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C^*-algebras associated with textile dynamical systems

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Abstract. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is a finite family $\{\rho_\alpha\}_{\alpha\in\Sigma}$ of endomorphisms of a C^*-algebra $\mathcal{A}$ with some conditions. It yields a C^*-algebra $\mathcal{O}_\rho$ from an associated Hilbert C^*-bimodule. In this paper, we will extend the notion of C^*-symbolic dynamical system to C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ which consists of two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ with certain commutation relations κ between their endomorphisms $\{\rho_\alpha\}_{\alpha\in\Sigma^\rho}$ and $\{\eta_a\}_{a\in\Sigma^\eta}$. C^*-textile dynamical systems yield two-dimensional subshifts and C^*-algebras $\mathcal{O}^\kappa_{\rho,\eta}$. We will study their structure of the algebras $\mathcal{O}^\kappa_{\rho,\eta}$ and present its K-theory formulae.

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1. Introduction

In [24], the author has introduced a notion of λ-graph system as presentations of subshifts. The λ-graph systems are labeled Bratteli diagram with shift transformation. They yield C^*-algebras so that its K-theory groups are related to topological conjugacy invariants of the underlying symbolic dynamical systems. The class of these C^*-algebras include the Cuntz–Krieger algebras. He has extended the notion of λ-graph system to *C^*-symbolic dynamical system*, which is a generalization of both a λ-graph system and an automorphism of a unital C^*-algebra. It is a finite family $\{\rho_\alpha\}_{\alpha\in\Sigma}$ of endomorphisms of a unital C^*-algebra $\mathcal{A}$ such that $\rho_\alpha(Z_\mathcal{A}) \subset Z_\mathcal{A}, \alpha \in \Sigma$ and $\sum_{\alpha\in\Sigma}\rho_\alpha(1) \geq 1$ where $Z_\mathcal{A}$ denotes the center of $\mathcal{A}$. A finite labeled graph $\mathcal{G}$ gives rise to a C^*-symbolic dynamical system $(\mathcal{A}_\mathcal{G}, \rho^\mathcal{G}, \Sigma)$ such that $\mathcal{A} = \mathbb{C}^N$ for some $N \in \mathbb{N}$. A λ-graph system $\mathfrak{L}$ is a generalization of a finite labeled graph and yields a C^*-symbolic dynamical system $(\mathcal{A}_\mathfrak{L}, \rho^\mathfrak{L}, \Sigma)$ such that $\mathcal{A}_\mathfrak{L}$ is $C(\Omega_\mathfrak{L})$ for some compact Hausdorff space $\Omega_\mathfrak{L}$ with $\dim\Omega_\mathfrak{L} = 0$. It also yields a C^*-algebra $\mathcal{O}_\mathfrak{L}$. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ provides a subshift Λ_ρ over Σ and a Hilbert C^*-bimodule $\mathcal{H}^\rho_\mathcal{A}$ over $\mathcal{A}$. The C^*-algebra $\mathcal{O}_\rho$ for $(\mathcal{A}, \rho, \Sigma)$ may be realized as a Cuntz–Pimsner algebra from the Hilbert C^*-bimodule $\mathcal{H}^\rho_\mathcal{A}$ ([27], cf. [15], [39]). We call the algebra $\mathcal{O}_\rho$ *the C^*-symbolic crossed product of $\mathcal{A}$ by the subshift* Λ_ρ. If $\mathcal{A} = C(X)$ with $\dim X = 0$, there exists a λ-graph system $\mathfrak{L}$ such that the subshift Λ_ρ is the subshift $\Lambda_\mathfrak{L}$ presented by $\mathfrak{L}$ and the C^*-algebra $\mathcal{O}_\rho$ is the C^*-algebra $\mathcal{O}_\mathfrak{L}$ associated with $\mathfrak{L}$. If in particular, $\mathcal{A} = \mathbb{C}^N$, the subshift Λ_ρ is a sofic shift and $\mathcal{O}_\rho$ is a Cuntz–Krieger algebra. If $\Sigma = \{\alpha\}$ an automorphism α of a unital C^*-algebra $\mathcal{A}$, the C^*-algebra $\mathcal{O}_\rho$ is the ordinary crossed product $\mathcal{A} \times_\alpha \mathbb{Z}$.

G. Robertson–T. Steger [43] have initiated a certain study of higher dimensional analogue of Cuntz–Krieger algebras from the view point of tiling systems of 2-dimensional plane. After their work, A. Kumjian–D. Pask [19] have generalized their construction to introduce the notion of higher rank graphs and its C^*-algebras. The C^*-algebras constructed from higher rank graphs are called the higher rank graph C^*-algebras. Since then, there have been many studies on these C^*-algebras by many authors (cf. [1], [9], [10], [11], [13], [16], [19], [36], [42], [43], etc.).

M. Nasu in [34] has introduced the notion of textile system which is useful in analyzing automorphisms and endomorphisms of topological Markov shifts. A textile system also gives rise to a two-dimensional tiling called Wang tiling. Among textile systems, LR textile systems have specific properties that consist of two commuting symbolic matrices. In [28], the author has extended the notion of textile systems to λ-graph systems and has defined a notion of textile systems on λ-graph systems, which are called textile λ-graph systems for short. C^*-algebras associated to textile systems have been initiated by V. Deaconu ([9]).

In this paper, we will extend the notion of C^*-symbolic dynamical system to C^*-textile dynamical system which is a higher dimensional analogue of C^*-symbolic dynamical system. The C^*-*textile dynamical system* $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ consists of two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ with the following commutation relations between ρ and η through κ. Set

$$\Sigma^{\rho\eta} = \{(\alpha, b) \in \Sigma^\rho \times \Sigma^\eta \mid \eta_b \circ \rho_\alpha \neq 0\},$$
$$\Sigma^{\eta\rho} = \{(a, \beta) \in \Sigma^\eta \times \Sigma^\rho \mid \rho_\beta \circ \eta_a \neq 0\}.$$

We require that there exists a bijection $\kappa : \Sigma^{\rho\eta} \longrightarrow \Sigma^{\eta\rho}$, which we fix and call a specification. Then the required commutation relations are

$$\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a \qquad \text{if } \kappa(\alpha, b) = (a, \beta). \tag{1.1}$$

A C^*-textile dynamical system provides a two-dimensional subshifts and a C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$. The C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ is defined to be the universal C^*-algebra $C^*(x, S_\alpha, T_a; x \in \mathcal{A}, \alpha \in \Sigma^\rho, a \in \Sigma^\eta)$ generated by $x \in \mathcal{A}$ and two families of partial isometries $S_\alpha, \alpha \in \Sigma^\rho$, $T_a, a \in \Sigma^\eta$ subject to the following relations called $(\rho, \eta; \kappa)$:

$$\sum_{\beta \in \Sigma^\rho} S_\beta S_\beta^* = 1, \qquad x S_\alpha S_\alpha^* = S_\alpha S_\alpha^* x, \qquad S_\alpha^* x S_\alpha = \rho_\alpha(x), \tag{1.2}$$

$$\sum_{b \in \Sigma^\eta} T_b T_b^* = 1, \qquad x T_a T_a^* = T_a T_a^* x, \qquad T_a^* x T_a = \eta_a(x), \tag{1.3}$$

$$S_\alpha T_b = T_a S_\beta \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta) \tag{1.4}$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$.

In Section 3, we will construct a tiling system in the plane from a C^*-textile dynamical system. The resulting tiling system is a two-dimensional subshift. In Section 4, we will study some basic properties of the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$. In Section 5, we will introduce a condition called (I) on $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ which will be studied as a generalization of the condition (I) on C^*-symbolic dynamical system [26] (cf. [8], [25]). In Section 6, we will realize the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ as a Cuntz–Pimsner algebra associated with a certain Hilbert C^*-bimodule in a concrete way. We will have the following theorem.

Theorem 1.1. *Let $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ be a C^*-textile dynamical system satisfying condition* (I). *Then the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ is a unique concrete C^*-algebra subject to the relations $(\rho, \eta; \kappa)$. If $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is irreducible, $\mathcal{O}^\kappa_{\rho,\eta}$ is simple.*

A C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to form square if the C^*-subalgebra of $\mathcal{A}$ generated by the projections $\rho_\alpha(1), \alpha \in \Sigma^\rho$ and the C^*-subalgebra of $\mathcal{A}$ generated by the projections $\eta_a(1), a \in \Sigma^\eta$ coincide. It is said to have trivial K_1 if $K_1(\mathcal{A}) = \{0\}$. In Section 7 and Section 8, we will restrict our interest to the C^*-textile dynamical systems forming square to prove the following K-theory formulae:

Theorem 1.2. *Suppose that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ forms square and has trivial K_1. Then there exist short exact sequences for $K_0(\mathcal{O}^\kappa_{\rho,\eta})$ and $K_1(\mathcal{O}^\kappa_{\rho,\eta})$ such that*

$$\begin{aligned} 0 &\longrightarrow K_0(\mathcal{A})/((\mathrm{id}-\lambda_\eta)K_0(\mathcal{A}) + (\mathrm{id}-\lambda_\rho)K_0(\mathcal{A})) \\ &\longrightarrow K_0(\mathcal{O}^\kappa_{\rho,\eta}) \\ &\longrightarrow \mathrm{Ker}(\mathrm{id}-\lambda_\eta) \cap \mathrm{Ker}(\mathrm{id}-\lambda_\rho) \ \text{in } K_0(\mathcal{A}) \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 &\longrightarrow (\mathrm{Ker}(\mathrm{id}-\lambda_\eta) \ \text{in } K_0(\mathcal{A}))/(\mathrm{id}-\lambda_\rho)(\mathrm{Ker}(\mathrm{id}-\lambda_\eta) \ \text{in } K_0(\mathcal{A})) \\ &\longrightarrow K_1(\mathcal{O}^\kappa_{\rho,\eta}) \\ &\longrightarrow \mathrm{Ker}(\mathrm{id}-\bar{\lambda}_\rho) \ \text{in } (K_0(\mathcal{A})/(\mathrm{id}-\lambda_\eta)K_0(\mathcal{A})) \longrightarrow 0 \end{aligned}$$

where the endomorphisms $\lambda_\rho, \lambda_\eta : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})$ are defined by

$$\lambda_\rho([p]) = \sum_{\alpha \in \Sigma^\rho} [\rho_\alpha(p)] \in K_0(\mathcal{A}) \ \text{for } [p] \in K_0(\mathcal{A}),$$

$$\lambda_\eta([p]) = \sum_{a \in \Sigma^\eta} [\eta_a(p)] \in K_0(\mathcal{A}) \ \text{for } [p] \in K_0(\mathcal{A})$$

and $\bar{\lambda}_\rho$ denotes an endomorphism on $K_0(\mathcal{A})/(1-\lambda_\eta)K_0(\mathcal{A})$ induced by λ_ρ.

