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Matrices centrally image partition regular near 0

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Abstract. Hindman and Leader first investigated Ramsey theoretic properties near 0 for dense subsemigroups of $(\mathbb{R}, +)$. Following them, the notion of image partition regularity near zero for matrices was introduced by De and Hindman. It was also shown there that like image partition regularity over $\mathbb{N}$, the main source of infinite image partition regular matrices near zero are Milliken–Taylor matrices. But except for constant multiples of the Finite Sum matrix, no other Milliken–Taylor matrices have images in central sets. In this regard the notion of centrally image partition regular matrices were introduced. In the present paper we propose the notion of matrices that are centrally image partition regular matrices near zero for dense subsemigroups of $(\mathbb{R}, +)$ and show that for infinite matrices these may be different from centrally image partition regular matrices, unlike the situation for finite matrices.

Contents

1. Introduction

Let us start this article with the following well known definition of image partition regularity.

Definition 1.1. Let $u, v \in \mathbb{N}$ and let M be a $u \times v$ matrix with entries from $\mathbb{Q}$. The matrix M is image partition regular over $\mathbb{N}$ if and only if whenever

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$r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, there exist $i \in \{1, 2, ..., r\}$ and $\vec{x} \in \mathbb{N}^v$ such that $M\vec{x} \in C_i^u$.

It is well known that for finite matrices, image partition regularity behaves well with respect to central subsets of the underlying semigroup. (Central sets were introduced by Furstenberg and enjoy very strong combinatorial properties [5, Proposition 8.21]). But the situation becomes totally different for infinite image partition regular matrices. It was shown in [8] that some of the interesting properties for finite image partition regularity could not be generalized for infinite image partition regular matrices. To handle these situations the notion of centrally image partition regular matrices were introduced [8], while both these notions become identical for finite matrices. The same problem occurs in the setup of image partition regularity near zero over a dense subsemigroup of $\big((0,\infty),+\big)$. Again from [1, Theorem 2.4], it follows that image partition regularity and image partition regularity near zero over a dense subsemigroup of $\big((0,\infty),+\big)$ are equivalent for finite matrices. This situation motivates us to introduce the notion of *centrally image partition regular near zero over a dense subsemigroup of* $\big((0,\infty),+\big)$ which involves the notion of central sets near zero. The notion of central set near zero was introduced by Hindman and Leader [6] and these sets also enjoy a rich combinatorial structure like central sets.

We shall present the notion central sets and central sets near zero after giving a brief description of the algebraic structure of βS for a discrete semigroup $(S,+)$. We take the points of βS to be the ultrafilters on S, identifying the principal ultrafilters with the points of S and thus pretending that $S \subseteq \beta S$. Given $A \subseteq S$ let us set, $\overline{A} = \{p \in \beta S : A \in p\}$. Then the set $\{\overline{A} : A \subseteq S\}$ is a basis for a topology on βS. The operation $+$ on S can be extended to the Stone–Čech compactification βS of S so that $(\beta S,+)$ is a compact right topological semigroup (meaning that for any $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q + p$ is continuous) with S contained in its topological center (meaning that for any $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x + q$ is continuous). Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$, where $-x + A = \{y \in S : x + y \in A\}$.

A nonempty subset I of a semigroup $(T,+)$ is called a *left ideal of S* if $T + I \subset I$, a *right ideal* if $I + T \subset I$, and a *two sided ideal* (or simply an *ideal*) if it is both a left and right ideal. A *minimal left ideal* is the left ideal that does not contain any proper left ideal. Similarly, we can define *minimal right ideal* and *smallest ideal*.

Any compact Hausdorff right topological semigroup $(T,+)$ has a smallest two sided ideal

$$\begin{aligned} K(T) &= \bigcup\{L : L \text{ is a minimal left ideal of } T\} \\ &= \bigcup\{R : R \text{ is a minimal right ideal of } T\}. \end{aligned}$$

Given a minimal left ideal L and a minimal right ideal R, $L \cap R$ is a group, and in particular contains an idempotent. An idempotent in $K(T)$ is called a *minimal* idempotent. If p and q are idempotents in T, we write $p \leq q$ if and only if $p + q = q + p = p$. An idempotent is minimal with respect to this relation if and only if it is a member of the smallest ideal. See [9] for an elementary introduction to the algebra of βS and for any unfamiliar details.

