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Multipliers in weighted settings and strong convergence of associated operator-valued Fourier series

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Abstract. This note describes the pleasant features that accrue in weighted settings when the partial sums of the operator-valued Fourier series corresponding to a multiplier function $\psi : \mathbb{T} \to \mathbb{C}$ are uniformly bounded in operator norm. This circle of ideas also includes a Tauberian-type condition on the multiplier function ψ sufficient to insure such uniform boundedness of partial sums. These considerations are shown to apply to Riemann's continuous, "sparsely differentiable," periodic function. In a larger sense, our considerations aim at showing how pillars of functional analysis and real-varable methods in Fourier analysis can be combined with "bread-and-butter" techniques from these subjects so as to reveal hitherto unnoticed useful tools in multiplier theory for weighted Lebesgue spaces.

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1. Introduction

This note concerns Fourier multiplier theory for the space $\ell^p(w)$, where $1 < p < \infty$, and $w \equiv \{w_k\}_{k=-\infty}^{\infty}$ is an A_p weight sequence (in symbols, $w \in A_p(\mathbb{Z})$), that is, $w \equiv \{w_k\}_{k=-\infty}^{\infty}$ is a bilateral sequence of positive real numbers for which there is a real constant C (called an $A_p\,(\mathbb{Z})$ *weight constant for* w) such that

$$\left(\frac{1}{M-L+1}\sum_{k=L}^{M} w_k\right)\left(\frac{1}{M-L+1}\sum_{k=L}^{M} w_k^{-1/(p-1)}\right)^{p-1} \le C,$$

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whenever $L \in \mathbb{Z}$, $M \in \mathbb{Z}$, and $L \leq M$. The space $\ell^p(w)$ is the corresponding Banach space consisting of all complex-valued sequences $x \equiv \{x_k\}_{k=-\infty}^{\infty}$ such that

$$\|x\|_{\ell^p(w)} \equiv \left\{ \sum_{k=-\infty}^{\infty} |x_k|^p \, w_k \right\}^{1/p} < \infty.$$

This introductory section sketches the requisite features of the multiplier theory for the spaces $\ell^p(w)$, since that theory is central to the treatment of $A_p(\mathbb{Z})$ weighted Lebesgue spaces. Some useful notation will be established in the process. We omit the background details regarding $A_1(\mathbb{Z})$, since they will not be needed below; however, for the explication of Remark 2.5(ii) and Theorem 2.6 below it is worthwhile to recall here that for $1 \leq r \leq u < \infty$, we have $A_r(\mathbb{Z}) \subseteq A_u(\mathbb{Z})$. Further relevant background details and references for the present discrete weighted setting can be found in, e.g., §8 of [13], §5 of [4], [6] (Theorem 3.3, Corollary 3.5, and Proposition 3.8 therein), and §5 of [3].

Except as otherwise noted below, our functional analysis approach combined with classical Fourier analysis methods is applied to obtain new outcomes for multiplier theory in the setting of weighted spaces. In particular, the Fourier multiplier aspects of Riemann's continuous "sparsely differentiable" function ((2.8) and Theorem 2.6) unearth novel features of this venerable icon of classical analysis.

A function $\psi \in L^\infty(\mathbb{T})$ is called a *(Fourier) multiplier* for $\ell^p(w)$ (in symbols, $\psi \in M_{p,w}(\mathbb{T})$) provided that convolution by its inverse Fourier transform defines a bounded operator on $\ell^p(w)$. Specifically, we require:

(i) For each $x \equiv \{x_k\}_{k=-\infty}^{\infty} \in \ell^p(w)$ and each $j \in \mathbb{Z}$, the series

$$\left(\psi^\vee * x\right)(j) \equiv \sum_{k=-\infty}^{\infty} \psi^\vee(j-k)\, x_k \text{ converges absolutely.}$$

(ii) The mapping $T_\psi^{(p,w)} : x \in \ell^p(w) \mapsto \psi^\vee * x$ is a bounded linear mapping of $\ell^p(w)$ into $\ell^p(w)$ (in symbols, $T_\psi^{(p,w)} \in \mathfrak{B}(\ell^p(w))$).

We then call $T_\psi^{(p,w)}$ the multiplier transform corresponding to ψ. The elements of $M_{p,w}(\mathbb{T})$ are identified modulo equality a.e. on $\mathbb{T}$. Straightforward reasoning shows that $M_{p,w}(\mathbb{T})$ is an algebra under pointwise operations, and that the mapping $\psi \in M_{p,w}(\mathbb{T}) \to T_\psi^{(p,w)}$ is an algebra isomorphism of $M_{p,w}(\mathbb{T})$ into $\mathfrak{B}(\ell^p(w))$. Hence $M_{p,w}(\mathbb{T})$ is a unital normed algebra under the norm

$$\|\psi\|_{M_{p,w}(\mathbb{T})} \equiv \left\| T_\psi^{(p,w)} \right\|_{\mathfrak{B}(\ell^p(w))},$$

and the algebra of multiplier transforms is commutative. Moreover, Theorem 2.10 of [5] shows that when $M_{p,w}(\mathbb{T})$ is furnished with the norm

$\|\cdot\|_{M_{p,w}(\mathbb{T})}$, it becomes a unital Banach algebra, with

$$\|\psi\|_{L^\infty(\mathbb{T})} \le \|\psi\|_{M_{p,w}(\mathbb{T})}\,, \text{ for all } \psi \in M_{p,w}(\mathbb{T})\,. \tag{1.1}$$

Regularization techniques for Fourier multipliers in the unweighted setting carry over to the weighted setting. Specifically, if $1 < p < \infty$, $w \in A_p(\mathbb{Z})$, $\psi \in M_{p,w}(\mathbb{T})$, and $\mathfrak{k} \in L^1(\mathbb{T})$, then the convolution $\mathfrak{k} * \psi \in M_{p,w}(\mathbb{T})$, and

$$\|\mathfrak{k} * \psi\|_{M_{p,w}(\mathbb{T})} \le \|\mathfrak{k}\|_{L^1(\mathbb{T})}\, \|\psi\|_{M_{p,w}(\mathbb{T})}\,.$$

In §2, we shall treat conditions engendering strong convergence of operator-valued Fourier series that arise in conjunction with multiplier transforms on $\ell^p(w)$. Remark 2.5(ii) and Theorem 2.6 apply these considerations to the multiplier properties of Riemann's continuous, "sparsely differentiable", periodic function.

