1,t}(e_{t-1,j}) = \alpha_1 \cdots \alpha_m$. By s's minimality, either $g_s(\alpha_i)$ has cancellation for some $1 \le i \le m$ or $Dg_s(\overline{\alpha_i}) = Dg_s(\alpha_{i+1})$ for some $1 \le i < m$. Since each g_s is a generator, no $g_s(\alpha_i)$ has cancellation. So, for some i, $Dg_s(\overline{\alpha_i}) = Dg_s(\alpha_{i+1})$. As we have proved (1) for all $k < s$, we know $g_{t-1,1}(e_{0,j})$ contains $e_{t-1,j}$. So $g_{s,1}(e_{0,j}) = g_{s,t}(g_{t-1,1}(e_{0,j}))$ contains cancellation, implying $g^p(e_{0,j}) = g_{pn,s+1}(g_{s,1}(e_{0,j})) = g_{s,t}(\dots e_{t-1,j} \dots)$ for some p (with $pn > s+1$) contains cancellation, contradicting that g is a train track map.

We now prove (3). Let $e_k^u = e_{k,l}$. By (2) we know that the edge path $g_{k-1,l}(e_{l-1,j})$ contains $e_{k-1,j}$. Let $e_1, \dots e_m \in \mathcal{E}_{k-1}$ be such that

$$g_{k-1,l}(e_{l-1,j}) = e_1 \dots e_{q-1} e_{k-1,j} e_{q+1} \dots e_m.$$

Then $g_{k,l}(e_{l-1,j}) = \gamma_1 \dots \gamma_{q-1} e_k^a e_k^u \gamma_{q+1} \dots \gamma_r$ where $\gamma_j = g_k(e_j)$ for all j. Thus $g_{k,l}(e_{k-1}^{pu})$ contains $\{\overline{d_k^a}, d_k^u\}$, as desired. □

Lemma 5.4 (Properties of $f_k = g_k \circ g_{k-1} \circ \cdots \circ g_{k+2} \circ g_{k+1} \colon \Gamma_k \to \Gamma_k$).

(a) *Each f_k represents the same ϕ. In particular, if g has $(r; (\frac{3}{2} - r))$ potential, then so does each f_k.*

(b) *If g is rotationless, then each f_k is rotationless (and all periodic directions are fixed). In particular, if g is pNp-free, then each f_k is rotationless.*

(c) *Each f_k has $2r-1$ gates (and thus fixed directions).*

(d) *For each k, $d_k^u \notin \mathcal{IM}(Df_k)$. Thus, d_k^u is the unique nonperiodic direction for Df_k.*

(e) *If*

$$\Gamma = \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n = \Gamma$$

is an ideal decomposition of g, then

$$\Gamma_k \xrightarrow{g_{k+1}} \Gamma_{k+1} \xrightarrow{g_{k+2}} \cdots \xrightarrow{g_{k-1}} \Gamma_{k-1} \xrightarrow{g_k} \Gamma_k$$

is an ideal decomposition of f_k.

Proof. Lemma 4.1 implies (a). If g is rotationless, then each f_k is rotationless, as it represents a rotationless ϕ. If g has no pNp's, then Proposition 4.3(IIc), held by ideal decompositions, implies that g is rotationless. This gives (b).

We prove (c). The number of gates is the number of periodic directions, which here would be the number of fixed directions. Suppose, for the sake of contradiction, that f_k had more gates than f_l. Let p_k be such that $D(f_k^{p_k})$ maps each gate of f_k to a single direction and let p_l be such that $D(f_l^{p_l})$ maps each gate of f_l to a single direction. Let $\{\mathcal{G}_1, \dots, \mathcal{G}_s\}$ be the set of gates for f_k, let α_i be the periodic direction of $\mathcal{G}_i$ for each $1 \le i \le s$, let $\{\mathcal{G}'_1, \dots, \mathcal{G}'_{s'}\}$ be the set of gates for f_l, and let α'_i be the periodic direction of $\mathcal{G}'_i$ for each $1 \le i \le s'$. Consider $f_k^{p_k+p_l+1} = f_{k,l+1} \circ f_l^{p_l} \circ f_{l,k+1} \circ f_k^{p_k}$. Let $\{d_1, \dots, d_t\} = D(f_{l,k+1} \circ f_k^{p_k})(\mathcal{D}_k)$. Then $\{d_1, \dots, d_t\}$ is mapped by $D(f_l^{p_l})$ into $\{\alpha'_1 \dots \alpha'_{s'}\}$ and, consequently, $D(f_l^{p_l} \circ f_{l,k+1} \circ f_k^{p_k})(\mathcal{D}_k) \subset \{\alpha'_1 \dots \alpha'_{s'}\}$. This implies that

$$\begin{aligned} D(f_{k,l+1})(D(f_l^{p_l} \circ f_{l,k+1} \circ f_k^{p_k})(\mathcal{D}_k)) &= D(f_k^{p_k+p_l+1})(\mathcal{D}_k) \\ &\subset D(f_{k,l+1})(\{\alpha'_1 \dots \alpha'_{s'}\}), \end{aligned}$$

which has at most s' elements. But this contradicts f_k having more gates that f_l. Thus, all f_k have the same number of gates.

We prove (d). Since f_k is rotationless, all periodic directions are fixed. By (c), Df_k has 2r-1 fixed directions. Since $d_k^u \notin \mathcal{IM}(Dg_k)$, it cannot be in $\mathcal{IM}(Df_k)$, so is the unique nonfixed direction. We prove (e). Ideal decomposition properties (I)–(IIb) hold for f_k's decomposition, as they hold for g's decomposition and the decompositions have the same Γ_i and g_i (renumbered). (IIc) holds for f_k's decomposition by (d). $\square$

We add to the notation already established: $t_k^R = \{\overline{d_k^a}, d_k^u\}$, $e_k^R = [t_k^R]$, *and* $T_k = \{d_k^{pa}, d_k^{pu}\}$.

Lemma 5.5. *The following hold for each* $T_k = \{d_k^{pa}, d_k^{pu}\}$.

(a) T_k *is an illegal turn for* g_{k+1} *and, thus, also for* f_k.

(b) *For each* k, T_k *contains* d_k^u.

Proof. Recall that $T_k = \{d_k^{pa}, d_k^{pu}\}$. Since

$$D^t g_{k+1}(\{d_k^{pa}, d_k^{pu}\}) = \{Dg_{k+1}(d_k^{pa}), Dg_k(d_k^{pu})\} = \{d_{k+1}^a, d_{k+1}^a\},$$

we have

$$\begin{aligned} D^t f_k(\{d_k^{pa}, d_k^{pu}\}) &= D^t(g_{k,k+2} \circ g_{k+1})(\{d_k^{pa}, d_k^{pu}\}) \\ &= D^t(g_{k,k+2})(D^t g_{k+1}(\{d_k^{pa}, d_k^{pu}\})) \\ &= D^t g_{k,k+2}(\{d_{k+1}^a, d_{k+1}^a\}) \\ &= \{D^t g_{k,k+2}(d_{k+1}^a), D^t g_{k,k+2}(d_{k+1}^a)\}, \end{aligned}$$

which is degenerate. So T_k is an illegal turn for f_k, proving (a).

For (b) suppose g has $2r-1$ periodic directions and, for contradiction's sake, the illegal turn T_k does not contain $d^u_k = d_{k,i}$. Let $d^u_{k+1} = d_{k+1,s}$ and $d^a_{k+1} = d_{k+1,t}$. Then $Dg_k(d_{k-1,s}) = d_{k,s}$ and $Dg_k(d_{k-1,t}) = d_{k,t}$, so

$$\begin{aligned} D^t(g_{k+1} \circ g_k)(\{d_{(k-1,s)}, d_{(k-1,t)}\}) \\ &= \{D(g_{k+1} \circ g_k)(d_{(k-1,s)}), D(g_{k+1} \circ g_k)(d_{(k-1,t)})\} \\ &= \{Dg_{k+1}(d_{k,s} = d^{pu}_k), Dg_{k+1}(d_{k,t} = d^{pa}_k)\} \\ &= \{d^a_{k+1}, d^a_{k+1}\}. \end{aligned}$$

So $d_{k-1,s}$ and $d_{k-1,t}$ share a gate. But $d_{k-1,i}$ already shares a gate with another element and we already established that $d_{k-1,i} \neq d_{k-1,s}$ and $d_{k-1,i} \neq d_{k-1,t}$. So f_{k-1} has at most $2r-2$ gates. Since each f_k has the same number of gates, this implies g has at most $2r-2$ gates, giving a contradiction. (b) is proved. □

Corollary 5.6. *For each* $1 \leq k \leq n$*:*

(a) $t^R_k = \{\overline{d^a_k}, d^u_k\}$*, must contain either* d^{pu}_k *or* d^{pa}_k*.*

(b) *The vertex labeled* d^u_k *in* G_k *is red and* $[t^R_k] = [\overline{d^a_k}, d^u_k]$ *is a red edge in* G_k*.*

Proof. We start with (a). Lemma 5.5 implies each T_k contains d^u_k. At the same time, we know $t^R_k = \{\overline{d^a_k}, d^u_k\}$, implying t^R_k contains d^u_k, thus either d^{pa}_k or d^{pu}_k. We now prove (b). By Lemma 5.4(d), d^u_k is not a periodic direction for Df_k, so is not a vertex of $\mathcal{SW}(f_k)$. Thus, d^u_k labels a red vertex in G_k. To show $[t^R_k]$ is in $\mathcal{LW}(f_k)$ it suffices to show t^R_k is in $f_k(e^u_k)$. Let $e^u_k = e_{k,l}$. By Lemma 5.3, the path $g_{k-1,k+1}(e^u_k = e_{k,l})$ contains $e_{k-1,l}$. Let $e_j \in \mathcal{E}_{l-1}$ be such that $g_{k-1,k+1}(e^u_k) = e_1 \dots e_{q-1} e_{k-1,l} e_{q+1} \dots e_m$. Then $f_k(e^u_k) = g_{k,k+1}(e^u_k) = \gamma_1 \dots \gamma_{q-1} e^a_k e^u_k \gamma_{q+1} \dots \gamma_m$ where $\gamma_j = g_k(e_{i_j})$ for all j. So $f_k(e^u_k)$ contains $\{\bar{d}^a_k, d^u_k\}$ and $\mathcal{LW}(f_k)$ contains $[t^R_k]$. Since $[\overline{d^a_k}, d^u_k]$ contains the red vertex d^u_k, it is red in G_k. □

Lemma 5.7. *If* $[d_{(l,i)}, d_{(l,j)}]$ *is in* $\mathcal{C}(G_l)$*, then* $[D^t g_{k,l+1}(\{d_{(l,i)}, d_{(l,j)}\})]$ *is a purple edge in* G_k*.*

Proof. It suffices to show two things:

(1) $D^t g_{k,l+1}(\{d_{(l,i)}, d_{(l,j)}\})$ is a turn in some edge path $f^p_l(e_{l,m})$ with $p \geq 1$.

