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A-polynomials of a family of two-bridge knots

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Abstract. The $J(k,l)$ knots, often called the double twist knots, are a subclass of two-bridge knots which contains the twist knots. We show that the A-polynomial of these knots can be determined by an explicit resultant. We present this resultant in two different ways. We determine a recursive definition for the A-polynomials of the $J(4,2n)$ and $J(5,2n)$ knots, and for the canonical component of the A-polynomials of the $J(2n,2n)$ knots. Our work also recovers the A-polynomials of the $J(1,2n)$ knots, and the recursive formulas for the A-polynomials of the $A(2,2n)$ and $A(3,2n)$ knots as computed by Hoste and Shanahan.

Contents

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1. Introduction

The A-polynomial of a 3-manifold M^3 with a single torus cusp was introduced in [4]. It is a two variable polynomial, usually written in terms of the variables M and L, which encodes how eigenvalues of a fixed meridian and longitude are related under representations from $\pi_1(M^3)$ into $\mathrm{SL}_2(\mathbb{C})$. This polynomial is closely related to the $\mathrm{SL}_2(\mathbb{C})$ character variety of M^3, and for hyperbolic manifolds it encodes information about the deformation of the hyperbolic structure of M^3, the existence of nonhyberbolic fillings of M^3, and can be used to determine boundary slopes of essential surfaces in M^3. Specifically, the boundary slopes of the Newton polygon of the A-polynomial are the boundary slopes detected by the $\mathrm{SL}_2(\mathbb{C})$ character variety [4], and the Newton polygon is dual to the fundamental polygon of the Culler–Shalen seminorm [7]. This seminorm can be used to classify finite and exceptional surgeries of M^3 [1].

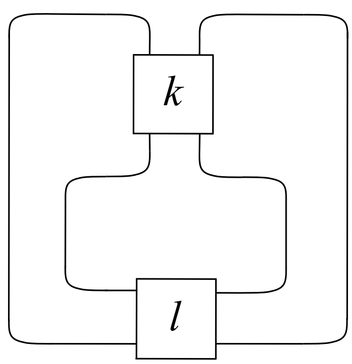

FIGURE 1. The knot $J(k, l)$

This polynomial has proven difficult to compute; data has been collected for most knots up to nine crossings (see Knot Info [3] and calculations by Marc Culler [6]), and a few families of knot complements in S^3 have proven amenable to computations. Tamura and Yokota [13] computed the A-polynomials of the $(-2, 3, 3+2n)$ pretzel knots. Garoufalidis and Mattman [9] studied these A-polynomials, showing that they satisfy a specific type of linear recurrance relation. Hoste and Shanahan [10] established a recursively defined formula for the A-polynomials of the twist knots and the $J(3, 2n)$ knots. This paper shows that the A-polynomial of the $J(k, l)$ knots can be computed as an explicit resultant, and recovers the work of Hoste and Shanahan. We also recursively determine the A-polynomial for the $J(4, 2n)$ and $J(5, 2n)$ knots, and the canonical component of the A-polynomial for the $J(2n, 2n)$ knots.

We consider the two-bridge knots $J(k, l)$ as described in Figure 1 where k and l are integers denoting the number of half twists in the labeled boxes; positive numbers correspond to right-handed twists and negative numbers correspond to left-handed twists. Such a projection determines a knot if kl is even, and we can reduce to considering the $J(k, 2n)$ knots as $J(k, l) = J(l, k)$. The knot $J(-k, -l)$ is the mirror image of the knot $J(k, l)$, and as a result the A-polynomial of $J(-k, -l)$ is the A-polynomial of $J(k, l)$ with each M replaced by a M^{-1} [5]. Therefore we may assume k or l is positive. As discussed in §3 the $J(k, l)$ knots are a particularly attractive family as the fundamental groups of their complements have a relatively simple form. In Section 5.1 we show that the contribution to the A-polynomial from reducible representations is the term $L - 1$. Therefore our main theorems focus on the term of the A-polynomials corresponding to irreducible representations. We write $A(k, 2n)$ to denote the contribution of factors of the A-polynomial corresponding to irreducible representations. (This is well-defined up to multiplication by elements in $\mathbb{Q}$ and by powers of M and L.)

Our first main theorem is the following, where the polynomials $F_{k,n}$ and $G_{k,n}$ are defined in Definition 4.6 and 4.12.

Theorem 1.1. *Assume $k \neq -1, 0, 1$ and $n \neq 0$. Then $A(k, 2n)$ is the common vanishing set of $F_{k,n}(r)$ and $G_{k,n}(r)$.*

The A-polynomial can be determined by the resultant of these two polynomials, eliminating the variable r. This can be done, for example, using the Sylvester matrix and has been implemented in many computer algebra programs. The degree of $F_{k,n}$ as function of r is roughly $\frac{1}{4}|n|k^2$ and the degree of $G_{k,n}$ is roughly $\frac{1}{2}|k|$. The next theorem demonstrates that the resultant can be computed using the polynomial $H_{k,n}$, of degree about $2|n|$, which is defined in Definition 6.8, in place of $F_{k,n}$. This resultant will differ from the resultant of $F_{k,n}$ and $G_{k,n}$ by factors of $\beta = M^2 + \ell$, $\gamma = (M^2 - 1)(\ell - 1)$, and $\delta = M^2\ell + 1$. As in Definition 3.3, $\ell = L$ if k is even and $\ell = LM^{4n}$ if k is odd.

Theorem 1.2. *Let $\epsilon = 1$ if k is positive and 0 if k is negative, let $\epsilon' = 1$ if k is even and 0 if k is odd, and let $\epsilon'' = (1-\epsilon)(1-\epsilon')$. For $k \neq -1, 0, 1$ and $n \neq 0$ we have*

$$\operatorname{Res}(H_{k,n}(r), G_{k,n}(r)) = \begin{cases} A(k,2n)\gamma^{|m|+\epsilon'|n|-\epsilon''} & nk < 0 \\ A(k,2n)\beta\gamma^{|m|+\epsilon'|n|-\epsilon''} & n,k > 0 \\ A(k,2n)\delta\gamma^{|m|+\epsilon'|n|-\epsilon''} & n,k < 0. \end{cases}$$

Theorem 1.1 is proven in Section 4.4 and Theorem 1.2 is proven in Section 6.

In addition, we perform explicit computations in some special cases. If $k = 0$, then $J(k,l)$ is the unknot and the A-polynomial is $L - 1$. The knots $J(\pm 1, 2n)$ are the torus knots whose A-polynomials are well known [4]. We present their A-polynomials in Theorem 7.1 for completeness. In Theorem 7.2 and Theorem 7.3 we recover the recursive formulas of Hoste and Shanahan [10] for the A-polynomials of the twist knots (the $J(\pm 2, 2n)$ knots) and the $J(\pm 3, 2n)$ knots. We also compute recursive formulas for the A-polynomials of the $J(\pm 4, 2n)$ knots and the $J(\pm 5, 2n)$ knots. These are given in Theorem 7.5 and Theorem 7.7, respectively. Finally, we consider the $J(2n, 2n)$ knots. These knots have an additional symmetry that the other $J(k,l)$ knots do not have, seen by flipping the corresponding four-plat upside down. This symmetry effectively factors the representation variety (see [11]) and this factorization can be seen on the level of the A-polynomial as well. A canonical component of the A-polynomial is an irreducible polynomial which contains the image of the discrete and faithful representation. We determine a recursive formula for the canonical component for these knots, given in Theorem 7.9.

2. The *A*-polynomial

We follow the construction of the A-polynomial given in [5]. We consider a knot K in S^3, and let μ be a natural (oriented) meridian of K and λ a natural (oriented) longitude. Let Γ be the fundamental group of the complement of the knot K in S^3. For an oriented loop $\alpha \in (S^3 - K)$ we write $[\alpha] \in \Gamma$ to be a base pointed homotopy class. We define R_U to be the subset of the affine algebraic variety

$$R = \operatorname{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$$

consisting of all representations ρ such that $\rho([\mu])$ and $\rho([\lambda])$ are upper triangular. The set R_U is an affine algebraic variety as well, since it simply has two additional equations specifying that these $(2,1)$ entries equal zero. Define the eigenvalue map

$$\xi = (\xi_\mu \times \xi_\lambda) : R_U \to \mathbb{C}^2$$

by setting $\xi(\rho) = (e_\mu, e_\lambda)$ where e_μ is an eigenvalue of $\rho([\mu])$, and e_λ is an eigenvalue of $p([\lambda])$, both chosen consistently. We let M be the $(1,1)$ entry of

$\rho([\mu])$ and L be the $(1,1)$ entry of $\rho([\lambda])$. We will use an alternate longitude, denoted by ℓ, which we define below.

If C is an algebraic component of R_U, then the Zariski closure of $\xi(C)$, $\overline{\xi(C)}$, is an algebraic subset of $\mathbb{C}^2$. If $\overline{\xi(C)}$ is a curve, then there is a polynomial that defines $\overline{\xi(C)}$ which is unique up to constant multiples. The A-polynomial is defined as the product of all such polynomials. This polynomial may be taken to have integral coefficients with content zero, so it is well-defined up to sign. (See [4].) There are an infinite number of abelian representations of Γ into $\mathrm{SL}_2(\mathbb{C})$. In such a representation every element of the commutator subgroup, including the longitude, is sent to the identity matrix. Therefore, $L-1$ is always a factor of the A-polynomial. This factor is often ignored in the definition of the A-polynomial. For the $J(k,l)$ knots, we will show that all reducible representations correspond only to the factor $L-1$, and we will usually omit this factor except when referring to the unknot.

3. The $J(k,l)$ knots

There are many relations amongst the $J(k,l)$ knots. The knot $J(k,l)$ is ambient isotopic to the knot $J(l,k)$, so we will consider the knots $J(k,2n)$ as if kl is odd then $J(k,l)$ is a two component link. The twist knots are the knots $J(\pm 2,l)$. The figure-eight and the trefoil are $J(2,-2)$ and $J(2,2)$, respectively. Also, $J(-k,-l)$ is the mirror image of the knot $J(k,l)$. The A-polynomial of a knot and its reflection differ only by replacing M with M^{-1} [5], so we may assume that k or l is positive. The $J(k,l)$ knots are hyperbolic unless $|k|$ or $|l|$ is less than 2 or $k=l=\pm 2$.

We turn to the fundamental group of $S^3 - J(k,l)$. (See [10, 11].)

Proposition 3.1. *The fundamental group of the complement of the knot $J(k,2n)$ in S^3 is isomorphic to the group*

$$\Gamma(k,2n) = \langle a,b : aw_k^n = w_k^n b\rangle$$

where

$$w_k = \begin{cases} (ab^{-1})^m(a^{-1}b)^m & \text{if } k=2m \\ (ab^{-1})^m ab(a^{-1}b)^m & \text{if } k=2m+1. \end{cases}$$

Among the relations mentioned above, the knot $J(k,l)$ is ambient isotopic to $J(l,k)$, so the corresponding groups are isomorphic, but the above presentations are different.

For a word $v \in \Gamma(k,2n)$ written in powers of a and b, let v^* refer to the word obtained by reading v backwards.

Definition 3.2. Let $\epsilon(v)$ to be the exponent sum of v, written as a word in a and b, and let $\epsilon(k,n)=\epsilon(w_k^n)$. (So that $\epsilon(2m,n)=0$ and $\epsilon(2m+1,n)=2n$.)

A natural meridian, μ, of the knot corresponds to a and a natural longitude, λ, to $w_k^n(w_k^n)^*a^{-2\epsilon(k,n)}$. That is, $[\mu]=a$ and $[\lambda]=w_k^n(w_k^n)^*a^{-2\epsilon(k,n)}$. We will also make use of an alternative longitude, which we now define. (See [10].)

Definition 3.3. Let λ_1 be the longitude corresponding to the word $w_k^n(w_k^n)^*$, so that $[\lambda_1] = w_k^n(w_k^n)^*$, and let ℓ be a preferred eigenvalue of $[\lambda_1]$. (This is chosen so that if $k = 2m$ we have $\ell = L$ and if $k = 2m+1$ then $\ell = LM^{4n}$.)

4. Representations

First, we define a few polynomials we will use throughout.

Definition 4.1. The j^{th} Fibonacci polynomial, f_j, is the Chebyshev polynomial defined recursively by the relation

$$f_{j+1}(x) + f_{j-1}(x) = xf_j(x)$$

and initial conditions $f_0(x) = 0$, and $f_1(x) = 1$. With the substitution $x = y + y^{-1}$ we have $f_j(y + y^{-1}) = (y^j - y^{-j})/(y - y^{-1})$.

Additionally, define the polynomials $g_j(x) = f_j(x) - f_{j-1}(x)$.

We will use the following several times.

Lemma 4.2. *If $j \neq 0, 1$ then the polynomials $f_j(x)$ and $f_{j-1}(x)$ share no common factors.*

Proof. Let $x = y + y^{-1}$ so that $f_j(x) = (y^j - y^{-j})/(y - y^{-1})$. A root of $f_j(x)$ determines a solution to $y^{2j} = 1$. Similarly, a root of $f_{j-1}(f)$ determines a solution to $y^{2j-2} = 1$. Since $\gcd(2j, 2j-2) = 2$ the only simultaneous solutions are $y = \pm 1$. It follows that if $f_j(x)$ and $f_{j-1}(x)$ share a root, it must be $x = \pm 2$. The Fibonacci recursion implies that $f_j(2) = j$ and $f_j(-2) = \pm j$. Therefore, as $j \neq \pm 1$, $x = \pm 2$ is not a root and the polynomials are relatively prime. $\square$

Furthermore, we define certain terms in M and L to shorten some expressions.