Historically one of the foremost and central examples of an $\ell^p(w)$ multiplier, where $1 < p < \infty$, has been the function $\psi_0 : \mathbb{T} \to \mathbb{C}$, which is specified by $\psi_0(1) = 0$, and $\psi_0\left(e^{it}\right) = -i(\pi - t)$, for $0 < t < 2\pi$, and whose inverse Fourier transform is the discrete Hilbert kernel: $\psi_0^\vee(0) = 0$, and $\psi_0^\vee(k) = 1/k$, for $k \in \mathbb{Z} \setminus \{0\}$. The pivotal role of ψ_0 in this discrete venue where $1 < p < \infty$ stems from Theorem 10 of [13], which characterizes $A_p(\mathbb{Z})$ weights w among weight sequences by the condition that $M_{p,w}(\mathbb{T})$ includes the function ψ_0. In this setup $T_{\psi_0}^{(p,w)}$ coincides with the discrete Hilbert transform $\mathfrak{H}_{p,w}$, which is specified as convolution by the discrete Hilbert kernel on $\ell^p(w)$. Stečkin's Theorem for $A_p(\mathbb{Z})$ weighted Lebesgue spaces, where $1 < p < \infty$, generalizes this property of ψ_0 by asserting that if $\phi : \mathbb{T} \to \mathbb{C}$ has bounded variation (in symbols, $\phi \in BV(\mathbb{T})$), then $\phi \in M_{p,w}(\mathbb{T})$. A straightforward proof of this weighted version of Stečkin's Theorem is readily seen by obvious transplanting of its classical unweighted proof, as presented in, e.g., [7], pp. 377,378, and includes the following estimate for all $\phi \in BV(\mathbb{T})$.

$$\left\|T_\phi^{(p,w)}\right\|_{\mathfrak{B}(\ell^p(w))} \le \left|\widehat{\phi}(0)\right| + (2\pi)^{-1}\, \|\mathfrak{H}_{p,w}\|_{\mathfrak{B}(\ell^p(w))} \operatorname{var}(\phi, \mathbb{T})\,.$$

The role played by $BV(\mathbb{T})$ as a multiplier class for our weighted setting of $\ell^p(w)$, where $1 < p < \infty$, can be extended to the class of Marcinkiewicz multipliers $\mathfrak{M}_1(\mathbb{T})$, which consists of all bounded functions $f : \mathbb{T} \to \mathbb{C}$ such that f has uniformly bounded variations on the dyadic arcs of $\mathbb{T}$. This weighted version of the classical Marcinkiewicz Multiplier Theorem is due to D.S. Kurtz (Theorem 2 of [15], which transplants from $\mathbb{R}$ to our setting of $\ell^p(w)$), and yields the estimate

$$\left\|T_f^{(p,w)}\right\|_{\mathfrak{B}(\ell^p(w))} \le K_{p,C}\, \|f\|_{\mathfrak{M}_1(\mathbb{T})}\,,$$

where C is an arbitrarily chosen $A_p(\mathbb{Z})$ weight constant for w. (Here and henceforth the symbol "K" with a possibly empty set of subscripts signifies a constant which depends only on those subscripts, and which may change in value from one occurrence to another.)

In connection with this circle of ideas, we recall that if $g \in BV(\mathbb{T})$ (respectively, $g \in \mathfrak{M}_1(\mathbb{T})$) then $\sup\{|k\widehat{g}(k)| : k \in \mathbb{Z}\} \le (2\pi)^{-1} \operatorname{var}(g, \mathbb{T})$ (respectively, $|\widehat{g}(k)| \le K \|\psi\|_{\mathfrak{M}_1(\mathbb{T})} |k|^{-1} \log(\pi|k|)$, for $k \in \mathbb{Z} \setminus \{0\}$). The first of these estimates is a familiar consequence of integration by parts. The estimate here for the decay rate for the Fourier transforms of $\mathfrak{M}_1(\mathbb{T})$ functions traces back to the discussion in §8 of [10], which states this order of upper estimate without proof in the form $O\left(|k|^{-1} \log(|k|)\right)$, for $|k| \ge 2$, the treatment in [10] being couched in what is now archaic nomenclature from an era prior to the advent of the original Marcinkiewicz Multiplier Theorem. (The proof of a more general class of estimates can be found in Theorem 3.2 of [3].) Their above-noted upper estimate for the decay rate of the Fourier transforms of $\mathfrak{M}_1(\mathbb{T})$ functions in [10] prompts Hardy and Littlewood to raise the question of whether $\log|k|$ can be dropped from it, as would be the case if g were of bounded variation on all of $\mathbb{T}$. They answer this question in the negative by explicitly constructing a counterexample in the form of a function $g_0 \in \mathfrak{M}_1(\mathbb{T})$ such that $g_0\left(e^{i(\cdot)}\right)$ is an even function on $\mathbb{R}$, and such that g_0 fails to satisfy

$$\widehat{g_0}(k) = O\left(|k|^{-1}\right), \quad \text{as } (|k|) \to \infty.$$