(2) $Dg_{k,l+1}(d_{l,i})$ and $Dg_{k,l+1}(d_{l,j})$ are periodic directions for f_l.

We use induction. Start with (1). For the base case assume $[d_{(k-1,i)}, d_{(k-1,j)}]$ is in $\mathcal{C}(G_{k-1})$, so

$$f^p_{k-1}(e_{k-1,t}) = s_1 \dots \overline{e_{(k-1,i)}} e_{(k-1,j)} \dots s_m \tag{5.1}$$

for some $e_{(k-1,t)}, s_1, \dots s_m \in \mathcal{E}_{k-1}$ and $p \geq 1$. By Lemma 5.3, $e_{k-1,t}$ is in the path $g_{k-1} \circ \dots \circ g_1 \circ g_n \circ \dots \circ g_{k+1}(e_{k,t})$. Thus, by (5.1), since no $g_{i,j}(e_{j-1,t})$ can have cancellation, $s_1 \dots \overline{e_{(k-1,i)}} e_{(k-1,j)} \dots s_m$ is a subpath of

$$f^p_{k-1} \circ g_{k-1} \circ \dots \circ g_1 \circ g_n \circ \dots \circ g_{k+1}(e_{k,t}).$$

Apply g_k to $f_{k-1}^p \circ g_{k-1} \circ \cdots \circ g_1 \circ g_n \circ \cdots \circ g_k(e_{k-1,t})$ to get $f_k^{p+1}(e_{k,t})$.

Suppose $Dg_k(e_{k-1,i}) = e_{k,i}$ and $Dg_k(e_{k-1,j}) = e_{k,j}$. Then

$$g_k(\dots \overline{e_{k-1,i}} e_{k-1,j} \dots) = \dots \overline{e_{(k,i)}} e_{(k,j)} \dots,$$

with possibly different edges before and after $\overline{e_{k,i}}$ and $e_{k,j}$ than before and after $\overline{e_{k-1,i}}$ and $e_{k-1,j}$. Thus, here, $f_k^{p+1}(\dots \overline{e_{(k-1,i)}} e_{(k-1,j)} \dots)$ contains $\{d_{(k,i)}, d_{(k,j)}\}$, which here is $D^t g_k(\{d_{(k-1,i)}, d_{(k-1,j)}\})$. So

$$[D^t g_k(\{d_{(k-1,i)}, d_{(k-1,j)}\})]$$

is an edge in G_k.

Suppose $g_k \colon e_{k-1,j} \mapsto e_{k,l} e_{k,j}$. Then

$$g_k(\dots \overline{e_{k-1,i}} e_{k-1,j} \dots) = \dots \overline{e_{k,i}} e_{k,l} e_{k,j} \dots,$$

(again with possibly different edges before and after $\overline{e_{k,i}}$ and $e_{k,j}$). So $g_k(\dots \overline{e_{(k-1,i)}} e_{(k-1,j)} \dots)$ contains $\{\overline{d_{(k,l)}}, d_{(k,j)}\}$, which here is

$$D^t g_k(\{d_{(k-1,i)}, d_{(k-1,j)}\}),$$

so $[D^t g_k(\{d_{(k-1,i)}, d_{(k-1,j)}\})]$ again is in G_k.

Finally, suppose g_k is defined by $e_{k-1,j} \mapsto e_{k,j} e_{k,l}$. Unless $\overline{e_{k-1,i}} = e_{(k-1,j)}$, we have

$$g_k(\dots \overline{e_{(k-1,i)}} e_{(k-1,j)} \dots) = \dots \overline{e_{(k,i)}} e_{(k,j)} e_{(k,l)} \dots,$$

containing $\{d_{(k,i)}, d_{(k,j)}\} = D^t g_k(\{d_{(k-1,i)}, d_{(k-1,j)}\})$. So

$$[D^t g_k(\{d_{(k-1,i)}, d_{(k-1,j)}\})]$$

is an edge in G_k here also.

If $\overline{e_{k-1,i}} = e_{k-1,j}$, we are in a reflection of the previous case. The other cases ($g_k : \overline{e_{k-1,i}} \mapsto \overline{e_{k,i}} e_{k,l}$ and $g_k : \overline{e_{k-1,i}} \mapsto e_{k,l} \overline{e_{k,i}}$) follow similarly by symmetry. The base case for (1) is complete.

We prove the base case of (2). Since

$$[D^t g_k(\{d_{(k-1,i)}, d_{(k-1,j)}\})] = [Dg_k(d_{(k-1,i)}), Dg_k(d_{(k-1,j)})],$$

both vertex labels of $[D^t g_k(\{d_{(k-1,i)}, d_{(k-1,j)}\})]$ are in $\mathcal{IM}(Dg_k)$. By Lemma 5.4(d), both vertices are periodic. So $[D^t g_k(\{d_{(k-1,i)}, d_{(k-1,j)}\})]$ is in $\mathcal{PI}(G_k)$. The base case is proved. Suppose inductively $[d_{(l,i)}, d_{(l,j)}]$ is an edge in $\mathcal{C}(G_l)$ and $[D^t g_{k-1,l+1}(\{d_{(l,i)}, d_{(l,j)}\})]$ is an edge in $\mathcal{PI}(G_{k-1})$. The base case implies $[D^t g_k(D^t g_{k-1,l+1}(\{d_{(l,i)}, d_{(l,j)}\})]$ is an edge in $\mathcal{PI}(G_k)$. But $D^t g_k(D^t g_{k-1,l+1}(\{d_{(l,i)}, d_{(l,j)}\})) = D^t g_{k,l+1}(\{d_{(l,i)}, d_{(l,j)}\})$. The lemma is proved. □

Lemma 5.8 (Properties of t_k^R and e_k^R)**.** *For each $1 \le l, k \le n$:*

(a) $[D^t g_{l,k}(\{\overline{d_{k-1}^a}, d_{k-1}^u\})]$ *is a purple edge in* G_l.

(b) $[\overline{d_k^a}, d_k^u]$ *is not in* $D^C g_k(G_{k-1})$.

Proof. By Lemma 5.7, it suffices to show for (a) that $[\overline{d^a_{k-1}}, d^u_{k-1}]$ is a colored edge of G_{k-1}. This was shown in Corollary 5.6(b). By Lemma 5.7, each edge in $\mathcal{C}(G_{k-1})$ is mapped to a purple edge in G_k. On the other hand, $[\overline{d^a_k}, d^u_k]$ is a red edge in G_k. Thus, $[\overline{d^a_k}, d^u_k]$ is not in $D^C g_k(G_{k-1})$ and (b) is proved. $\square$

Each G_k has a unique red edge ($e^R_k = [t^R_k] = [\overline{d^a_k}, d^u_k]$):

Lemma 5.9. *$\mathcal{C}(G_k)$ can have at most 1 edge segment connecting the nonperiodic direction red vertex d^u_k to the set of purple periodic direction vertices.*

Proof. First note that the nonperiodic direction d^u_k labels the red vertex in G_k. If $g_k(e_{k-1,i}) = e_{k,i}e_{k,j}$, then the red vertex in G_k is $\overline{d_{k,i}}$ (where $d_{k,i} = D_0(e_{k,i})$ and $d_{k,j} = D_0(e_{k,j})$). The vertex $\overline{d_{k,i}}$ will be adjoined to the vertex for $d_{k,j}$ and only $d_{k,j}$: each occurrence of $e_{k-1,i}$ in the image under $g_{k-1,1}$ of any edge has been replaced by $e_{k,i}e_{k,j}$ and every occurrence of $\overline{e_{k,i}}$ has been replaced by $\overline{e_{k,i}e_{k,j}}$, i.e., there are no copies of $e_{k,j}$ without $e_{k,i}$ following them and no copies of $\overline{e_{k,i}}$ without $\overline{e_{k,j}}$ preceding them. $\square$

The red edge and vertex of G_k determine g_k:

Lemma 5.10. *Suppose that the unique red edge in G_k is $[t^R_k] = [d_{(k,j)}, \overline{d_{(k,i)}}]$ and that the vertex representing $d_{k,j}$ is red. Then $g_k(e_{k-1,j}) = e_{k,i}e_{k,j}$ and $g_k(e_{k-1,t}) = e_{k,t}$ for $e_{k-1,t} \neq (e_{k-1,j})^{\pm 1}$, where $D_0(e_{s,t}) = d_{s,t}$ and $D_0(\overline{e_{s,t}}) = \overline{d_{s,t}}$ for all s, t.*