Definition 4.3. Let

$$\begin{aligned}
\alpha &= (M^2 + 1)(\ell + 1), \\
\beta &= M^2 + \ell, \\
\gamma &= (M^2 - 1)(\ell - 1), \\
\delta &= M^2\ell + 1, \\
\sigma &= \beta^2 + \delta^2
\end{aligned}$$

and

$$\tau = (\ell - 1)^2 M^{-2} + 4\ell + 2(\ell + 1)^2 M^2 + 4M^4\ell + (\ell - 1)^2 M^6.$$

In many of the calculations to follow, we will use the Cayley–Hamilton theorem, which we now explicitly state for matrices in $\mathrm{SL}_2(\mathbb{C})$.

Theorem 4.4 (Cayley–Hamilton)**.** *Let $X \in \mathrm{SL}_2(\mathbb{C})$ with trace x. For any integer j we have*

$$X^j = f_j(x)X - f_{j-1}(x)I$$

where I is the 2×2 identity matrix.

For a matrix X, we will use X_{ij} to denote the $(i,j)^{th}$ entry of X.

As the eigenvalue map is invariant under conjugation, we consider representations $\rho : \Gamma(k, 2n) \to \mathrm{SL}_2(\mathbb{C})$ up to conjugation. A representation $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ is called reducible if all images share a one-dimensional eigenspace. Otherwise, a representation is called irreducible.

4.1. Reducible representations. A reducible representation of $\Gamma(k, 2n)$ can be conjugated so that it is upper triangular with

$$\rho(a) = A = \begin{pmatrix} M & s \\ 0 & M^{-1} \end{pmatrix} \quad \text{and} \quad \rho(b) = B = \begin{pmatrix} M & t \\ 0 & M^{-1} \end{pmatrix}.$$

We include the calculation of these terms of the A-polynomial for completeness.

Proposition 4.5. *The contribution to the A-polynomial from the reducible representations is the factor $L - 1$.*

Proof. We use the presentation of the fundamental group from Proposition 3.1. Let $W_k = \rho(w_k)$, $W_k^n = \rho(w_k^n)$, and let $\mathbf{0}$ denote the 2×2 zero matrix.

First, consider $k = 2m$. The alternate longitude corresponds to $\rho([\lambda_1]) = W_k^n(W_k^n)^*$ and is the identity in this case. That is, $\ell = 1$ for these representations. The defining word, in terms of matrices is $AW_{2m}^n - W_{2m}^n B = \mathbf{0}$. The matrix on the left is identically zero except for the $(1,2)$ entry which is $(nmM^2 + (1 - 2nm) + nmM^{-2})(s - t)$. This equals zero for infinitely many representations, and M can be any value independent of ℓ. Therefore, these representations contribute the factor $\ell - 1$, which is $L - 1$ to the A-polynomial.

Similarly, consider $k = 2m+1$. The relation gives $AW_k^n - W_k^n B$, a matrix whose only nonzero entry is the $(1,2)$ entry which is

$$(s - t)\big(M^{2n-1} + M^{-2n+1} - m(M - M^{-1})(M^{2n} - M^{-2n})\big)/(M + M^{-1})$$

We conclude that there are infinitely many representations, and that M can be any value, independent of ℓ. The $(1,1)$ entry of $\rho([\lambda_1])$ is always M^{4n}, so $\ell = M^{4n}$. This is $L = 1$, and gives the factor $L - 1$. $\square$

4.2. Irreducible representations. To compute the A-polynomial, we restrict attention to the irreducible representations $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ which map the meridian and longitude to upper triangular matrices. For such a representation, there are M and L in $\mathbb{C}^*$ such that

$$A = \rho([\mu]) = \rho(a) = \begin{pmatrix} M & \star \\ 0 & M^{-1} \end{pmatrix}$$

and

$$\Lambda = \rho([\lambda]) = \rho(w_k^n (w_k^n)^* a^{-2\epsilon(k,n)}) = \begin{pmatrix} L & \star \\ 0 & L^{-1} \end{pmatrix}.$$

The alternative longitude, λ_1, corresponds to a matrix with $(1,1)$ entry $\ell = LM^{2\epsilon(k,n)}$. Let $W_k = \rho(w_k)$, $W = \rho(w_k^n)$ and $W^* = \rho((w_k^n)^*)$.

Up to conjugation all irreducible representations $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ are given by

$$A = \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix} \quad \text{and} \quad B = \rho([b]) = \begin{pmatrix} M & 0 \\ 2-r & M^{-1} \end{pmatrix}$$

where $r \neq 2$. The (2,1) entry of $W = \rho(w_k^n)$, W_{21}, equals $(2-r)W_{12}$ for these groups. Therefore, the words W and W^* can be written as

$$W = \begin{pmatrix} W_{11} & W_{12} \\ (2-r)W_{12} & W_{22} \end{pmatrix} \quad \text{and} \quad W^* = \begin{pmatrix} W'_{22} & W'_{12} \\ (2-r)W'_{12} & W'_{11} \end{pmatrix}$$

where W'_{ij} is W_{ij} with all M's exchanged with M^{-1}'s. (This follows from [10].)

The relation $aw_k^n = w_k^n b$ implies that, on the level of matrices, $AW = WB$. It follows by direct computation that the variables M and r define a valid representation if

$$(M - M^{-1})W_{12} + W_{22} = 0.$$

(In fact, Riley [12] shows that a similar, more general statement is true for all two-bridge knots.)

Definition 4.6. Let $F_{k,n}(r,t)$ be defined by

$$F_{k,n}(r) = f_n(t_k(r))F_{k,1}(r) - f_{n-1}(t_k(r))$$

with $F_{k,1}(r)$ defined by

$$F_{2m,1}(r) = f_m(r)g_m(r)(M^2 + M^{-2} - r) + 1$$
$$F_{2m+1,1}(r) = -f_m(r)g_{m+1}(r)(M^2 + M^{-2} - r) + 1$$

and $t = t_k(r)$ defined by

$$t_{2m}(r) = -f_m(r)(g_{m+1}(r) - g_m(r))(M^2 + M^{-2} - r) + 2$$
$$t_{2m+1}(r) = g_{m+1}(r)^2(M^2 + M^{-2} - r) + 2.$$

By [11] (Proposition 3.7) the condition for ρ to be a representation can be encoded by these polynomials. Specifically, we have the following.

Proposition 4.7. *The map ρ above determines a representation of $\Gamma(k, 2n)$ into $\mathrm{SL}_2(\mathbb{C})$ if and only if $F_{k,n}(r) = 0$.*

4.3. The longitude. The equation $F_{k,n}(r) = 0$ with $r \neq 2$ holds if and only if ρ is an irreducible representation of $\Gamma(k, 2n)$. We now turn to the condition that the boundary subgroup is upper triangular. A natural meridian corresponds to the group element a, which was already taken to map to an upper triangular matrix A. Therefore, the remaining condition is to ensure that an element of $\Gamma(k, 2n)$ corresponding to a longitude, either λ or λ_1, also maps to an upper triangular matrix. The image $\rho([\lambda])$ can be explicitly computed, as stated in the following lemma. We will let $\Lambda = \rho([\lambda])$ henceforth.

Lemma 4.8. *With $W = \rho(w_k^n)$, $\Lambda = \rho([\lambda])$,*

$$\begin{aligned}
\Lambda_{11} &= \big(W_{11}W'_{22} + W_{12}W'_{12}(2-r)\big)M^{-2\epsilon(k,n)} \\
\Lambda_{12} &= \big(W_{11}W'_{22} + W_{12}W'_{12}(2-r)\big)f_{-2\epsilon(k,n)}(M + M^{-1}) \\
&\quad + \big(W_{11}W'_{12} + W_{12}W'_{11}\big)M^{2\epsilon(k,n)} \\
\Lambda_{21} &= (2-r)\big(W_{12}W'_{22} + W_{22}W'_{12}\big)M^{-2\epsilon(k,n)} \\
\Lambda_{22} &= (2-r)\big(W_{12}W'_{22} + W_{22}W'_{12}\big)f_{-2\epsilon(k,n)}(M + M^{-1}) \\
&\quad + \big(W_{22}W'_{11} + W_{12}W'_{12}(2-r)\big)M^{2\epsilon(k,n)}.
\end{aligned}$$

Proof. Since $[\lambda] = w_k^n(w_k^n)^*a^{-2\epsilon(k,n)}$ one can compute $\Lambda = WW^*A^{-2\epsilon(k,n)}$ explicitly. A direct computation using the Cayley–Hamilton theorem confirms the above. □

From the lemma above, we now deduce an algebraic condition for Λ to be upper triangular.

Lemma 4.9. *For an irreducible representation ρ, $\Lambda = \rho([\lambda])$ is upper triangular if and only if $W_{12}\ell + W'_{12} = 0$, where $\ell = LM^{2\epsilon(k,n)}$.*

Proof. By Lemma 4.8, $\Lambda_{21} = 0$ reduces to

$$W_{12}W'_{22} + W_{22}W'_{12} = 0.$$

Moreover, as $L = \Lambda_{11}$ by Lemma 4.8,

$$L = \big(W_{11}W'_{22} + (2-r)W_{12}W'_{12}\big)M^{-2\epsilon(k,n)}$$

from which it follows that $M^{2\epsilon(k,n)}L = W_{11}W'_{22} + (2-r)W_{12}W'_{12}$. The alternate longitude is related by $[\lambda_1] = a^{2\epsilon(k,n)}[\lambda]$. By the Cayley–Hamilton theorem $M^{2\epsilon(k,n)} = (A^{2\epsilon(k,n)})_{11}$, so that as $\rho(a)$ and $\rho([\lambda])$ are upper triangular

$$\ell = \rho([\lambda_1])_{11} = M^{2\epsilon(k,n)}L.$$

Therefore,

$$\ell = W_{11}W'_{22} + (2-r)W_{12}W'_{12}. \tag{1}$$

We consider this as the defining equation for ℓ.

Now we explore the condition for Λ to be upper triangular. We follow [10]. We multiply equation (1) by W_{12}, and have

$$W_{12}\ell = W_{11}W_{12}W'_{22} + (2-r)W_{12}^2W'_{12}.$$

Since $W_{12}W'_{22} = -W'_{12}W_{22}$ we rewrite this as

$$W_{12}\ell = -W_{11}W'_{12}W_{22} + (2-r)W_{12}^2W'_{12}.$$

Next, since $\det(W) = W_{11}W_{22} - (2-r)W_{12}^2 = 1$

$$W_{12}\ell = -W_{11}W'_{12}W_{22} + (W_{11}W_{22} - 1)W'_{12} = -W'_{12}.$$

Combining this, $W_{12}\ell + W'_{12} = 0$. As $\ell, M \in \mathbb{C}^*$ this condition is equivalent to $\Lambda_{21} = 0$. $\square$

Now we show that the condition for Λ to be upper triangular can be expressed in terms of the entries of W_k instead of $W = (W_k)^n$.

Lemma 4.10. *For an irreducible representation ρ, $\Lambda = \rho([\lambda])$ is upper triangular if and only if $(W_k)_{12}\ell + (W_k)'_{12} = 0$, where $\ell = LM^{2\epsilon(k,n)}$.*

Proof. Using the Cayley–Hamilton theorem,

$$W = (W_k)^n = f_n(t_k(r))W_k - f_{n-1}(t_k(r))I.$$

Therefore, $W_{12} = f_n(t_k(r))(W_k)_{12}$ and $W'_{12} = f_n(t_k(r))(W_n)'_{12}$, as $f_n(t_k(r))$ is symmetric in M and M^{-1}. As a result,

$$W_{12}\ell + W'_{12} = f_n(t_k(r))\big((W_k)_{12}\ell + (W_k)'_{12}\big).$$

It suffices to show that $f_n(t_k(r))$ is not zero. If $f_n(t_k(r)) = 0$ then by Definition 4.6 $F_{k,n}(r) = -f_{n-1}(t_k(r)) = 0$ and so $f_{n-1}(t_k(r)) = 0$. This cannot occur if $n \neq -1, 0, 1$ as by Lemma 4.2 f_n and f_{n-1} are relatively prime. If $n = -1, 0, 1$ then either $f_n(t_k(r))$ or $f_{n-1}(t_k(r))$ equals ± 1 and cannot be zero. $\square$

4.4. Proof of Theorem 1.1. We now use the Cayley–Hamilton theorem to determine explicit equations for the polynomials above. First, we determine explicit equations for the entries of W_k.

