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Convexity and concavity of the ground state energy

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Abstract. This note proves convexity (resp. concavity) of the ground state energy of one dimensional Schrödinger operators as a function of an endpoint of the interval for convex (resp. concave) potentials.

Contents

1. Main result and context

Let $I = (a, b) \subset \mathbb{R}$ be an open interval, $V \in C(a, b)$ be a convex or concave potential with $\liminf_{t\to-\infty} V = \infty$ if $a = -\infty$. Consider for $t \in (a, b]$ the energy

$$E_t(u) = \int_a^t u_x^2 + V u^2 dx.$$

There is a unique positive minimizer $u \in H_0^1(a, t)$ under the constraint $\|u\|_{L^2(a,t)} = 1$. It satisfies the Euler–Lagrange equation

$$-u_{xx} + V u = \lambda(t) u \tag{1}$$

on (a, t) with boundary conditions $u(a) = u(t) = 0$ (and obvious modifications if $a = -\infty$). Here $\lambda(t)$ is the Lagrangian multiplier, and $\lambda(t) = E_t(u)$. The map $t \to \lambda(t)$ is the main object of interest.

Theorem 1. *The map $(a, b] \ni t \to \lambda(t)$ is twice differentiable, strictly decreasing and $\lim_{t\to a} \lambda(t) = \infty$. The map $t \to \lambda(t)$ is convex if V is convex, strictly convex if V is convex and not affine. If $a = -\infty$ it is concave if V is concave and strictly concave if V is concave and not affine.*

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The convexity part follows from a much stronger celebrated result by Brascamp and Lieb [3, 4]. It is related to a weaker statement in Friedland and Hayman [6] with a computer based proof there. These statements found considerable interest and use in the context of monotonicity formulas beginning with the seminal work of Alt, Caffarelli and Friedman [1]. Caffarelli and Kenig [5] prove a related monotonicity formula using the results by Brascamp–Lieb [3]. They attribute an analytic proof to Beckner, Kenig and Pipher [2] which the author has never seen. To the best knowledge of the author the concavity statements are new.

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2. A short elementary proof

Proof. Monotonicity and $\lim_{t\to a}\lambda(t) = \infty$ are an immediate consequence of the definition. We consider Equation (1) on the interval (a,t) and denote by $u(x) = u(x,t)$ the unique L^2 normalized non negative ground state with ground state energy $\lambda = \lambda(t)$. Differentiability with respect to x and t is an elementary property of ordinary differential equations. We argue at a formal level and do not check existence of integrals resp. derivatives below, which follows from standard arguments. We differentiate the equation with respect to t, denote the derivative of with respect to t by $\dot u$ and obtain

$$-\dot u_{xx} + V\dot u - \lambda\dot u = \dot\lambda u \tag{2}$$

with boundary conditions $\dot u(a) = 0$ and $\dot u(t) = -u_x(t)$. We multiply (2) by u, integrate and integrate by parts. Then most terms drop out by (1). Since $\|u\|_{L^2} = 1$ we obtain

$$\dot\lambda = \dot u(t)u_x(t) = -u_x^2(t). \tag{3}$$

Due to the normalization $\dot u$ is orthogonal to u, i.e., $\int_a^t u\dot u dx = 0$. The quotient $w = \frac{\dot u}{u}$ satisfies

$$w_{xx} + \frac{u_x}{u}w_x - \frac{u_x^2}{u^2}w = \dot\lambda < 0.$$

In particular w has no nonpositive local minimum. Since $w \to \infty$ as $x \to t$ there can be at most one sign change. Since $\dot u$ is orthogonal to u there is exactly one sign change of $\dot u$, lets say at $a < t_0 < t$. Since also $\dot u(a) = 0$ if $a > -\infty$ we have $\dot u_x(a) \le 0$ if $a > -\infty$. We multiplying (1) by u_x and integrate to get

$$\dot\lambda = -u_x(t)^2 = \int_a^t V'u^2 dx - u_x^2(a) \tag{4}$$

where we omit the last term here and below if $a = -\infty$.

We differentiate (4) with respect to t and use the orthogonality $\int_a^t u\dot{u}dx = 0$ to obtain a partly implicit formula for the second derivative of λ with respect to t:

$$\begin{aligned}\ddot{\lambda} =& 2\int_a^t (V'(x) - V'(t_0))u\dot{u}dx - 2u_x(a)\dot{u}_x(a)\\ =& 2\int_a^t (V'(x) - V'(t_0))wu^2 dx - 2u_x(a)\dot{u}_x(a).\end{aligned}$$

Recall that $u_x(a) > 0$ and $\dot{u}_x(a) \leq 0$ and hence the second term on the right hand side is nonnegative. By the choice of t_0 the first term is nonnegative if V is convex, nonpositive if it is concave, positive if V is convex and not affine, and negative if V is concave and not affine. Thus $t \to \lambda$ is convex if V is convex, it satisfies $\ddot{\lambda} > 0$ if V is convex and not affine (i.e., V' is not constant), if $a = -\infty$ it is concave if V is concave and $\ddot{\lambda} < 0$ if V is concave and not affine. □

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