Definition 1.2. Let $(S, +)$ be an infinite discrete semigroup. A set $C \subseteq S$ is central if and only if there is some minimal idempotent p in $(\beta S, +)$ such that $C \in p$. C is called central* set if it belongs to every minimal idempotent of $(\beta S, +)$.

We will be considering semigroups which are dense in $\big((0, \infty), +\big)$. Here "dense" means with respect to the usual topology on $\big((0, \infty), +\big)$. When passing to the Stone–Čech compactification of such a semigroup S, we deal with S_d which is the set S with the discrete topology.

Definition 1.3. If S is a dense subsemigroup of $\big((0, \infty), +\big)$, then

$$0^+(S) = \{p \in \beta S_d : (\forall \epsilon > 0)(S \cap (0, \epsilon) \in p)\}.$$

It was proved in [6, Lemma 2.5], that $0^+(S)$ is a compact right topological subsemigroup of $(\beta S_d, +)$. It was also noted therein $0^+(S)$ is disjoint from $K(\beta S_d)$, and hence gives some new information which is not available from $K(\beta S_d)$. Being a compact right topological semigroup, $0^+(S)$ contains minimal idempotents. In [1], the authors applied the algebraic structure of $0^+(S)$ in their investigation of image partition regularity near zero of finite and infinite matrices. Moreover in [3], the algebraic structure of $0^+(\mathbb{R})$ has been used to investigate image partition regularity of matrices with real entries from $\mathbb{R}$.

Definition 1.4. Let S be a dense subsemigroup of $\big((0, \infty), +\big)$, A set $C \subset S$ is central near 0 if and only if there is some minimal idempotent p in $0^+(S)$ such that $C \in p$. C is called central* set near zero if it belongs to every minimal idempotent of $0^+(S)$.

Next we present some well known characterizations of image partition regularity of matrices.

Theorem 1.1 ([7, Theorem 2.10])**.** *Let $u, v \in \mathbb{N}$ and let M be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) *M is image partition regular.*
(b) *For every central subset C of $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ such that*

$$M\vec{x} \in C^u.$$

(c) *For every central subset C of $\mathbb{N}$, $\{\vec{x} \in \mathbb{N}^v$: such that $M\vec{x} \in C^u\}$ is central in $\mathbb{N}^v$.*

(d) *For each* $\vec{r} \in \mathbb{Q}^v \setminus \{\vec{0}\}$ *there exists* $b \in \mathbb{Q} \setminus 0$ *such that*

$$\begin{pmatrix} b\vec{r} \\ M \end{pmatrix}$$

is image partition regular.

(e) *For every central subset* C *of* $\mathbb{N}$*, there exists* $\vec{x} \in \mathbb{N}^v$ *such that*

$$\vec{y} = M\vec{x} \in C^u,$$

all entries of $\vec{x}$ *are distinct, and for all* $i, j \in \{1, 2, \ldots, u\}$*, if rows* i *and* j *of* M *are unequal, then* $y_i \neq y_j$*.*

In the paper [8], the authors presented some contrasts between finite and infinite image partition regular matrices and showed that some of the interesting properties of finite image partition regular matrices could not be generalized for infinite image partition regular matrices.

It is interesting to observe from Theorem 1.1(b) that, if M and N are finite image partition regular matrices, then the matrix

$$\begin{pmatrix} M & \mathbf{O} \\ \mathbf{O} & N \end{pmatrix}$$

is also image partition regular. But this property does not hold for infinite matrices as was shown in [8, Theorem 2.2].

Definition 1.5. Let $\vec{a}$ be a finite or infinite sequence in $\mathbb{Q}^+$ with only finitely many nonzero entries. Then $c(\vec{a})$ is the sequence obtained from $\vec{a}$ by deleting all zeroes and then deleting all adjacent repeated entries. The sequence $c(\vec{a})$ is the *compressed form* of $\vec{a}$. If $\vec{a} = c(\vec{a})$, then $\vec{a}$ is a *compressed sequence*.