Lemma 4.11. *With $W_k = \rho(w_k)$ then $(W_k)_{21} = (2-r)(W_k)_{12}$. If $k = 2m$ then*

$$\begin{aligned}
(W_k)_{11} &= f_m(r)^2(2-r)M^2 + [f_m(r)(r-1) - f_{m-1}(r)]^2\\
(W_k)_{12} &= f_m(r)^2[M + M^{-1} - rM^{-1}] + f_m(r)f_{m-1}(r)[M^{-1} - M]\\
(W_k)_{22} &= f_m(r)^2M^{-2}(2-r) + [f_m(r) - f_{m-1}(r)]^2.
\end{aligned}$$

If $k = 2m+1$ then

$$\begin{aligned}
(W_k)_{11} = {}& f_m(r)^2[M^2(r-1)^2 - (r-2)r^2]\\
&+ 2f_m(r)f_{m-1}(r)[(1-r)M^2 + r(r-2)] + f_{m-1}(r)^2[2 + M^2 - r]\\
(W_k)_{12} = {}& -f_m(r)^2[M(r-1) + (r-r^2)M^{-1}]\\
&- f_m(r)f_{m-1}(r)[(2r-1)M^{-1} - M] + f_{m-1}(r)^2M^{-1}
\end{aligned}$$

$$(W_k)_{22} = f_m(r)^2[(r-1)^2M^{-2} + 2 - r] - 2f_m(r)f_{m-1}(r)(r-1)M^{-2} + f_{m-1}(r)^2M^{-2}.$$

Proof. We calculate

$$AB^{-1} = \begin{pmatrix} r-1 & M \\ (r-2)M^{-1} & 1 \end{pmatrix} \qquad A^{-1}B = \begin{pmatrix} r-1 & -M^{-1} \\ (2-r)M & 1 \end{pmatrix}$$

both of which have trace r, so that by the Cayley–Hamilton theorem

$$(AB^{-1})^m = \begin{pmatrix} (r-1)f_m(r) - f_{m-1}(r) & Mf_m(r) \\ (r-2)M^{-1}f_m(r) & f_m(r) - f_{m-1}(r) \end{pmatrix}$$

$$(A^{-1}B)^m = \begin{pmatrix} (r-1)f_m(r) - f_{m-1}(r) & -M^{-1}f_m(r) \\ (2-r)Mf_m(r) & f_m(r) - f_{m-1}(r) \end{pmatrix}.$$

As $W_k = \rho(w_k)$, using Proposition 3.1 for $k = 2m$ we have

$$W_k = (AB^{-1})^m(A^{-1}B)^m$$

and for $k = 2m+1$,

$$W_k = (AB^{-1})^m AB(A^{-1}B)^m.$$

Upon multiplying we obtain the stated expressions. □

In light of Lemma 4.10 we now compute $(W_k)_{12}\ell + (W_k)'_{12}$ using these equations.

Definition 4.12. Let $G_{k,n}(r)$ be defined by

$$G_{2m,n}(r) = f_m(r)(r\delta - \alpha) - \gamma f_{m+1}(r)$$
$$G_{2m+1,n}(r) = \beta f_{m+1}(r) - \delta f_m(r).$$

Using the Fibonacci identities, one can write these polynomials in different ways. For example, when $k = 2m$ we also have

$$G_{k,n}(r) = -(f_m(r)(\alpha - r\beta) - \gamma f_{m-1}(r)).$$

Lemma 4.13. *For an irreducible representation ρ, the condition for $\Lambda = \rho([\lambda])$ to be upper triangular is equivalent to $G_{k,n}(r) = 0$.*

Proof. By Lemma 4.10 it suffices to consider $(W_k)_{12}\ell + (W_k)'_{12}$. With Lemma 4.11 we see that for $k = 2m$, $(W_k)_{12}\ell + (W_k)'_{12}$ is exactly $-f_m(r)M^{-1}$ times the expression for $G_{k,n}$. When $k = 2m+1$, by Lemma 4.11 similar to the above, $(W_k)_{12}\ell + (W_k)'_{12}$ is $g_{m+1}(r)M^{-1}$ times the expression for $G_{k,n}$.

It suffices to show that $f_m(r) \neq 0$ when $k = 2m$ and that $g_{m+1}(r) \neq 0$ when $k = 2m+1$. First, consider the even case. If $f_m(r) = 0$ then $F_{k,n}(r)$ can be explicitly computed using Proposition 4.7 and is the A-polynomial. This implies that $F_{k,1}(r) = 1$ and $t_k(r) = 2$. Therefore, $F_{k,n}(r) = f_n(2) - f_{n-1}(2)$. As $f_j(2) = j$ for all j, we conclude that $F_{k,n}(r) = 1$. Therefore the A-polynomial is $L - 1$, with the inclusion of the reducible factor. By [2, 8] the only knot in S^3 with A-polynomial equal to $L - 1$ is the unknot. Similarly, if k is odd we conclude that $F_{k,1}(r) = 1$ and $t = 2$ so that $F_{k,n}(r) = 1$. □

Lemma 4.14. *For $k \neq -1, 0, 1$ the polynomial $G_{k,n}(r)$ is an irreducible nonconstant polynomial in $\mathbb{Q}[M^{\pm 1}, \ell^{\pm 1}]$.*

Proof. Let $R = \mathbb{Q}[M^{\pm 1}, \ell^{\pm 1}]$. Consider the case when $k = 2m$, so that

$$G_{2m,n}(r) = -f_m(r)((M^2+1)(\ell+1) - (M^2+\ell)r) + f_{m-1}(r)(M^2-1)(\ell-1).$$

Modulo the ideal generated by $M^2 - 1$, R can be identified with $\mathbb{Q}[\ell]$ and $G_{2m,n}(r) = (\ell+1)f_m(r)(r-2)$. As a result, any factorization of $G_{2m,n}(r)$ in $R[r]$ modulo M^2-1, gives a factorization of $f_m(r)(r-2)$ in $\mathbb{Q}[r]$. Modulo the ideal generated by $M^2+\ell$ we have

$$G_{2m,n}(r) = (\ell+1)(\ell-1)\big(f_{m+1}(r) - f_m(r)\big).$$

Similarly, a factorization modulo $M^2+\ell$ gives a factorization of $f_m(r) - f_{m+1}(r)$ in $\mathbb{Q}[r]$. As $f_m(r)(r-2)$ and $f_m(r) - f_{m+1}(r)$ share no common roots by Lemma 4.2 we see that there is no factorization. We conclude that $G_{2m,n}(r)$ is irreducible.

Next consider $k = 2m+1$, so that

$$G_{2m+1,n}(r) = -(M^2\ell+1)f_m(r) + f_{m+1}(r)(M^2+\ell).$$

First, assume that $2m+1 > 0$, reducing modulo $M^2\ell+1$ we have

$$G_{2m+1,n}(r) = (M^2+\ell)f_{m+1}(r),$$

and reducing modulo $(M^2+1)(\ell+1)$,

$$G_{2m+1,n}(r) = (M^2+\ell)\big(f_m(r) + f_{m+1}(r)\big).$$

Again, a factorization modulo either $M^2\ell+1$ or $(M^2+1)(\ell+1)$ gives a factorization of either $f_{m+1}(r)$ or $f_m(r) + f_{m+1}(r)$ in $\mathbb{Q}[r]$. But these are relatively prime by Lemma 4.2. Therefore $G_{2m+1,n}(r)$ is irreducible in this case as well. Now consider the case when $2m+1 < 0$. Reducing modulo $M^2+\ell$, $G_{2m+1,n}(r) = -(M^2\ell+1)f_m(r)$ and reducing modulo $(M^2-1)(\ell-1)$, $G_{2m+1,n}(r) = -(M^2+\ell)\big(f_m(r) - f_{m+1}(r)\big)$. As before, we conclude that $G_{2m+1,n}(r)$ is irreducible. $\square$

Theorem 1.1 now follows from Proposition 4.7, Lemma 4.13, and Lemma 4.14.

Remark 4.15. Let $K = \mathbb{Q}(M, \ell)$. To determine the A-polynomial by eliminating the variable r from $F_{k,n}(t_k(r))$ and $G_{k,n}(r)$ using resultants, we require $F_{k,n}(t_k(r))$ and $G_{k,n}(r)$ to be nonconstant polynomials in $K[r]$. The polynomial $G_{k,n}(r)$ is a nonconstant polynomial in $K[r]$ unless $k = -1, 0, 1$. For these values we have $G_{-1,n}(r) = \delta$, $G_{0,n}(r) = -\gamma$, and $G_{1,n}(r) = \beta$.

The polynomial $F_{k,1}(x)$ is a nonconstant polynomial in $K[x]$ unless $k = 0, 1$. We have $F_{0,1}(x) = F_{1,1}(x) = 1$. The defining equation for $t_k(r)$ is nonconstant in $K[r]$ unless $k = 0$, in which case $t_0(r) = 2$. (It is nonlinear if $|k| > 1$ and if $k = \pm 1$ then $t = M^2 + M^{-2} + 2 - r$.) It follows that $F_{k,n}(t_k(r))$ is constant only when $n = 0$, $k = 0$, or when $n = k = 1$. For these values there are no irreducible representations. These correspond to $J(k,0) = J(0,2n) = J(1,2)$ which are all the unknot.

We conclude that one of $F_{k,n}(t_k(r))$ and $G_{k,n}(r)$ is constant when $k = -1, 0, 1$ or $n = 0$. These values of k are the excluded values in Theorem 1.1.

5. Resultants

In this section we collect facts about resultants that will be used later. If $p(x)$ and $q(x)$ are polynomials with leading coefficients P and Q, the resultant of $p(x)$ and $q(x)$ is

$$\operatorname{Res}\big(p(x), q(x)\big) = P^{\deg q} Q^{\deg p} \prod (r_p - r_q)$$

where the product ranges over all roots r_p of $p(x)$ and r_q of $q(x)$. First, we summarize some basic facts about resultants.

Lemma 5.1. *Assume that $p(x), q(x)$, and $r(x)$ are polynomials and the leading coefficient of $q(x)$ is Q.*

(1) $\operatorname{Res}(p,q) = (-1)^{\deg p + \deg q} \operatorname{Res}(q,p)$.

(2) $\operatorname{Res}(pr,q) = \operatorname{Res}(p,q)\operatorname{Res}(r,q)$.

(3) $Q^{\deg(p+rq)}\operatorname{Res}\big(p,q\big) = Q^{\deg p}\operatorname{Res}\big(p+rq,q\big)$.

The following polynomial will be useful in our calculations, as rational functions will naturally come out of our calculation.

Definition 5.2. For a fixed x and y, and a polynomial φ, define the polynomial

$$\overline{\varphi}(x,y) = y^{\deg(\varphi)}\varphi\big(\tfrac{x}{y}\big).$$

We now collect a few useful lemmas.

Lemma 5.3. *Let $\varphi(z)$ be a polynomial of degree d and assume that*

$$tp_1 = p_2 + gp_3.$$

Then $p_1^d\varphi(t) = \overline{\varphi}(p_2,p_1) + gp_4$ for some polynomial p_4.

Proof. Notice that if $\varphi(z) = c_d z^d + c_{d-1}z^{d-1} + \cdots + c_1 z + c_0$ then since $tp_1 = p_2 + gp_3$ we have

$$\begin{aligned} p_1^d\varphi(t) &= c_d(p_1t)^d + c_{d-1}p_1(p_1t)^{d-1} + \cdots + c_1p_1^{d-1}(p_1t) + c_0p_1^d \\ &= c_d(p_2+gp_3)^d + c_{d-1}p_1(p_2+gp_3)^{d-1} + \cdots + c_1p_1^{d-1}(p_2+gp_3) \\ &\quad + c_0p_1^d. \end{aligned}$$

For each $n > 0$ the term $(p_2+gp_3)^n = p_2^n + gp$ where p is a polynomial. Therefore,

$$\begin{aligned} p_1^d\varphi(t) &= c_dp_2^d + c_{d-1}p_1p_2^{d-1} + \cdots + c_1p_1^{d-1}p_2 + c_0p_1^d + gp_4 \\ &= p_1^d\varphi\big(\tfrac{p_2}{p_1}\big) + gp_4 = \overline{\varphi}(p_2,p_1) + gp_4. \end{aligned}$$

□

The following can be verified directly from the Fibonacci recursion.

Lemma 5.4. *Let A and B be constants. Then*

$$p_k(x,y) = A\overline{f}_k(x,y) + B\overline{f}_{k-1}(x,y)$$

satisfies the following recursion. We have

$$p_1(x,y) = A, \quad p_2(x,y) = Ax + B$$

and for $k > 2$ the polynomial

$$p_k(x,y) = xp_{k-1}(x,y) - y^2 p_{k-2}(x,y).$$

We have $p_0(x,y) = -B$, $p_{-1}(x,y) = -A - xB$ and for $k < -1$ the polynomial

$$p_k(x,y) = xp_{k+1}(x,y) - y^2 p_{k+2}(x,y).$$

5.1. A special resultant. In this section we compute the resultant

$$R_j = \operatorname{Res}\Big(f_j(x)(x+A) + f_{j-1}(x)B, p(x)\Big)$$

where $p(x) = x^2 - ax + b$ and a, b, A and B are constants. This type of resultant will be used several times in our computations. With $X_j = R_j + R_{j+1}$ we will first show that X_j satisfies a recursion, from which a recursion for R_j follows. All of the proofs will rely on the recursive definition of the Fibonacci polynomials, $f_{j+1}(x) + f_{j-1}(x) = xf_j(x)$. Let x_1 and x_2 be the two roots of $p(x)$, so that $x_1x_2 = b$ and $x_1 + x_2 = a$. Let

$$\begin{aligned}
\mathcal{F}_j &= f_j(x_1)f_j(x_2) = \operatorname{Res}(f_j(x), p(x)) \\
\mathcal{G}_j &= f_j(x_1)f_{j-1}(x_2) + f_j(x_2)f_{j-1}(x_1) \\
\mathcal{H}_j &= x_1 f_j(x_1)f_{j-1}(x_2) + x_2 f_j(x_2)f_{j-1}(x_1).
\end{aligned}$$

In this section, let $d_1 = b$, $d_2 = (2 - a^2 + 2b)$, $d_3 = b$ and $d_4 = -1$.