For each $k \in \mathbb{Z}$, let us denote by $\mathfrak{e}_k$ the the character of $\mathbb{T}$ specified by $\mathfrak{e}_k(z) \equiv z^k$. Then $\mathfrak{e}_k \in M_{p,w}(\mathbb{T})$, and

$$T_{\mathfrak{e}_k}^{(p,w)} = \mathcal{L}^k, \tag{1.2}$$

where $\mathcal{L} \in \mathfrak{B}(\ell^p(w))$ is the left bilateral shift, defined on $\ell^p(w)$ by putting $\mathcal{L}x \equiv \{x_{k+1}\}_{k=-\infty}^{\infty}$, for each $x = \{x_k\}_{k=-\infty}^{\infty} \in \ell^p(w)$. $\mathcal{L}$ is positivity-preserving, invertible, disjoint, and mean-bounded in the sense that

$$\varsigma_{p,w} \equiv \sup\left\{\left\|\frac{1}{2N+1}\sum_{j=-N}^{N}\mathcal{L}^j\right\|_{\mathfrak{B}(\ell^p(w))} : N \ge 0\right\} < \infty. \tag{1.3}$$

The inverse of $\mathcal{L}$ is the right bilateral shift $\mathcal{R}$ specified on $\ell^p(w)$ by writing $\mathcal{R}x = \{x_{k-1}\}_{k=-\infty}^{\infty}$. It is readily seen that, for each $n \in \mathbb{Z}$,

$$\|\mathcal{L}^n\|_{\mathfrak{B}(\ell^p(w))} = \sup\left\{\left(\frac{w_{k-n}}{w_k}\right)^{1/p} : k \in \mathbb{Z}\right\}. \tag{1.4}$$

A family of concrete examples for the foregoing scenario can be formulated as follows. Let $\eta \in \mathbb{R}$, with $0 < \eta < p-1$. Then (as covered in Proposition 3.8 of [6]) the $A_p(\mathbb{Z})$ condition is satisfied by the weight sequence $\mathfrak{w}^{(\eta)}$ specified by: $\mathfrak{w}^{(\eta)}(0) = 1$, and $\mathfrak{w}^{(\eta)}(k) = |k|^{\eta}$, for $k \in \mathbb{Z}\setminus\{0\}$. Elementary calculations proceeding from (1.4) readily show that for all $n \in \mathbb{Z}$,

$$\|\mathcal{L}^n\|_{\mathfrak{B}\left(\ell^p\left(\mathfrak{w}^{(\eta)}\right)\right)} = (|n|+1)^{\eta/p}. \tag{1.5}$$

(In particular, $\mathcal{L} \in \mathfrak{B}\left(\ell^p\left(w^{(\eta)}\right)\right)$ is not power-bounded.)

For each $z \in \mathbb{T}$, we define the linear isometry V_z of $\ell^p(w)$ onto $\ell^p(w)$ by writing for each $x \equiv \{x_k\}_{k=-\infty}^{\infty} \in \ell^p(w)$,

$$V_z(x) = \left\{z^{-k}x_k\right\}_{k=-\infty}^{\infty}. \tag{1.6}$$

It is elementary to verify by direct calculations that for each $\psi \in M_{p,w}(\mathbb{T})$ and $z \in \mathbb{T}$, the rotated function $\psi_{z^{-1}}(\cdot) = \psi\left((\cdot)\,z^{-1}\right)$ belongs to $M_{p,w}(\mathbb{T})$, with

$$T_{\psi_{z^{-1}}}^{(p,w)} = V_{z^{-1}} T_{\psi}^{(p,w)} V_z,$$

whence

$$\left\|T_{\psi_{z^{-1}}}^{(p,w)}\right\|_{\mathfrak{B}(\ell^p(w))} = \left\|T_{\psi}^{(p,w)}\right\|_{\mathfrak{B}(\ell^p(w))}. \tag{1.7}$$

Moreover, the reasoning on pages 151, 152 of [3] goes over *mutatis mutandis* to the present circumstances to show that the function $\Psi : z \in \mathbb{T} \mapsto T_{\psi_z}^{(p,w)}$ is continuous with respect to the strong operator topology of $\mathfrak{B}(\ell^p(w))$.

It will be convenient to record here our context's version of the following universal workhorse for multiplier theory. This proposition also has a few pleasant consequences regarding the convergence of multiplier transforms that seem to have been overlooked in the lore of multiplier theory, and which will be treated below in the form of Theorem 2.2.

Proposition 1.1. *Suppose* $1 < p < \infty$, $w \in A_p(\mathbb{Z})$, $\{\phi_n\}_{n=1}^{\infty} \subseteq M_{p,w}(\mathbb{T})$, *and*

$$\mathfrak{s} = \sup_{n \in \mathbb{N}} \|\phi_n\|_{M_{p,w}(\mathbb{T})} < \infty. \tag{1.8}$$

Suppose further that there is a function $\phi : \mathbb{T} \to \mathbb{C}$ *such that, with respect to Haar measure on* $\mathbb{T}$,

$$\phi_n \to \phi \ \textit{a.e.on}\ \mathbb{T}, \tag{1.9}$$

then $\phi \in M_{p,w}(\mathbb{T})$, *and*

$$\|\phi\|_{M_{p,w}(\mathbb{T})} \le \sup_{n \in \mathbb{N}} \|\phi_n\|_{M_{p,w}(\mathbb{T})}. \tag{1.10}$$

Proof. Notice first that by virtue of (1.1), we have, in the notation of (1.8),

$$\sup_{n \in \mathbb{N}} \|\phi_n\|_{L^\infty(\mathbb{T})} \le \mathfrak{s} < \infty, \tag{1.11}$$

and so in view of (1.9), $\phi \in L^\infty(\mathbb{T})$, and, for each $k \in \mathbb{Z}$,

$$\lim_n (\phi_n)^\vee(k) = \phi^\vee(k). \tag{1.12}$$

Here and henceforth we denote by ℓ_0 the linear space of all finitely supported, complex-valued bilateral sequences. Now let $x = \{x_k\}_{k=-\infty}^{\infty} \in \ell_0$. We infer from (1.12) that for each $k \in \mathbb{Z}$,

$$\left((\phi_n)^{\vee} * x\right)(k) \to \left(\phi^{\vee} * x\right)(k). \tag{1.13}$$