Theorem 1.2 ([8, Theorem 2.2])**.** *Let* $\vec{b}$ *be a compressed sequence with entries from* $\mathbb{N}$ *such that* $\vec{b} \neq (n)$ *for any* $n \in \mathbb{N}$*. Let* M *be a matrix whose rows are all rows* $\vec{a} \in \mathbb{Q}^\omega$ *with only finitely many nonzero entries such that* $c(\vec{a}) = \vec{b}$*. Let* N *be the finite sums matrix.*

(a) *The matrices* M *and* N *are image partition regular.*

(b) *There is a subset* C *of* $\mathbb{N}$ *which is a member of every idempotent in* $\beta\mathbb{N}$ *(and is thus, in particular, central) such that for no* $\vec{x} \in \mathbb{N}^\omega$ *does one have* $M\vec{x} \in C^\omega$*.*

(c) *The matrix*

$$\begin{pmatrix} M & \boldsymbol{O} \\ \boldsymbol{O} & N \end{pmatrix}$$

is not image partition regular.

To overcome the above situation the following notion was introduced in [8, Definition 2.7].

Definition 1.6. Let M be an $\omega \times \omega$ matrix with entries from $\mathbb{Q}$. Then M is *centrally image partition regular* if and only if whenever C is a central set in $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^\omega$ such that $M\vec{x} \in C^\omega$.

Note that Definition 1.6 has a natural generalization for an arbitrary subsemigroup S of $\big((0,\infty),+\big)$, and hence forth we will abbreviate this by CIPR/S. Motivation behind the introduction of this new notion was that many good properties of finite image partition regular matrices could not be extended with respect to infinite image partition regular matrices.

It is easy to see that whenever M and N are *centrally image partition regular* matrices over any subsemigroup S of $\big((0,\infty),+\big)$, then so is

$$\begin{pmatrix} M & \mathbf{O} \\ \mathbf{O} & N \end{pmatrix}.$$

The above observation tells us that centrally image partition regular matrices are more natural candidate to generalize the properties of finite image partition regular matrices to infinite matrices.

In this paper we introduce another natural candidate to generalize the properties of finite image partition regularity near zero in case of infinite matrices. First we recall the following definition.

Definition 1.7. Let S be a dense subsemigroup of $(0,\infty),+)$ and let $u,v\in\mathbb{N}$ and let M be a $u\times v$ matrix with entries from $\mathbb{Q}$. The matrix M is image partition regular near zero over S if and only if whenever $r\in\mathbb{N}$, $\epsilon>0$ and $S=\bigcup_{i=1}^{r} C_i$, there exist $i\in\{1,2,...,r\}$ and $\vec{x}\in\mathbb{N}^v$ such that $M\vec{x}\in(C_i\cap(0,\epsilon))^u$.

Definition 1.8. Let M be an $\omega\times\omega$ matrix with entries from $\mathbb{Q}$ and let S be a dense subsemigroup of $\big((0,\infty),+\big)$. Then M is *centrally image partition regular near zero* over S if whenever C is a central set near zero in S, there exists $\vec{x}\in S^{\omega}$ such that $M\vec{x}\in C^{\omega}$.

Henceforth for arbitrary subsemigroup S of $\big((0,\infty),+\big)$, we will abbreviate *centrally image partition regular near zero* over S by CIPR/S_0.

It is a simple fact that if M and N are two centrally image partition regular near zero matrices over a dense subsemigroup S of $\big((0,\infty,+\big)$, then the diagonal sum

$$\begin{pmatrix} M & \mathbf{O} \\ \mathbf{O} & N \end{pmatrix}$$

is also a centrally image partition regular near zero matrix over S.

The following examples show that there exist infinite matrices which are centrally image partition regular over $\mathbb{Q}^+$ but not centrally image partition regular near zero over $\mathbb{Q}^+$ and vice versa.

Example 1.3. *Let*

$$M=\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & \dots \\ 4 & 0 & 1 & 0 & \dots \\ 8 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then M is CIPR/$\mathbb{Q}^+$ but is not CIPR/$\mathbb{Q}_0^+$.

Proof. To see that M is centrally image partition regular matrix, let C be any central set in $\mathbb{Q}^+$ and pick a sequence $\langle y_n\rangle_{n=0}^{\infty}$ in C such that for each $n \in \mathbb{N}$, $y_n > 2^n y_0$. Let $x_0 = y_0$ and for each $n \in \mathbb{N}$, let $x_n = y_n - 2^n y_0$. Then $M\vec{x} = \vec{y}$.