Lemma 5.5. *The following identities hold.*

(1) $\mathcal{H}_j = -\mathcal{F}_{j+1} + b\mathcal{F}_j + \mathcal{F}_{j-1}$
(2) $\mathcal{H}_j - \mathcal{H}_{j-2} = \mathcal{F}_{j-1}(a^2 - 2b) - 2b\mathcal{F}_{j-2}$
(3) $\mathcal{F}_j = b\mathcal{F}_{j-1} + (2 - a^2 + 2b)\mathcal{F}_{j-2} + b\mathcal{F}_{j-3} - \mathcal{F}_{j-4}$
(4) $\mathcal{G}_j = a\mathcal{F}_{j-1} - \mathcal{G}_{j-1}$.

Proof. For the first identity,

$$\begin{aligned}
\mathcal{F}_j &= (x_1 f_{j-1}(x_1) - f_{j-2}(x_1))(x_2 f_{j-1}(x_2) - f_{j-2}(x_2)) \\
&= x_1x_2\mathcal{F}_{j-1} + \mathcal{F}_{j-2} - (x_1 f_{j-1}(x_1)f_{j-2}(x_2) + x_2 f_{j-1}(x_2)f_{j-2}(x_1)) \\
&= b\mathcal{F}_{j-1} + \mathcal{F}_{j-2} - \mathcal{H}_{j-1}.
\end{aligned}$$

This proves the first assertion.

Next, consider

$$\begin{aligned}
\mathcal{H}_j &= x_1(x_1 f_{j-1}(x_1) - f_{j-2}(x_1))f_{j-1}(x_2) + x_2(x_2 f_{j-1}(x_2) \\
&\qquad - f_{j-2}(x_2))f_{j-1}(x_1) \\
&= \mathcal{F}_{j-1}(x_1^2 + x_2^2) - \big(x_1 f_{j-1}(x_2)f_{j-2}(x_1) + x_2 f_{j-1}(x_1)f_{j-2}(x_2)\big)
\end{aligned}$$

$$\begin{aligned}
&= \mathcal{F}_{j-1}(a^2-2b) - \big(x_1 f_{j-2}(x_1)(x_2 f_{j-2}(x_2) - f_{j-3}(x_2)) \\
&\quad + x_2 f_{j-2}(x_2)(x_1 f_{j-2}(x_1) - f_{j-3}(x_1))\big) \\
&= \mathcal{F}_{j-1}(a^2-2b) - 2x_1x_2\mathcal{F}_{j-2} + x_1 f_{j-2}(x_1) f_{j-3}(x_2) \\
&\quad + x_2 f_{j-3}(x_1) f_{j-2}(x_2) \\
&= \mathcal{F}_{j-1}(a^2-2b) - 2b\mathcal{F}_{j-2} + \mathcal{H}_{j-2}.
\end{aligned}$$

This shows the second assertion. With the above, we conclude that the third assertion holds. Finally,

$$\begin{aligned}
\mathcal{G}_j &= f_{j-1}(x_2) f_j(x_1) + f_{j-1}(x_1) f_j(x_2) \\
&= f_{j-1}(x_2)(x_1 f_{j-1}(x_1) - f_{j-2}(x_1)) + f_{j-1}(x_1)(x_2 f_{j-1}(x_2) - f_{j-2}(x_2)) \\
&= (x_1+x_2) f_{j-1}(x_2) f_{j-1}(x_1) - (f_{j-1}(x_2) f_{j-2}(x_1) + f_{j-1}(x_1) f_{j-2}(x_2)) \\
&= a\mathcal{F}_{j-1} - \mathcal{G}_{j-1}
\end{aligned}$$

proving the final identity. □

Lemma 5.6. *The following identity holds*

$$\begin{aligned}
X_j &= -B\mathcal{F}_{j+2} + (b + Aa + A^2 + bB - B)\mathcal{F}_{j+1} \\
&\qquad + \mathcal{F}_j(b + Aa + A^2 + bB + aAB + B^2 + B) + (B^2+B)\mathcal{F}_{j-1}.
\end{aligned}$$

Proof. The resultant R_j is

$$\begin{aligned}
R_j &= \Big(f_j(x_1)(x_1+A) + f_{j-1}(x_1)B\Big)\Big(f_j(x_2)(x_2+A) + f_{j-1}(x_2)B\Big) \\
&= \mathcal{F}_j(b + Aa + A^2) + B^2\mathcal{F}_{j-1} + AB\mathcal{G}_j + B\mathcal{H}_j.
\end{aligned}$$

By Lemma 5.5, we can substitute $\mathcal{H}_j$ and rewrite R_j as

$$R_j = -B\mathcal{F}_{j+1} + \mathcal{F}_j(b + Aa + A^2 + bB) + (B^2+B)\mathcal{F}_{j-1} + AB\mathcal{G}_j.$$

By another application of Lemma 5.5 we can substitute $\mathcal{G}_j$ so that X_j is as stated. □

Lemma 5.7. *The term X_j satisfies the recursion*

$$X_j = bX_{j-1} + (2 - a^2 + 2b)X_{j-2} + bX_{j-3} - X_{j-4}.$$

Since $X_j = R_j + R_{j-1}$ this implies the following recursion for R_j.

Proposition 5.8. *The resultant R_j of $f_j(x)(x+A)+f_{j-1}(x)B$ and x^2-ax+b satisfies the following recursion.*

$$R_j = (b-1)R_{j-1} + (2-a^2+3b)R_{j-2} + (2-a^2+3b)R_{j-3} + (b-1)R_{j-4} - R_{j-5}$$

with initial conditions

$$\begin{aligned}
R_{-2} &= b^2 + 2Bb + 2Bb^2 + B^2 + 2bB^2 + B^2b^2 + Aab - AaB + aBbA - a^2B \\
&\quad - a^2B^2 + bA^2 \\
R_{-1} &= A^2 + Aa + AaB + b + 2Bb + bB^2 \\
R_0 &= B^2
\end{aligned}$$

$$R_1 = b + Aa + A^2$$
$$R_2 = B^2 - 2Bb + b^2 + AaB + Aab + a^2B + bA^2.$$

Proof. A recursion for X_j of the form

$$X_j = c_1X_{j-1} + c_2X_{j-2} + c_3X_{j-2} + c_4X_{j-4}$$

determines a recursion

$$R_{j+1} = (c_1-1)R_j+(c_1+c_2)R_{j-1}+(c_2+c_3)R_{j-2}+(c_3+c_4)R_{j-3}+c_4R_{j-4}. \square$$

The following is immediate.

Proposition 5.9. *If $S_j = cd^{|j|}R_j$ then for $j > 5$, S_j satisfies the recursion*

$$S_j = d(b-1)S_{j-1} + d^2(2-a^2+3b)S_{j-2} + d^3(2-a^2+3b)S_{j-3} + d^4(b-1)S_{j-4} - d^5S_{j-5}$$

and for $j \le -5$, S_j satisfies the recursion

$$S_j = d(b-1)S_{j+1} + d^2(2-a^2+3b)S_{j+2} + d^3(2-a^2+3b)S_{j+3} + d^4(b-1)S_{j+4} - d^5S_{j+5}.$$

The initial conditions are determined by $S_j = cd^{|j|}R_j$ for $j > 5$ and $j \le -5$.

6. Proof of Theorem 1.2

By Theorem 1.1 the A-polynomial can be computed as

$$\mathrm{Res}(F_{k,n}(t_k(r)), G_{k,n}(r)).$$

The degree of $F_{k,n}$ is approximately $\frac{1}{4}|n|k^2$ and the degree of $G_{k,n}$ is about $\frac{1}{2}|k|$. In this section we undergo a series of reductions replace $F_{k,n}$ with a polynomial of smaller degree, approximately $2|n|$. The terms α, β, γ and δ are defined in Definition 4.3.

Definition 6.1. Let $q_1(r) = \delta\beta r - \sigma$ and $q_2(r) = \sigma r - \tau$, and let g_0 be the leading coefficient of $G_{k,n}(r)$. (Specifically, $g_0 = \beta$ if $k > 0$ and δ if $k < 0$.)

The following can be verified using the Fibonacci recursion.

Lemma 6.2. *We have the following:*

$$G_{k,n}\left(\tfrac{\sigma}{\delta\beta}\right) = \begin{cases} -\gamma\left(\dfrac{\beta}{\delta}\right)^m & \text{if } k = 2m \\ \dfrac{\beta^{m+1}}{\delta^m} & \text{if } k = 2m+1 > 0 \\ \dfrac{\delta^{|m|}}{\beta^{|m|-1}} & \text{if } k = 2m+1 < 0. \end{cases}$$

In the next few lemmas we rewrite key terms. Essentially we rewrite these terms as a multiple of $G_{k,n}(r)$ plus a remainder. This will allow us to simplify the resultant of $F_{k,n}(r)$ and $G_{k,n}(r)$ using Lemma 5.1. Let $F = \mathbb{Q}(M, L)$. First, we address $F_{k,1}(r)$. Let $F'_{k,1} = \gamma^{-1}(\alpha - \beta t_k(r))$ and let $F'_{k,n}(r) = f_n(t_k(r))F'_{k,1} - f_{n-1}(t_k(r))$.

Lemma 6.3. *For a fixed $k \neq -1, 0, 1$, and $n \neq 0$ there are polynomials $R_1(r)$ and $R_2(r)$ in $F[r]$ such that*

$$F_{k,1}(r) = F'_{k,1} - R_1(r)G_{k,n}(r)$$

and

$$F_{k,n}(r) = F'_{k,n}(r) - R_2(r)G_{k,n}(r).$$

Proof. If $k = 2m$, let $R_1(r) = \gamma^{-1} f_m(r)(M^2 + M^{-2} - r)$. If $k = 2m + 1$ let $R_1(r) = -\gamma^{-1} g_{m+1}(r)(M^2 + M^{-2} - r)$. In each case, the first identity can be directly verified using the definition of $F_{k,1}(r)$ and $G_{k,n}(r)$ with the substitution $r = s + s^{-1}$ so that $f_j(r) = (s^j - s^{-j})/(s - s^{-1})$. The second assertion follows from the definition of $F_{k,n}(r)$ and $F'_{k,n}(r)$ letting $R_2(r) = f_n(t_k(r))R_1(r)$. □

This allows us to calculate the resultant of $F_{k,n}(r)$ and $G_{k,n}(r)$ in terms of $F'_{k,n}(r)$. Let $A_1 = \operatorname{Res}(F_{k,n}(r), G_{k,n}(r))$ and $A_2 = \operatorname{Res}(F'_{k,n}(r), G_{k,n}(r))$.

Lemma 6.4. *For a fixed $k \neq -1, 0, 1$, and $n \neq 0$ then*

$$A_2 = \begin{cases} A_1 & nk < 0 \\ \pm\beta A_1 & n, k > 0 \\ (-1)^k \delta A_1 & n, k < 0. \end{cases}$$

Proof. When $n > 0$ by Lemma 6.3 and Lemma 5.1 up to sign, we see that

$$A_1 g_0^{\deg(F'_{k,n}(r))} = g_0^{\deg(F_{k,n}(r))} A_2.$$

Since $\deg(F_{k,n}(r)) = \deg(t_k(r))(|n| - 1) + \deg(F_{k,1}(r))$ and $\deg(F'_{k,n}(r)) = \deg(t_k(r))|n|$ we have

$$A_1 g_0^{\deg t_k(r)} = A_2 g_0^{\deg(F_{k,1}(r))}.$$

The statement follows from calculating the degrees of $t_k(r)$ and $F_{k,1}(r)$.

The other cases are similar. □

Now we turn to the defining equation for $t_k(r)$.

Lemma 6.5. *Fix $k \neq -1, 0, 1$ and $n \neq 0$. Then there is a polynomial $R_3(r) \in F[r]$ such that*

$$t_k(r)q_1(r) = q_2(r) - G_{k,n}(r)R_3(r).$$

Proof. When $k = 2m$ let

$$R_3(r) = (M^2 + M^{-2} - r)(f_m(r)(\beta r - \alpha) + \gamma f_{m+1}(r))$$

and when $k = 2m + 1$ let

$$R_3(r) = (r - 2)(M^2 + M^{-2} - r)(\beta f_m(r) - \delta f_{m+1}(r)).$$

The statement follows immediately. $\square$

Let $A_3 = \mathrm{Res}(q_1(r)^{|n|}\gamma F'_{k,n}(r), G_{k,n}(r))$.

Lemma 6.6. *Fix $k \neq -1, 0, 1$ and $n \neq 0$. We have*

$$A_3 = \begin{cases} A_1\gamma^{|m|-\epsilon_2}\Big((\delta\beta)^{|m|-\epsilon_2}G_{k,n}(\frac{\sigma}{\delta\beta})\Big)^{|n|} & nk < 0 \\ \pm A_1\beta\gamma^{|m|-\epsilon_2}\Big((\delta\beta)^{|m|-\epsilon_2}G_{k,n}(\frac{\sigma}{\delta\beta})\Big)^{|n|} & n, k > 0 \\ \pm A_1\delta\gamma^{|m|-\epsilon_2}\Big((\delta\beta)^{|m|-\epsilon_2}G_{k,n}(\frac{\sigma}{\delta\beta})\Big)^{|n|} & n, k < 0 \end{cases}$$

where $\epsilon_2 = 0$ unless k is odd and negative, in which case $\epsilon_2 = 1$.