Proof. By the ideal decomposition definition, g_k is defined by $e_{k-1,j} \mapsto e_{k,i}e_{k,j}$. Corollary 5.6 implies $D_0(e_{k,j}) = d^u_k$, i.e., the direction associated to the red vertex of G_k. So the second index of d^u_k uniquely determines the index j, so $e_{k-1,j} = e^{pu}_{k-1}$ and $e_{k,i} = e^a_k$. Additionally, the proof of Corollary 5.6 implies $[\overline{d_{(k,i)}}, d_{(k,j)}]$ is the red edge of G_k. So $e_{k,i} = e^a_k$. And g_k must be defined by $e^{pu}_{k-1} \mapsto e^a_k e^u_k$, i.e $e_{k-1,j} \mapsto e_{k,i}e_{k,j}$. $\square$

Lemma 5.11 (Induced maps of ltt structures)**.**

(a) *$D^C f_k$ maps $\mathcal{PI}(G_k)$ isomorphically onto itself via a label-preserving isomorphism.*

(b) *The set of purple edges of G_{k-1} is mapped by $D^C g_k$ injectively into the set of purple edges of G_k.*

(c) *For each $0 \leq l, k \leq n$, $Dg_{l,k+1}$ induces an isomorphism from $\mathcal{SW}(f_k)$ onto $\mathcal{SW}(f_l)$.*

Proof. (a) Lemma 5.7 implies that $D^C f_k$ maps $\mathcal{PI}(G_k)$ into itself. However, Df_k fixes all directions labeling vertices of $\mathcal{SW}(f_k) = \mathcal{PI}(G_k)$. Thus, $D^C f_k$ restricted to $\mathcal{PI}(G_k)$ is a label-preserving graph isomorphism onto its image.

(b) Since d^a_k is the only direction with more than one Dg_k preimage, and these two preimages are d^{pa}_{k-1} and d^{pu}_{k-1}, the $[d_{(k,i)}, d^a_k]$ are the only edges in G_k with more than one $D^C g_k$ preimage. The two preimages are the edges $[d_{(k-1,i)}, d^{pa}_{k-1}]$ and $[d_{(k-1,i)}, d^{pu}_{k-1}]$ in G_{k-1}. However, by Lemma 5.5, either

$e^u_{k-1} = e^{pu}_{k-1}$ or $e^u_{k-1} = e^{pa}_{k-1}$. So one of the preimages of d^a_k is actually d^u_{k-1}, i.e., one of the preimage edges is actually $[d_{(k-1,i)}, d^u_{k-1}]$. Since $[t^R_{k-1}]$ is the only edge of $\mathcal{C}(G_{k-1})$ containing d^u_{k-1}, one of the preimages of $[d_{(k,i)}, d^a_k]$ must be $[t^R_{k-1}]$, leaving only one possible purple preimage.

(c) By (b), the set of G_k's purple edges is mapped injectively by $D^C g_{l,k+1}$ into the set of G_l's purple edges. Likewise, the set of G_l's purple edges is mapped injectively by $D^C g_{k,l+1}$ into G_k. (a) implies

$$D^C f_k = (D^C g_{k,l+1}) \circ (D^C g_{l,k+1})$$

and $D^C f_l = (D^C g_{l,k+1}) \circ (D^C g_{k,l+1})$ are bijections. So, the map $D^C g_{l,k+1}$ induces on the set of G_k's purple edges is a bijection. It is only left to show that two purple edges share a vertex in G_k if and only if their $D^C g_{l,k+1}$ images share a vertex in G_l.

If $[x, d_1]$ and $[x, d_2]$ are in $\mathcal{PI}(G_k)$,

$$\begin{aligned} D^C g_{l,k+1}([x, d_1]) &= [Dg_{l,k+1}(x), Dg_{l,k+1}(d_1)] \quad \text{and} \\ D^t g_{l,k+1}([x, d_2]) &= [Dg_{l,k+1}(x), Dg_{l,k+1}(d_2)] \end{aligned}$$

share $Dg_{l,k+1}(x)$. On the other hand, if $[w, d_3]$ and $[w, d_4]$ in $\mathcal{PI}(G_l)$ share w, then

$$\begin{aligned} [D^t g_{k,l+1}(\{w, d_3\})] &= [Dg_{k,l+1}(w), Dg_{k,l+1}(d_3)] \quad \text{and} \\ [D^t g_{k,l+1}(\{w, d_4\})] &= [Dg_{k,l+1}(w), Dg_{k,l+1}(d_4)] \end{aligned}$$

share $Dg_{k,l+1}(w)$. Since $D^C f_l$ is an isomorphism on $\mathcal{PI}(G_l)$, $D^C g_{l,k+1}$ and $D^C g_{k,l+1}$ act as inverses. So the preimages of $[w, d_3]$ and $[w, d_4]$ under $D^C g_{l,k+1}$ share a vertex in G_l. □

Lemma 5.12 gives properties stemming from irreducibility (though not proving irreducibility):

Lemma 5.12. *For each $1 \leq j \leq r$:*

(a) *There exists a k such that either $e^u_k = E_{k,j}$ or $e^u_k = \overline{E_{k,j}}$.*
(b) *There exists a k such that either $e^a_k = E_{k,j}$ or $e^a_k = \overline{E_{k,j}}$.*

Proof. (a) For contradiction's sake suppose there is some j so that $e^u_k \neq E^{\pm 1}_{k,j}$ for all k. We inductively show $g(E_{0,j}) = E_{0,j}$, implying g's reducibility. Induction will be on the k in $g_{k-1,1}$.

For the base case, we need $g_1(E_{0,j}) = E_{1,j}$ if $e^u_1 \neq E^{\pm 1}_{1,j}$. g_1 is defined by $e^{pu}_0 \mapsto e^a_1 e^u_1$. Since $e^u_1 \neq E_{1,j}$ and $\overline{e^u_1} \neq \overline{E_{(1,j)}}$, we know $e^{pu}_0 \neq E^{\pm 1}_{(0,j)}$. Thus, $g_1(E_{0,j}) = E_{(1,j)}$, as desired. Now inductively suppose $g_{k-1,1}(E_{0,j}) = E_{k-1,j}$ and $e^u_k \neq E^{\pm 1}_{k,j}$. Then $e^{pu}_{k-1} \neq E^{\pm 1}_{k-1,j}$. Thus, since $e^{pu}_{k-1} \mapsto e^a_k e^u_k$ defines g_k, we know $g_k(E_{k-1,j}) = E_{k,j}$. So

$$g_{k,1}(E_{0,j}) = g_k(g_{k-1,1}(E_{0,j})) = g_k(E_{k-1,j}) = E_{k,j}.$$

Inductively, this proves $g(E_{0,j}) = E_{0,j}$, we have our contradiction, and (a) is proved.

(b) For contradiction's sake, suppose that, for some $1 \leq j \leq r$, $e_k^a \neq E_{k,j}$ and $e_k^a \neq \overline{E_{k,j}}$ for each k. The goal will be to inductively show that, for each $E_{0,i}$ with $E_{0,i} \neq E_{0,j}$ and $E_{0,i} \neq \overline{E_{(0,j)}}$, $g(E_{0,i})$ does not contain $E_{0,j}$ and does not contain $\overline{E_{0,j}}$ (contradicting irreducibility).

We prove the base case. g_1 is defined by $e_0^{pu} \mapsto e_1^a e_1^u$. First suppose $E_{0,j} = (e_0^{pu})^{\pm 1}$. Then $e_0^{pu} \neq E_{0,i}^{\pm 1}$ (since $E_{0,i} \neq E_{0,j}^{\pm 1}$). So $g_1(E_{0,i}) = E_{1,i}$, which does not contain $E_{1,j}^{\pm 1}$. Now suppose that $E_{0,j} \neq e_0^{pu}$ and $E_{0,j} \neq \overline{e_0^{pu}}$. Then $e_1^a e_1^u$ does not contain $E_{1,j}$ or $\overline{E_{1,j}}$ (since $e_k^a \neq (E_{k,j})^{\pm 1}$ by assumption). So $E_{1,j}^{\pm 1}$ are not in the image of $E_{0,i}$ if $E_{0,i} = e_0^{pu}$ (since the image of $E_{0,i}$ is then $e_1^a e_1^u$) and are not in the image of $\overline{E_{0,i}}$ (since the image is $\overline{e_1^u e_1^a}$) and are not in the image $E_{0,i}$ if $E_{0,i} \neq (e_0^{pu})^{\pm 1}$ (since the image is $E_{1,i}$ and $E_{1,i} \neq E_{1,j}^{\pm 1}$). The base case is proved.

Inductively suppose $g_{k-1,1}(E_{0,i})$ does not contain $E_{k-1,j}^{\pm 1}$. Similar analysis as above shows $g_k(E_{k-1,i})$ does not contain $E_{k,j}^{\pm 1}$ for any $E_{k,i} \neq E_{k,j}^{\pm 1}$. Since $g_{k-1,1}(E_{k-1,i})$ does not contain $E_{k-1,j}^{\pm 1}$, $g_{k-1,1}(E_{0,i}) = e_1 \ldots e_m$ with each $e_i \neq E_{k-1,j}^{\pm 1}$. Thus, no $g_k(e_i)$ contains $E_{k,j}^{\pm 1}$. So

$$g_{k,1}(E_{0,i}) = g_k(g_{k-1,1}(E_{0,i})) = g_k(e_1) \ldots g_k(e_m)$$

does not contain $E_{k,j}^{\pm 1}$. This completes the inductive step, thus (b). $\square$

Remark 5.13. Lemma 5.12 is necessary, but not sufficient, for g to be irreducible. For example, the composition of $a \mapsto ab$, $b \mapsto ba$, $c \mapsto cd$, and $d \mapsto dc$ satisfies Lemma 5.12, but is reducible.