Moreover, (1.8) shows that for each $n \in \mathbb{N}$,

$$\left\|(\phi_n)^{\vee} * x\right\|_{\ell^p(w)} \leq \mathfrak{s} \left\| x\right\|_{\ell^p(w)}. \tag{1.14}$$

Applying (1.13) and (1.14) in conjunction with Fatou's Lemma in $\ell^p(w)$, we see that every $x = \{x_k\}_{k=-\infty}^{\infty} \in \ell_0$ has the property that $(\phi^{\vee} * x) \in \ell^p(w)$, with

$$\left\| \phi^{\vee} * x\right\|_{\ell^p(w)} \leq \mathfrak{s} \left\| x\right\|_{\ell^p(w)}. \tag{1.15}$$

The proof of Proposition 1.1 follows readily from (1.15) via the density of ℓ_0 in $\ell^p(w)$ with respect to the norm topology. $\square$

Given a complex-valued function $F \in L^1(\mathbb{T})$, and an integer $n \geq 0$, we shall symbolize the $n^{\underline{th}}$ partial sum of the Fourier series of F by $s_n(F, (\cdot)) \equiv \sum_{k=-n}^{n} \widehat{F}(k)\, \mathfrak{e}_k$. In view of (1.2) this definition can be rephrased in terms of multiplier transforms by writing

$$T_{s_n(F,(\cdot))}^{(p,w)} = \sum_{k=-n}^{n} \widehat{F}(k)\, \mathcal{L}^k. \tag{1.16}$$

We shall designate by $\sigma_N(F, (\cdot))$ the $N^{\underline{th}}$ $(C, 1)$ mean of the Fourier series of F. Thus, $\sigma_N(F, (\cdot)) \equiv (N+1)^{-1} \sum_{n=0}^{N} s_n(F, (\cdot))$. It is elementary that for all $z \in \mathbb{T}$,

$$\sigma_N(F, z) = (\kappa_N * F)(z),$$

where κ_N is the Fejér kernel for $\mathbb{T}$ of order N, specified by

$$\kappa_N(z) \equiv \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) z^k,$$

and satisfying $\kappa_N \geq 0$, and $\|\kappa_N\|_{L^1(\mathbb{T})} = 1$.

In terms of this notation, we can directly deduce the following corollary of Proposition 1.1 by appeal to the Fejér-Lebesgue Theorem (XI.(44.1) on pg. 631 of [12]). (The "if" part of its equivalence is immediately seen by convolving ϕ with the Fejér kernel for $\mathbb{T}$.)

Corollary 1.2. *Suppose that $1 < p < \infty$, $w \in A_p(\mathbb{Z})$, $\phi \in L^1(\mathbb{T})$. Then*

$$\sup_{N \geq 0} \|\sigma_N(\phi, (\cdot))\|_{M_{p,w}(\mathbb{T})} < \infty$$

if and only if $\phi \in M_{p,w}(\mathbb{T})$. If this is the case, then

$$\|\phi\|_{M_{p,w}(\mathbb{T})} \leq \sup_{N \geq 0} \|\sigma_N(\phi, (\cdot))\|_{M_{p,w}(\mathbb{T})}.$$

We close this introductory section with a pair of handy consequences of (1.16). Specifically, for each $\psi \in M_{p,w}(\mathbb{T})$, $z \in \mathbb{T}$, and integer $n \geq 0$, we can use the elementary relationship $\widehat{\psi_z}(k) = z^k \widehat{\psi}(k)$, valid for all $k \in \mathbb{Z}$, to infer that:

$$T^{(p,w)}_{s_n(\psi_z,(\cdot))} = \sum_{k=-n}^{n} z^k \widehat{\psi}(k) \mathcal{L}^k,$$

whence (1.7) yields

$$\begin{aligned} (1.17) \qquad \left\| T^{(p,w)}_{s_n(\psi_z,(\cdot))} \right\|_{\mathfrak{B}(\ell^p(w))} &= \left\| T^{(p,w)}_{(s_n(\psi,(\cdot)))_z} \right\|_{\mathfrak{B}(\ell^p(w))} \\ &= \left\| T^{(p,w)}_{s_n(\psi,(\cdot))} \right\|_{\mathfrak{B}(\ell^p(w))}. \end{aligned}$$

2. Expansion of multiplier transforms in Fourier series convergent in the strong operator topology

The stage is now set for our results on operator-valued Fourier series presented in the following theorems.

Theorem 2.1. *Suppose that* $1 < p < \infty$, $w \in A_p(\mathbb{Z})$, $\psi \in L^\infty(\mathbb{T})$, *and*

$$\mathfrak{S} \equiv \sup_{n \geq 0} \|s_n(\psi, (\cdot))\|_{M_{p,w}(\mathbb{T})} < \infty. \tag{2.1}$$

Then the following conclusions hold:

(i) $\psi \in M_{p,w}(\mathbb{T})$, *and* $\|\psi\|_{M_{p,w}(\mathbb{T})} \leq \mathfrak{S}$.

(ii) *For each* $z \in \mathbb{T}$, *the series*

$$\sum_{k=-\infty}^{\infty} z^k \widehat{\psi}(k) \mathcal{L}^k$$

converges in the strong operator topology of $\mathfrak{B}(\ell^p(w))$ *to*

$$\Psi(z) = T^{(p,w)}_{\psi_z},$$

which, as already noted in §1 is a continuous function of $z \in \mathbb{T}$, *with respect to the strong operator topology of* $\mathfrak{B}(\ell^p(w))$.