Proof. We use the identities in Lemma 5.1. First, notice that since

$$\deg G_{k,n} = |m| - \epsilon_2,$$

we have

$$A_2\gamma^{\deg(F_{k,n}(r))}A_2\gamma^{|m|-\epsilon_2} = \mathrm{Res}(\gamma F'_{k,n}(r), G_{k,n}(r)).$$

Next,

$$\mathrm{Res}(\gamma F'_{k,n}(r), G_{k,n}(r))\mathrm{Res}(q_1(r), G_{k,n}(r))^{|n|} = A_3.$$

Therefore, $A_3 = \mathrm{Res}(q_1(r), G_{k,n}(r))^{|n|}\gamma^{|m|-\epsilon_2}A_2$. By definition, $q_1(r) = (\delta\beta)(r - \frac{\sigma}{\delta\beta})$. It follows that

$$\mathrm{Res}(q_1(r), G_{k,n}(r)) = (\delta\beta)^{|m|-\epsilon_2}G_{k,n}(\tfrac{\sigma}{\delta\beta}).$$

This simplifies to

$$A_2\gamma^{|m|-\epsilon_2}\Big((\delta\beta)^{|m|-\epsilon_2}G_{k,n}(\tfrac{\sigma}{\delta\beta})\Big)^{|n|} = A_3.$$

The lemma follows from combining this with the identity from Lemma 6.4. $\square$

The term $F_{k,n}(r)$ has a factor of $M^{2|n|}$ in the denominator from the M^{-2} in the defining equation for $t_k(r)$. The A-polynomial is well-defined up to multiples of M, so we will multiply by powers of M to make the powers of M all positive. To remove this factor of $M^{2|n|}$, we compute

$$A_0 = \mathrm{Res}(M^{2|n|}F_{k,n}(r), G_{k,n}(r)).$$

Therefore, $A_0 = (M^{2|n|})^{\deg G_{k,n}(r)}A_1$.

By Lemma 6.6 we have

$$(M^{2|n|})^{\deg G_{k,n}(r)} A_3 = \begin{cases} A_0\gamma^{|m|-\epsilon_2}\Big((\delta\beta)^{|m|-\epsilon_2}G_{k,n}(\frac{\sigma}{\delta\beta})\Big)^{|n|} & nk<0 \\ \pm A_0\beta\gamma^{|m|-\epsilon_2}\Big((\delta\beta)^{|m|-\epsilon_2}G_{k,n}(\frac{\sigma}{\delta\beta})\Big)^{|n|} & n,k>0 \\ \pm A_0\delta\gamma^{|m|-\epsilon_2}\Big((\delta\beta)^{|m|-\epsilon_2}G_{k,n}(\frac{\sigma}{\delta\beta})\Big)^{|n|} & n,k<0. \end{cases}$$

Since $A_3 = \mathrm{Res}(q_1(r)^{|n|}\gamma F'_{k,n}(r), G_{k,n}(r))$, we conclude that

$$(M^{2|n|})^{\deg G_{k,n}(r)} A_3 = \mathrm{Res}(M^{2|n|}q_1(r)^{|n|}\gamma F'_{k,n}(r), G_{k,n}(r)).$$

We conclude the following, with $A_4 = \mathrm{Res}(M^{2|n|}q_1(r)^{|n|}\gamma F'_{k,n}(r), G_{k,n}(r))$.

Lemma 6.7. *With ϵ_2 defined in Lemma 6.6,*

$$A_4 = \begin{cases} A_0\gamma^{|m|-\epsilon_2}\Big((\delta\beta)^{|m|-\epsilon_2}G_{k,n}(\frac{\sigma}{\delta\beta})\Big)^{|n|} & nk<0 \\ \pm A_0\beta\gamma^{|m|-\epsilon_2}\Big((\delta\beta)^{|m|-\epsilon_2}G_{k,n}(\frac{\sigma}{\delta\beta})\Big)^{|n|} & n,k>0 \\ \pm A_0\delta\gamma^{|m|-\epsilon_2}\Big((\delta\beta)^{|m|-\epsilon_2}G_{k,n}(\frac{\sigma}{\delta\beta})\Big)^{|n|} & n,k<0. \end{cases}$$

Now we define the polynomials $H_{k,n}(r)$ which differ from $M^{2|n|}q_1^{|n|}\gamma F'_{k,n}(r)$ by a multiple of $G_{k,n}(r)$.

Definition 6.8. Let $q_1(r)$ and $q_2(r)$ be as in Definition 6.1. If n is positive let

$$H_{k,n}(r) = q_2(r)M^2H_{k,n-1}(r) - q_1(r)^2M^4H_{k,n-2}(r)$$

with initial conditions $H_{k,1}(r) = M^2(\alpha q_1(r) - \beta q_2(r))$, and

$$H_{k,2}(r) = M^4(\alpha q_1(r)q_2(r) - \gamma q_1(r)^2 - \beta q_2(r)^2).$$

If n is negative let

$$H_{k,n}(r) = q_2(r)M^2H_{k,n+1}(r) - q_1(r)^2M^4H_{k,n+2}(r)$$

with initial conditions $H_{k,0}(r) = \gamma$ and

$$H_{k,-1}(r) = M^2(\beta q_2(r) - \alpha q_1(r) + \gamma q_2(r)).$$

The following follows directly from Lemma 5.4.

Lemma 6.9. *If n is positive, then*

$$H_{k,n}(r) = M^{2|n|}\Big(\overline{f_n}(q_2,q_1)\big(\alpha q_1(r) - \beta q_2(r)\big) - \gamma q_1(r)^2\overline{f_{n-1}}(q_2,q_1)\Big)$$

and if n is negative, then

$$H_{k,n}(r) = M^{2|n|}\Big(\overline{f_n}(q_2,q_1)\big(\alpha q_1(r) - \beta q_2(r)\big) - \gamma\overline{f_{n-1}}(q_2,q_1)\Big).$$

Lemma 6.10. *Fix $k \neq -1,0,1$ and $n \neq 0$. Then there is a polynomial $R_4(r) \in F[r]$ such that*

$$M^{2|n|}q_1(r)^{|n|}\gamma F'_{k,n}(r) = H_{k,n}(r) + G_{k,n}(r)R_4(r).$$

Proof. By Lemma 5.3 and Lemma 6.5 since $tq_1(r) = q_2(r) - G_{k,n}R_3(r)$ we have when $n > 0$

$$\begin{aligned} &q_1(r)^n\big(f_n(t)\big(\alpha - \beta t\big) - \gamma f_{n-1}(t)\big) \\ &\qquad = q_1(r)^{n-1} f_n(t)\big(\alpha q_1(r) - \beta q_2(r) + \beta G_{k,n} R_3\big) - \gamma q_1(r)^n f_{n-1}(t) \\ &\qquad = H_{k,n}(r) M^{-2|n|} + G_{k,n} R_4. \end{aligned}$$

Here

$$R_4 = \beta R_3 q_1^{n-1} f_n(t_k(r)) + P_4(\alpha q_1 - \beta q_2) - \gamma q_1^2 P_4'$$

with P_4 and P_4' from Lemma 5.3.

If n is negative,

$$\begin{aligned} &q_1(r)^{|n|}\big(f_n(t)\big(\alpha - \beta t\big) - \gamma f_{n-1}(t)\big) \\ &\qquad = q_1(r)^{|n|-1} f_n(t)\big(\alpha q_1 - \beta t q_1\big) - \gamma q_1(r)^{|n|} f_{n-1}(t) \\ &\qquad = H_{k,n}(r) M^{-2|n|} + G_{k,n} R_4. \end{aligned}$$

Here

$$R_4 = \beta R_3 q_1^{|n|-1} f_n(t_k(r)) + P_4(\alpha q_1 - \beta q_2) - \gamma P_4'$$

with P_4 and P_4' as in the previous case. $\square$

We are now ready to prove Theorem 1.2. Let $A_5 = \operatorname{Res}(H_{k,n}(r), G_{k,n}(r))$. It is enough to show that up to sign,

$$A_5 = \begin{cases} A_0 \gamma^{|m| + \epsilon'|n| - (1-\epsilon)(1-\epsilon')} & nk < 0 \\ A_0 \beta \gamma^{|m| + \epsilon'|n| - (1-\epsilon)(1-\epsilon')} & n, k > 0 \\ A_0 \delta \gamma^{|m| + \epsilon'|n| - (1-\epsilon)(1-\epsilon')} & n, k < 0 \end{cases}$$

where $\epsilon' = 1$ if k is even and $\epsilon' = 0$ if k is odd, and $\epsilon = 1$ if k is positive and 0 if k is negative. Lemma 6.10 with Lemma 5.1 implies that

$$A_4 g_0^{\deg H_{k,n}(r)} = g_0^{|n| + \deg(F'_{k,n}(r))} A_5.$$

Since the degree of $H_{k,n}(r)$ is $|n|$, we have $A_4 = g_0^{\deg(F'_{k,n}(r))} A_5$. The degree of $\deg(F'_{k,n}(r)) = |n| \deg(t_k(r)) = |nk|$ so that

$$A_4 = g_0^{|nk|} A_5.$$

Using Lemma 6.7 it follows that

$$g_0^{|nk|} A_5 = \begin{cases} A_0 \gamma^{|m| - \epsilon_2} \Big((\delta\beta)^{|m| - \epsilon_2} G_{k,n}(\frac{\sigma}{\delta\beta}) \Big)^{|n|} & nk < 0 \\ \pm A_0 \beta \gamma^{|m| - \epsilon_2} \Big((\delta\beta)^{|m| - \epsilon_2} G_{k,n}(\frac{\sigma}{\delta\beta}) \Big)^{|n|} & n, k > 0 \\ \pm A_0 \delta \gamma^{|m| - \epsilon_2} \Big((\delta\beta)^{|m| - \epsilon_2} G_{k,n}(\frac{\sigma}{\delta\beta}) \Big)^{|n|} & n, k < 0. \end{cases}$$

When $k = 2m$ then $\epsilon_2 = 0$ and $\Big((\delta\beta)^{|m| - \epsilon_2} G_{k,n}(\frac{\sigma}{\delta\beta}) \Big)^{|n|} = \pm \gamma^{|n|} g_0^{|nk|}$. This reduces to the stated form. When $k = 2m + 1$, then ϵ_2 depends on the sign of k. In both cases, $\left((\delta\beta)^{|m|-\epsilon_2}G_{k,n}(\frac{\sigma}{\delta\beta})\right)^{|n|} = \pm g_0^{|nk|}$. This reduces to the stated form. This completes the proof of Theorem 1.2.

7. Explicit computations for $J(k, 2n)$ knots for small k

When k is small, we can recursively compute these resultants for all n. Here we calculate the A-polynomials for the $J(1, 2n)$, $J(2, 2n)$, $J(3, 2n)$, $J(4, 2n)$ and $J(5, 2n)$ knots. Since $J(-k, -l)$ is the mirror image of $J(k, l)$, the A-polynomial of $J(-k, -l)$ can be determined from the A-polynomial of the $J(k, l)$ knot by replacing every M with an M^{-1}. Therefore, these calculations also determine the A-polynomials of the $J(-1, -2n)$, $J(-2, -2n)$, $J(-3, -2n)$, $J(-4, -2n)$, and $J(-5, -2n)$ knots.

It is sufficient to determine the contribution from the irreducible representations, as the reducible representations contribute the factor $L - 1$ to the A-polynomial.

7.1. $J(1, 2n)$, torus knots.

$J(1, 2n)$ is the torus knot $T(2, 2n - 1)$ corresponding to the Schubert pair $(p, q) = (-1, 1 - 2k)$.

By Remark 4.15 when $k = \pm 1$, $G_{k,n}(r)$ is constant. In particular, $G_{1,n} = \beta$ and $G_{-1,n} = \delta$. Therefore this determines the irreducible contribution of the A-polynomial. (The algebraic condition $F_{1,k}(r) = 0$ merely determines valid r values for ρ to be an irreducible representation. For $n = 0$, or 1 however, $F_{1,0}(r)$ and $F_{1,1}(r)$ are ± 1. This indicates that there are no irreducible representations, which is clear as $J(1, 0) = J(1, 2)$ are the unknot.) Here $\ell = LM^{4n}$ and since $M \neq 0$, $G_{1,k}(r) = \beta = 0$ determines the factor $\beta = M^2 + \ell = M^2 + LM^{4n}$. Normalizing, by dividing by M^2, this is equivalent to the factor of $1 + LM^{4n-2}$. We obtain the following, which has been observed before [4].

Theorem 7.1. *The A-polynomial of the $J(1, 2n)$ torus knot is given by*

$$A(1, 2n) = \begin{cases} 1 + LM^{4n-2} & \text{if } n > 1 \\ M^{2-4n} + L & \text{if } n \leq -1. \end{cases}$$

7.2. $J(2, 2n)$, the twist knots.