Now $(0,1) \cap \mathbb{Q}^+$ is a central set near zero in $\mathbb{Q}^+$. Suppose one has $\vec{x} \in (\mathbb{Q}^+)^\omega$ such that $\vec{y} = M\vec{x} \in \big((0,1)\cap\mathbb{Q}^+\big)^\omega$. Then $x_0 = y_0 > 0$. Pick $k \in \mathbb{N}$ such that $2^k x_0 > 1$. Then $y_k = 2^k x_0 + x_k > 1$, a contradiction. □

Example 1.4. *Let*

$$M = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & \dots \\ 1/3 & 0 & -1 & 0 & 0 & \dots \\ 1/5 & 0 & 0 & -1 & 0 & \dots \\ 1/7 & 0 & 0 & 0 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then M is CIPR/$\mathbb{Q}_0^+$ but is not CIPR/$\mathbb{Q}^+$.

Proof. To see that M is not CIPR/$\mathbb{Q}^+$, we have to find a central set C in $\mathbb{Q}^+$ such that for no $\vec{x} \in (\mathbb{Q}^+)^\omega$ we have $\vec{y} = M\vec{x} \in C^\omega$. Let us take $C = \{x \in \mathbb{Q} : x > 1\}$. Then C is an ideal of $(\mathbb{Q}^+, +)$ and so, by [9, Theorem 2.19], $\overline{C}$ is an ideal of βQ_d^+ and therefore C is central, in fact central*. Suppose one has $\vec{x} \in (\mathbb{Q}^+)^\omega$ with $\vec{y} = M\vec{x} \in C^\omega$. Pick $n \in \mathbb{N}$ such that $2n-1 > x_0$. Then $y_{n-1} = \big(1/(2n-1)\big)x_0 - x_n < 1$, a contradiction.

To see that M is CIPR/$\mathbb{Q}_0^+$ near zero let C be a central set near zero in $\mathbb{Q}^+$. Note that $0 \in c\ell C$ and pick a sequence $\langle y_n\rangle_{n=0}^{\infty}$ in C which converges to 0. We may assume that for each n, $y_n < 1/(2n+1)$. Let $x_0 = 1$ and for $n \in \mathbb{N}$, let $x_n = 1/(2n-1) - y_{n-1}$. Then $M\vec{x} = \vec{y} \in C^\omega$. □

In [8], we have seen that finite image partition regular matrices satisfy some interesting properties that are not satisfied by infinite image partition regular matrices. In this paper, we will show this behaviour is also true in the case of image partition regularity near zero. This is why we introduced the notion of centrally image partition regularity near zero. In Section 2 of this paper, we first prove that for any two infinite image partition regular matrices near zero M and N, over $\mathbb{D}^+$, their diagonal sum

$$\begin{pmatrix} M & \mathbf{O} \\ \mathbf{O} & N \end{pmatrix}$$

may not be image partition regular near zero over $\mathbb{D}^+$. Also we show that infinite matrices which are image partition regular near zero can be extended by finite ones. Also we will show in Proposition 2.6 how new types of centrally infinite image partition regular matrices near zero are constructed from old ones.

In Section 3, we prove that a special type of infinite image partition regular matrices (i.e. segmented image partition regular matrices) are also centrally image partition regular near zero.

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2. Matrices centrally image partition regularity near zero

In Theorem 1.2 we have observed that there exist two infinite image partition regular matrices M and N over $\mathbb{N}$ such that their diagonal sum

$$\begin{pmatrix} M & \mathbf{O} \\ \mathbf{O} & N \end{pmatrix}$$

is not image partition regular matrix over $\mathbb{N}$. The central tool to prove the above Theorem is the Milliken–Taylor separating theorem [4, Theorem 3.2]. Recently in [11], a Milliken–Taylor separating theorem has been proved for dyadic rational numbers which we will employ to prove a generalization generalization of Theorem 1.2. First we recall some Definitions from [11].

Definition 2.1. The set of dyadic rational numbers is
$\mathbb{D} = \{\frac{m}{2^t} : m \in \mathbb{Z} \text{ and } t \in \omega\}$.

We will be considering $\mathbb{D}^+$, the set of positive numbers contained in $\mathbb{D}$.

Definition 2.2. Let $x \in \mathbb{D}^+$. The *support* of x, denoted $\text{supp}(x)$, is the unique finite nonempty subset of $\mathbb{Z}$ such that $x = \sum_{t \in \text{supp}(x)} 2^t$.