Proof of Proposition 5.1. $\mathcal{AM}$I follows from Proposition 3.7 and Lemma 5.4, $\mathcal{AM}$II from Lemma 5.5, $\mathcal{AM}$III from Corollary 5.6, $\mathcal{AM}$IV from Lemma 5.7, $\mathcal{AM}$V from Lemma 5.9 and Corollary 5.6, $\mathcal{AM}$VI from Lemma 5.10, $\mathcal{AM}$VII from Lemma 5.11, and $\mathcal{AM}$VIII from Lemma 5.12. $\square$

6. Lamination train track (ltt) structures

In Subection 2.5 we defined ltt structures for ideally decomposed representatives with $(r; (\frac{3}{2} - r))$ potential. Both for defining $\mathcal{ID}$ diagrams and for applying the Birecurrency Condition, we need abstract definitions of ltt structures motivated by the $\mathcal{AM}$ properties of Section 5.

6.1. Abstract lamination train track structures.

Definition 6.1. (See Example 2.6) A *lamination train track* (ltt) *structure* G is a pair-labeled colored train track graph (black edges will be included, but not considered colored) satisfying:

(ltt1) Vertices are either purple or red.

(ltt2) Edges are of 3 types ($\mathcal{E}_b$ comprises the black edges and $\mathcal{E}_c$ comprises the red and purple edges):

Black Edges: A single black edge connects each pair of (edge-pair)-labeled vertices. There are no other black edges. In particular, each vertex is contained in a unique black edge.

Red Edges: A colored edge is red if and only if at least one of its endpoint vertices is red.

Purple Edges: A colored edge is purple if and only if both endpoint vertices are purple.

(ltt3) No pair of vertices is connected by two distinct colored edges.

The purple subgraph of G will be called the *potential ideal Whitehead graph associated to* G, denoted $\mathcal{PI}(G)$. For a finite graph $\mathcal{G} \cong \mathcal{PI}(G)$, we say *$G$ is an ltt structure for $\mathcal{G}$*.

An $(r;(\frac{3}{2}-r))$ *ltt structure* is an ltt structure G for a $\mathcal{G} \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$ such that:

(ltt4) G has precisely 2r-1 purple vertices, a unique red vertex, and a unique red edge.

Ltt structures are *equivalent* that differ by an ornamentation-preserving (label and color preserving), homeomorphism.

Standard Notation/Terminology 6.2 (Ltt structures)**.** For an ltt structure G:

(1) An edge connecting a vertex pair $\{d_i, d_j\}$ will be denoted $[d_i, d_j]$, with interior (d_i, d_j).

(While the notation $[d_i, d_j]$ may be ambiguous when there is more than one edge connecting the vertex pair $\{d_i, d_j\}$, we will be clear in such cases as to which edge we refer to.)

(2) $[e_i]$ will denote $[d_i, \overline{d_i}]$

(3) Red vertices and edges will be called *nonperiodic*.

(4) Purple vertices and edges will be called *periodic*.

(5) $\mathcal{C}(G)$ will denote the colored subgraph of G, called the *colored subgraph associated to* (or *of*) G.

(6) G will be called *admissible* if it is birecurrent.

For an $(r;(\frac{3}{2}-r))$ ltt structure G for $\mathcal{G}$, additionally:

(1) d^u will label the unique red vertex and be called the *unachieved direction*.

(2) $e^R = [t^R]$, will denote the unique red edge and $\overline{d^a}$ its purple vertex's label. So $t^R = \{d^u, \overline{d^a}\}$ and $e^R = [d^u, \overline{d^a}]$.

(3) $\overline{d^a}$ is contained in a unique black edge, which we call the *twice-achieved edge*.

(4) d^a will label the other twice-achieved edge vertex and be called the *twice-achieved direction*.

(5) If G has a subscript, the subscript carries over to all relevant notation. For example, in G_k, d_k^u will label the red vertex and e_k^R the red edge.

A $2r$-element set of the form $\{x_1, \overline{x_1}, \dots, x_r, \overline{x_r}\}$, with elements paired into *edge pairs* $\{x_i, \overline{x_i}\}$, will be called a *rank-r edge pair labeling set*. It will then be standard to say $\overline{\overline{x_i}} = x_i$. A graph with vertices labeled by an edge pair labeling set will be called a *pair-labeled* graph. If an indexing is prescribed, it will be called an *indexed pair-labeled* graph.

Definition 6.3. For an ltt structure to be considered *indexed pair-labeled*, we require:

(1) It is index pair-labeled (of rank r) as a graph.
(2) The vertices of the black edges are indexed by edge pairs.

Index pair-labeled ltt structures are *equivalent* that are equivalent as ltt structures via an equivalence preserving the indexing of the vertex labeling set.

By index pair-labeling (with rank r) an $(r; (\frac{3}{2} - r))$ ltt structure G and edge-indexing the edges of an r-petaled rose Γ, one creates an identification of the vertices in G with $\mathcal{D}(v)$, where v is the vertex of Γ. With this identification, we say G is *based* at Γ. In such a case it will be standard to use the notation $\{d_1, d_2, \dots, d_{2r-1}, d_{2r}\}$ for the vertex labels (instead of $\{x_1, x_2, \dots, x_{2r-1}, x_{2r}\}$). Additionally, $[e_i]$ will denote $[D_0(e_i), D_0(\overline{e_i})] = [d_i, \overline{d_i}]$ for each edge $e_i \in E(\Gamma)$.

A $\mathcal{G} \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$ will be called *(index) pair-labeled* if its vertices are labeled by a $2r-1$ element subset of the rank r (indexed) edge pair labeling set.

6.2. Maps of lamination train track structures. *Let G and G' be rank-r indexed pair-labeled $(r; (\frac{3}{2} - r))$ ltt structures, with bases Γ and Γ', and $g : \Gamma \to \Gamma'$ a tight homotopy equivalence taking edges to nondegenerate edge-paths.*

Recall that Dg induces a map of turns $D^t g : \{a, b\} \mapsto \{Dg(a), Dg(b)\}$. Dg additionally induces a map on the corresponding edges of $\mathcal{C}(G)$ and $\mathcal{C}(G')$ if the appropriate edges exist in $\mathcal{C}(G')$:

Definition 6.4. When the map sending

(1) the vertex labeled d in G to that labeled by $Dg(d)$ in G' and
(2) the edge $[d_i, d_j]$ in $\mathcal{C}(G)$ to the edge $[Dg(d_i), Dg(d_j)]$ in $\mathcal{C}(G')$ also satisfies that
(3) each $\mathcal{PI}(G)$ is mapped isomorphically onto $\mathcal{PI}(G')$,

we call it the *map of colored subgraphs induced by g* and denote it

$$D^C(g) : \mathcal{C}(G) \to \mathcal{C}(G').$$

When it exists, the map $D^T(g) : G \to G'$ *induced by* g is the extension of $D^C(g) : \mathcal{C}(G) \to \mathcal{C}(G')$ taking the interior of the black edge of G corresponding to the edge $E \in E(\Gamma)$ to the interior of the smooth path in G' corresponding to $g(E)$.

6.3. ltt structures are ltt structures. By showing that the ltt structures of Subsection 2.5 are indeed abstract ltt structures, we can create a finite list of ltt structures for a particular $\mathcal{G} \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$ to apply the birecurrency condition to.

Lemma 6.5. *Let $g : \Gamma \to \Gamma$ be a representative of $\phi \in Out(F_r)$, with $(r;(\frac{3}{2}-r))$ potential, such that $\mathcal{IW}(g) \cong \mathcal{G}$. Then $G(g)$ is an $(r;(\frac{3}{2}-r))$ ltt structure with base graph Γ. Furthermore, $\mathcal{PI}(G(g)) \cong \mathcal{G}$.*

Proof. This is more or less just direct applications of the lemmas above. [Pfa12] gives a detailed proof of a more general lemma. □

6.4. Generating triples. Since we deal with representatives decomposed into Nielsen generators, we use an abstract notion of an "indexed generating triple."

Definition 6.6. A *triple* (g_k, G_{k-1}, G_k) will be an ordered set of three objects where $g_k : \Gamma_{k-1} \to \Gamma_k$ is a proper full fold of roses and, for $i = k-1, k$, G_i is an ltt structure with base Γ_i.

Definition 6.7. A *generating triple* is a triple (g_k, G_{k-1}, G_k) where:

(gtI) $g_k : \Gamma_{k-1} \to \Gamma_k$ is a proper full fold of edge-indexed roses defined by:
- (a) $g_k(e_{k-1,j_k}) = e_{k,i_k} e_{k,j_k}$ where $d_k^a = D_0(e_{k,i_k})$, $d_k^u = D_0(e_{k,j_k})$, and $e_{k,i_k} \neq (e_{k,j_k})^{\pm 1}$.
- (b) $g_k(e_{k-1,t}) = e_{k,t}$ for all $e_{k-1,t} \neq (e_{k,j_k})^{\pm 1}$.

(gtII) G_i is an indexed pair-labeled $(r;(\frac{3}{2}-r))$ ltt structure with base Γ_i for $i = k-1, k$.

(gtIII) The induced map of based ltt structures $D^T(g_k) : G_{k-1} \to G_k$ exists and, in particular, restricts to an isomorphism from $\mathcal{PI}(G_{k-1})$ to $\mathcal{PI}(G_k)$.

Standard Notation/Terminology 6.8 (Generating triples)**.** For a generating triple (g_k, G_{k-1}, G_k):

(1) We call G_{k-1} the *source ltt structure* and G_k the *destination ltt structure.*

(2) g_k will be called the *(ingoing) generator* and will sometimes be written $g_k : e_{k-1}^{pu} \mapsto e_k^a e_k^u$ ("p" is for "pre"). Thus, d_{k-1,j_k} will sometimes be written d_{k-1}^{pu}.

(3) e_{k-1}^{pa} denotes e_{k-1,i_k} (again "p" is for "pre").

(4) If G_k and G_{k-1} are indexed pair-labeled $(r;(\frac{3}{2}-r))$ ltt structures for $\mathcal{G}$, then (g_k, G_{k-1}, G_k) will be a generating triple *for* $\mathcal{G}$.