We will directly compute the A-polynomial as the common vanishing set of $F_{2,n}$ and $G_{2,n}$ using Theorem 1.1. Here $m = 1$ and we have the following:

$$\begin{aligned} F_{2,n}(r) &= f_n(t_2(r))F_{2,1}(r) - f_{n-1}(t_2(r)) \\ F_{2,1}(r) &= M^2 + M^{-2} - r + 1 \\ t_2(r) &= (2 - r)(M^2 + M^{-2} - r) + 2 \\ G_{2,n}(r) &= (M^2 + \ell)r - (M^2 + 1)(\ell + 1). \end{aligned}$$

Since $G_{2,n}(r)=0$ we conclude that $r=(M^2+1)(\ell+1)/(M^2+\ell)$. It follows that

$$F_{2,1}(r)=\frac{M^6+\ell}{M^2(M^2+\ell)}$$

and

$$t_2=\frac{n_2}{d_2}=\frac{(1-\ell)M^8+2\ell M^6+(\ell+1)^2M^4+2\ell M^2+\ell^2-\ell}{(M^2+\ell)^2M^2}.$$

Therefore,

$$F_{2,n}(r)=f_n(t_2)\frac{M^6+\ell}{M^2(M^2+\ell)}-f_{n-1}(t_2).$$

One can easily verify the following:

$$\begin{aligned}
F_{2,-1}&=\big(-M^8\ell+M^6\ell+M^4(\ell+1)^2+M^2\ell-\ell\big)/M^2(M^2+\ell)^2\\
F_{2,0}&=1\\
F_{2,1}&=(M^6+\ell)/M^2(M^2+\ell)\\
F_{2,2}&=\big((1-\ell)M^{14}+2\ell M^{12}+(\ell^2+2\ell)M^{10}-M^8\ell^2-\ell M^6\\
&\quad+(2\ell^2+\ell)M^4+2M^2\ell^2+\ell^3-\ell^2\big)/(M^4(M^2+\ell)^3).
\end{aligned}$$

The A-polynomial is given by the numerator of $F_{k,n}$. Using Lemma 5.4 we conclude that the A-polynomials satisfy the recursion given by Hoste and Shanahan [10]. In this case $\ell=L$.

Theorem 7.2. *Let n_2 and d_2 be defined as above. For n positive*

$$A(2,2n)=n_2A(2,2n-2)-d_2^2A(2,2n-4)$$

with initial conditions

$$\begin{aligned}
A(2,2)&=M^6+\ell\\
A(2,4)&=(1-\ell)M^{14}+2\ell M^{12}+(\ell^2+2\ell)M^{10}-M^8\ell^2-\ell M^6\\
&\quad+(2\ell^2+\ell)M^4+2M^2\ell^2+\ell^3-\ell^2.
\end{aligned}$$

For negative n

$$A(2,2n)=n_2A(2,2n+2)-d_2^2A(2,2n+4)$$

with initial conditions

$$\begin{aligned}
A(2,0)&=1\\
A(2,-2)&=-M^8\ell+\ell M^6+(2\ell+\ell^2+1)M^4+M^2\ell-\ell.
\end{aligned}$$

7.3. $J(3,2n)$. We will directly compute the A-polynomial as the common vanishing set of $F_{3,n}$ and $G_{3,n}$ using Theorem 1.1. Here,

$$\begin{aligned} F_{3,n}(r) &= f_n(t_3(r))F_{3,1}(r) - f_{n-1}(t_3(r)) \\ F_{3,1}(r) &= -(r-1)(M^2+M^{-2}-r)+1 \\ t_3(r) &= (r-1)^2(M^2+M^{-2}-r)+2 \\ G_{3,n}(r) &= (M^2+\ell)r-(M^2\ell+1). \end{aligned}$$

Since $G_{3,n}(r)=0$ we have $r=(M^2\ell+1)/(M^2+\ell)$. Using this,

$$F_{3,1}(r) = \frac{(1-\ell)M^8+\ell M^6+2M^4\ell+M^2\ell+\ell^2-\ell}{(M^2+\ell)^2M^2}.$$

Therefore, $F_{3,n}$ is defined by

$$F_{3,n}(r) = f_n(t_3)\Big(\frac{(1-\ell)M^8+\ell M^6+2M^4\ell+M^2\ell+\ell^2-\ell}{(M^2+\ell)^2M^2}\Big) - f_{n-1}(t_3)$$

and $t=n_3/d_3$ where

$$\begin{aligned} n_3 = {} & (\ell-1)^2M^{10}+2\ell(2-\ell)M^8+(\ell^2+4\ell+1)M^6+\ell(\ell^2+4\ell+1)M^4 \\ & + 2\ell(2\ell-1)M^2+\ell(\ell-1)^2 \end{aligned}$$

and $d_3=(M^2+\ell)^3M^2$. The A-polynomial is given by the numerator of $F_{k,n}$. Using Lemma 5.4 we see that the numerators satisfy the recursion given by Hoste and Shanahan [10]. Recall that since $k=3$ we have $\ell=LM^{4n}$. The results that helped us determine the recursion for the resultant rely on the recursive definitions of $F_{k,n}(r)$ and $G_{k,n}(r)$ as polynomials in $\ell, M^{\pm 2}$ and r. Therefore, to write $A(3,2n)$ in terms of the variables L and M, one uses the recursive definition below to determine the polynomial in ℓ and M and then substitutes $\ell=LM^{4n}$.

Theorem 7.3. *Let n_3 and d_3 be defined as above. For n positive*

$$A(3,2n) = n_3A(3,2n-2) - d_3^2A(3,2n-4)$$

with initial conditions

$$\begin{aligned} A(3,2) = {} & (1-\ell)M^8+\ell M^6+2M^4\ell+M^2\ell+\ell^2-\ell \\ A(3,4) = {} & (1-3\ell+3\ell^2-\ell^3)M^{18}+(-8\ell^2+3\ell^3+5\ell)M^{16} \\ & + (-3\ell^2-\ell^3+5\ell)M^{14}+(-5\ell^3-2\ell+13\ell^2-\ell^4)M^{12} \\ & + (-\ell+2\ell^4+12\ell^2-3\ell^3)M^{10}+(12\ell^3-3\ell^2+2\ell-\ell^4)M^8 \\ & + (13\ell^3-\ell-5\ell^2-2\ell^4)M^6+(-3\ell^3-\ell^2+5\ell^4)M^4 \\ & + (-8\ell^3+5\ell^4+3\ell^2)M^2-3\ell^4+\ell^5+3\ell^3-\ell^2. \end{aligned}$$

For n negative

$$A(3,2n) = n_3A(3,2n+2) - d_3^2A(3,2n+4)$$

with initial conditions

$$\begin{aligned} A(3,0) &= 1 \\ A(3,-2) &= (\ell^2-\ell)M^{10} + (-\ell^2+2\ell)M^8 + (1+2\ell)M^6 + (2\ell^2+\ell^3)M^4 \\ &\quad + (-\ell+2\ell^2)M^2 + \ell - \ell^2. \end{aligned}$$

7.4. $\boldsymbol{J(4,2n)}$. We will show that the resultant defining $A(4,2n)$ is of the form considered in Section 5.1. Since $k=4=2m$ we have $\ell = L$ for all of these knots. By Theorem 1.1 we can compute the A-polynomial as

$$\mathrm{Res}(F_{4,n}(r), G_{4,n}(r))$$

where

$$\begin{aligned} F_{4,n}(r) &= f_n(t_4(r))F_{4,1}(r) - f_{n-1}(t_4(r)) \\ F_{4,1}(r) &= r(r-1)(M^2+M^{-2}-r)+1 \\ t_4(r) &= -r(r^2-2r)(M^2+M^{-2}-r)+2 \\ G_{4,n}(r) &= (M^2+L)r^2 - (L+1)(M^2+1)r - (L-1)(M^2-1). \end{aligned}$$

Moreover, for this section, let

$$\begin{aligned} a &= \beta^{-4}M^{-2}\Big(L(L-1)(2L^2-L+1) + 2L(1-4L+5L^2)M^2 \\ &\quad + L(2L^3+5L^2+10L-1)M^4 + 4L(1+4L+L^2)M^6 \\ &\quad + (2+5L+10L^2-L^3)M^8 + 2L(5-4L+L^2)M^{10} \\ &\quad + (1-L)(2-L+L^2)M^{12})\Big) \\ b &= \beta^{-4}M^{-4}\Big(L(L-1)^3 + 2L(L-1)(3L-1)M^2 \\ &\quad + 2L(L^3+L^2+L-1)M^4 + 2L(L+1)(3L+1)M^6 \\ &\quad + (1+4L+14L^2+4L^3+L^4)M^8 + 2L(L+1)(L+3)M^{10} \\ &\quad + 2(1+L+L^2-L^3)M^{12} + 2L(L-1)(L-3)M^{14} + (1-L)3M^{16}\Big) \\ c &= M^4\beta^7L^{-2}(L-1)^{-2}(M^2-1)^{-6}(M^2+1)^{-4} \\ A &= -\alpha/\beta \\ C &= -\beta/\gamma \\ B &= -1/C. \end{aligned}$$

First we show that $\mathrm{Res}(F_{4,n}(r), G_{4,n}(r))$ is of the form considered in Proposition 5.9. Let

$$R_n = \mathrm{Res}\big(f_n(x)(x+A) + f_{n-1}(x)B, x^2-ax+b\big)$$

and let

$$A_1 = \mathrm{Res}(F_{4,n}(r), G_{4,n}(r)).$$

Lemma 7.4. *Let $\epsilon = 1$ if $n > 0$ and $\epsilon = 0$ if $n < 0$. We have*

$$A_1(L-1)^2(M^2-1)^2 = \beta^{4|n|+2-\epsilon}R_n$$

Proof. First, we define the following

$$P_1 = \frac{M^6 + L - rM^2\beta}{\beta^2 M^2}$$

$$P_2 = \frac{M^2r^2\beta^2 - (M^2+1)(M^4+L)\beta r - (L-1)(M^2-1)(M^6+L)}{\beta^3 M^2}$$

$$t' = 2 + \frac{(L-1)^2(M^2-1)^2(M^6+L) - (L-1)(M^2-1)^3(M^2+1)^2Lr}{M^2\beta^3}$$

$$F'_{4,1} = \frac{(L-1)(L-M^8) + LM^2(M^2+1)^2 + L(M^4-1)^2r}{M^2\beta^2}.$$

With this,

$$F_{4,1} = F'_{4,1} + G_{4,n}P_1, \qquad \text{and} \qquad t = t' + G_{4,n}P_2.$$

Moreover, in terms of t' we have $F'_{4,1} = C(t'+A)$, and $G_{4,n} = c(t'^2 - at' + b)$. Additionally, let

$$F'_{4,n}(r) = f_n(t_4)F'_{4,1} - f_{n-1}(t_4), \text{ and } F''_{4,n}(r) = f_n(t')F'_{4,1} - f_{n-1}(t').$$

Let $A_2 = \operatorname{Res}(F'_{4,n}(r), G_{4,n})$. By the above, we have that

$$F_{4,n} = F'_{4,n}(r) + f_n(t_4)G_{4,n}P_1$$

and we conclude that

$$A_1 g_0^{\max\{4|n|-3, 4|n-1|-4\}} = A_2 g_0^{\max\{4|n-1|-4, 4|n|-1\}}.$$

Therefore, when $n > 0$, $A_1 = A_2\beta^2$ and when $n < 0$, $A_1 = A_2$.

We also have

$$\begin{aligned} F'_{4,n}(r) &= f_n(t' + G_{4,n}P_2)F'_{4,1} - f_{n-1}(t' + G_{4,n}P_2) \\ &= F''_{4,n}(r) + G_{4,n}P_3. \end{aligned}$$

Let $A_3 = \operatorname{Res}(F''_{4,n}(r), G_{k,n}(r))$ (as a function of r). The degree of $F'_{4,n}(r)$ is $\max\{4|n|-3, 4|n-1|-4\}$ and the degree of $F''_{4,n}(r)$ is $|n|$ so we conclude that when $n > 0$ we have $A_2 = \beta^{3n-3}A_3$ and when $n < 0$, $A_2 = \beta^{3|n|}A_3$.