(iii) *For each* $x \in \ell^p(w)$, *and each* $k \in \mathbb{Z}$, *the* $k^{\underline{th}}$ *Fourier coefficient of the* $\ell^p(w)$*-valued continuous function* $\Psi(\cdot)x$ *is expressed by*

$$\widehat{\Psi(\cdot)\,x}(k) = \widehat{\psi}(k) \mathcal{L}^k x. \tag{2.2}$$

Proof. The proof of (i) is an immediate consequence of Corollary 1.2: since the $\sigma_N(\psi, (\cdot))$ are averages of the partial sums $s_n(\psi, (\cdot))$, we have

$$\sup_{N \geq 0} \|\sigma_N(\psi, (\cdot))\|_{M_{p,w}(\mathbb{T})} \leq \mathfrak{S} < \infty.$$

To establish (ii), for each fixed $m \in \mathbb{Z}$, let us denote by

$$\tau^{(m)} = \left\{ \tau_k^{(m)} \right\}_{k=-\infty}^{\infty} \in \ell^p(w)$$

the vector whose coordinates are defined in terms of Kronecker's delta by $\tau_k^{(m)} = \delta_{m,k}$. (Thus $\tau^{(m)} = \mathcal{L}^{-m}\tau^{(0)}$.) Notice in particular that for each $z \in \mathbb{T}$,

$$\ell^p(w) \ni T_{\psi_{z^{-1}}}^{(p,w)}\left(\tau^{(m)}\right) = \left\{z^{k-m}\psi^{\vee}(k-m)\right\}_{k=-\infty}^{\infty}.$$

Specializing to $m = 0$, $z = 1$, we see that the sequence

$$\psi^{\vee} \equiv \left\{\psi^{\vee}(k)\right\}_{k=-\infty}^{\infty} \in \ell^p(w).$$

Write χ_n for the characteristic function, defined on $\mathbb{Z}$, of $\{k \in \mathbb{Z} : |k| \le n\}$. Then, with convergence in the norm topology of $\ell^p(w)$,

$$\sum_{k=-n}^{n} \psi^{\vee}(k)\mathcal{L}^{-k}\left(\tau^{(0)}\right) = \left\{\psi^{\vee}(k)\chi_n(k)\right\}_{k=-\infty}^{\infty} \to \psi^{\vee} = T_{\psi}^{(p,w)}\left(\tau^{(0)}\right),$$

as $n \to \infty$. We can rewrite this convergence result in the form

$$\sum_{k=-n}^{n} \widehat{\psi}(k)\mathcal{L}^{k}\left(\tau^{(0)}\right) \to T_{\psi}^{(p,w)}\left(\tau^{(0)}\right). \tag{2.3}$$

For each $j \in \mathbb{Z}$, the multiplier transform $\mathcal{L}^{-j} = T_{\mathfrak{e}_{-j}}^{(p,w)}$ commutes with the operators occurring in both members of (2.3), and so upon application of $\mathcal{L}^{-j}$ to (2.3) we see that for all $j \in \mathbb{Z}$,

$$\left\|\sum_{k=-n}^{n} \widehat{\psi}(k)\mathcal{L}^{k}\left(\tau^{(j)}\right) - T_{\psi}^{(p,w)}\left(\tau^{(j)}\right)\right\|_{\ell^p(w)} \to 0, \text{ as } n \to \infty.$$

Since the vectors $\tau^{(j)}$, $j \in \mathbb{Z}$, span a dense linear manifold in the Banach space $\ell^p(w)$, it follows from this last and the uniform boundedness of the operators

$$\left\{\sum_{k=-n}^{n} \widehat{\psi}(k)\mathcal{L}^{k}\right\}_{n=0}^{\infty} = \left\{T_{s_n(\psi,(\cdot))}^{(p,w)}\right\}_{n=0}^{\infty}$$

stated by our hypothesis (2.1) that the sequence

$$\left\{\sum_{k=-n}^{n} \widehat{\psi}(k)\mathcal{L}^{k}\right\}_{n=0}^{\infty}$$

converges to $T_{\psi}^{(p,w)}$ in the strong operator topology of $\mathfrak{B}\left(\ell^p(w)\right)$. The desired conclusion (ii) is immediate from this, since for each $z \in \mathbb{T}$, we can replace ψ by ψ_z therein by virtue of (1.17).

For the demonstration of (iii), we need only utilize the convergence in (ii), while applying Bounded Convergence to the relevant $\ell^p(w)$-valued Bochner integrals arising. $\square$

Although we shall not need the next theorem, it fits neatly into the theme of convergence of multiplier transforms at a general level, and is consequently included here.

Theorem 2.2. *Under the hypotheses and notation of Proposition 1.1, the following two conclusions are valid:*

(i) *The sequence of multiplier transforms*

$$\left\{T_{\phi_n}^{(p,w)}\right\}_{n=1}^{\infty}$$

converges in the weak operator topology of $\mathfrak{B}\left(\ell^p(w)\right)$ *to* $T_{\phi}^{(p,w)}$.

(ii) *There is a subsequence*

$$\left\{T_{\phi_{n_k}}^{(p,w)}\right\}_{k=1}^{\infty} \quad \text{of} \quad \left\{T_{\phi_n}^{(p,w)}\right\}_{n=1}^{\infty}$$

such that the sequence of $(C,1)$ *averages of*

$$\left\{T_{\phi_{n_k}}^{(p,w)}\right\}_{k=1}^{\infty}$$

converges in the strong operator topology of $\mathfrak{B}\left(\ell^p(w)\right)$ *to* $T_{\phi}^{(p,w)}$.