Let A, B be mutually commuting $N \times N$ matrices with entries in nonnegative integers. Let $G_A = (V_A, E_A), G_B = (V_B, E_B)$ be directed graphs with common vertex set $V_A = V_B$, whose transition matrices are A, B respectively. Let $\mathcal{M}_A, \mathcal{M}_B$ denote symbolic matrices for G_A, G_B whose components consist of formal sums of the directed edges of G_A, G_B respectively. Let Σ^{AB}, Σ^{BA} be the sets of the pairs of the concatenated directed edges in $E_A \times E_B, E_B \times E_A$ respectively. By the condition $AB = BA$, one may take a bijection $\kappa : \Sigma^{AB} \longrightarrow \Sigma^{BA}$ which gives rise to a specified equivalence $\mathcal{M}_A\mathcal{M}_B \overset{\kappa}{\cong} \mathcal{M}_B\mathcal{M}_A$. We then have a C^*-textile dynamical system written as $(\mathcal{A}, \rho^A, \rho^B, \Sigma^A, \Sigma^B, \kappa)$. The associated C^*-algebra is denoted by $\mathcal{O}^\kappa_{A,B}$. The C^*-algebra $\mathcal{O}^\kappa_{A,B}$ is realized as a 2-graph C^*-algebra constructed by Kumjian–Pask ([19]). It is also seen in Deaconu's paper [9]. We will see the following proposition in Section 9.

Proposition 1.3. *Keep the above situations. There exist short exact sequences for $K_0(\mathcal{O}^\kappa_{A,B})$ and $K_1(\mathcal{O}^\kappa_{A,B})$ such that*

$$\begin{aligned} 0 &\longrightarrow \mathbb{Z}^N/((1-A)\mathbb{Z}^N + (1-B)\mathbb{Z}^N) \\ &\longrightarrow K_0(\mathcal{O}^\kappa_{A,B}) \\ &\longrightarrow \mathrm{Ker}(1-A) \cap \mathrm{Ker}(1-B) \ \text{in } \mathbb{Z}^N \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned}0 &\longrightarrow (\operatorname{Ker}(1-B) \text{ in } \mathbb{Z}^N)/(1-A)(\operatorname{Ker}(1-B) \text{ in } \mathbb{Z}^N) \\ &\longrightarrow K_1(\mathcal{O}^\kappa_{A,B}) \\ &\longrightarrow \operatorname{Ker}(1-\bar{A}) \text{ in } (\mathbb{Z}^N/(1-B)\mathbb{Z}^N) \longrightarrow 0,\end{aligned}$$

where $\bar{A}$ is an endomorphism on the abelian group $\mathbb{Z}^N/(1-B)\mathbb{Z}^N$ induced by the matrix A.

Throughout the paper, we will denote by $\mathbb{Z}_+$ the set of nonnegative integers and by $\mathbb{N}$ the set of positive integers.

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2. λ-graph systems, C^*-symbolic dynamical systems and their C^*-algebras

In this section, we will briefly review λ-graph systems and C^*-symbolic dynamical systems. Throughout the section, Σ denotes a finite set with its discrete topology, that is called an alphabet. Each element of Σ is called a symbol. Let $\Sigma^{\mathbb{Z}}$ be the infinite product space $\prod_{i\in\mathbb{Z}}\Sigma_i$, where $\Sigma_i = \Sigma$, endowed with the product topology. The transformation σ on $\Sigma^{\mathbb{Z}}$ given by $\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$ is called the full shift over Σ. Let Λ be a shift invariant closed subset of $\Sigma^{\mathbb{Z}}$ i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|_\Lambda)$ is called a two-sided subshift, written as Λ for brevity. A word $\mu = (\mu_1, \dots, \mu_k)$ of Σ is said to be admissible for Λ if there exists $(x_i)_{i\in\mathbb{Z}} \in \Lambda$ such that $\mu_1 = x_1, \dots, \mu_k = x_k$. Let us denote by $|\mu|$ the length k of μ. Let $B_k(\Lambda)$ be the set of admissible words of Λ with length k. The union $\cup_{k=0}^\infty B_k(\Lambda)$ is denoted by $B_*(\Lambda)$ where $B_0(\Lambda)$ denotes the empty word. For two words $\mu = (\mu_1, \dots, \mu_k), \nu = (\nu_1, \dots, \nu_n)$, we write a new word $\mu\nu = (\mu_1, \dots, \mu_k, \nu_1, \dots, \nu_n)$.

There is a class of subshifts called sofic shifts, that are presented by finite labeled graphs ([14], [17], [18]). λ-graph systems are generalization of finite labeled graphs. Any subshift is presented by a λ-graph system. Let

$$\mathfrak{L} = (V, E, \lambda, \iota)$$

be a λ-graph system over Σ with vertex set $V = \cup_{l\in\mathbb{Z}_+} V_l$ and edge set $E = \cup_{l\in\mathbb{Z}_+} E_{l,l+1}$ that is labeled with symbols in Σ by a map $\lambda : E \to \Sigma$, and that is supplied with surjective maps $\iota (= \iota_{l,l+1}) : V_{l+1} \to V_l$ for $l \in \mathbb{Z}_+$. Here the vertex sets $V_l, l \in \mathbb{Z}_+$ and the edge sets $E_{l,l+1}, l \in \mathbb{Z}_+$ are finite disjoint sets for each $l \in \mathbb{Z}_+$. An edge e in $E_{l,l+1}$ has its source vertex $s(e)$ in V_l and its terminal vertex $t(e)$ in V_{l+1} respectively. Every vertex in V has a successor and every vertex in V_l for $l \in \mathbb{N}$ has a predecessor. It is then required that for vertices $u \in V_{l-1}$ and $v \in V_{l+1}$, there exists a bijective correspondence between the set of edges $e \in E_{l,l+1}$ such that $t(e) = v, \iota(s(e)) = u$ and the set of edges $f \in E_{l-1,l}$ such that $s(f) = u, t(f) = \iota(v)$, preserving their labels ([24]). We assume that $\mathfrak{L}$ is left-resolving, which means that $t(e) \neq t(f)$

whenever $\lambda(e) = \lambda(f)$ for $e, f \in E_{l,l+1}$. Let us denote by $\{v_1^l, \dots, v_{m(l)}^l\}$ the vertex set V_l at level l. For $i = 1, 2, \dots, m(l),\ j = 1, 2, \dots, m(l+1),\ \alpha \in \Sigma$ we put

$$A_{l,l+1}(i,\alpha,j) = \begin{cases} 1 & \text{if } s(e) = v_i^l, \lambda(e) = \alpha, t(e) = v_j^{l+1} \text{ for some } e \in E_{l,l+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{l,l+1}(i,j) = \begin{cases} 1 & \text{if } \iota_{l,l+1}(v_j^{l+1}) = v_i^l, \\ 0 & \text{otherwise.} \end{cases}$$

The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with $\mathfrak{L}$ is the universal C^*-algebra generated by partial isometries $S_\alpha, \alpha \in \Sigma$ and projections $E_i^l, i = 1, 2, \dots, m(l),\ l \in \mathbb{Z}_+$ subject to the following operator relations called $(\mathfrak{L})$:

$$\sum_{\beta \in \Sigma} S_\beta S_\beta^* = 1, \tag{2.1}$$

$$\sum_{i=1}^{m(l)} E_i^l = 1, \qquad E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j) E_j^{l+1}, \tag{2.2}$$

$$S_\alpha S_\alpha^* E_i^l = E_i^l S_\alpha S_\alpha^*, \tag{2.3}$$

$$S_\alpha^* E_i^l S_\alpha = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i,\alpha,j) E_j^{l+1}, \tag{2.4}$$

for $i = 1, 2, \dots, m(l), l \in \mathbb{Z}_+, \alpha \in \Sigma$. If $\mathfrak{L}$ satisfies λ-condition (I) and is λ-irreducible, the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is simple and purely infinite ([25], [26]).

Let $\mathcal{A}_{\mathfrak{L},l}$ be the C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by the projections $E_i^l, i = 1, \dots, m(l)$. We denote by $\mathcal{A}_{\mathfrak{L}}$ the C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by all the projections $E_i^l, i = 1, \dots, m(l), l \in \mathbb{Z}_+$. As $\mathcal{A}_{\mathfrak{L},l} \subset \mathcal{A}_{\mathfrak{L},l+1}$ and $\cup_{l \in \mathbb{Z}_+} \mathcal{A}_{\mathfrak{L},l}$ is dense in $\mathcal{A}$, the algebra $\mathcal{A}_{\mathfrak{L}}$ is a commutative AF-algebra. For $\alpha \in \Sigma$, put

$$\rho_\alpha^{\mathfrak{L}}(X) = S_\alpha^* X S_\alpha \qquad \text{for} \qquad X \in \mathcal{A}_{\mathfrak{L}}.$$

Then $\{\rho_\alpha^{\mathfrak{L}}\}_{\alpha \in \Sigma}$ yields a family of $*$-endomorphisms of $\mathcal{A}_{\mathfrak{L}}$ such that $\rho_\alpha^{\mathfrak{L}}(1) \neq 0$, $\sum_{\alpha \in \Sigma} \rho_\alpha^{\mathfrak{L}}(1) \geq 1$ and for any nonzero $x \in \mathcal{A}_{\mathfrak{L}}$, $\rho_\alpha^{\mathfrak{L}}(x) \neq 0$ for some $\alpha \in \Sigma$.