Definition 2.3. Given a binary number, an *even* 0-*block* is the occurrence of a positive even number of consecutive zeros between two consecutive ones.

For $x \in \mathbb{D}^+$, define the *start* of x as the position of the first 1 appearing in x moving from left to right and the *end* as the position of the last 1. The formal definition is the following.

Definition 2.4. Let $x \in \mathbb{D}^+$. Then $x = \sum_{t \in \text{supp}(x)} 2^t$ where

$$\text{supp}(x) \in \mathcal{P}_f(\mathbb{Z}).$$

Define the *start* of x as $\max\text{supp}(x)$ and the *end* as $\min\text{supp}(x)$.

Now we present the following Proposition from [11, Proposition 2.12] that plays the key role to prove the following Theorem 2.2. Let us first introduce the following definition.

Definition 2.5. Let $m \in \omega$, let $\vec{a} = \langle a_i \rangle_{i=0}^m$ be a sequence in $\mathbb{Q}^+$, and let $\vec{x} = \langle x_n \rangle_{n=0}^\infty$ be a sequence in the dense subsemigroup S of $\mathbb{R}^+$. The

Milliken–Taylor system determined by $\vec{a}$ and $\vec{x}$ will be denoted by $MT(\vec{a},\vec{x})$, and defined as

$$\Bigg\{\sum_{i=0}^{m} a_i \cdot \sum_{t\in F_i} x_t : \text{ each } F_i \in \mathcal{P}_f(\omega) \text{ and if } i<m, \text{ then } \max F_i < \min F_{i+1}\Bigg\}.$$

Proposition 2.1 ([11, Proposition 2.12])**.** *Let $\varphi(z)$ be the number of even* 0*-blocks between the start and end of z for any $z \in \mathbb{D}\cap(0,2)$. For $i \in \{0,1,2\}$, let*

$$C_i = \{c \in \mathbb{D}\cap(0,2) : \varphi(c) \equiv i \bmod 3\}.$$

Then $\{C_0, C_1, C_2\}$ is a partition of $\mathbb{D}\cap(0,2)$ such that no C_i contains $MT(\langle 1\rangle, \langle x_i\rangle_{i=1}^{\infty}) \cup MT(\langle 1,2\rangle, \langle y_i\rangle_{i=1}^{\infty})$ for any sequences $\langle x_i\rangle_{i=1}^{\infty}$ and $\langle y_i\rangle_{i=1}^{\infty}$ in $\mathbb{D}\cap(0,2)$.

Definition 2.6. Let $\vec{a}$ be a compressed sequence in $\mathbb{Q}^+$. A *Milliken–Taylor matrix determined by $\vec{a}$* is an $\omega\times\omega$ matrix M such that the rows of M are all rows which have only finitely many non zero entries whose compressed form is equal to $\vec{a}$.

Theorem 2.2. *Let M be the finite sum matrix and N be the Milliken–Taylor matrix determined by compressed sequence $\langle 1,2\rangle$. Then:*

(1) *The matrices M and N are image partition regular near zero over $\mathbb{D}^+$.*

(2) *The matrix*

$$\begin{pmatrix} M & \boldsymbol{O} \\ \boldsymbol{O} & N \end{pmatrix}$$

is not image partition regular near zero over $\mathbb{D}^+$.

(3) *The matrix N is not centrally image partition regular near zero over $\mathbb{D}^+$.*

Proof. Statement (1) follows from [1, Theorem 5.7].

(2) From Proposition 2.1 the matrix

$$\begin{pmatrix} M & \mathbf{O} \\ \mathbf{O} & N \end{pmatrix}$$

is not image partition regular near zero over $\mathbb{D}^+$.

(3) Suppose that N is centrally image partition regular near zero. Again M is centrally image partition regular near zero follows from [6, Theorem 3.1]. Then the matrix

$$\begin{pmatrix} M & \mathbf{O} \\ \mathbf{O} & N \end{pmatrix}$$

is centrally image partition regular near 0 and hence also image partition regular near 0. But this is a contradiction. Therefore, N is not centrally image partition regular near zero over $\mathbb{D}^+$. $\square$

Next, we show that infinite image partition regular near zero matrices can be extended by finite ones. The proof of the following Theorem is adapted from the proof of [8, Lemma 2.3] which is due to V. Rödl.