Proof. We continue with the notation used in the proof of Theorem 2.1. Under the present hypotheses, we obviously have for each $n \in \mathbb{N}$,

$$\ell^p(w) \ni T_{\phi_n}^{(p,w)}\left(\tau^{(0)}\right) = \left\{\phi_n^{\vee}(k)\right\}_{k=-\infty}^{\infty}.$$

Moreover, it is clear by the pointwise a.e. Bounded Convergence of $\{\phi_n\}_{n=1}^{\infty}$ that for each $\nu \in \mathbb{Z}$,

$$\lim_n \phi_n^{\vee}(\nu) = \phi^{\vee}(\nu). \tag{2.4}$$

Since the sequence $\left\{T_{\phi_n}^{(p,w)}\left(\tau^{(0)}\right)\right\}_{n=1}^{\infty}$ is norm-bounded in the reflexive space $\ell^p(w)$, we can apply the Eberlein–Smulian Theorem (see, e.g., p. 458 of [8] for the Eberlein–Smulian Theorem) to infer that every subsequence of $\left\{T_{\phi_n}^{(p,w)}\left(\tau^{(0)}\right)\right\}_{n=1}^{\infty}$ has in turn a subsequence $\left\{T_{\phi_{n_{m_j}}}^{(p,w)}\left(\tau^{(0)}\right)\right\}_{j=1}^{\infty}$ weakly convergent in $\ell^p(w)$ to a corresponding $y \in \ell^p(w)$. However, since the coordinate functionals are bounded linear functionals on $\ell^p(w)$, we have for each $\nu \in \mathbb{Z}$,

$$\lim_j \phi_{n_{m_j}}^{\vee}(\nu) = y(\nu),$$

whence by (2.4) y is uniquely determined in all the above instances to be $\phi^{\vee} = T_{\phi}^{(p,w)}\left(\tau^{(0)}\right)$. An elementary reductio ad absurdum argument now establishes that, relative to the weak topology of $\ell^p(w)$, we have:

$$\left\{T_{\phi_n}^{(p,w)}\left(\tau^{(0)}\right)\right\}_{n=1}^{\infty} \to T_{\phi}^{(p,w)}\left(\tau^{(0)}\right). \tag{2.5}$$

It follows by the Banach–Saks Theorem ([17, p. 101]) that $\left\{T_{\phi_n}^{(p,w)}\left(\tau^{(0)}\right)\right\}_{n=1}^{\infty}$ contains a subsequence $\left\{T_{\phi_{n_k}}^{(p,w)}\left(\tau^{(0)}\right)\right\}_{k=1}^{\infty}$ such that, relative to the norm topology of $\ell^p(w)$,

$$\left\{T_{\phi_{n_k}}^{(p,w)}\left(\tau^{(0)}\right)\right\}_{k=1}^{\infty} \text{ converges } (C,1) \text{ to } T_{\phi}^{(p,w)}\left(\tau^{(0)}\right). \tag{2.6}$$

For each $j \in \mathbb{Z}$, $\tau^{(j)} = \mathcal{L}^{-j}\tau^{(0)}$.), and we can apply $\mathcal{L}^{-j}$ separately to (2.5) and (2.6) in order to deduce that the assertion in each of these items remains true if we replace $\tau^{(0)}$ by $\tau^{(j)}$ therein. Thus each of (2.5) and (2.6) continues to hold if $\tau^{(j)}$ is replaced by any finitely supported vector. Since the linear subspace ℓ_0 consisting of the finitely supported vectors is norm dense in $\ell^p(w)$, and $\sup_n \left\|T_{\phi_n}^{(p,w)}\right\|_{\mathfrak{B}(\ell^p(w))} < \infty$, the validity of this theorem's conclusions (i) and (ii) is now apparent. □

Remark 2.3. The assumption (2.1) is not a necessary condition for ψ to belong to $M_{p,w}(\mathbb{T})$. A counterexample to necessity can be found in the special ("unweighted") setting $w(k) \equiv 1$, with $p = 2$. (whence $M_{p,w}(\mathbb{T}) = L^{\infty}(\mathbb{T})$, with equality of norms). In this setting the requisite counterexample is furnished by the classical example of a continuous function $f : \mathbb{T} \to \mathbb{C}$ such that the sequence $\{s_n(f,1)\}_{n=0}^{\infty}$ is unbounded. (For details on the existence of such a function f, see, e.g., Theorem II.2.1 on pg. 51 of [14].)

We now take up a Tauberian-like converse for Theorem 2.1. This result is reminiscent of, but structurally distinct from, a Tauberian Theorem of G.H. Hardy regarding Fourier series convergence (see pgs. 52,53 of [14] for the latter).

Theorem 2.4. *Suppose that* $1 < p < \infty$, $w \in A_p(\mathbb{Z})$, $\psi \in M_{p,w}(\mathbb{T})$, *and*

$$\sup\left\{\left|k\widehat{\psi}(k)\right| : k \in \mathbb{Z}\right\} < \infty. \tag{2.7}$$

Then $\sup_{n\geq 0} \|s_n(\psi,(\cdot))\|_{M_{p,w}(\mathbb{T})} < \infty$. *(Hence the conclusions of Theorem 2.1 hold here.)*

Proof. For $n \geq 0$, $\|\kappa_n * \psi\|_{M_{p,w}(\mathbb{T})} \leq \|\psi\|_{M_{p,w}(\mathbb{T})}$, and

$$\kappa_n * \psi = s_n(\psi,(\cdot)) - (n+1)^{-1} \sum_{0<|k|\leq n} |k|\,\widehat{\psi}(k)\,\mathfrak{e}_k.$$

So it suffices for the proof of this theorem to show that for all $n \geq 0$, and all $x \in \ell^p(w)$,

$$\left\|(n+1)^{-1} \sum_{0<|k|\leq n} |k|\,\widehat{\psi}(k)\,\mathcal{L}^k x\right\|_{\ell^p(w)} \leq K_{p,w,\psi}\,\|x\|_{\ell^p(w)}.$$

This can be seen as follows by virtue of our hypothesis in (2.7) and the mean-boundedness of $\mathcal{L}$ expressed by (1.3). We have, pointwise on $\mathbb{Z}$,

$$\left|(n+1)^{-1}\sum_{0<|k|\leq n}|k|\,\widehat{\psi}\,(k)\,\mathcal{L}^k x\right| \\ \leq \left(\sup\left\{\left|j\widehat{\psi}\,(j)\right| : j\in\mathbb{Z}\right\}\right)(n+1)^{-1}\sum_{k=-n}^{n}\mathcal{L}^k\left(|x|\right),$$

and so, with $\varsigma_{p,w}$ as in (1.3) we see that

$$\left\|(n+1)^{-1}\sum_{0<|k|\leq n}|k|\,\widehat{\psi}\,(k)\,\mathcal{L}^k x\right\|_{\ell^p(w)} \\ \leq \left(\sup\left\{\left|j\widehat{\psi}\,(j)\right| : j\in\mathbb{Z}\right\}\right)2\varsigma_{p,w}\,\|x\|_{\ell^p(w)} \quad \square$$

Remark 2.5.