The situations above are generalized to C^*-symbolic dynamical systems as follows. Let $\mathcal{A}$ be a unital C^*-algebra. In what follows, an endomorphism of $\mathcal{A}$ means a $*$-endomorphism of $\mathcal{A}$ that does not necessarily preserve the unit $1_{\mathcal{A}}$ of $\mathcal{A}$. The unit $1_{\mathcal{A}}$ is denoted by 1 unless we specify. Denote by $Z_{\mathcal{A}}$ the center of $\mathcal{A}$. Let $\rho_\alpha, \alpha \in \Sigma$ be a finite family of endomorphisms of $\mathcal{A}$ indexed by symbols of a finite set Σ. We assume that $\rho_\alpha(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}, \alpha \in \Sigma$. The family $\rho_\alpha, \alpha \in \Sigma$ of endomorphisms of $\mathcal{A}$ is said to be *essential* if $\rho_\alpha(1) \neq 0$ for all $\alpha \in \Sigma$ and $\sum_\alpha \rho_\alpha(1) \geq 1$. It is said to be *faithful* if for any nonzero $x \in \mathcal{A}$ there exists a symbol $\alpha \in \Sigma$ such that $\rho_\alpha(x) \neq 0$.

Definition 2.1 (cf. [27]). A C^*-*symbolic dynamical system* is a triplet $(\mathcal{A}, \rho, \Sigma)$ consisting of a unital C^*-algebra $\mathcal{A}$ and an essential and faithful finite family $\{\rho_\alpha\}_{\alpha\in\Sigma}$ of endomorphisms of $\mathcal{A}$.

As in the above discussion, we have a C^*-symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ from a λ-graph system $\mathfrak{L}$. In [27], [29], [30], we have defined a C^*-symbolic dynamical system in a less restrictive way than the above definition. Instead of the above condition $\sum_{\alpha\in\Sigma}\rho_\alpha(1) \geq 1$ with $\rho_\alpha(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}, \alpha \in \Sigma$, we have used the condition in the papers that the closed ideal generated by $\rho_\alpha(1), \alpha \in \Sigma$ coincides with $\mathcal{A}$. All of the examples appeared in the papers [27], [29], [30] satisfy the condition $\sum_{\alpha\in\Sigma}\rho_\alpha(1) \geq 1$ with $\rho_\alpha(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}, \alpha \in \Sigma$, and all discussions in the papers well work under the above new definition.

A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ yields a subshift Λ_ρ over Σ such that a word $(\alpha_1, \dots, \alpha_k)$ of Σ is admissible for Λ_ρ if and only if

$$(\rho_{\alpha_k} \circ \cdots \circ \rho_{\alpha_1})(1) \neq 0$$

([27, Proposition 2.1]). We say that *a subshift* Λ *acts on a* C^*-*algebra* $\mathcal{A}$ if there exists a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated subshift Λ_ρ is Λ.

The C^*-algebra $\mathcal{O}_\rho$ associated with a C^*-symbolic dynamical system

$$(\mathcal{A}, \rho, \Sigma)$$

has been originally constructed in [27] as a C^*-algebra by using the Pimsner's general construction of C^*-algebras from Hilbert C^*-bimodules [39] (cf. [15] etc.). It is realized as the universal C^*-algebra $C^*(x, S_\alpha; x \in \mathcal{A}, \alpha \in \Sigma)$ generated by $x \in \mathcal{A}$ and partial isometries $S_\alpha, \alpha \in \Sigma$ subject to the following relations called (ρ):

$$\sum_{\beta\in\Sigma} S_\beta S_\beta^* = 1, \qquad xS_\alpha S_\alpha^* = S_\alpha S_\alpha^* x, \qquad S_\alpha^* x S_\alpha = \rho_\alpha(x)$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma$. The C^*-algebra $\mathcal{O}_\rho$ is a generalization of the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with the λ-graph system $\mathfrak{L}$.

A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is said to be *free* if there exists a unital increasing sequence $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}$ of C^*-subalgebras of $\mathcal{A}$ such that:

(1) $\rho_\alpha(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$ for all $l \in \mathbb{Z}_+$ and $\alpha \in \Sigma$.
(2) $\cup_{l\in\mathbb{Z}_+}\mathcal{A}_l$ is dense in $\mathcal{A}$.
(3) For $j \leq l$ there exists a projection $q \in \mathcal{D}_\rho \cap \mathcal{A}_l{}'$ such that:
  (i) $qx \neq 0$ for $0 \neq x \in \mathcal{A}_l$,
  (ii) $\phi_\rho^n(q)q = 0$ for all $n = 1, 2, \dots, j$,
  where $\mathcal{D}_\rho$ is the C^*-subalgebra of $\mathcal{O}_\rho$ generated by elements

$$S_{\mu_1} \cdots S_{\mu_k} x S_{\mu_k}^* \cdots S_{\mu_1}^*$$

for $(\mu_1, \dots, \mu_k) \in B_*(\Lambda_\rho)$ and $x \in \mathcal{A}$, and

$$\phi_\rho(X) = \sum_{\alpha \in \Sigma} S_\alpha X S_\alpha^*, \quad X \in \mathcal{D}_\rho.$$

The freeness has been called condition (I) in [30]. If in particular, one may take the above subalgebras $\mathcal{A}_l \subset \mathcal{A}$, $l = 0, 1, 2, \dots$ to be of finite dimensional, then $(\mathcal{A}, \rho, \Sigma)$ is said to be *AF-free.* $(\mathcal{A}, \rho, \Sigma)$ is said to be *irreducible* if there is no nontrivial ideal of $\mathcal{A}$ invariant under the positive operator λ_ρ on $\mathcal{A}$ defined by $\lambda_\rho(x) = \sum_{\alpha \in \Sigma} \rho_\alpha(x)$, $x \in \mathcal{A}$. It has been proved that if $(\mathcal{A}, \rho, \Sigma)$ is free and irreducible, then the C^*-algebra $\mathcal{O}_\rho$ is simple ([30]).

3. C^*-textile dynamical systems and two-dimensional subshifts

Let Σ be a finite set. The two-dimensional full shift over Σ is defined to be

$$\Sigma^{\mathbb{Z}^2} = \{(x_{i,j})_{(i,j) \in \mathbb{Z}^2} \mid x_{i,j} \in \Sigma\}.$$

An element $x \in \Sigma^{\mathbb{Z}^2}$ is regarded as a function $x : \mathbb{Z}^2 \longrightarrow \Sigma$ which is called a configuration on $\mathbb{Z}^2$. For $x \in \Sigma^{\mathbb{Z}^2}$ and $F \subset \mathbb{Z}^2$, let x_F denote the restriction of x to F. For a vector $m = (m_1, m_2) \in \mathbb{Z}^2$, let $\sigma^m : \Sigma^{\mathbb{Z}^2} \longrightarrow \Sigma^{\mathbb{Z}^2}$ be the translation along vector m defined by

$$\sigma^m((x_{i,j})_{(i,j) \in \mathbb{Z}^2}) = (x_{i+m_1, j+m_2})_{(i,j) \in \mathbb{Z}^2}.$$

A subset $X \subset \Sigma^{\mathbb{Z}^2}$ is said to be translation invariant if $\sigma^m(X) = X$ for all $m \in \mathbb{Z}^2$. It is obvious to see that a subset $X \subset \Sigma^{\mathbb{Z}^2}$ is translation invariant if ond only if X is invariant only both horizontally and vertically, that is, $\sigma^{(1,0)}(X) = X$ and $\sigma^{(0,1)}(X) = X$. For $k \in \mathbb{Z}_+$, put

$$[-k, k]^2 = \{(i, j) \in \mathbb{Z}^2 \mid -k \le i, j \le k\} = [-k, k] \times [-k, k].$$

A metric d on $\Sigma^{\mathbb{Z}^2}$ is defined by for $x, y \in \Sigma^{\mathbb{Z}^2}$ with $x \ne y$

$$d(x, y) = \frac{1}{2^k} \quad \text{if} \quad x_{(0,0)} = y_{(0,0)},$$

where $k = \max\{k \in \mathbb{Z}_+ \mid x_{[-k,k]^2} = y_{[-k,k]^2}\}$. If $x_{(0,0)} \ne y_{(0,0)}$, put $k = -1$ on the above definition. If $x = y$, we set $d(x, y) = 0$. A two-dimensional subshift X is defined to be a closed, translation invariant subset of $\Sigma^{\mathbb{Z}^2}$ (cf. [21, p.467]). A finite subset $F \subset \mathbb{Z}^2$ is said to be a shape. A pattern f on a shape F is a function $f : F \longrightarrow \Sigma$. For a list $\mathfrak{F}$ of patterns, put

$$X_{\mathfrak{F}} = \{(x_{i,j})_{(i,j) \in \mathbb{Z}^2} \mid \sigma^m(x)|_F \notin \mathfrak{F} \text{ for all } m \in \mathbb{Z}^2 \text{ and } F \subset \mathbb{Z}^2\}.$$

It is well-known that a subset $X \subset \Sigma^{\mathbb{Z}^2}$ is a two-dimensional subshift if and only if there exists a list $\mathfrak{F}$ of patterns such that $X = X_{\mathfrak{F}}$.