By definition of A, B, C, a, b, and c,

$$F''_{4,n}(r) = C[f_n(t')(t'+A) + Bf_{n-1}(t')] \text{ and } G_{4,n}(r) = c(t'^2 - at' + b).$$

Therefore, $A_3 = A_4$ where

$$A_4 = \operatorname{Res}(C[f_n(t')(t'+A) + Bf_{n-1}(t')], c(t'^2 - at' + b)),$$

and the resultant is taken with respect to the variable r. Let

$$A_5 = \operatorname{Res}(C[f_n(x)(x+A) + Bf_{n-1}(x)], c(x^2 - ax + b))$$

with respect to the variable x. The leading coefficient of t' is

$$t'_0 = L(1-L)(M^2-1)^3(M^2+1)^2/(M^2\beta^3)$$

and we conclude that these two resultants differ by this leading coefficient to the product of the powers of the two polynomials, $A_4 = A_5(t_0')^{2|n|}$. Explicitly, this is

$$A_4 M^{4|n|}\beta^{6|n|} = A_5 L^{2|n|}(1-L)^{2|n|}(M^2-1)^{6|n|}(M^2+1)^{4|n|}.$$

Finally, since $R_n = \mathrm{Res}\big(f_n(x)(x+A)+f_{n-1}(x)B, x^2-ax+b\big)$ we conclude that $R_n = c^{\deg(f_n(x)(x+A)+f_{n-1}(x)B)}C^{\deg(x^2-ax+b)}A_5$, which reduces to the stated formulas. □

The M^{-2} factor in $t_4(r)$ introduces a factor of $M^{4|n|}$ into A_1. After normalization, $A(4,2n) = M^{4|n|}A_1$ so that by Lemma 7.4

$$A(4,2n)(L-1)^2(M^2-1)^2 = M^{4|n|}\beta^{4|n|+2-\epsilon}R_n.$$

We obtain a recursive formula for $A(4,2n)$ using Proposition 5.9. Let $c_1 = b-1$, and $c_2 = 2-a^2+3b$. Specifically, for $n>5$,

$$\begin{aligned} A(4,2n) &= c_1(M^4\beta^4)A(4,2(n-1)) + c_2(M^4\beta^4)^2A(4,2(n-2)) \\ &\quad + c_2(M^4\beta^4)^3A(4,2(n-3)) + c_1(M^4\beta^4)^4A(4,2(n-4)) \\ &\quad - (M^4\beta^4)^5A(4,2(n-5)). \end{aligned}$$

where the initial conditions are $A(4,2n) = M^{4n}\mathrm{Res}(F_{4,n}(r), G_{4,n}(r))$ for $1 \le n \le 5$. (in fact, setting $A(4,0) = \beta^{-1}$ we can begin the recursion one step earlier.) When $n \le -5$, using $A(4,0)=1$ the recursion is

$$\begin{aligned} A(4,2n) &= c_1(M^4\beta^4)A(4,2(n+1)) + c_2(M^4\beta^4)^2A(4,2(n+2)) \\ &\quad + c_2(M^4\beta^4)^3A(4,2(n+3)) + c_1(M^4\beta^4)^4A(4,2(n+4)) \\ &\quad - (M^4\beta^4)^5A(4,2(n+5)). \end{aligned}$$

where initial terms $A(4,2n)$ for $-4 \le n \le 0$ are

$$A(4,2n) = M^{4|n|}\mathrm{Res}(F_{4,n}(r), G_{4,n}(r)).$$

Finally, we write this as a self contained theorem, computing these coefficients explicitly.

Theorem 7.5. *With d_1, d_2, d_3, d_4, and d_5 as in the appendix, we have the following. For n positive*

$$\begin{aligned} A(4,2n) &= d_1A(4,2(n-1)) + d_2A(4,2(n-2)) + d_3A(4,2(n-3)) \\ &\quad + d_4A(4,2(n-4) + d_5A(4,2(n-5)). \end{aligned}$$

The initial conditions are $A(4,2n) = \mathrm{Res}(M^{2n}F_{4,n}(r), G_{4,n}(r))$ for $0 < n \le 5$.

For n negative

$$\begin{aligned} A(4,2n) &= d_1A(4,2(n+1)) + d_2A(4,2(n+2)) + d_3A(4,2(n+3)) \\ &\quad + d_4A(4,2(n+4)) + d_5A(4,2(n+5)). \end{aligned}$$

The initial conditions are $A(4,2n) = \mathrm{Res}(M^{2|n|}F_{4,n}(r), G_{4,n}(r))$ *for* $-4 \leq n \leq 0$.

7.5. $J(5, 2n)$. By Theorem 1.1 the A-polynomial of $J(5, 2n)$ may be computed as as

$$\mathrm{Res}(F_{5,n}(r), G_{5,n}(r))$$

where

$$\begin{aligned}
F_{5,n}(r) &= f_n(t)F_{5,1}(r) - f_{n-1}(t)\\
F_{5,1}(r) &= -r(r^2 - r - 1)(M^2 + M^{-2} - r) + 1\\
t_5(r) &= (r^2 - r - 1)^2(M^2 + M^{-2} - r) + 2\\
G_{5,n}(r) &= (M^2 + \ell)r^2 - (M^2\ell + 1)r - M^2 - \ell.
\end{aligned}$$

We proceed in the same fashion as the $A(4, 2n)$ case. For this section, we define the following

$$\begin{aligned}
a = \Big(&(\ell^4 - 2\ell^3 + 3\ell^2 - 4\ell + 2)M^{14} + (9\ell^3 + 13\ell - 16\ell^2 - 2\ell^4)M^{12}\\
&+ (\ell^4 - 12\ell^3 + 23\ell^2 + 6\ell + 2)M^{10} + (5\ell^3 + 5\ell + 30\ell^2)M^8\\
&+ (30\ell^3 + 5\ell^2 + 5\ell^4)M^6 + (6\ell^4 + 23\ell^3 - 12\ell^2 + \ell + 2\ell^5)M^4\\
&+ (-16\ell^3 - 2\ell + 9\ell^2 + 13\ell^4)M^2 + 3\ell^3 - 2\ell^2 + 2\ell^5 + \ell - 4\ell^4\Big)M^{-2}\beta^{-5}\\
b = \Big(&(6\ell^2 + \ell^4 - 4\ell^3 + 1 - 4\ell)M^{18} + (-2\ell^4 - 18\ell^2 + 8\ell + 12\ell^3)M^{16}\\
&+ (2 + 2\ell + 2\ell^4 - 6\ell^3 + 4\ell^2)M^{14} + (8\ell - 2\ell^4 - 8\ell^3 + 22\ell^2)M^{12}\\
&+ (6\ell^3 + 1 + \ell^4 + 26\ell^2 + 6\ell)M^{10} + (\ell + \ell^5 + 26\ell^3 + 6\ell^4 + 6\ell^2)M^8\\
&+ (-8\ell^2 - 2\ell + 8\ell^4 + 22\ell^3)M^6 + (2\ell^5 + 4\ell^3 - 6\ell^2 + 2\ell^4 + 2\ell)M^4\\
&+ (12\ell^2 - 2\ell - 18\ell^3 + 8\ell^4)M^2 + 6\ell^3 + \ell - 4\ell^4 + \ell^5 - 4\ell^2\Big)M^{-4}\beta^{-5}\\
c &= \beta^9 M^4(\ell - 1)^{-4}(M^2 + 1)^{-4}(M^2 - 1)^{-8}\ell^{-2}\\
A &= \alpha/\beta\\
B &= \gamma/\beta\\
C &= -\beta/\gamma.
\end{aligned}$$

In addition, let

$$R_n = \mathrm{Res}(f_n(x)(x + A) + f_{m-1}(x)B, x^2 - ax + b)$$

and

$$A_1 = \mathrm{Res}(F_{5,n}(r), G_{5,n}(r)).$$

Lemma 7.6. *Let* ϵ *equal 1 when* $n > 0$ *and 2 when* $n < 0$. *Then*

$$A_1\gamma^2 = \beta^{5|n|+\epsilon}R_n.$$

Proof. Let

$$\begin{aligned}
P_1 &= \Big(r^2M^2(M^2+\ell)^2 - r(1+M^2)(M^4+\ell)(M^2+\ell)\\
&\quad - (\ell-1)(M^2-1)(M^6+\ell)\Big)/\beta^3M^2\\
P_2 &= \Big(-r^3M^2(M^2+\ell)^3 + (M^2+\ell)^2(\ell+2M^2\ell+2M^4+M^6)r^2\\
&\quad + (M^2+\ell)(M^6+\ell)(M^2\ell-2\ell-2M^2+1)r\\
&\quad + (M^2\ell+1)(-2M^8+M^8\ell-2\ell M^6-2M^4\ell-2M^2\ell\\
&\quad - 2\ell^2+\ell)\Big)/M^2\beta^4\\
F'_{5,1} &= \Big(-\ell(1+M^2)^2(M^2-1)^3(\ell-1)r\\
&\quad - (M^2+\ell)(M^8\ell-M^8-\ell M^6-2M^4\ell-M^2\ell+\ell-\ell^2)\Big)/\beta^3M^2\\
t' &= \Big(\ell(\ell-1)^2(1+M^2)^2(M^2-1)^4r\\
&\quad + (M^2+\ell)(M^{10}-2M^{10}\ell+M^{10}\ell^2-2M^8\ell^2+4M^8\ell+4\ell M^6\\
&\quad + M^6+\ell^2M^6+M^4\ell^3+M^4\ell+4M^4\ell^2-2M^2\ell+4M^2\ell^2\\
&\quad + \ell-2\ell^2+\ell^3)\Big)/M^2\beta^4.
\end{aligned}$$

Moreover, let

$$F'_{5,n}(r) = f_n(t_5)F'_{5,1} - f_{n-1}(t_5) \quad\text{and}\quad F''_{5,n}(r) = f_n(t')F'_{5,1} - f_{n-1}(t').$$

A direct calculation shows that

$$F_{5,1} = F'_{5,1} + G_{5,n}P_1, \qquad\text{and}\qquad t = t' + G_{5,n}P_2.$$

Let $A_2 = \mathrm{Res}(F'_{5,n}(r), G_{5,n})$. We conclude that

$$A_1 g_0^{\max\{5|n|-4,5|n-1|-5\}} = A_2 g_0^{\max\{5|n|-1,5|n-1|-5\}}.$$

Therefore when $n>0$ we have $A_1 = \beta^3 A_2$ and when $n \le 0$, $A_1 = A_2$.

Next, we see that

$$\begin{aligned}
F'_{5,n}(r) &= f_n(t'+G_{5,n}P_2)F'_{5,1} - f_{n-1}(t'+G_{5,n}P_2)\\
&= F''_{5,n}(r) + G_{5,n}P_3.
\end{aligned}$$

The degree of $F'_{5,n}(r)$ is $\max\{5|n|-4, 5|n-1|-5\}$ and the degree of $F''_{5,n}(r)$ is $|n|$. Let $A_3 = \mathrm{Res}(F''_{5,n}(r), G_{5,n}(r))$ (in terms of the variable r). Then $A_2 = A_3\beta^{4(n-1)}$ when $n>0$ and $A_2 = A_3\beta^{4|n|}$ when $n \le 0$.

With the terms as defined in the statement of the lemma,

$$G_{5,n} = c\big(t'^2 - at' + b\big),$$

and

$$F''_{5,n}(r) = C\big(f_n(t')(t'+A) + Bf_{n-1}(t')\big).$$

Let $A_4 = \operatorname{Res}\big(C(f_n(t')(t'+A) + Bf_{n-1}(t')), c(t'^2 - at' + b)\big)$, where the resultant is taken with respect to the variable r. Therefore, $A_3 = A_4$. Let $A_5 = \operatorname{Res}\big(C(f_n(x)(x+A) + Bf_{n-1}(x)), c(x^2 - ax + b)\big)$. The leading coefficient of t' is $\ell(\ell-1)^2(M^2+1)^2(M^2-1)^4M^{-2}\beta^{-4}$. Since the degree of $C(f_n(x)(x+A) + Bf_{n-1}(x))$ is $|n|$ we conclude that

$$A_4(M^2\beta^4)^{2|n|} = \big(\ell(\ell-1)^2(M^2+1)^2(M^2-1)^4\big)^{2|n|}A_5.$$

With $A_6 = \operatorname{Res}(f_n(x)(x+A) + Bf_{n-1}(x)), x^2 - ax + b)$ we see that $A_5 = A_6C^2c^{|n|}$ which is

$$A_5\gamma^2(\ell-1)^{4|n|}(M^2+1)^{4|n|}(M^2-1)^{8|n|}\ell^{2|n|} = A_6\beta^{9|n|+2}M^{4|n|}.$$

The lemma follows. $\square$

Because of the M^{-2} in the $t_5(r)$ equation, A_1 is a polynomial divided by $M^{4|n|}$. Therefore, we normalize such that the A-polynomial is $A(5,2n) = M^{4|n|}A_1$. By Lemma 7.6,

$$A(5,2n)\gamma^2 = M^{4|n|}\beta^{5|n|+\epsilon}R_n.$$

It now suffices to modify the recursion using Proposition 5.9 to remove the factors of γ^2 and $\beta^\epsilon(\beta^5)^{|n|}$. Let $c_1 = b-1$ and $c_2 = 2 - a^2 + 3b$. As a result, for $n > 5$

$$\begin{aligned} A(5,2n) = {} & (\beta^5M^4)c_1A(5,2(n-1)) + (\beta^5M^4)^2c_2A(5,2(n-2)) \\ & + (\beta^5M^4)^3c_2A(5,2(n-3)) + (\beta^5M^4)^4c_1A(5,2(n-4) \\ & - (\beta^5M^4)^5A(5,2(n-5)). \end{aligned}$$

(In fact, setting $A(5,0) = \beta^{-1}$ we can begin the recursion one term earlier.)

For negative n, we have

$$A(5,2n)\gamma^2 = M^{4|n|}\beta^{5|n|+2}R_n$$

and

$$\begin{aligned} A(5,2n) = {} & (\beta^5M^4)c_1A(5,2(n+1)) + (\beta^5M^4)^2c_2A(5,2(n+2)) \\ & + (\beta^5M^4)^3c_2A(5,2(n+3)) + (\beta^5M^4)^4c_1A(5,2(n+4)) \\ & - (\beta^5M^4)^5A(5,2(n+5)). \end{aligned}$$

(As for the positives, setting $A(5,0) = \beta^{-1}$ we can begin the recursion one term earlier.)

Finally, we write this as a self contained theorem. Recall that since $k=5$ we have $\ell = LM^{4n}$. Similar to the case when $k=3$, the results that helped us determine the recursion for the resultant rely on the recursive definitions of $F_{5,n}(r)$ and $G_{5,n}(r)$ as polynomials in $\ell, M^{\pm 2}$ and r. Therefore, to write $A(5,2n)$ in terms of the variables L and M, one uses the recursive definition below to determine the polynomial in ℓ and M and then substitutes $\ell = LM^{4n}$.