(i) For $1<p<\infty$, $w\in A_p(\mathbb{Z})$, Theorem 2.4 clearly applies to all functions in the class $BV(\mathbb{T})$. In the equivalent restatement of (1.16) for multiplier norms of partial sums s_n, the conclusion of Theorem 2.4 for $BV(\mathbb{T})$ is known as a specialization to $\mathcal{L}$ of the more general result for trigonometrically well-bounded operators on super-reflexive spaces expressed by Theorem 4.4 of [1]. The class of $\phi\in M_{p,w}(\mathbb{T})$ for which the condition $\sup_{n\geq 0}\|s_n(\phi,(\cdot))\|_{M_{p,w}(\mathbb{T})}<\infty$ holds can be considerably expanded from $BV(\mathbb{T})$ to functions of higher variation by way of the discussion in Remark 3.1(i) of [2], taken in conjunction with Theorem 4.1(b) of [2]. When applied to the left shift $\mathcal{L}$ in our context, this approach shows that corresponding to p and w there is a constant $\beta_{p,w}$ such that $1<\beta_{p,w}<\infty$, and such that for every r belonging to the open interval $(1,\beta_{p,w})$, and every function ϕ having finite r-variation on $\mathbb{T}$ (written $\phi\in V_r(\mathbb{T})$) we have $\sup_{n\geq 0}\left\|\sum_{k=-n}^{n}\widehat{\phi}(k)\mathcal{L}^k\right\|_{\mathfrak{B}(\ell^p(w))}<\infty$. However, as described in Remark 2.8(ii) of [1], each class $V_r(\mathbb{T})$, where $r>1$, includes a corresponding Hardy–Weierstrass type of continuous, nowhere differentiable function ψ_r such that (2.7) fails to hold for ψ_r.

(ii) For an example of a function satisfying (2.7) and belonging to $M_{p,w}(\mathbb{T})\setminus BV(\mathbb{T})$ in some settings, we turn to Riemann's continuous "almost nowhere" differentiable function $\mathcal{R}:\mathbb{R}\to\mathbb{R}$ specified by writing

$$\mathcal{R}\,(x)=\sum_{k=1}^{\infty}k^{-2}\sin\left(k^2x\right).$$

The set of points where $\mathcal{R}$ is differentiable is precisely

$$\left\{\pi\left(2m+1\right)\left(2n+1\right)^{-1} : m\in\mathbb{Z}, n\in\mathbb{Z}\right\}$$

([9]). Clearly this sparse differentiability shows that the (2π)-periodic function $\mathcal{R}$ cannot be regarded as an element of $BV(\mathbb{T})$ (for our purposes it will be convenient to regard the Riemann function $\mathcal{R}$ as the function $\mathfrak{R}$ defined on $\mathbb{T}$ by putting

$$\mathfrak{R}\left(z\right)=\sum_{k=1}^{\infty}\left(2i\right)^{-1}k^{-2}z^{k^2}-\sum_{k=1}^{\infty}\left(2i\right)^{-1}k^{-2}z^{-k^2}.) \tag{2.8}$$

Obviously the definition of $\mathfrak{R}$ shows that it satisfies the condition (2.7). For its multiplier status in appropriate weighted settings, we next fix p in the range $2\le p<\infty$, and suppose that $w\in A_{p/2}(\mathbb{Z})$. For this setting Theorem 5.1 of [5] furnishes us with a corresponding s, $2<s<\infty$, such that whenever $1\le q<s$ and ψ belongs to the Marcinkiewicz q-class $\mathfrak{M}_q(\mathbb{T})$ (in particular whenever $\psi\in V_q(\mathbb{T})$), then $\psi\in M_{p,w}(\mathbb{T})$. We now proceed to show that $\mathfrak{R}\in V_q(\mathbb{T})$ for some $q\in(1,s)$. Choose and fix a number γ such that $2<\gamma<s$ and then choose $\lambda>2$ sufficiently large enough to ensure that

$$2<\frac{\lambda\gamma}{\lambda-1}<\frac{\lambda\gamma}{\lambda-2}<s.$$

Now let $\alpha=\dfrac{\lambda-1}{\lambda\gamma}$. In particular $\alpha<\dfrac{1}{2}$; so $2\left(1-\alpha\right)>1$, and we can now define the continuous function $\mathfrak{f}$ on $\mathbb{T}$ by putting

$$\begin{aligned}\mathfrak{f}\left(z\right)&=\sum_{k=1}^{\infty}\left(ik^2\right)^{\alpha}\left(2i\right)^{-1}k^{-2}z^{k^2}-\sum_{k=1}^{\infty}\left(-ik^2\right)^{\alpha}\left(2i\right)^{-1}k^{-2}z^{-k^2}\\&=\sum_{k=1}^{\infty}i^{\alpha}\left(2i\right)^{-1}k^{-2(1-\alpha)}z^{k^2}-\sum_{k=1}^{\infty}\left(-i\right)^{\alpha}\left(2i\right)^{-1}k^{-2(1-\alpha)}z^{-k^2}.\end{aligned}$$

Clearly $\mathfrak{R}$ is the $\alpha^{\underline{th}}$ fractional integral of $\mathfrak{f}$ according to Weyl's formulation of fractional integration for periodic functions whose integral vanishes over a period (see §2 of [16], wherein "the origin of integration" is taken to be $-\infty$). Since $\mathfrak{f}\in L^{\lambda\gamma}(\mathbb{T})$, and $\lambda>2$ implies that $\left(\lambda\gamma\right)^{-1}<\alpha$, an application of Theorem 13 of [11] now shows that, in particular, $\mathfrak{R}$ is a Lipschitz function on $\mathbb{T}$ of order

$$\left(\alpha-\frac{1}{\lambda\gamma}\right)=\frac{\lambda-2}{\lambda\gamma},$$

and hence $\mathfrak{R}\in V_q(\mathbb{T})$, where $q=\dfrac{\lambda\gamma}{\lambda-2}<s$.