We will define a certain property of two-dimensional subshift as follows:

Definition 3.1. A two-dimensional subshift X is said to have the *diagonal property* if for $(x_{i,j})_{(i,j)\in\mathbb{Z}^2}, (y_{i,j})_{(i,j)\in\mathbb{Z}^2} \in X$, the conditions

$$x_{i,j} = y_{i,j}, \qquad\qquad x_{i+1,j-1} = y_{i+1,j-1}$$

imply

$$x_{i,j-1} = y_{i,j-1}, \qquad\qquad x_{i+1,j} = y_{i+1,j}.$$

A two-dimensional subshift having the diagonal property is called *a textile dynamical system.*

Lemma 3.2. *If a two dimensional subshift X has the diagonal property, then for $x \in X$ and $(i,j) \in \mathbb{Z}^2$, the configuration x is determined by the diagonal line $(x_{i+n,j-n})_{n\in\mathbb{Z}}$ through (i,j).*

Proof. By the diagonal property, the sequence $(x_{i+n,j-n})_{n\in\mathbb{Z}}$ determines both the sequences $(x_{i+1+n,j-n})_{n\in\mathbb{Z}}$ and $(x_{i-1+n,j-n})_{n\in\mathbb{Z}}$. Repeating this way, the sequence $(x_{i+n,j-n})_{n\in\mathbb{Z}}$ determines the whole configuration x. $\square$

Let $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ be a C^*-*textile dynamical system.* It consists of two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ with common unital C^*-algebra $\mathcal{A}$ and commutation relations between their endomorphisms $\rho_\alpha, \alpha \in \Sigma^\rho, \eta_a, a \in \Sigma^\eta$ through a bijection κ between the following sets $\Sigma^{\rho\eta}$ and $\Sigma^{\eta\rho}$, where

$$\Sigma^{\rho\eta} = \{(\alpha, b) \in \Sigma^\rho \times \Sigma^\eta \mid \eta_b \circ \rho_\alpha \neq 0\},$$
$$\Sigma^{\eta\rho} = \{(a, \beta) \in \Sigma^\eta \times \Sigma^\rho \mid \rho_\beta \circ \eta_a \neq 0\}.$$

The given bijection $\kappa : \Sigma^{\rho\eta} \longrightarrow \Sigma^{\eta\rho}$ is called a specification. The required commutation relations are

$$\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a \qquad \text{if } \kappa(\alpha, b) = (a, \beta). \tag{3.1}$$

A C^*-textile dynamical system will yield a two-dimensional subshift $X^\kappa_{\rho,\eta}$. We set

$$\Sigma_\kappa = \{\omega = (\alpha, b, a, \beta) \in \Sigma^\rho \times \Sigma^\eta \times \Sigma^\eta \times \Sigma^\rho \mid \kappa(\alpha, b) = (a, \beta)\}.$$

For $\omega = (\alpha, b, a, \beta)$, since $\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a$ as endomorphisms on $\mathcal{A}$, one may identify the quadruplet (α, b, a, β) with the endomorphism $\eta_b \circ \rho_\alpha (= \rho_\beta \circ \eta_a)$ on $\mathcal{A}$ which we will denote by simply ω. Define maps $t(= top), b(= bottom) : \Sigma_\kappa \longrightarrow \Sigma^\rho$ and $l(= left), r(= right) : \Sigma_\kappa \longrightarrow \Sigma^\rho$ by setting

$$t(\omega) = \alpha, \quad b(\omega) = \beta, \quad l(\omega) = a, \quad r(\omega) = b.$$

$$\begin{array}{ccc} \cdot & \xrightarrow{\alpha = t(\omega)} & \cdot \\ {\scriptstyle a = l(\omega)}\downarrow & & \downarrow{\scriptstyle b = r(\omega)} \\ \cdot & \xrightarrow[\beta = b(\omega)]{} & \cdot \end{array}$$

A configuration $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \Sigma_\kappa^{\mathbb{Z}^2}$ is said to be *paved* if the conditions

$$
\begin{aligned}
t(\omega_{i,j}) &= b(\omega_{i,j+1}), & r(\omega_{i,j}) &= l(\omega_{i+1,j}),\\
l(\omega_{i,j}) &= r(\omega_{i-1,j}), & b(\omega_{i,j}) &= t(\omega_{i,j-1})
\end{aligned}
$$

hold for all $(i,j) \in \mathbb{Z}^2$. We set

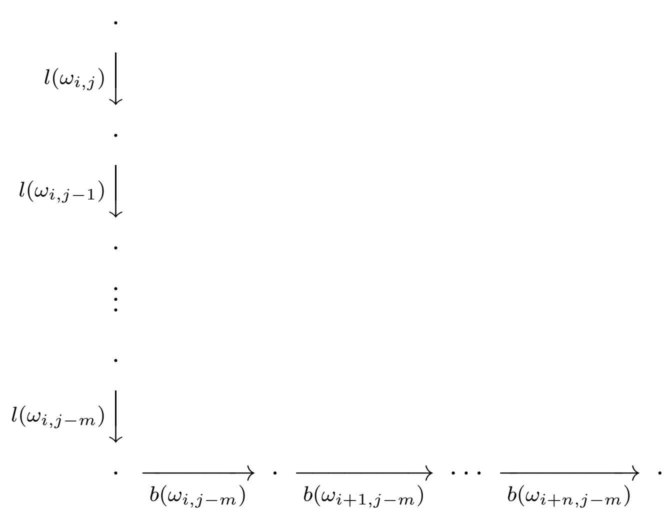
$$
\begin{aligned}
X_{\rho,\eta}^\kappa = \{(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \Sigma_\kappa^{\mathbb{Z}^2} \mid (\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \text{ is paved and}\\
\omega_{i+n,j-n} \circ \omega_{i+n-1,j-n+1} \circ \cdots \circ \omega_{i+1,j-1} \circ \omega_{i,j} \neq 0\\
\text{for all } (i,j) \in \mathbb{Z}^2, n \in \mathbb{N}\},
\end{aligned}
$$

where $\omega_{i+n,j-n} \circ \omega_{i+n-1,j-n+1} \circ \cdots \circ \omega_{i+1,j-1} \circ \omega_{i,j}$ is the compositions as endomorphisms on $\mathcal{A}$.

Lemma 3.3. *Suppose that a configuration $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \Sigma_\kappa^{\mathbb{Z}^2}$ is paved. Then $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in X_{\rho,\eta}^\kappa$ if and only if*

$$
\rho_{b(\omega_{i+n,j-m})} \circ \cdots \circ \rho_{b(\omega_{i+1,j-m})} \circ \rho_{b(\omega_{i,j-m})} \circ \eta_{l(\omega_{i,j-m})} \circ \cdots \eta_{l(\omega_{i,j-1})} \circ \eta_{l(\omega_{i,j})} \neq 0
$$

for all $(i,j) \in \mathbb{Z}^2$, $n,m \in \mathbb{Z}_+$.



Proof. Suppose that $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in X_{\rho,\eta}^\kappa$. For $(i,j) \in \mathbb{Z}^2$, $n,m \in \mathbb{Z}_+$, we may assume that $m \geq n$. Since

$$
\begin{aligned}
0 \neq & \omega_{i+m,j-m} \circ \cdots \circ \omega_{i+n+1,j-m} \circ \omega_{i+n,j-m} \circ \cdots \circ \omega_{i,j-m}\\
& \circ \cdots \circ \omega_{i+1,j-1} \circ \omega_{i,j}\\
= & \omega_{i+m,j-m} \circ \cdots \circ \omega_{i+n+1,j-m} \circ \rho_{b(\omega_{i+n,j-m})} \circ \cdots \circ \rho_{b(\omega_{i+1,j-m})} \circ \rho_{b(\omega_{i,j-m})}\\
& \circ \eta_{l(\omega_{i,j-m})} \cdots \circ \eta_{l(\omega_{i,j-m})} \circ \cdots \circ \eta_{l(\omega_{i,j-1})} \circ \eta_{l(\omega_{i,j})},
\end{aligned}
$$

one has

$$
\rho_{b(\omega_{i+n,j-m})} \circ \cdots \circ \rho_{b(\omega_{i+1,j-m})} \circ \rho_{b(\omega_{i,j-m})} \circ \eta_{l(\omega_{i,j-m})} \circ \cdots \eta_{l(\omega_{i,j-1})} \circ \eta_{l(\omega_{i,j})} \neq 0.
$$

The converse implication is clear by the equality:

$$\begin{aligned}&\omega_{i+n,j-n}\circ\cdots\circ\omega_{i,j-n}\circ\cdots\circ\omega_{i,j-1}\circ\omega_{i,j}\\&\quad=\rho_{b(\omega_{i+n,j-n})}\circ\cdots\circ\rho_{b(\omega_{i,j-n})}\circ\eta_{l(\omega_{i,j-n})}\cdots\circ\eta_{l(\omega_{i,j-1})}\circ\eta_{l(\omega_{i,j})}.\end{aligned}$$ □

Proposition 3.4. *$X^\kappa_{\rho,\eta}$ is a two-dimensional subshift having diagonal property, that is, $X^\kappa_{\rho,\eta}$ is a textile dynamical system.*

Proof. It is easy to see that the set

$$E=\{(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2}\in\Sigma_\kappa^{\mathbb{Z}^2}\mid(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2}\text{ is paved}\}$$

is closed, because its complement is open in $\Sigma_\kappa^{\mathbb{Z}^2}$. The following set

$$\begin{aligned}U=\{(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2}\in\Sigma_\kappa^{\mathbb{Z}^2}\mid{}&\omega_{k+n,l-n}\circ\omega_{k+n-1,l-n+1}\\&\circ\cdots\circ\omega_{k+1,l-1}\circ\omega_{k,l}=0\text{ for some }(k,l)\in\mathbb{Z}^2,n\in\mathbb{N}\}\end{aligned}$$

is open in $\Sigma_\kappa^{\mathbb{Z}^2}$. As the equality $X^\kappa_{\rho,\eta}=E\cap U^c$ holds, the set $X^\kappa_{\rho,\eta}$ is closed. It is also obvious that $X^\kappa_{\rho,\eta}$ is translation invariant so that $X^\kappa_{\rho,\eta}$ is a two-dimensional subshift. It is easy to see that $X^\kappa_{\rho,\eta}$ has diagonal property. □

We call $X^\kappa_{\rho,\eta}$ the textile dynamical system associated with

$$(\mathcal{A},\rho,\eta,\Sigma^\rho,\Sigma^\eta,\kappa).$$

Let us now define a (one-dimensional) subshift X_{δ^κ} over Σ_κ, which consists of diagonal sequences of $X^\kappa_{\rho,\eta}$ as follows:

$$X_{\delta^\kappa}=\{(\omega_{n,-n})_{n\in\mathbb{Z}}\in\Sigma_\kappa^{\mathbb{Z}}\mid(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2}\in X^\kappa_{\rho,\eta}\}.$$

By Lemma 3.2, an element $(\omega_{n,-n})_{n\in\mathbb{Z}}$ of X_{δ^κ} may be extended to

$$(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2}\in X^\kappa_{\rho,\eta}$$

in a unique way. Hence the one-dimensional subshift X_{δ^κ} determines the two-dimensional subshift $X^\kappa_{\rho,\eta}$. Therefore we have:

Lemma 3.5. *The two-dimensional subshift $X^\kappa_{\rho,\eta}$ is not empty if and only if the one-dimensional subshift X_{δ^κ} is not empty.*

For $(\mathcal{A},\rho,\eta,\Sigma^\rho,\Sigma^\eta,\kappa)$, we will have a C^*-symbolic dynamical system $(\mathcal{A},\delta^\kappa,\Sigma_\kappa)$ in Section 4. It presents the subshift X_{δ^κ}. Since a subshift presented by a C^*-symbolic dynamical system is always not empty, one sees

Proposition 3.6. *The two-dimensional subshift $X^\kappa_{\rho,\eta}$ is not empty.*

4. C^*-textile dynamical systems and their C^*-algebras

The C^*-algebra $\mathcal{O}^{\kappa}_{\rho,\eta}$ is defined to be the universal C^*-algebra

$$C^*(x, S_\alpha, T_a; x \in \mathcal{A}, \alpha \in \Sigma^\rho, a \in \Sigma^\eta)$$

generated by $x \in \mathcal{A}$ and partial isometries $S_\alpha, \alpha \in \Sigma^\rho$, $T_a, a \in \Sigma^\eta$ subject to the following relations called $(\rho, \eta; \kappa)$:

$$\sum_{\beta \in \Sigma^\rho} S_\beta S_\beta^* = 1, \qquad x S_\alpha S_\alpha^* = S_\alpha S_\alpha^* x, \qquad S_\alpha^* x S_\alpha = \rho_\alpha(x), \tag{4.1}$$

$$\sum_{b \in \Sigma^\eta} T_b T_b^* = 1, \qquad x T_a T_a^* = T_a T_a^* x, \qquad T_a^* x T_a = \eta_a(x), \tag{4.2}$$

$$S_\alpha T_b = T_a S_\beta \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta) \tag{4.3}$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$. We will study the algebra $\mathcal{O}^{\kappa}_{\rho,\eta}$. For $(\alpha, b, a, \beta) \in \Sigma^\rho \times \Sigma^\eta \times \Sigma^\eta \times \Sigma^\rho$, we set

$$\begin{aligned} RB(\alpha, a) &= \{(b, \beta) \in \Sigma^\eta \times \Sigma^\rho \mid \kappa(\alpha, b) = (a, \beta)\}, \\ R(\alpha, a, \beta) &= \{b \in \Sigma^\eta \mid \kappa(\alpha, b) = (a, \beta)\}, \\ R(\alpha, a) &= \bigcup_{\beta \in \Sigma^\rho} R(\alpha, a, \beta). \end{aligned}$$

Lemma 4.1. *For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, one has $T_a^* S_\alpha \neq 0$ if and only if $RB(\alpha, a) \neq \emptyset$.*

Proof. Suppose that $T_a^* S_\alpha \neq 0$. As $T_a^* S_\alpha = \sum_{b' \in \Sigma^\eta} T_a^* S_\alpha T_{b'} T_{b'}^*$, there exists $b' \in \Sigma^\eta$ such that $T_a^* S_\alpha T_{b'} \neq 0$. Hence $\eta_{b'} \circ \rho_\alpha \neq 0$ so that $(\alpha, b') \in \Sigma^{\rho\eta}$. Then one may find $(a', \beta') \in \Sigma^\rho$ such that $\kappa(\alpha, b') = (a', \beta')$ and hence $S_\alpha T_{b'} = T_{a'} S_{\beta'}$. Since $0 \neq T_a^* S_\alpha T_{b'} = T_a^* T_{a'} S_{\beta'}$, one sees that $a = a'$ so that $(b', \beta') \in RB(\alpha, a)$.

Suppose next that $\kappa(\alpha, b) = (a, \beta)$ for some $(b, \beta) \in \Sigma^\eta \times \Sigma^\rho$. Since $\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a \neq 0$, one has $0 \neq S_\alpha T_b = T_a S_\beta$. It follows that

$$S_\beta^* T_a^* S_\alpha T_b = (T_a S_\beta)^* T_a S_\beta$$

so that $T_a^* S_\alpha \neq 0$. □

Lemma 4.2. *For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, we have*

$$T_a^* S_\alpha = \sum_{(b,\beta) \in RB(\alpha, a)} S_\beta \eta_b(\rho_\alpha(1)) T_b^* \tag{4.4}$$

and hence

$$S_\alpha^* T_a = \sum_{(b,\beta) \in RB(\alpha, a)} T_b \rho_\beta(\eta_a(1)) S_\beta^*. \tag{4.5}$$

Proof. We may assume that $T_a^* S_\alpha \neq 0$. One has

$$T_a^* S_\alpha = \sum_{b' \in \Sigma^\eta} T_a^* S_\alpha T_{b'} T_{b'}^*.$$

For $b' \in \Sigma^\eta$ with $(\alpha, b') \in \Sigma^{\rho\eta}$, take $(a', \beta') \in \Sigma^{\eta\rho}$ such that $\kappa(\alpha, b') = (a', \beta')$ so that

$$T_a^* S_\alpha T_{b'} T_{b'}^* = T_a^* T_{a'} S_{\beta'} T_{b'}^*.$$

Hence $T_a^* S_\alpha T_{b'} T_{b'}^* \neq 0$ implies $a = a'$. Since $T_a^* T_a = \eta_a(1)$ which commutes with $S_{\beta'} S_{\beta'}^*$, we have

$$T_a^* T_a S_{\beta'} T_{b'}^* = S_{\beta'} S_{\beta'}^* T_a^* T_a S_{\beta'} T_{b'}^* = S_{\beta'} \rho_{\beta'}(\eta_a(1)) T_{b'}^* = S_{\beta'} \eta_{b'}(\rho_\alpha(1)) T_{b'}^*.$$

It follows that

$$T_a^* S_\alpha = \sum_{(b',\beta') \in RB(\alpha,a)} T_a^* T_a S_{\beta'} T_{b'}^* = \sum_{(b',\beta') \in RB(\alpha,a)} S_{\beta'} \eta_{b'}(\rho_\alpha(1)) T_{b'}^*. \qquad \square$$

Hence we have:

Lemma 4.3. *For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, we have*

$$T_a T_a^* S_\alpha S_\alpha^* = \sum_{b \in R(\alpha,a)} S_\alpha T_b T_b^* S_\alpha^*.$$

Hence $T_a T_a^*$ commutes with $S_\alpha S_\alpha^*$.

Proof. By (4.4), we have

$$\begin{aligned} T_a T_a^* S_\alpha S_\alpha^* &= \sum_{(b,\beta) \in RB(\alpha,a)} T_a S_\beta \eta_b(\rho_\alpha(1)) T_b^* S_\alpha^* \\ &= \sum_{b \in R(\alpha,a)} S_\alpha T_b \eta_b(\rho_\alpha(1)) T_b^* S_\alpha^* \\ &= \sum_{b \in R(\alpha,a)} S_\alpha \rho_\alpha(1) T_b T_b^* S_\alpha^* \\ &= \sum_{b \in R(\alpha,a)} S_\alpha T_b T_b^* S_\alpha^*. \end{aligned} \qquad \square$$

Recall that $Z_{\mathcal{A}}$ denotes the center of $\mathcal{A}$ which consists of elements of $\mathcal{A}$ commuting with all elements of $\mathcal{A}$.

Lemma 4.4. *For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and $x, y \in Z_{\mathcal{A}}$, $T_a y T_a^*$ commutes with $S_\alpha x S_\alpha^*$.*

Proof. By (4.4), we have

$$\begin{aligned}
T_a y T_a^* S_\alpha x S_\alpha^* &= T_a y \sum_{(b,\beta)\in RB(\alpha,a)} S_\beta \eta_b(\rho_\alpha(1)) T_b^* x S_\alpha^* \\
&= \sum_{(b,\beta)\in RB(\alpha,a)} T_a S_\beta S_\beta^* y S_\beta \eta_b(\rho_\alpha(1)) T_b^* x T_b T_b^* S_\alpha^* \\
&= \sum_{(b,\beta)\in RB(\alpha,a)} S_\alpha T_b \rho_\beta(y) \eta_b(\rho_\alpha(1)) \eta_b(x) S_\beta^* T_a^* \\
&= \sum_{(b,\beta)\in RB(\alpha,a)} S_\alpha T_b \eta_b(x) \eta_b(\rho_\alpha(1)) \rho_\beta(y) S_\beta^* T_a^* \\
&= \sum_{(b,\beta)\in RB(\alpha,a)} S_\alpha x \rho_\alpha(1) T_b S_\beta^* y T_a^* \\
&= \sum_{(b,\beta)\in RB(\alpha,a)} S_\alpha x S_\alpha^* S_\alpha T_b S_\beta^* T_a^* T_a y T_a^* \\
&= \sum_{b\in R(\alpha,a)} S_\alpha x \cdot S_\alpha^* S_\alpha T_b T_b^* S_\alpha^* T_a \cdot y T_a^*.
\end{aligned}$$

Now if $(\alpha, b') \not\in \Sigma^{\rho,\eta}$, then $S_\alpha T_{b'} = 0$. Hence

$$\sum_{b\in R(\alpha,a)} S_\alpha^* S_\alpha T_b T_b^* S_\alpha^* T_a = \sum_{b\in\Sigma^\eta} S_\alpha^* S_\alpha T_b T_b^* S_\alpha^* T_a = S_\alpha^* T_a.$$

Therefore we have

$$T_a y T_a^* S_\alpha x S_\alpha^* = S_\alpha x S_\alpha^* T_a y T_a^*. \qquad \square$$

For words $\mu = (\mu_1, \dots, \mu_j) \in B_j(\Lambda_\rho), \zeta = (\zeta_1, \dots, \zeta_k) \in B_k(\Lambda_\eta)$, we set

$$S_\mu = S_{\mu_1} \cdots S_{\mu_j}, \qquad T_\zeta = T_{\zeta_1} \cdots T_{\zeta_k}.$$