Theorem 7.7. *With d_1, d_2, d_3, d_4, and d_5 as in the appendix, we have the following. For n positive*

$$\begin{aligned} A(5,2n) &= d_1 A(5,2(n-1)) + d_2 A(5,2(n-2)) + d_3 A(5,2(n-3)) \\ &\quad + d_4 A(5,2(n-4) + d_5 A(5,2(n-5)). \end{aligned}$$

The initial conditions are $A(5,2n) = \mathrm{Res}(M^{2n} F_{5,n}(r), G_{5,n}(r))$ for $0 < n \leq 5$.

For n negative

$$\begin{aligned} A(5,2n) &= d_1 A(5,2(n+1)) + d_2 A(5,2(n+2)) + d_3 A(5,2(n+3)) \\ &\quad + d_4 A(5,2(n+4)) + d_5 A(5,2(n+5)). \end{aligned}$$

The initial conditions are $A(5,2n) = \mathrm{Res}(M^{2|n|} F_{5,n}(r), G_{5,n}(r))$ for $-5 \leq n \leq 0$.

7.6. $J(2m,2m)$. The knots $J(2m,2m)$ exhibit an additional symmetry, seen by turning the 4-plat diagram for these two-bridge knots upside down. This effectively factors the character variety, and the canonical component is birational to $\mathbb{C}$ (see [11]). Moreover, the canonical component of the A-polynomial is determined by the resultant of $(t_m(r) - r)$ and $G_{2m,m}$. In this case, $\ell = L$. The equations are, up to sign

$$\begin{aligned} t_m(r) &= -f_m(r)(g_{m+1}(r) - g_m(r))(M^2 + M^{-2} - r) + 2 \\ G_{2m,m}(r) &= f_m(r)(\alpha - \beta r) - \gamma f_{m-1}(r). \end{aligned}$$

The polynomial $t_m(r) - r$ is reducible. Since $t_m(r) - r$ is given by

$$-f_m(r)(g_{m+1}(r) - g_m(r))(M^2 + M^{-2} - r) + (2 - r)$$

it suffices to see that $2 - r$ is a factor of $g_{m+1}(r) - g_m(r)$. Since $f_j(2) = j$ for all j, it follows that $g_j(2) = f_j(2) - f_{j-1}(2) = j - (j-1) = 1$ and the fact that for all j, $g_{j+1}(2) - g_j(2) = 0$ follows. Since $r = 2$ corresponds to the reducible representations, we do not want to include the contribution of these to the A-polynomial. However, the format of $t(r) - r$ as above is easy to use, so we will compute the resultant of $G_{2m,m}(r)$ and $t(r) - r$ and then divide by the extra factor which corresponds to this extra term. The contribution of this extra factor is $\mathrm{Res}(r - 2, G_{2m,m}(r)) = G_{2m,m}(2) = \gamma$.

For small values of m, the resultant can be computed directly using a computer algebra program. We show that this resultant can be computed recursively using Proposition 5.9. Let

$$a = \tfrac{2\sigma}{\beta\delta},\ b = \tfrac{\tau}{\beta\delta},\ A = \tfrac{-\alpha}{\beta},\ \text{ and }\ B = \tfrac{\gamma}{\beta}.$$

Let $R_m = \mathrm{Res}(r^2 - ar + b, f_m(r)(r + A) + B f_{m-1}(r))$ and let

$$A_1 = \mathrm{Res}(t_m(r) - r, G_{2m,m}(r)).$$

Lemma 7.8. *For $m \neq 0$ we have*

$$\gamma g_0 \mathrm{Res}(t_m(r) - r, G_{2m,m}(r)) = \beta^2 (\beta\delta)^{|m|} R_m$$

where g_0 is the leading coefficient of $G_{2m,m}(r)$.

Proof. Let $A_2 = \mathrm{Res}((\delta\beta r - \sigma)(t_m(r) - r), G_{2m,m})$. First, notice that

$$A_2 = A_1 \mathrm{Res}(\delta\beta r - \sigma, G_{2m,m})$$

and since $\deg(G_{2m,m}) = |m|$ using Lemma 6.2 we have

$$\mathrm{Res}(\delta\beta r - \sigma, G_{2m,m}) = \gamma g_0^{2|m|}.$$

Therefore,

$$A_2 = \gamma g_0^{2|m|} A_1.$$

By Lemma 6.5 there is a polynomial P_3 such that

$$t_m(r)(\delta\beta r - \sigma) + (\tau - \sigma r) + G_{2m,m}(r)P_3(r) = 0.$$

It follows that

$$(\delta\beta r - \sigma)(t_m(r) - r) = \big(- \beta\delta r^2 + 2\sigma r - \tau\big) - G_{2m,m}(r)P_3(r).$$

Let $A_3 = \mathrm{Res}(\beta\delta r^2 - 2\sigma r + \tau, G_{2m,m}(r))$.

By Lemma 5.1 if $|m| > 0$ then

$$A_2 = g_0^{2|m|-1} A_3.$$

We conclude that

$$\gamma g_0 A_1 = A_3.$$

We have $\beta\delta r^2 - 2\sigma r + \tau = \beta\delta(r^2 - ar + b)$ and

$$G_{2m,m}(r) = -\beta(f_m(r)(r + A) - Bf_{m-1}(r)),$$

so that

$$A_3 = (\beta\delta)^{|m|}\mathrm{Res}(r^2 - ar + b, G_{2m,m}(r))$$

and

$$\beta^2 R_m = \mathrm{Res}(r^2 - ar + b, G_{2m,m}(r)).$$

(When $m > 0$ we are factoring out the leading term, β. When $m < 0$ we are factoring out β even though it is not the leading term.) Therefore,

$$A_3 = (\beta\delta)^{|m|}\beta^2 R_m.$$

As a result,

$$\gamma g_0 A_1 = \beta^2(\beta\delta)^{|m|}R_m. \qquad \square$$

Since $J(-2m, -2m)$ is the mirror image of $J(2m, 2m)$ we now assume that $m > 0$ so that $g_0 = \beta$. Because of the M^{-2} factor in $t_m(r)$, A_1 is a polynomial in M and L divided by M^{2m}. Therefore, the A-polynomial is

$$A(2m, 2m) = M^{2m}\tfrac{1}{\gamma}A_1 = \frac{\beta^2}{\gamma^2 g_0}(\beta\delta M^2)^{|m|}R_m.$$

Using $M^{2m}\frac{1}{\gamma}\mathrm{Res}(t_m(r) - r, G_{2m,m}(r))$ as the base cases, $A(2m, 2m)$ satisfies the R_m recursion. Specifically, we have the following, by Proposition 5.9

with $c_1 = (b-1)$ and $c_2 = (2 - a^2 + 3b)$. The A-polynomial $A(2m, 2m)$ satisfies the recursion

$$\begin{aligned} A(2m,2m) = {} & (\beta\delta M^2)c_1 A(2(m-1),2(m-1)) \\ & + (\beta\delta M^2)^2 c_2 A(2(m-2),2(m-2)) \\ & + (\beta\delta M^2)^3 c_2 A(2(m-3),2(m-3)) \\ & + (\beta\delta M^2)^4 c_1 A(2(m-4),2(m-4)) \\ & - (\beta\delta M^2)^5 A(2(m-5),2(m-5)). \end{aligned}$$

The initial conditions are determined by

$$A(2m,2m) = M^{2|m|}\gamma^{-1}\mathrm{Res}(t_m(r) - r, G_{2m,m}(r)) = M^{2|m|}\frac{\beta^2}{\gamma^2 g_0}(\beta\delta)^{|m|}R_m$$

for $|m| < 5$.

Finally, we write this as a self contained theorem.

Theorem 7.9. *With d_1, d_2, d_3, d_4, and d_5 as in the appendix, for $m > 5$*

$$\begin{aligned} A(2m,2m) = {} & d_1 A(2(m-1),2(m-1)) + d_2 A(2(m-2),2(m-2)) \\ & + d_3 A(2(m-3),2(m-3)) + d_4 A(2(m-4),2(m-4)) \\ & + d_5 A(2(m-5),2(m-5)). \end{aligned}$$

The initial conditions are $A(2m,2m) = M^{2m}\mathrm{Res}(M^2(t_m(r) - r), G_{2m,m}(r))$ for $0 < |m| \leq 5$.

8. Appendix

8.1. $A(4, 2n)$ terms. The coefficients for the recursion for $A(4, 2n)$ given in Theorem 7.5 are as follows. Let $\beta = M^2 + L$ and a and b be as in Section 7.4:

$$\begin{aligned} a = {} & \beta^{-4}M^{-2}\Big(L(L-1)(2L^2 - L + 1) + 2L(1 - 4L + 5L^2)M^2 \\ & + L(2L^3 + 5L^2 + 10L - 1)M^4 + 4L(1 + 4L + L^2)M^6 \\ & + (2 + 5L + 10L^2 - L^3)M^8 + 2L(5 - 4L + L^2)M^{10} \\ & + (1 - L)(2 - L + L^2)M^{12})\Big) \end{aligned}$$

$$\begin{aligned} b = {} & \beta^{-4}M^{-4}\Big(L(L-1)^3 + 2L(L-1)(3L-1)M^2 \\ & + 2L(L^3 + L^2 + L - 1)M^4 + 2L(L+1)(3L+1)M^6 \\ & + (1 + 4L + 14L^2 + 4L^3 + L^4)M^8 + 2L(L+1)(L+3)M^{10} \\ & + 2(1 + L + L^2 - L^3)M^{12} + 2L(L-1)(L-3)M^{14} + (1-L)3M^{16}\Big) \end{aligned}$$

We define

$$d = M^4\beta^4$$

$$\begin{aligned}
d_1 &= d(b-1)\\
d_2 &= d^2(2-a^2+3b)\\
d_3 &= d^3(2-a^2+3b)\\
d_4 &= d^4(b-1)\\
d_5 &= -d^5.
\end{aligned}$$

8.2. $A(5, 2n)$ terms. The coefficients for the recursion for $A(5, 2n)$ given in Theorem 7.7 are as follows. Let $\beta = M^2 + \ell$, and a and b be as in Section 7.5:

$$\begin{aligned}
a = \Big(&(\ell^4 - 2\ell^3 + 3\ell^2 - 4\ell + 2)M^{14} + (9\ell^3 + 13\ell - 16\ell^2 - 2\ell^4)M^{12}\\
&+ (\ell^4 - 12\ell^3 + 23\ell^2 + 6\ell + 2)M^{10} + (5\ell^3 + 5\ell + 30\ell^2)M^8\\
&+ (30\ell^3 + 5\ell^2 + 5\ell^4)M^6 + (6\ell^4 + 23\ell^3 - 12\ell^2 + \ell + 2\ell^5)M^4\\
&+ (-16\ell^3 - 2\ell + 9\ell^2 + 13\ell^4)M^2 + 3\ell^3 - 2\ell^2 + 2\ell^5 + \ell - 4\ell^4\Big)M^{-2}\beta^{-5}\\
b = \Big(&(6\ell^2 + \ell^4 - 4\ell^3 + 1 - 4\ell)M^{18} + (-2\ell^4 - 18\ell^2 + 8\ell + 12\ell^3)M^{16}\\
&+ (2 + 2\ell + 2\ell^4 - 6\ell^3 + 4\ell^2)M^{14} + (8\ell - 2\ell^4 - 8\ell^3 + 22\ell^2)M^{12}\\
&+ (6\ell^3 + 1 + \ell^4 + 26\ell^2 + 6\ell)M^{10} + (\ell + \ell^5 + 26\ell^3 + 6\ell^4 + 6\ell^2)M^8\\
&+ (-8\ell^2 - 2\ell + 8\ell^4 + 22\ell^3)M^6 + (2\ell^5 + 4\ell^3 - 6\ell^2 + 2\ell^4 + 2\ell)M^4\\
&+ (12\ell^2 - 2\ell - 18\ell^3 + 8\ell^4)M^2 + 6\ell^3 + \ell - 4\ell^4 + \ell^5 - 4\ell^2\Big)M^{-4}\beta^{-5}.
\end{aligned}$$

We define

$$\begin{aligned}
d &= M^4\beta^5\\
d_1 &= d(b-1)\\
d_2 &= d^2(2-a^2+3b)\\
d_3 &= d^3(2-a^2+3b)\\
d_4 &= d^4(b-1)\\
d_5 &= -d^4.
\end{aligned}$$

8.3. $A(2m, 2m)$ terms. The coefficients for the recursion for $A(2m, 2m)$ given in Theorem 7.9 are as follows. Let $\beta = M^2 + L$, $\delta = M^2L + 1$, $\sigma = \beta^2 + \delta^2$, and $\tau = (L-1)^2M^{-2} + 4L + 2(L+1)^2M^2 + 4M^4L + (L-1)^2M^6$. Let a and b be as in Section 7.6:

$$\begin{aligned}
a &= 2\sigma/(\beta\delta)\\
b &= \tau/(\beta\delta).
\end{aligned}$$

We define

$$d = \beta\delta M^2$$

$$\begin{aligned} d_1 &= d(b-1) \\ d_2 &= d^2(2-a^2+3b) \\ d_3 &= d^3(2-a^2+3b) \\ d_4 &= d^4(b-1) \\ d_5 &= -d^4. \end{aligned}$$

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