Remark 6.9. While d_i^u is determined by the red vertex of G_i (and does not rely on other information in the triple), d_{k-1}^{pu} and d_{k-1}^{pa} actually rely on (gtI), and cannot be determined by knowing only G_{k-1}.

Example 6.10. The triple (g_2, G_1, G_2) of Example 4.5 is an example of a generating triple where x denotes both $E_{(1,1)}$ and $E_{(2,1)}$, y denotes both $E_{(1,2)}$ and $E_{(2,2)}$, and z denotes both $E_{(1,3)}$ and $E_{(2,3)}$.

Definition 6.11. Suppose (g_i, G_{i-1}, G_i) and $(g_i', G_{i-1}', G_i)'$ are generating triples. Let $g_i^T : G_{i-1} \to G_i$ be induced by $g_i : \Gamma_{i-1} \to \Gamma_i$ and $g_i^T : G_{i-1}' \to G_i'$ by $g_i : \Gamma_{i-1}' \to \Gamma_i'$. We say (g_i, G_{i-1}, G_i) and (g_i', G_{i-1}', G_i') are *equivalent* if there exist indexed pair-labeled graph equivalences $H_{i-1} : \Gamma_{i-1} \to \Gamma_{i-1}'$ and $H_i : \Gamma_i \to \Gamma_i'$ such that:

(1) For $k = i, i-1$, $H_i : \Gamma_i \to \Gamma_i'$ induces an indexed pair-labeled ltt structure equivalence of G_i and G_i'.
(2) $H_i \circ g_i = g_i' \circ H_{i-1}$.

7. Peels, extensions, and switches

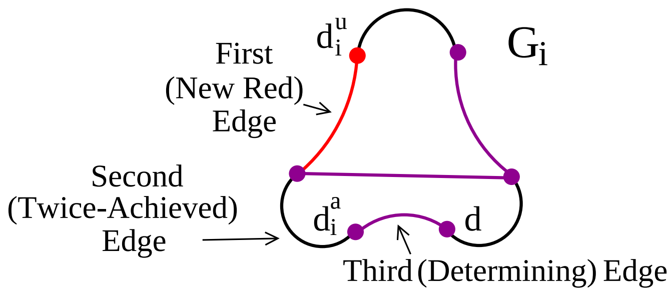
Suppose $\mathcal{G} \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$. By Section 4, if there is a $\phi \in \mathcal{AF}_r$ with $\mathcal{IW}(\phi) \cong \mathcal{G}$, then there is an ideally decomposed $(r;(\frac{3}{2}-r))$-potential representative g of a power of ϕ. By Section 5, such a representative would satisfy the $\mathcal{AM}$ properties. Thus, if we can show that a representative satisfying the properties does not exist, we have shown there is no $\phi \in \mathcal{AF}_r$ with $\mathcal{IW}(\phi) \cong \mathcal{G}$ (we use this fact in Section 9). In this section we show what triples (g_k, G_{k-1}, G_k) satisfying the $\mathcal{AM}$ properties must look like. We prove in Proposition 7.8 that, if the structure G_k and a purple edge $[d, d_k^a]$ in G_k are set, then there is only one g_k possibility and at most two G_{k-1} possibilities (one generating triple possibility will be called a "switch" and the other an "extension"). Extensions and switches are used here only to define ideal decomposition diagrams but have interesting properties used (and proved) in [Pfa13a] and [Pfa13b].

7.1. Peels.

As a warm-up, we describe a geometric method for visualizing "switches" and "extensions" as moves, "peels," transforming an ltt structure G_i into an ltt structure G_{i-1}.

Each peel of an ltt structure G_i involves three directed edges of G_i:

- The *First Edge of the Peel* (*New Red Edge* in G_i): the red edge from d_i^u to $\overline{d_i^a}$.
- The *Second Edge of the Peel* (*Twice-Achieved Edge* in G_i): the black edge from $\overline{d_i^a}$ to d_i^a.
- The *Third Edge of the Peel* (*Determining Edge* for the peel): a purple edge from d_i^a to d. (In G_{i-1}, this vertex d will be the red edge's attaching vertex, labeled $\overline{d_{i-1}^a}$).



For each determining edge choice $[d_i^a, d]$ in G_i, there is one "peel switch" (Figure 8) and one "peel extension" (Figure 7). When G_i has only a single purple edge at d_i^a, the switch and extension differ by a color switch of two edges and two vertices. We start by explaining this case. After, we explain the preliminary step necessary for any switch where more than one purple edge in G_i contains d_i^a.

We describe how, when G_i has only a single purple edge at d_i^a, the two peels determined by $[d_i^a, d]$ transform G_i into G_{i-1}. While keeping d fixed, starting at vertex $\overline{d_i^a}$, peel off black edge $[\overline{d_i^a}, d_i^a]$ and the third edge $[d_i^a, d]$, leaving copies of $[\overline{d_i^a}, d_i^a]$ and $[d_i^a, d]$ and creating a new edge $[d_i^u, d]$ from the concatenation of the peel's first, second, and third edges (Figure 7 or 8).

In a *peel extension*: $[d_i^u, \overline{d_i^a}]$ disappears into the concatenation and does not exist in G_{i-1}, the copy of $[\overline{d_i^a}, d_i^a]$ left behind stays black in G_{i-1}, the copy of $[d_i^a, d]$ left behind stays purple in G_{i-1}, the edge $[d_i^u, d]$ formed from the concatenation is red in G_{i-1}, and nothing else changes from G_i to G_{i-1} (if one ignores the first indices of the vertex labels). The triple (g_i, G_{i-1}, G_i), with g_i as in $\mathcal{AM}$VI, will be called the *extension determined by* $[d_i^a, d]$.

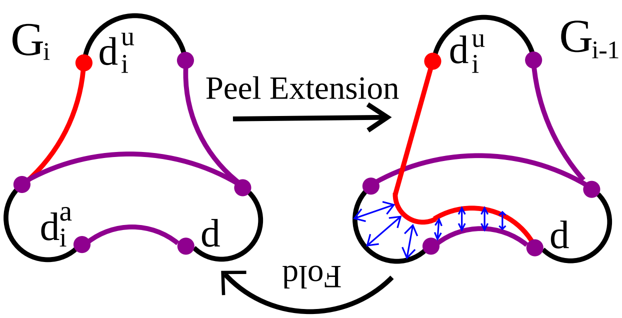


FIGURE 7. Peel Extension: Note that the first, second, and third edges of the peel concatenate to form the red edge $[d_i^u, d]$ in G_{i-1} and that copies of $[\overline{d_i^a}, d_i^a]$ and $[d_i^a, d]$ remain in G_{i-1}.

In a *peel switch* (where $[d_i^a, d]$ was the only purple edge in G_i containing d_i^a): Again $[d_i^u, \overline{d_i^a}]$ has disappeared into the concatenation and the copy of

$[\overline{d_i^a}, d_i^a]$ left behind stays black in G_{i-1}. But now the edge $[d_i^u, d]$ formed from the concatenation is purple in G_{i-1}, the copy of $[d_i^a, d]$ left behind and the vertex d_i^a are both red in G_{i-1} (so that d_i^a is now actually d_{i-1}^u), and the vertex d_i^u is purple in G_{i-1}. The triple (g_i, G_{i-1}, G_i), with g_i as in $\mathcal{AM}$VI, will be called the *switch determined by* $[d_i^a, d]$.

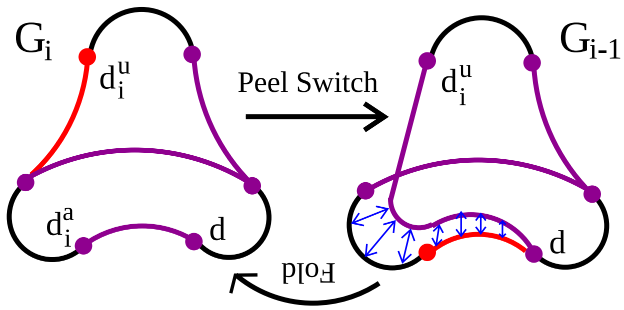


FIGURE 8. Peel Switch (when d_i^a only belongs to one purple edge in G_{i-1}): The first, second, and third edges of the peel concatenate to form a purple edge $[d_i^u, d]$ in G_{i-1}. The determining edge $[d_i^a, d]$ is the red edge of G_{i-1}, with red vertex d_i^a.

Preliminary step for a switch where purple edges other than the determining edge $[d_i^a, d]$ contain vertex d_i^a in G_i: For each purple edge $[d_i^a, d']$ in G_i where $d \neq d'$, form a purple concatenated edge $[d', d_i^u]$ in G_{i-1} by concatenating $[d', d_i^a]$ with a copy of $[d_i^a, \overline{d_i^a}, d_i^u]$, created by splitting open, as in Figure 9, $[d_i^a, \overline{d_i^a}]$ from d_i^a to $\overline{d_i^a}$ and $[\overline{d_i^a}, d_i^u]$ from $\overline{d_i^a}$ to d_i^u.

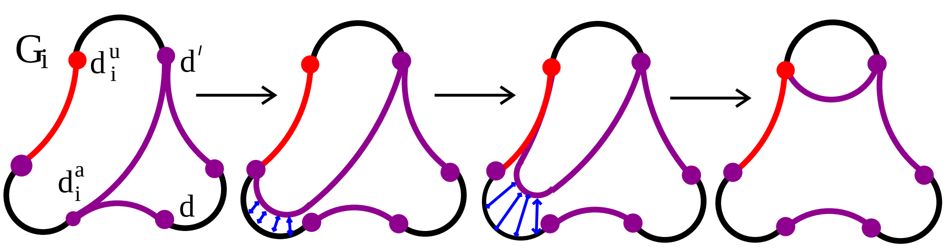


FIGURE 9. Peel Switch Preliminary Step: For each purple edge $[d_i^a, d']$ in G_i, the peeler peels a copy of $[d_i^a, \overline{d_i^a}, d_i^u]$ off to concatenate with $[d_i^a, d']$ and form the purple edge $[d_i^u, d']$.