For a subset F of $\mathcal{O}^\kappa_{\rho,\eta}$, denote by $C^*(F)$ the C^*-subalgebra of $\mathcal{O}^\kappa_{\rho,\eta}$ generated by the elements of F. We define C^*-subalgebras $\mathcal{D}_{\rho,\eta}, \mathcal{D}_{j,k}$ of $\mathcal{O}^\kappa_{\rho,\eta}$ by

$$\begin{aligned}
\mathcal{D}_{\rho,\eta} =& C^*(S_\mu T_\zeta x T_\zeta^* S_\mu^* : \mu \in B_*(\Lambda_\rho), \zeta \in B_*(\Lambda_\eta), x \in \mathcal{A}), \\
\mathcal{D}_{j,k} =& C^*(S_\mu T_\zeta x T_\zeta^* S_\mu^* : \mu \in B_j(\Lambda_\rho), \zeta \in B_k(\Lambda_\eta), x \in \mathcal{A}) \quad \text{for } j,k \in \mathbb{Z}_+.
\end{aligned}$$

By the commutation relation (4.3), one sees that

$$\mathcal{D}_{j,k} = C^*(T_\xi S_\nu x S_\nu^* T_\xi^* : \nu \in B_j(\Lambda_\rho), \xi \in B_k(\Lambda_\eta), x \in \mathcal{A}).$$

The identities

$$\begin{aligned}
S_\mu T_\zeta x T_\zeta^* S_\mu^* &= \sum_{a\in\Sigma^\eta} S_\mu T_{\zeta a} \eta_a(x) T_{\zeta a}^* S_\mu^*, \\
T_\xi S_\nu x S_\nu^* T_\xi^* &= \sum_{\alpha\in\Sigma^\rho} T_\xi S_{\nu\alpha} \rho_\alpha(x) S_{\nu\alpha}^* T_\xi^*
\end{aligned}$$

for $x \in \mathcal{A}$ and $\mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta)$ yield the embeddings

$$\mathcal{D}_{j,k} \hookrightarrow \mathcal{D}_{j,k+1}, \qquad \mathcal{D}_{j,k} \hookrightarrow \mathcal{D}_{j+1,k}$$

respectively such that $\cup_{j,k\in\mathbb{Z}_+}\mathcal{D}_{j,k}$ is dense in $\mathcal{D}_{\rho,\eta}$.

Proposition 4.5. *If $\mathcal{A}$ is commutative, so is $\mathcal{D}_{\rho,\eta}$.*

Proof. The preceding lemma tells us that $\mathcal{D}_{1,1}$ is commutative. Suppose that the algebra $\mathcal{D}_{j,k}$ is commutative for fixed $j,k\in\mathbb{N}$. We will show that the both algebras $\mathcal{D}_{j+1,k}$ and $\mathcal{D}_{j,k+1}$ are commutative. The algebra $\mathcal{D}_{j+1,k}$ consists of the linear span of elements of the form:

$$S_\alpha x S_\alpha^* \qquad \text{for } x\in\mathcal{D}_{j,k}, \alpha\in\Sigma^\rho.$$

For $x,y\in\mathcal{D}_{j,k},\alpha,\beta\in\Sigma^\rho$, we will show that $S_\alpha x S_\alpha^*$ commutes with both $S_\beta y S_\beta^*$ and y. If $\alpha=\beta$, it is easy to see that $S_\alpha x S_\alpha^*$ commutes with $S_\alpha y S_\alpha^*$, because $\rho_\alpha(1)\in\mathcal{A}\subset\mathcal{D}_{j,k}$. If $\alpha\neq\beta$, both $S_\alpha x S_\alpha^* S_\beta y S_\beta^*$ and $S_\beta y S_\beta^* S_\alpha x S_\alpha^*$ are zeros. Since $S_\alpha^* y S_\alpha\in\mathcal{D}_{j-1,k}\subset\mathcal{D}_{j,k}$, one sees $S_\alpha^* y S_\alpha$ commutes with x. One also sees that $S_\alpha S_\alpha^*\in\mathcal{D}_{j,k}$ commutes with y. It follows that

$$S_\alpha x S_\alpha^* y = S_\alpha x S_\alpha^* y S_\alpha S_\alpha^* = S_\alpha S_\alpha^* y S_\alpha x S_\alpha^* = y S_\alpha x S_\alpha^*.$$

Hence the algebra $\mathcal{D}_{j+1,k}$ is commutative, and similarly so is $\mathcal{D}_{j,k+1}$. By induction, the algebras $\mathcal{D}_{j,k}$ are all commutative for all $j,k\in\mathbb{N}$. Since $\cup_{j,k\in\mathbb{N}}\mathcal{D}_{j,k}$ is dense in $\mathcal{D}_{\rho,\eta}$, $\mathcal{D}_{\rho,\eta}$ is commutative. □

Proposition 4.6. *Let $\mathcal{O}_{\rho,\eta}^{alg}$ be the dense $*$-subalgebra of $\mathcal{O}_{\rho,\eta}^{\kappa}$ algebraically generated by elements $x\in\mathcal{A}$, $S_\alpha,\alpha\in\Sigma^\rho$ and $T_a, a\in\Sigma^\eta$. Then each element of $\mathcal{O}_{\rho,\eta}^{alg}$ is a finite linear combination of elements of the form:*

$$S_\mu T_\zeta x T_\xi^* S_\nu^* \qquad \text{for } x\in\mathcal{A},\mu,\nu\in B_*(\Lambda_\rho),\zeta,\xi\in B_*(\Lambda_\eta). \tag{4.6}$$

Proof. For $\alpha,\beta\in\Sigma^\rho$, $a,b\in\Sigma^\eta$ and $x\in\mathcal{A}$, we have

$$S_\alpha^* S_\beta = \begin{cases}\rho_\alpha(1)\in\mathcal{A} & \text{if } \alpha=\beta,\\ 0 & \text{otherwise,}\end{cases} \qquad T_a^* T_b = \begin{cases}\eta_a(1)\in\mathcal{A} & \text{if } a=b,\\ 0 & \text{otherwise,}\end{cases}$$

$$S_\alpha^* T_a = \sum_{(b,\beta)\in RB(\alpha,a)} T_b\rho_\beta(\eta_a(1))S_\beta^*, \qquad T_a^* S_\alpha = \sum_{(b,\beta)\in RB(\alpha,a)} S_\beta\eta_b(\rho_\alpha(1))T_b^*,$$

$$S_\alpha^* x = \rho_\alpha(x) S_\alpha, \qquad T_a^* x = \eta_a(x) T_a^*.$$

And also

$$S_\beta^* T_a^* = \begin{cases} T_b^* S_\alpha^* & \text{if } (a,\beta)\in\Sigma^{\eta\rho} \text{ and } (a,\beta)=\kappa(\alpha,b),\\ 0 & \text{if } (a,\beta)\notin\Sigma^{\eta\rho}.\end{cases}$$

Therefore we conclude that any element of $\mathcal{O}_{\rho,\eta}^{alg}$ is a finite linear combination of elements of the form of (4.6). □

Similarly we have:

Proposition 4.7. *Each element of $\mathcal{O}_{\rho,\eta}^{alg}$ is a finite linear combination of elements of the form:*

$$T_\zeta S_\mu x S_\nu^* T_\xi^* \qquad \text{for } x\in\mathcal{A},\mu,\nu\in B_*(\Lambda_\rho),\zeta,\xi\in B_*(\Lambda_\eta). \tag{4.7}$$

In the rest of this section, we will have a C^*-symbolic dynamical system $(\mathcal{A}, \delta^\kappa, \Sigma_\kappa)$ from $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$, which presents the one-dimensional subshift X_{δ^κ} described in the previous section. For $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$, define an endomorphism δ^κ_ω on $\mathcal{A}$ for $\omega \in \Sigma_\kappa$ by setting

$$\delta^\kappa_\omega(x) = \eta_b(\rho_\alpha(x))(= \rho_\beta(\eta_a(x))), \qquad x \in \mathcal{A}, \quad \omega = (\alpha, b, a, \beta) \in \Sigma_\kappa.$$

Lemma 4.8. *$(\mathcal{A}, \delta^\kappa, \Sigma_\kappa)$ is a C^*-symbolic dynamical system that presents X_{δ^κ}.*

Proof. We will show that δ^κ is essential and faithful. Now both C^*-symbolic dynamical systems $(\mathcal{A}, \eta, \Sigma^\eta)$ and $(\mathcal{A}, \rho, \Sigma^\eta)$ are essential. Since $\rho_\alpha(Z_\mathcal{A}) \subset Z_\mathcal{A}$ and $\eta_a(Z_\mathcal{A}) \subset Z_\mathcal{A}$, it is clear that $\delta^\kappa_\omega(Z_\mathcal{A}) \subset Z_\mathcal{A}$. By the inequalities

$$\sum_{\omega \in \Sigma_\kappa} \delta^\kappa_\omega(1) = \sum_{b \in \Sigma^\eta} \sum_{\alpha \in \Sigma^\rho} \eta_b(\rho_\alpha(1)) \geq \sum_{b \in \Sigma^\eta} \eta_b(1) \geq 1$$

$\{\delta^\kappa\}_{\omega \in \Sigma_\kappa}$ is essential. For any nonzero $x \in \mathcal{A}$, there exists $\alpha \in \Sigma^\rho$ such that $\rho_\alpha(x) \neq 0$ and there exists $b \in \Sigma^\eta$ such that $\eta_b(\rho_\alpha(x)) \neq 0$. Hence δ^κ is faithful so that $(\mathcal{A}, \delta^\kappa, \Sigma_\kappa)$ is a C^*-symbolic dynamical system. It is obvious that the subshift presented by $(\mathcal{A}, \delta^\kappa, \Sigma_\kappa)$ is X_{δ^κ}. □

Put

$$\widehat{X}^\kappa_{\rho,\eta} = \{(\omega_{i,-j})_{(i,j) \in \mathbb{N}^2} \in \Sigma_\kappa^{\mathbb{N}^2} \mid (\omega_{i,j})_{(i,j) \in \mathbb{Z}^2} \in X^\kappa_{\rho,\eta}\}$$

and

$$\widehat{X}_{\delta^\kappa} = \{(\omega_{n,-n})_{n \in \mathbb{N}} \in \Sigma_\kappa^{\mathbb{N}} \mid (\omega_{i,j})_{(i,j) \in \mathbb{N}^2} \in \widehat{X}^\kappa_{\rho,\eta}\}.$$

The latter set $\widehat{X}_{\delta^\kappa}$ is the right one-sided subshift for X_{δ^κ}.