Theorem 2.3. *Let M be a finite image partition regular matrix over $\mathbb{N}$ of order $u \times v$ and N be an infinite image partition regular near zero matrix over a dense subsemigroup S of $\big((0,\infty,+\big)$. Then*

$$\begin{pmatrix} M & \boldsymbol{O} \\ \boldsymbol{O} & N \end{pmatrix}$$

is image partition regular near zero over S.

Proof. Let $r \in \mathbb{N}$ and let $\varphi : S \to \{1, 2, \ldots, r\}$ be an r-coloring of S. For $i \in \{1, 2, \ldots, r\}$, let $C_i = \{x \in S : \varphi(x) = i\}$. Let $\epsilon > 0$. By a standard compactness argument (see [9, Section 5.5]) there exists $k \in \mathbb{N}$ such that whenever $\{1, 2, \cdots, k\} = \bigcup_{i=1}^{r} D_i$ there exists $\vec{x} \in \{1, 2, \cdots, k\}^v$ and $i \in \{1, 2, \cdots, r\}$ such that $M\vec{x} \in (D_i)^u$. Pick $z \in S \cap (0, \epsilon/k)$.

Now color S with r^k colors via ψ as $S = \bigcup_{i=1}^{r^k} F_i$, where $\psi(x) = \psi(y)$ if and only if for all $t \in \{1, 2, \cdots, k\}$, $\varphi(tx) = \varphi(ty)$. Choose $\vec{y} \in S^\omega$ such that the entries of $N\vec{y}$ are in $F_i \cap (0, z)$ for some $i \in \{1, 2, \cdots, r^k\}$. Pick an entry a of $N\vec{y}$ and for each $i \in \{1, 2, \cdots, r\}$ let us set $D_i = \{t \in \{1, 2, \cdots, k\} : ta \in C_i\}$. Then $\{1, 2, \cdots, k\} = \bigcup_{i=1}^{r} D_i$. Note that since $a \in (0, z)$, $ta \in (0, \epsilon)$ for all $t \in \{1, 2, \cdots, k\}$. If we express this coloring as

$$\gamma : \{1, 2, \cdots, k\} \to \{1, 2, \cdots, r\},$$

then $\gamma(p) = \varphi(ap)$. So there exists $\vec{u} \in \{1, 2, \cdots, k\}^v$ and $i \in \{1, 2, \cdots, r\}$ such that $M\vec{u} \in (D_i)^u$ so that $a(M\vec{u}) \in (C_i)^u$. Now $a(M\vec{u}) = M(a\vec{u})$. Put $\vec{x} = a\vec{u}$. Then $M\vec{x} \in (C_i \cap (0, \epsilon))^u$. Choose an entry l of $M\vec{u}$ and let $j = \gamma(l)$.

Let $\vec{z} = \begin{pmatrix} a\vec{u} \\ l\vec{y} \end{pmatrix}$. We claim that for any row $\vec{w}$ of $\begin{pmatrix} M & O \\ O & N \end{pmatrix}$, $\varphi(\vec{w} \cdot \vec{z}) = j$. To observe this first assume that $\vec{w}$ is a row of $\begin{pmatrix} M & O \end{pmatrix}$, so that $\vec{w} = \vec{s}^\frown \vec{0}$, where $\vec{s}$ is a row of M. Then $\vec{w} \cdot \vec{z} = \vec{s} \cdot (a\vec{u}) = a(\vec{s} \cdot \vec{u})$. Therefore

$$\varphi(\vec{w} \cdot \vec{z}) = \varphi(a(\vec{s} \cdot \vec{u})) = \gamma(\vec{s} \cdot \vec{u}) = j.$$

Next assume that $\vec{w}$ is a row of $\begin{pmatrix} O & N \end{pmatrix}$, so that $\vec{w} = \vec{0}^\frown \vec{s}$ where $\vec{s}$ is a row of N. Then $\vec{w} \cdot \vec{z} = l(\vec{s} \cdot \vec{y})$. Now $\psi(\vec{s} \cdot \vec{y}) = \psi(a)$. So

$$\varphi(l(\vec{s} \cdot \vec{y})) = \varphi(la) = \gamma(l) = j. \qquad \square$$

We now present the following theorem and corollary in order to prove Proposition 2.6.