This discussion in Remark 2.5 has demonstrated the following application of Theorem 2.4.

Theorem 2.6. *Riemann's continuous "sparsely differentiable" function*

$$\mathfrak{R} : \mathbb{T} \to \mathbb{R}$$

defined in (2.8) *above satisfies a Lipschitz condition on* $\mathbb{T}$ *of order less than* 2^{-1}. *If* $2 \leq p < \infty$, *and* $w \in A_{p/2}(\mathbb{Z})$, *then* $\mathfrak{R} \in M_{p,w}(\mathbb{T})$, *and the conclusions of Theorem* 2.4 *above are valid for* $\mathfrak{R}$ *in the context of* $\ell^p(w)$.

References

[1] Berkson, Earl. Spectral theory and operator ergodic theory on super-reflexive Banach spaces. *Studia Math.* **200** (2010), no. 3, 221–246. MR2733267 (2012c:47009), Zbl 1206.26015, doi: 10.4064/sm200-3-2.

[2] Berkson, Earl. Rotation methods in operator ergodic theory. *New York J. Math.* **17** (2011), 21–39. MR2781906 (2012f:47029), Zbl 1230.47029.

[3] Berkson, Earl. Marcinkiewicz multipliers of higher variation and summability of operator-valued Fourier series. *Studia Math.* **222** (2014), no. 2, 123–155. MR3223321, Zbl 06303629, doi: 10.4064/sm222-2-2.

[4] Berkson, Earl; Gillespie, T. A. Mean-boundedness and Littlewood–Paley for separation-preserving operators. *Trans. Amer. Math. Soc.* **349** (1997), no. 3, 1169–1189. MR1407694 (98b:42011), Zbl 0888.42004, doi: 10.1090/S0002-9947-97-01896-5.

[5] Berkson, Earl; Gillespie, T. A. Multipliers for weighted L^p-spaces, transference, and the q-variation of functions. *Bull. Sci. Math.* **122** (1998), no. 6, 427–454. MR1649414 (99h:46042), Zbl 0935.42005, doi: 10.1016/S0007-4497(98)80002-X.

[6] Berkson, Earl; Gillespie, T. A. Operator means and spectral integration of Fourier multipliers. *Houston J. Math.* **30** (2004), no. 3, 767–814. MR2083875 (2005h:42013), Zbl 1056.42008.

[7] Dowson, H. R. Spectral theory of linear operators. London Mathematical Society Monographs, 12. *Academic Press, Inc., London-New York*, 1978. xii+422 pp. ISBN: 0-12-220950-8. MR0511427 (80c:47022), Zbl 0384.47001.

[8] Dunford, Nelson; Schwartz, Jacob T. Linear operators. I: General theory. Pure and Applied Mathematics, 7. *Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London*, 1958. xiv+858 pp. MR0117523 (22 #8302), Zbl 0084.10402.

[9] Gerver, Joseph. More on the differentiability of the Riemann function. *Amer. J. Math.* **93** (1971), 33–41. MR0276422 (43 #2169), Zbl 0228.26008, doi: 10.2307/2373445.

[10] Hardy, G. H.; Littlewood, J. E. Notes on the theory of series (VII): On Young's convergence criterion for Fourier series. *Proc. London Math. Soc.* (2) **28** (1928), no. 1, 301–311. MR1575856, JFM 54.0302.01.

[11] Hardy, G. H.; Littlewood, J. E. Some properties of fractional integrals. I. *Math. Z.* **27** (1928), no. 1, 565–606. MR1544927, JFM 54.0275.05, doi: 10.1007/BF01171116.

[12] Hewitt, Edwin; Ross, Kenneth A. Abstract harmonic analysis. II. Structure and analysis for compact groups. Analysis on locally compact Abelian groups. Die Grundlehren der mathematischen Wissenschaften, Band 152. *Springer-Verlag, New York-Berlin*, 1970. ix+771 pp. MR0262773 (41 #7378), Zbl 0213.40103, doi: 10.1007/978-3-642-62008-9.

[13] Hunt, Richard; Muckenhoupt, Benjamin; Wheeden, Richard. Weighted norm inequalities for the conjugate function and Hilbert transform. *Trans. Amer. Math. Soc.* **176** (1973), 227-251. MR0312139 (47 #701), Zbl 0262.44004, doi: 10.2307/1996205.

[14] Katznelson, Yitzhak. An introduction to harmonic analysis. Second corrected edition. *Dover Publications, Inc., New York*, 1976. xiv+264 pp. MR0422992 (54 #10976), Zbl 0352.43001.

[15] Kurtz, Douglas S. Littlewood–Paley and multiplier theorems on weighted L^p spaces. *Trans. Amer. Math. Soc.* **259** (1980), no. 1, 235–254. MR0561835 (80f:42013), Zbl 0436.42012, doi: 10.2307/1998156.

[16] Weyl, H. Bemerkungen zum Begriff des Differentialquotienten gebrochener Ordnung. Vierteljschr. Naturforsch. *Ges. Zürich* **62** (1917), 296–302. JFM 46.0437.01.

[17] Wojtaszczyk, P. Banach spaces for analysts. Cambridge Studies in Advanced Mathematics, 25. *Cambridge University Press, Cambridge*, 1991. xiv+382 pp. ISBN: 0-521-35618-0. MR1144277 (93d:46001), Zbl 0724.46012, doi: 10.1017/CBO9780511608735.

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