Lemma 4.9. *A configuration $(\omega_{i,-j})_{(i,j) \in \mathbb{N}^2} \in \widehat{X}^\kappa_{\rho,\eta}$ extends to a whole configuration $(\omega_{i,j})_{(i,j) \in \mathbb{Z}^2} \in X^\kappa_{\rho,\eta}$.*

Proof. For $(\omega_{i,-j})_{(i,j) \in \mathbb{N}^2} \in \widehat{X}^\kappa_{\rho,\eta}$, put $x_i = \omega_{i,-i}, i \in \mathbb{N}$ so that $x = (x_i)_{i \in \mathbb{N}} \in \widehat{X}_{\delta^\kappa}$. Since $\widehat{X}_{\delta^\kappa}$ is a one-sided subshift, there exists an extension $\tilde{x} \in X_{\delta^\kappa}$ to two-sided sequence such that $\tilde{x}_i = x_i$ for $i \in \mathbb{N}$. By the diagonal property, $\tilde{x}$ determines a whole configuration $\tilde{\omega}$ to $\mathbb{Z}^2$ such that $\tilde{\omega} \in X^\kappa_{\delta,\eta}$ and $(\tilde{\omega}_{i,-i})_{i \in \mathbb{N}} = \tilde{x}$. Hence $\tilde{\omega}_{i,-j} = \omega_{i,-j}$ for all $i, j \in \mathbb{N}$. □

Let $\mathfrak{D}_{\rho,\eta}$ be the C^*-subalgebra of $\mathcal{D}_{\rho,\eta}$ defined by

$$\begin{aligned}\mathfrak{D}_{\rho,\eta} &= C^*(S_\mu T_\zeta T^*_\zeta S^*_\mu : \mu \in B_*(\Lambda_\rho), \zeta \in B_*(\Lambda_\eta)) \\ &= C^*(T_\xi S_\nu S^*_\nu T^*_\xi : \nu \in B_*(\Lambda_\rho), \xi \in B_*(\Lambda_\eta))\end{aligned}$$

which is a commutative C^*-subalgebra of $\mathcal{D}_{\rho,\eta}$. Put for $\mu = (\mu_1, \ldots, \mu_n) \in B_*(\Lambda_\rho)$, $\zeta = (\zeta_1, \cdots, \zeta_m) \in B_*(\Lambda_\eta)$ the cylinder set

$$\begin{aligned}U_{\mu,\zeta} = \{&(\omega_{i,-j})_{(i,j) \in \mathbb{N}^2} \in \widehat{X}^\kappa_{\rho,\eta} \mid \\ &t(\omega_{i,-1}) = \mu_i, i = 1, \ldots, n, r(\omega_{n,-j}) = \zeta_j, j = 1, \ldots, m\}.\end{aligned}$$

The following lemma is direct.

Lemma 4.10. *$\mathfrak{D}_{\rho,\eta}$ is isomorphic to $C(\widehat{X}^{\kappa}_{\rho,\eta})$ through the correspondence such that $S_\mu T_\zeta T_\zeta^* S_\mu^*$ goes to $\chi_{U_{\mu,\zeta}}$, where $\chi_{U_{\mu,\zeta}}$ is the characteristic function for the cylinder set $U_{\mu,\zeta}$ on $\widehat{X}^{\kappa}_{\rho,\eta}$.*

5. Condition (I) for C^*-textile dynamical systems

The notion of condition (I) for finite square matrices with entries in $\{0,1\}$ has been introduced in [8]. The condition has been generalized by many authors to corresponding conditions for generalizations of the Cuntz–Krieger algebras (cf. [12], [15], [20], [41], etc.). The condition (I) for C^*-symbolic dynamical systems (including λ-graph systems) has been also defined in [29] (cf. [25], [26]). All of these conditions give rise to the uniqueness of the associated C^*-algebras subject to some operator relations among certain generating elements.

In this section, we will introduce the notion of condition (I) for C^*-textile dynamical systems to prove the uniqueness of the C^*-algebras $\mathcal{O}^{\kappa}_{\rho,\eta}$ under the relation $(\rho,\eta;\kappa)$.

Let $(\mathcal{A},\rho,\eta,\Sigma^\rho,\Sigma^\eta,\kappa)$ be a C^*-symbolic dynamical system over Σ and $X^{\kappa}_{\rho,\eta}$ the associated two-dimensional subshift. Denote by $\Lambda_\rho,\Lambda_\eta$ the associated subshifts to the C^*-symbolic dynamical systems $(\mathcal{A},\rho,\Sigma^\rho),(\mathcal{A},\eta,\Sigma^\eta)$ respectively. For $\mu=(\mu_1,\dots,\mu_j)\in B_j(\Lambda_\rho),\zeta=(\zeta_1,\dots,\zeta_k)\in B_k(\Lambda_\eta)$, we put $\rho_\mu=\rho_{\mu_j}\circ\cdots\circ\rho_{\mu_1},\eta_\zeta=\eta_{\zeta_k}\circ\cdots\circ\eta_{\zeta_1}$ respectively. Recall that $|\mu|,|\zeta|$ denotes the lengths j,k respectively. In the algebra $\mathcal{O}^{\kappa}_{\rho,\eta}$, we set the subalgebras

$$\mathcal{F}_{\rho,\eta} = C^*(S_\mu T_\zeta x T_\xi^* S_\nu^* : \mu,\nu\in B_*(\Lambda_\rho),\zeta,\xi\in B_*(\Lambda_\eta),|\mu|=|\nu|,|\zeta|=|\xi|,x\in\mathcal{A})$$

and for $j,k\in\mathbb{Z}_+$,

$$\mathcal{F}_{j,k} = C^*(S_\mu T_\zeta x T_\xi^* S_\nu^* : \mu,\nu\in B_j(\Lambda_\rho),\zeta,\xi\in B_k(\Lambda_\eta),x\in\mathcal{A}).$$

We notice that

$$\mathcal{F}_{j,k} = C^*(T_\zeta S_\mu x S_\nu^* T_\xi^* : \mu,\nu\in B_j(\Lambda_\rho),\zeta,\xi\in B_k(\Lambda_\eta),x\in\mathcal{A}).$$

The identities

$$S_\mu T_\zeta x T_\xi^* S_\nu^* = \sum_{a\in\Sigma^\eta} S_\mu T_{\zeta a}\eta_a(x) T_{\xi a}^* S_\nu^*, \tag{5.1}$$

$$T_\zeta S_\mu x S_\nu^* T_\xi^* = \sum_{\alpha\in\Sigma^\rho} T_\zeta S_{\mu\alpha}\rho_\alpha(x) S_{\nu\alpha}^* T_\xi^* \tag{5.2}$$

for $x\in\mathcal{A}$ and $\mu,\nu\in B_j(\Lambda_\rho),\zeta,\xi\in B_k(\Lambda_\eta)$ yield the embeddings

$$\iota_{*,+1}:\mathcal{F}_{j,k}\hookrightarrow\mathcal{F}_{j,k+1}, \qquad \iota_{+1,*}:\mathcal{F}_{j,k}\hookrightarrow\mathcal{F}_{j+1,k} \tag{5.3}$$

respectively, such that $\cup_{j,k\in\mathbb{Z}_+}\mathcal{F}_{j,k}$ is dense in $\mathcal{F}_{\rho,\eta}$.

By the universality of $\mathcal{O}^{\kappa}_{\rho,\eta}$ subject to the relations $(\rho,\eta;\kappa)$, we may define an action $\theta : \mathbb{T}^2 \longrightarrow \mathrm{Aut}(\mathcal{O}^{\kappa}_{\rho,\eta})$ of the two-dimensional torus group

$$\mathbb{T}^2 = \{(z,w) \in \mathbb{C}^2 \mid |z| = |w| = 1\}$$

to $\mathcal{O}^{\kappa}_{\rho,\eta}$ by setting

$$\theta_{z,w}(S_\alpha) = zS_\alpha, \quad \theta_{z,w}(T_a) = wT_a, \quad \theta_{z,w}(x) = x$$

for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, $x \in \mathcal{A}$ and $z,w \in \mathbb{T}$. We call the action $\theta : \mathbb{T}^2 \longrightarrow \mathrm{Aut}(\mathcal{O}^{\kappa}_{\rho,\eta})$ the gauge action of $\mathbb{T}^2$ on $\mathcal{O}^{\kappa}_{\rho,\eta}$. The fixed point algebra of $\mathcal{O}^{\kappa}_{\rho,\eta}$ under θ is denoted by $(\mathcal{O}^{\kappa}_{\rho,\eta})^\theta$. Let $\mathcal{E}_{\rho,\eta} : \mathcal{O}^{\kappa}_{\rho,\eta} \longrightarrow (\mathcal{O}^{\kappa}_{\rho,\eta})^\theta$ be the conditional expectation defined by

$$\mathcal{E}_{\rho,\eta}(X) = \int_{(z,w)\in\mathbb{T}^2} \theta_{z,w}(X)\, dzdw, \qquad X \in \mathcal{O}^{\kappa}_{\rho,\eta}$$

where $dzdw$ means the normalized Haar measure on $\mathbb{T}^2$. The following lemma is routine.

Lemma 5.1. $(\mathcal{O}^{\kappa}_{\rho,\eta})^\theta = \mathcal{F}_{\rho,\eta}$.