Theorem 2.4. *Let S be a subsemigroup of $\big((0,\infty,+\big)$ such that for any $\epsilon > 0$, $|(0,\epsilon)\cap S| = |S|$. Let $p \in K\big(0^+(S)\big)$, let $C \in p$, and let R be the minimal right ideal of $0^+(S)$ to which p belongs. Then there are at least $2^{\mathfrak{c}}$ idempotents in $K(0^+(S)) \cap R \cap \overline{C}$.*

Proof. Let $\mathcal{A} = \{(0,1/n)\cap S : n \in \mathbb{N}\}$ and apply [2, Theorem 2.3]. □

Corollary 2.5. *Let S be a subsemigroup of $\big((0,\infty,+\big)$ such that for any $\epsilon > 0$, $|(0,\epsilon)\cap S| = |S|$ and let C be a central set near zero. Then there exists a sequence $\langle C_n\rangle_{n=1}^{\infty}$ of pairwise disjoint central sets near zero in S with $\bigcup_{n=1}^{\infty} C_n \subseteq C$.*

Proof. By Theorem 2.4, there are at infinitely many idempotents in $\overline{C}$, hence $\overline{C}$ contains an infinite strongly discrete subset. (Alternatively, there are two minimal idempotents in $\overline{C}$ so that C can be split into two central sets near zero, C_1 and D_1. Then D_1 can be split into two central sets near zero, C_2 and D_2, and so on.) □

Proposition 2.6. *Let S be a subsemigroup of $\big((0,\infty,+\big)$ such that for any $\epsilon > 0$, $|(0,\epsilon)\cap S| = |S|$. For each $n \in \mathbb{N}$, let M_n be a centrally image partition regular near zero matrix over S. Then the matrix*

$$M = \begin{pmatrix} M_1 & 0 & 0 & \dots \\ 0 & M_2 & 0 & \dots \\ 0 & 0 & M_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

is also centrally image partition regular near zero .

Proof. Let C be a central sets near zero and choose by Corollary 2.5 a sequence $\langle C_n\rangle_{n=1}^{\infty}$ of pairwise disjoint central sets near zero in S with $\bigcup_{n=1}^{\infty} C_n \subseteq C$. For each $n \in \mathbb{N}$ choose $\vec{x}^{(n)} \in S^{\omega}$ such that $\vec{y}^{(n)} = M_n\vec{x}^{(n)} \in C_n^{\omega}$. Let

$$\vec{z} = \begin{pmatrix} \vec{x}^{(1)} \\ \vec{x}^{(2)} \\ \vdots \end{pmatrix}.$$

Then all entries of $M\vec{z}$ are in C. □

3. A class of infinite matrices that are centrally image partition regular near zero

We now present a class of image partition regular matrices, called the segmented image partition regular matrices, which were first introduced in [10]. Here, we show that these matrices are also centrally image partition regular. Let us first recall the definition of a first entry matrix.

Definition 3.1. Let M be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then M is a first entries matrix if:

(1) No row of M is $\vec{0}$.
(2) The first nonzero entry of each row is positive.
(3) If the first nonzero entries of any two rows occur in the same column, then they are equal.

If M is a first entries matrix and c is the first nonzero entry of some row of M, then c is called a first entry of M.

Definition 3.2. Let M be an $\omega \times \omega$ matrix with entries from $\mathbb{Q}$. Then M is a segmented image partition regular matrix if and only if:

(1) No row of M is $\vec{0}$.
(2) For each $i \in \omega$, $\{j \in \omega : a_{i,j} \neq 0\}$ is finite.
(3) There is an increasing sequence $\langle \alpha_n \rangle_{n=0}^{\infty}$ in ω such that $\alpha_0 = 0$ and for each $n \in \omega$, $\{\langle a_{i,\alpha_n}, a_{i,\alpha_n+1}, a_{i,\alpha_n+2}, \ldots, a_{i,\alpha_{n+1}-1} \rangle : i \in \omega\} \setminus \{\vec{0}\}$ is empty or is the set of rows of a finite image partition regular matrix.

If each of these finite image partition regular matrices is a first entries matrix, then M is a segmented first entries matrix. If also the first nonzero entry of each $\langle a_{i,\alpha_n}, a_{i,\alpha_n+1}, a_{i,\alpha_n+2}, \ldots, a_{i,\alpha_{n+1}-1} \rangle$, if any, is 1, then M is a monic segmented first entries matrix.