To check the peel switch was performed correctly, one can: remove G_i's red edge, lift vertex d_i^a (with purple edges containing it dangling from one's fingers), and drop vertex d_i^a in the spot of vertex d_i^u, while leaving behind a copy of $[d_i^a, d]$ to become the new red edge of G_{i-1} (with d_{i-1}^{pa} as the red vertex).

7.2. Extensions and switches. In this subsection, we describe "moves" one applies to ltt structures in defining edges of the $\mathcal{ID}$ diagrams.

Throughout this section G_k will be an indexed pair-labeled $(r;(\frac{3}{2}-r))$ ltt structure for a $\mathcal{G} \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$ with rose base graph Γ_k. We use the standard notation.

We define extensions and switches "entering" an indexed pair-labeled admissible $(r;(\frac{3}{2}-r))$ ltt structure G_k for $\mathcal{G}$. However, we first prove that determining edges exist.

Lemma 7.1. *There exists a purple edge with vertex d^a_k, so that it may be written $[d^a_k, d_{k,l}]$.*

Proof. If d^a_k were red, e^R_k would be $[d^a_k, \overline{d^a_k}]$, violating that $\mathcal{G} \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$. d^a_k must be contained in an edge $[d^a_k, d_{k,l}]$ or $\mathcal{G}$ would not have 2r-1 vertices. If $d_{k,l}$ were red, i.e., $d_{k,l} = d^u_k$, then both $[d^u_k, \overline{d^a_k}]$ and $[d^u_k, d^a_k]$ would be red, violating (ltt4). So $[d^a_k, d_{k,l}]$ must be purple. $\square$

Definition 7.2 (See Figure 10)**.** For a purple edge $[d^a_k, d_{k,l}]$ in G_k, the *extension determined by* $[d^a_k, d_{k,l}]$, is the generating triple (g_k, G_{k-1}, G_k) for $\mathcal{G}$ satisfying:

(extI) The restriction of $D^T(g_k)$ to $\mathcal{PI}(G_{k-1})$ is defined by sending, for each j, the vertex labeled $d_{k-1,j}$ to the vertex labeled $d_{k,j}$ and extending linearly over edges.

(extII) $d^u_{k-1} = d^{pu}_{k-1}$, i.e., $d^{pu}_{k-1} = d_{k-1,j_k}$ labels the single red vertex in G_{k-1}.

(extIII) $\overline{d^a_{k-1}} = d_{k-1,l}$.

Remark 7.3. (extIII) implies that the single red edge $e^R_{k-1} = [d^u_{k-1}, \overline{d^a_{k-1}}]$ of G_{k-1} can be written, among other ways, as $[d^{pu}_{k-1}, d_{(k-1,l)}]$.

Explained in Section 7.1, but with this section's notation, an extension transforms ltt structures as:

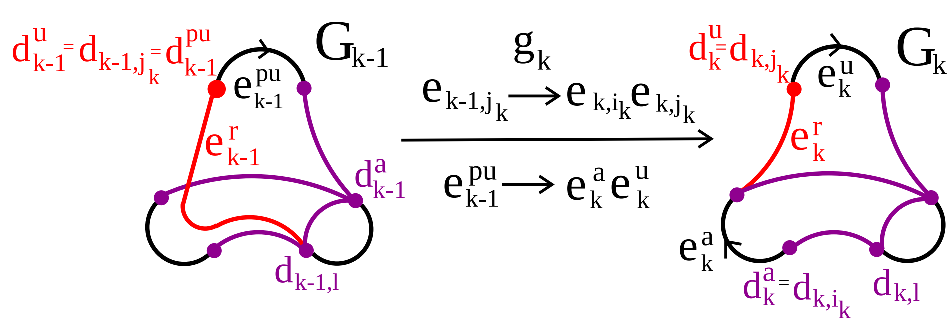


FIGURE 10. Extension.

Lemma 7.4. *The extension (g_k, G_{k-1}, G_k) determined by an edge $[d^a_k, d_{k,l}]$ in $\mathcal{PI}(G_k)$ is unique.*

(I) *G_{k-1} can be obtained from G_k by the following steps:*
   (1) *removing the interior of the red edge from G_k;*
   (2) *replacing each vertex label $d_{k,i}$ with $d_{k-1,i}$ and each vertex label $\overline{d_{k,i}}$ with $\overline{d_{k-1,i}}$; and*
   (3) *adding a red edge e^R_{k-1} connecting the red vertex to $d_{k-1,l}$.*

(II) *The fold is such that the corresponding homotopy equivalence maps the oriented $e_{k-1,j_k} \in \mathcal{E}_{k-1}$ over the path $e_{k,i_k}e_{k,j_k}$ in Γ_k and then each oriented $e_{k-1,t} \in \mathcal{E}_{k-1}$ with $e_{k-1,t} \neq e^{\pm 1}_{k-1,j_k}$ over $e_{k,t}$.*

Proof. The proof is an unraveling of definitions. A full presentation can be found in [Pfa12]. $\square$

Definition 7.5 (See Figure 11)**.** The *switch* determined by a purple edge $[d^a_k, d_{(k,l)}]$ in G_k is the generating triple (g_k, G_{k-1}, G_k) for $\mathcal{G}$ satisfying:

(swI) $D^T(g_k)$ restricts to an isomorphism from $\mathcal{PI}(G_{k-1})$ to $\mathcal{PI}(G_k)$ defined by

$$\mathcal{PI}(G_{k-1}) \xrightarrow{d^{pu}_{k-1} \mapsto d^a_k = d_{k,i_k}} \mathcal{PI}(G_k)$$

($d_{k-1,t} \mapsto d_{k,t}$ for $d_{k-1,t} \neq d^{pu}_{k-1}$) and extended linearly over edges.

(swII) $d^{pa}_{k-1} = d^u_{k-1}$.

(swIII) $\overline{d^a_{k-1}} = d_{k-1,l}$.

Remark 7.6. (swII) implies that the red edge $e^R_{k-1} = [d^u_{k-1}, d^a_{k-1}]$ of G_{k-1} can be written $[d^{pa}_{k-1}, \overline{d^a_{k-1}}]$, among other ways. (swIII) implies that e^R_{k-1} can be written $[d_{(k-1,i_k)}, d_{(k-1,l)}]$.

Explained in Section 7.1, but with this section's notation, a switch transforms ltt structures as follows:

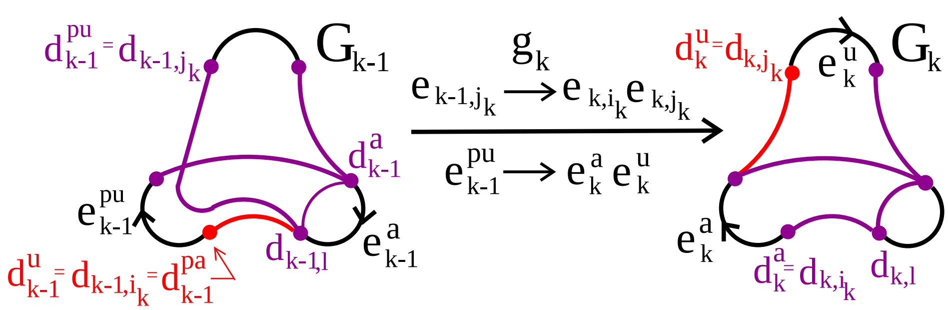


FIGURE 11. Switch.

Lemma 7.7. *Given an edge $[d^a_k, d_{k,l}]$ in $\mathcal{PI}(G_k)$, the switch (g_k, G_{k-1}, G_k) determined by $[d^a_k, d_{k,l}]$ is unique.*

(I) *G_{k-1} can be obtained from G_k by the following steps:*
   (1) *Start with $\mathcal{PI}(G_k)$.*
   (2) *Replace each vertex label $d_{k,i}$ with $d_{k-1,i}$.*

(3) *Switch the attaching (purple) vertex of the red edge to be* $d_{k-1,l}$.
(4) *Switch the labels* $d_{(k-1,j_k)}$ *and* $d_{(k-1,i_k)}$, *so that the red vertex of* G_{k-1} *will be* d_{k-1,i_k} *and the red edge of* G_{k-1} *will be* $[d_{(k-1,i_k)}, d_{(k-1,l)}]$.
(5) *Include black edges connecting inverse pair labeled vertices (there is a black edge* $[d_{(k-1,i)}, d_{(k-1,j)}]$ *in* G_{k-1} *if and only if there is a black edge* $[d_{(k,i)}, d_{(k,j)}]$ *in* G_k*).*

(II) *The fold is such that the corresponding homotopy equivalence maps the oriented* $e_{k-1,j_k} \in \mathcal{E}_{k-1}$ *over the path* $e_{k,i_k}e_{k,j_k}$ *in* Γ_k *and then each oriented* $e_{k-1,t} \in \mathcal{E}_{k-1}$ *with* $e_{k-1,t} \neq e^{\pm 1}_{k-1,j_k}$ *over* $e_{k,t}$.

Proof. The proof is an unraveling of definitions. A full presentation can be found in [Pfa12]. □

Recall (Proposition 5.1) that each triple in an ideal decomposition satisfies $\mathcal{AM}$I–$\mathcal{AM}$VII. Thus, to construct a diagram realizing any ideally decomposed $(r;(\frac{3}{2}-r))$-potential representative with ideal Whitehead graph $\mathcal{G}$, we want edges of the diagram to correspond to triples satisfying $\mathcal{AM}$I–$\mathcal{AM}$VII. Proposition 7.8 tells us each such triple is either an admissible switch or admissible extension.