The proof of the following theorem is adapted from the proof of [10, Theorem 3.2].

Theorem 3.1. *Let S be a dense subsemigroup of $\big((0,\infty,+\big)$ for which cS is central* near zero for every $c \in \mathbb{N}$ and let M be a segmented image partition regular matrix with entries from ω. Then M is centrally image partition regular near zero.*

Proof. Let $\vec{c}_0, \vec{c}_1, \vec{c}_2, \ldots$ denote the columns of M. Let $\langle \alpha_n \rangle_{n=0}^{\infty}$ be as in the definition of a segmented image partition regular matrix. For each $n \in \omega$, let M_n be the matrix whose columns are $\vec{c}_{\alpha_n}, \vec{c}_{\alpha_n+1}, \ldots, \vec{c}_{\alpha_{n+1}-1}$. Then the set of nonzero rows of M_n is finite and, if nonempty, is the set of rows of a finite image partition regular matrix. Let $B_n = (M_0\ M_1 \ldots M_n)$.

Let C be a central set near zero over S. Then there exists a minimal idempotent $p \in 0^+(S)$ such that $C \in p$. Let $C^\star = \{x \in C : -x + C \in p\}$. Then $C^\star \in p$ and, for every $x \in C^\star$, $-x + C^\star \in p$ by [9, Lemma 4.14]. Now the set of nonzero rows of M_n is finite and, if nonempty, is the set of rows of a finite image partition regular matrix over $\mathbb{N}$ and hence by [1, Theorem 2.3] IPR/S_0. Then by [1, Theorem 4.10], we can choose $\vec{x}^{(0)} \in S^{\alpha_1 - \alpha_0}$ such that, if $\vec{y} = M_0 \vec{x}^{(0)}$, then $y_i \in C^\star$ for every $i \in \omega$ for which the i^{th} row of M_0 is nonzero.

Assume inductively that for some $m \in \omega$, we have chosen

$$\vec{x}^{(0)}, \vec{x}^{(1)}, \ldots, \vec{x}^{(m)}$$

such that $\vec{x}^{(i)} \in S^{\alpha_{i+1}-\alpha_i}$ for every $i \in \{0, 1, \ldots, m\}$, and if

$$\vec{y} = B_m \begin{pmatrix} \vec{x}^{(0)} \\ \vec{x}^{(1)} \\ \vdots \\ \vec{x}^{(m)} \end{pmatrix},$$

then $y_j \in C^\star$ for every $j \in \omega$ for which the j^{th} row of B_m is nonzero.

Let $D = \{j \in \omega : \text{ row } j \text{ of } B_{m+1} \text{ is not } \vec{0}\}$ and note that for each $j \in \omega$, $-y_j + C^\star \in p$. (Either $y_j = 0$ or $y_j \in C^\star$.) By [1, Theorem 4.10] we can choose $\vec{x}^{(m+1)} \in S^{\alpha_{m+2}-\alpha_{m+1}}$ such that, if $\vec{z} = M_{m+1}\vec{x}^{(m+1)}$, then $z_j \in C^\star \cap \bigcap_{t \in D}(-y_t + C^\star)$ for every $j \in D$.

Thus we can choose an infinite sequence $\langle \vec{x}^{(i)} \rangle_{i \in \omega}$ such that, for every $i \in \omega$, $\vec{x}^{(i)} \in S^{\alpha_{i+1}-\alpha_i}$, and, if

$$\vec{y} = B_i \begin{pmatrix} \vec{x}^{(0)} \\ \vec{x}^{(1)} \\ \vdots \\ \vec{x}^{(i)} \end{pmatrix},$$

then $y_j \in C^\star$ for every $j \in \omega$ for which the j^{th} row of B_i is nonzero.

Let

$$\vec{x} = \begin{pmatrix} \vec{x}^{(0)} \\ \vec{x}^{(1)} \\ \vec{x}^{(2)} \\ \vdots \end{pmatrix}$$

and let $\vec{y} = M\vec{x}$. We note that, for every $j \in \omega$, there exists $m \in \omega$ such that y_j is the j^{th} entry of

$$B_i \begin{pmatrix} \vec{x}^{(0)} \\ \vec{x}^{(1)} \\ \vdots \\ \vec{x}^{(i)} \end{pmatrix}$$

whenever $i > m$. Thus all the entries of $\vec{y}$ are in $C^\star$. □

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