Proposition 7.8. *Suppose* (g_k, G_{k-1}, G_k) *is a triple for* $\mathcal{G}$ *such that:*

(1) $\mathcal{G} \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$.
(2) G_i *is an indexed pair-labeled* $(r;(\frac{3}{2}-r))$ *ltt structure for* $\mathcal{G}$ *with base graph* Γ_i, *for* $i = k, k-1$.

Then (g_k, G_{k-1}, G_k) *satisfies* $\mathcal{AM}$I–$\mathcal{AM}$VII *if and only if it is either an admissible switch or an admissible extension.*

In particular, in the circumstance where $d^u_{k-1} = d^{pa}_{k-1}$, *the triple is a switch and, in the circumstance where* $d^u_{k-1} = d^{pu}_{k-1}$, *the triple is an extension.*

Proof. For the forward direction, assume (g_k, G_{k-1}, G_k) satisfies $\mathcal{AM}$I–$\mathcal{AM}$VII and (1)–(2) above. We show the triple is either a switch or an extension ($\mathcal{AM}$I gives birecurrency). Assumption (1) implies (gtII).

By $\mathcal{AM}$VI, g_k is defined by $g_k(e^{pu}_{k-1}) = e^a_k e^u_k$ and $g_k(e_{k-1,i}) = e_{k,i}$ for $e_{k-1,i} \neq (e^{pu}_{k-1})^{\pm 1}$, $D_0(e^u_k) = d^u_k$, $D_0(\overline{e^a_k}) = \overline{d^a_k}$, and $e^{pu}_{k-1} = e_{(k-1,j)}$, where $e^u_k = e_{k,j}$. We have (gtI).

By $\mathcal{AM}$VII, Dg_k induces on isomorphism from $\mathcal{SW}(G_{k-1})$ to $\mathcal{SW}(G_k)$. Since the only direction whose second index is not fixed by Dg_k is d^{pu}_{k-1}, the only vertex label of $\mathcal{SW}(G_{k-1})$ not determined by this isomorphism is the preimage of d^a_k (which $\mathcal{AM}$IV dictates to be either d^{pu}_{k-1} or d^{pa}_{k-1}). When the preimage is d^{pa}_{k-1}, this gives (extI). When the preimage is d^{pu}_{k-1}, this gives (swI). For the isomorphism to extend linearly over edges, we need that images of edges in G_{k-1} are edges in G_k, i.e., $[Dg_k(d_{k-1,i}), Dg_k(d_{k-1,j})]$ is an edge in G_k for each edge $[d_{(k-1,i)}, d_{(k-1,j)}]$ in G_{k-1}. This follows from $\mathcal{AM}$IV. We have (gtIII).

$\mathcal{AM}$II gives either $d^u_{k-1} = d^{pa}_{k-1}$ or $d^u_{k-1} = d^{pu}_{k-1}$. In the switch case, the above arguments imply d^{pu}_{k-1} labels a purple vertex. So $d^u_{k-1} = d^{pa}_{k-1}$ (since $\mathcal{AM}$III tells us d^u_{k-1} is red). This gives (swII) once one appropriately coordinates notation. In the extension case, the above arguments give instead that d^{pa}_{k-1} labels a purple vertex, meaning $d^u_{k-1} = d^{pu}_{k-1}$ (again since $\mathcal{AM}$III tells us d^u_{k-1} is red). This gives us (extII). We are left with (extIII) and (swIII). What we need is that $[d^a_k, d_{k,l}]$ is a purple edge in G_k where $\overline{d^a_{k-1}} = d_{k-1,l}$.

By $\mathcal{AM}$V, G_{k-1} has a single red edge $[t^R_{k-1}] = [\overline{d^a_{k-1}}, d^u_{k-1}]$. By $\mathcal{AM}$IV, $D^C g_k([t^R_{k-1}])$ is in $\mathcal{PI}(G_k)$. First consider what we established is the switch case, i.e., assume $d^u_{k-1} = d^{pa}_{k-1}$. The goal is to determine

$$[t^R_{k-1}] = [d_{(k-1,i_k)}, d_{(k-1,l)}],$$

where $d^a_k = d_{k,i_k}$ $(d_{k-1,i_k} = d^{pa}_{k-1})$ and $[d^a_k, d_{k,l}]$ is in $\mathcal{PI}(G_k)$ (making (g_k, G_{k-1}, G_k) the switch determined by $[d^a_k, d_{k,l}]$). Since $d^u_{k-1} = d^{pa}_{k-1}$, we know $[t^R_{k-1}] = [\overline{d^a_{k-1}}, d^u_{k-1}] = [\overline{d^a_{k-1}}, d^{pa}_{k-1}]$. We know $\overline{d^a_{k-1}} \neq d^{pa}_{k-1}$ (since (tt2) implies $\overline{d^a_{k-1}} \neq d^u_{k-1}$, which equals d^{pa}_{k-1}). Thus, $\mathcal{AM}$VI says $D^C g_k([t^R_{k-1}]) = D^C g_k([\overline{d^a_{k-1}}, d^{pa}_{k-1}]) = [d_{(k,l)}, d^a_k]$ where $\overline{d^a_{k-1}} = e_{k-1,l}$. So $[d_{(k,l)}, d^a_k]$ is in $\mathcal{PI}(G_k)$. We thus have (swIII). Now consider what we established is the extension case, i.e., assume $d^u_{k-1} = d^{pu}_{k-1}$. We need $[t^R_{k-1}] = [d_{(k-1,j_k)}, d_{(k-1,l)}]$, where $d^u_{k-1} = d_{k-1,j_k}$ and $[d^a_k, d_{k,l}]$ is in $\mathcal{PI}(G_k)$ (making (g_k, G_{k-1}, G_k) the extension determined by $[d^a_k, d_{k,l}]$). Since $d^u_{k-1} = d^{pu}_{k-1}$, we know $[t^R_{k-1}] = [\overline{d^a_{k-1}}, d^u_{k-1}] = [\overline{d^a_{k-1}}, d^{pu}_{k-1}]$. We know $\overline{d^a_{k-1}} \neq d^{pu}_{k-1}$ (since (tt2) implies $\overline{d^a_{k-1}} \neq d^u_{k-1}$, which equals d^{pu}_{k-1}). Thus, by $\mathcal{AM}$VI, $D^C g_k([t^R_{k-1}]) = D^C g_k([\overline{d^a_{k-1}}, d^{pu}_{k-1}]) = [d_{(k,l)}, d^a_k]$, where $\overline{d^a_{k-1}} = e_{k-1,l}$. We have (extIII) and the forward direction.

For the converse, assume (g_k, G_{k-1}, G_k) is either an admissible switch or extension. Since we required extensions and switches be admissible, G_{k-1} and G_k are birecurrent. We have $\mathcal{AM}$I.

The first and second parts of $\mathcal{AM}$II are equivalent and the second part holds by (extII) for an extension and (swII) for a switch. For $\mathcal{AM}$III note that there is only a single red vertex (labeled d^u_k) in G_k and is only a single red vertex (labeled d^u_{k-1}) in G_{k-1} because of the requirement in (gtII) that G_k and G_{k-1} are $(r; (\frac{3}{2}-r))$ ltt structures (see the standard notation for why this is notationally consistent with the $\mathcal{AM}$ properties). What is left of $\mathcal{AM}$III is that the edge $[t^R_k] = [d^u_k, \overline{d^a_k}]$ in G_k and the edge $[t^R_{k-1}] = [d^u_{k-1}, \overline{d^a_{k-1}}]$ in G_{k-1} are both red. This follows from (gtI) combined with (extII) for an extension and (swII) for a switch.

(gtIII) implies $\mathcal{AM}$IV. For $\mathcal{AM}$V, note: $\mathcal{AM}$III implies e^R_k is a red edge containing the red vertex d^u_k. (ltt4) implies the uniqueness of both the red edge and direction.

Since $\mathcal{AM}$VI follows from (gtI), combined with (extII) for an extension and (swII) for a switch, and $\mathcal{AM}$VII follows from (gtIII), we have proved the converse. $\square$

Definition 7.9. In light of Proposition 7.8, an *admissible map* will mean a triple for a $\mathcal{G} \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$ that is an admissible switch or admissible extension or (equivalently) satisfies $\mathcal{AM}$I–$\mathcal{AM}$VII.

8. Ideal decomposition ($\mathcal{ID}$) diagrams

Throughout this section $\mathcal{G} \in \mathcal{PI}_{(r;(\frac{3}{2}-r))}$. We define the "ideal decomposition ($\mathcal{ID}$) diagram" for $\mathcal{G}$, as well as prove that representatives with $(r;(\frac{3}{2}-r))$ potential are realized as loops in these diagrams. We use $\mathcal{ID}$ diagrams to prove Theorem 9.1 and to construct examples in [Pfa13b].

Definition 8.1. A *preliminary ideal decomposition diagram for* $\mathcal{G}$ is the directed graph where:

(1) The nodes correspond to equivalence classes of admissible indexed pair-labeled $(r;(\frac{3}{2}-r))$ ltt structures for $\mathcal{G}$.
(2) For each equivalence class of an admissible generator triple (g_i, G_{i-1}, G_i) for $\mathcal{G}$, there exists a directed edge $E(g_i, G_{i-1}, G_i)$ from the node $[G_{i-1}]$ to the node $[G_i]$.

The disjoint union of the maximal strongly connected subgraphs of the preliminary ideal decomposition diagram for $\mathcal{G}$ will be called the *ideal decomposition* ($\mathcal{ID}$) *diagram for* $\mathcal{G}$ (or $\mathcal{ID}(\mathcal{G})$).

Remark 8.2. [Pfa12] gives a procedure for constructing $\mathcal{ID}$ diagrams (there called "AM Diagrams").

We say an ideal decomposition $\Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{k-1}} \Gamma_{k-1} \xrightarrow{g_k} \Gamma_k$ of a tt map g with indexed $(r;(\frac{3}{2}-r))$ ltt structures $G_0 \to G_1 \to \cdots \to G_{k-1} \to G_k$ for $\mathcal{G}$ is *realized* by $E(g_1, G_0, G_1) * \cdots * E(g_k, G_{k-1}, G_k)$ in $\mathcal{ID}(\mathcal{G})$ if the oriented path $E(g_1, G_0, G_1) * \cdots * E(g_k, G_{k-1}, G_k)$ in $\mathcal{ID}(\mathcal{G})$ from $[G_0]$ to $[G_k]$, traversing the $E(g_i, G_{i-1}, G_i)$ in order of increasing i (from $E(g_1, G_0, G_1)$ to $E(g_k, G_{k-1}, G_k)$), exists.

Proposition 8.3. *If* $g = g_k \circ \cdots \circ g_1$, *with ltt structures*

$$G_0 \to G_1 \to \cdots \to G_{k-1} \to G_k,$$

is an ideally decomposed representative of $\phi \in Out(F_r)$, *with* $(r;(\frac{3}{2}-r))$ *potential, such that* $\mathcal{IW}(\phi) = \mathcal{G}$, *then* $E(g_1, G_0, G_1) * \cdots * E(g_k, G_{k-1}, G_k)$ *exists in* $\mathcal{ID}(\mathcal{G})$ *and forms an oriented loop.*

Proof. This follows from Proposition 7.8 and Proposition 5.1. □

Corollary 8.4. *If no loop in* $\mathcal{ID}(\mathcal{G})$ *gives a potentially-*$(r;(\frac{3}{2}-r))$ *representative of a* $\phi \in Out(F_r)$ *with* $\mathcal{IW}(\phi) = \mathcal{G}$, *such a* ϕ *does not exist. In particular, any of the following* $\mathcal{ID}(\mathcal{G})$ *properties would prove such a representative does not exist:*

(1) *For at least one edge pair* $\{d_i, \overline{d_i}\}$, *where* $e_i \in E(\Gamma)$, *no red vertex in* $\mathcal{ID}(\mathcal{G})$ *is labeled by* $d_i^{\pm 1}$.
(2) *The representative corresponding to each loop in* $\mathcal{ID}(\mathcal{G})$ *has a pNp.*

As a result of Corollary 8.4(1) we define:

Definition 8.5 (Irreducibility Potential Test)**.** Check whether, in each connected component of $\mathcal{ID}(\mathcal{G})$, for each edge vertex pair $\{d_i, \overline{d_i}\}$, there is a node N in the component such that either d_i or $\overline{d_i}$ labels the red vertex in the structure N. If it holds for no component, $\mathcal{G}$ is unachieved.

Remark 8.6. Let $\{x_1, \overline{x_1}, \dots, x_{2r}, \overline{x_{2r}}\}$ be a rank-r edge pair labeling set. We call a permutation of the indices $1 \leq i \leq 2r$ combined with a permutation of the elements of each pair $\{x_i, \overline{x_i}\}$ an *Edge Pair (EP) Permutation.* Edge-indexed graphs will be considered *Edge Pair Permutation (EPP) isomorphic* if there is an EP permutation making the labelings identical (this still holds even if only a subset of $\{x_1, \overline{x_1}, \dots, x_{2r}, \overline{x_{2r}}\}$ is used to label the vertices, as with a graph in $\mathcal{PI}_{(r;(\frac{3}{2}-r))}$).

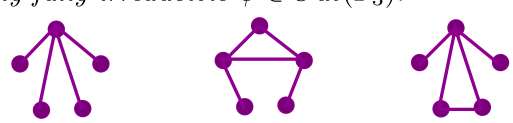
When checking for irreducibility, it is only necessary to look at one EPP isomorphism class of each component (where two components are in the same class if one can be obtained from the other by applying the same EPP isomorphism to each triple in the component).

9. Several unachieved ideal Whitehead graphs

Theorem 9.1. *For each $r \geq 3$, let $\mathcal{G}_r$ be the graph consisting of $2r-2$ edges adjoined at a single vertex.*

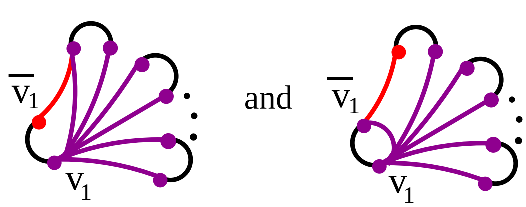
(I) *For no fully irreducible $\phi \in Out(F_r)$ is $\mathcal{IW}(\phi) \cong \mathcal{G}_r$.*

(II) *The following connected graphs are not the ideal Whitehead graph $\mathcal{IW}(\phi)$ for any fully irreducible $\phi \in Out(F_3)$:*



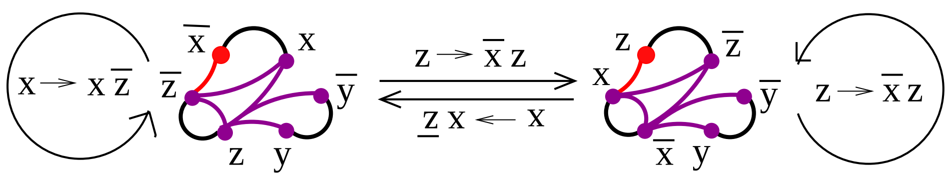
Proof. Notice that, if any of the graphs in (I) or (II) were realized by a fully irreducible $\phi \in Out(F_r)$, then ϕ would have index sum $\frac{3}{2} - r$, and hence would be ageometric.

We first prove (I). Recall that, by Proposition 4.3, if $\phi \in Out(F_r)$ is ageometric fully irreducible and $\mathcal{IW}(\phi)$ is a connected $(2r-1)$-vertex graph (such as the graph $\mathcal{G}_r$), then some positive power of ϕ admits a pNp-free tt representative on the r-petaled rose. By Proposition 3.7, it suffices to show that no admissible $(r;(\frac{3}{2}-r))$ ltt structure for $\mathcal{G}_r$ is birecurrent. Up to EPP-isomorphism, there are two such ltt structures to consider, neither birecurrent):



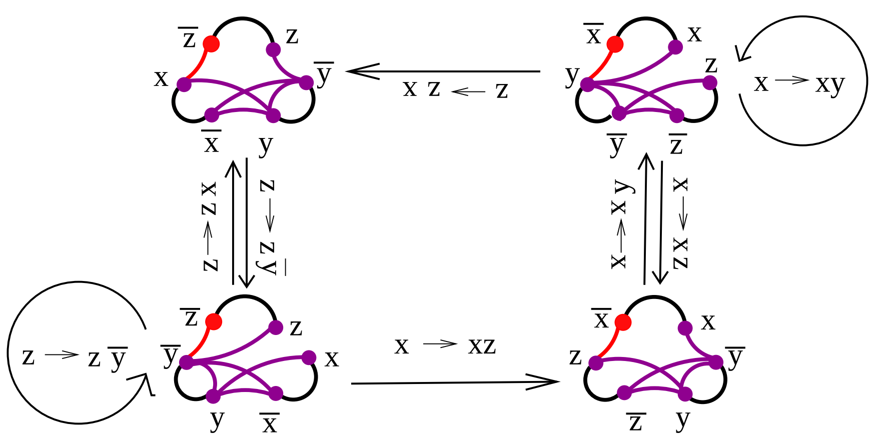
These are the only structures worth considering as follows: Call the valence-$(2r-2)$ vertex v_1. Either (1) some valence-1 vertex is labeled by $\overline{v_1}$ or (2) the set of valence-1 vertices $\{x_1, \overline{x_1}, \dots, x_{r-1}, \overline{x_{r-1}}\}$ consists of $r-1$ edge-pairs. Suppose (2) holds. The red edge cannot be attached in such a way that it is labeled with an edge-pair or is a loop and attaching it to any other vertex yields an EPP-isomorphic ltt structure to that on the left. Suppose (1) holds. Let x_i label the red vertex. The valence-1 vertex labels will be $\{\overline{v_1}, x_2, \overline{x_2}, \dots, x_{i-1}, \overline{x_{i-1}}, \overline{x_i}, x_{i+1}, \overline{x_{i+1}}, \overline{x_i} \dots, x_r, \overline{x_r}\}$. The red edge cannot be attached at $\overline{x_i}$. So either it will be attached at v_1, $\overline{v_1}$, or some x_j with $x_j \neq x_i^{\pm 1}$. Unless it is attached at $\overline{v_1}$, $\overline{v_1}$ is a valence-1 vertex of $[v_1, \overline{v_1}]$ in the local Whitehead graph, making $[v_1, \overline{v_1}]$ an edge only traversable once by a smooth line. If the red edge is attached at $\overline{v_1}$, we have the structure on the right.

We prove (II). The left graph is covered by (I). The following is a representative of the EPP isomorphism class of the only significant component of $\mathcal{ID}(\mathcal{G})$ where $\mathcal{G}$ is the right-most structure:



Since $\mathcal{ID}(\mathcal{G})$ contains only red vertices labeled z and $\bar{x}$ (leaving out $\{y, \overline{y}\}$), unless some other component contains all 3 edge vertex pairs ($\{x, \overline{x}\}$, $\{y, \overline{y}\}$, and $\{z, \overline{z}\}$), the middle graph would be unachieved. Since no other component does contain all 3 edge vertex pairs as vertex labels (all components are EPP-isomorphic), the middle graph is indeed unachieved.

Again, for the right-hand, the $\mathcal{ID}$ Diagram lacks irreducibility potential. A component of the $\mathcal{ID}$ diagram is given below (all components are EPP-isomorphic). The only edge pairs labeling red vertices of this component are $\{x, \overline{x}\}$ and $\{z, \overline{z}\}$:



□

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(Catherine Pfaff) Department of Mathematics, University of California, Santa Barbara, CA 93106-3080
`catherine.pfaff@gmail.com`

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