Define homomorphisms $\phi_\rho, \phi_\eta : \mathcal{D}_{\rho,\eta} \longrightarrow \mathcal{D}_{\rho,\eta}$ by setting

$$\phi_\rho(X) = \sum_{\alpha\in\Sigma^\rho} S_\alpha X S_\alpha^*, \qquad \phi_\eta(X) = \sum_{a\in\Sigma^\eta} T_a X T_a^*, \qquad X \in \mathcal{D}_{\rho,\eta}.$$

It is easy to see that by (4.3)

$$\phi_\rho \circ \phi_\eta = \phi_\eta \circ \phi_\rho \quad \text{ on } \mathcal{D}_{\rho,\eta}.$$

Definition 5.2. A C^*-textile dynamical system $(\mathcal{A},\rho,\eta,\Sigma^\rho,\Sigma^\eta,\kappa)$ is said to satisfy *condition* (I) if there exists a unital increasing sequence

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}$$

of C^*-subalgebras of $\mathcal{A}$ such that:

(1) $\rho_\alpha(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$, $\eta_a(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$ for all $l \in \mathbb{Z}_+, \alpha \in \Sigma^\rho, a \in \Sigma^\eta$.
(2) $\cup_{l\in\mathbb{Z}_+}\mathcal{A}_l$ is dense in $\mathcal{A}$.
(3) For $\epsilon > 0$, $j,k,l \in \mathbb{N}$ with $j+k \le l$ and

$$X_0 \in \mathcal{F}^l_{j,k} = C^*(S_\mu T_\zeta x T_\xi^* S_\nu^* : \mu,\nu \in B_j(\Lambda_\rho), \zeta,\xi \in B_k(\Lambda_\eta), x \in \mathcal{A}_l),$$

there exists an element

$$g \in \mathcal{D}_{\rho,\eta} \cap \mathcal{A}_l{}'(= \{y \in \mathcal{D}_{\rho,\eta} \mid ya = ay \ \text{ for } a \in \mathcal{A}_l\})$$

with $0 \le g \le 1$ such that:
(i) $\|X_0 \phi_\rho^j \circ \phi_\eta^k(g)\| \ge \|X_0\| - \epsilon$,
(ii) $\phi_\rho^n(g)\phi_\eta^m(g) = \phi_\rho^n(\phi_\eta^m(g))g = \phi_\rho^n(g)g = \phi_\eta^m(g)g = 0$ for all $n = 1,2,\dots,j$, $m = 1,2,\dots,k$.

If in particular, one may take the above subalgebras $\mathcal{A}_l \subset \mathcal{A}$, $l = 0,1,2,\dots$ to be of finite dimensional, then $(\mathcal{A},\rho,\eta,\Sigma^\rho,\Sigma^\eta,\kappa)$ is said to satisfy *AF-condition* (I). In this case, $\mathcal{A} = \overline{\cup_{l=0}^\infty \mathcal{A}_l}$ is an AF-algebra.

As the element g above belongs to the diagonal subalgebra $\mathcal{D}_{\rho,\eta}$ of $\mathcal{F}_{\rho,\eta}$, the condition (I) of $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is intrinsically determined by itself by virtue of Lemma 5.5 below.

We will also introduce the following condition called *free*, which will be stronger than condition (I) but easier to confirm than condition (I).

Definition 5.3. A C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to be *free* if there exists a unital increasing sequence $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}$ of C^*-subalgebras of $\mathcal{A}$ such that:

(1) $\rho_\alpha(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$, $\eta_a(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$ for all $l \in \mathbb{Z}_+, \alpha \in \Sigma^\rho, a \in \Sigma^\eta$.
(2) $\cup_{l\in\mathbb{Z}_+}\mathcal{A}_l$ is dense in $\mathcal{A}$.
(3) For $j, k, l \in \mathbb{N}$ with $j + k \le l$ there exists a projection $q \in \mathcal{D}_{\rho,\eta} \cap {\mathcal{A}_l}'$ such that:
  (i) $qa \ne 0$ for $0 \ne a \in \mathcal{A}_l$.
  (ii) $\phi_\rho^n(q)\phi_\eta^m(q) = \phi_\rho^n(\phi_\eta^m(q))q = \phi_\rho^n(q)q = \phi_\eta^m(q)q = 0$ for all $n = 1, 2, \dots, j$, $m = 1, 2, \dots, k$.

If in particular, one may take the above subalgebras $\mathcal{A}_l \subset \mathcal{A}$, $l = 0, 1, 2, \dots$ to be of finite dimensional, then $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to be *AF-free*.

Proposition 5.4. *If a C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is free (resp. AF-free), then it satisfies condition (I) (resp. AF-condition (I)).*

Proof. Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is free. Take an increasing sequence $\mathcal{A}_l, l \in \mathbb{N}$ of C^*-subalgebras of $\mathcal{A}$ satisfying the above conditions (1), (2), (3) of freeness. For $j, k, l \in \mathbb{N}$ with $j + k \le l$ there exists a projection $q \in \mathcal{D}_{\rho,\eta} \cap {\mathcal{A}_l}'$ satisfying the above two conditions (3i) and (3ii). Put

$$Q^l_{j,k} = \phi_\rho^j(\phi_\eta^k(q)).$$

For $x \in \mathcal{A}_l, \mu, \nu \in B_j(\Lambda_\rho), \xi, \zeta \in B_k(\Lambda_\eta)$, one has the equality

$$Q^l_{j,k} S_\mu T_\zeta x T_\xi^* S_\nu^* = S_\mu T_\zeta x T_\xi^* S_\nu^*$$

so that $Q^l_{j,k}$ commutes with all of elements of $\mathcal{F}^l_{j,k}$. By using the condition (3i) for q one directly sees that $S_\mu T_\zeta x T_\xi^* S_\nu^* \ne 0$ if and only if

$$Q^l_{j,k} S_\mu T_\zeta x T_\xi^* S_\nu^* \ne 0.$$

Hence the map

$$X \in \mathcal{F}^l_{j,k} \longrightarrow XQ^l_{j,k} \in \mathcal{F}^l_{j,k} Q^l_{j,k}$$

defines a homomorphism, that is proved to be injective by a similar proof to the proof of [30, Proposition 3.7]. Hence we have $\|XQ^l_{j,k}\| = \|X\| \ge \|X\| - \epsilon$ for all $X \in \mathcal{F}^l_{j,k}$. □

Let $\mathcal{B}$ be a unital C^*-algebra. Suppose that there exist an injective $*$-homomorphism $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ preserving their units and two families

$$s_\alpha \in \mathcal{B}, \alpha \in \Sigma^\rho \quad \text{and} \quad t_a \in \mathcal{B}, a \in \Sigma^\eta$$

of partial isometries satisfying

$$\sum_{\beta\in\Sigma^\rho} s_\beta s_\beta^* = 1, \qquad \pi(x)s_\alpha s_\alpha^* = s_\alpha s_\alpha^* \pi(x), \qquad s_\alpha^*\pi(x)s_\alpha = \pi(\rho_\alpha(x)),$$

$$\sum_{b\in\Sigma^\eta} t_b t_b^* = 1, \qquad \pi(x)t_a t_a^* = t_a t_a^* \pi(x), \qquad t_a^*\pi(x)t_a = \pi(\eta_a(x)),$$

$$s_\alpha t_b = t_a s_\beta \quad \text{if} \quad \kappa(\alpha, b) = (a, \beta)$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma^\rho$, $a \in \Sigma^\eta$. Put $\widetilde{\mathcal{A}} = \pi(\mathcal{A})$ and

$$\tilde\rho_\alpha(\pi(x)) = \pi(\rho_\alpha(x)), \quad \tilde\eta_a(\pi(x)) = \pi(\eta_a(x)), \quad x \in \mathcal{A}.$$

It is easy to see that $(\widetilde{\mathcal{A}}, \tilde\rho, \tilde\eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is a C^*-textile dynamical system such that the presented textile dynamical system $X^\kappa_{\tilde\rho,\tilde\eta}$ is the same as the one $X^\kappa_{\rho,\eta}$ presented by $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$. Let $\mathcal{O}_{\pi,s,t}$ be the C^*-subalgebra of $\mathcal{B}$ generated by $\pi(x)$ and s_α, t_a for $x \in \mathcal{A}, \alpha \in \Sigma^\rho, a \in \Sigma^\eta$. Let $\mathcal{F}_{\pi,s,t}$ be the C^*-subalgebra of $\mathcal{O}_{\pi,s,t}$ generated by $s_\mu t_\zeta \pi(x) t_\xi^* s_\nu^*$ for $x \in \mathcal{A}$ and $\mu, \nu \in B_*(\Lambda_\rho), \zeta, \xi \in B_*(\Lambda_\eta)$ with $|\mu| = |\nu|, |\zeta| = |\xi|$. By the universality of the algebra $\mathcal{O}^\kappa_{\rho,\eta}$, the correspondence

$$x \in \mathcal{A} \longrightarrow \pi(x) \in \widetilde{A}, \qquad S_\alpha \longrightarrow s_\alpha, \;\; \alpha \in \Sigma^\rho, \qquad T_a \longrightarrow t_a, \;\; a \in \Sigma^\eta$$

extends to a surjective $*$-homomorphism $\tilde\pi : \mathcal{O}^\kappa_{\rho,\eta} \longrightarrow \mathcal{O}_{\pi,s,t}$.

Lemma 5.5. *The restriction of $\tilde\pi$ to the subalgebra $\mathcal{F}_{\rho,\eta}$ is a $*$-isomorphism from $\mathcal{F}_{\rho,\eta}$ to $\mathcal{F}_{\pi,s,t}$. Hence if $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I) (resp. is free), $(\widetilde{\mathcal{A}}, \tilde\rho, \tilde\eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I) (resp. is free).*

Proof. It suffices to show that $\tilde\pi$ is injective on $\mathcal{F}_{j,k}$ for all $j, k \in \mathbb{Z}$. Suppose

$$\sum_{\mu,\nu\in B_j(\Lambda_\rho),\zeta,\xi\in B_k(\Lambda_\eta)} s_\mu t_\zeta \pi(x_{\mu,\zeta,\xi,\nu}) t_\xi^* s_\nu^* = 0$$

with $x_{\mu,\zeta,\xi,\nu} \in \mathcal{A}$. For $\mu', \nu' \in B_j(\Lambda_\rho), \zeta', \xi' \in B_k(\Lambda_\eta)$, one has

$$\begin{aligned}\pi(\eta_{\zeta'}(\rho_{\mu'}(1))&x_{\mu',\zeta',\xi',\nu'}\eta_{\xi'}(\rho_{\nu'}(1))) \\ &= t_{\zeta'}^* s_{\mu'}^* \left(\sum_{\mu,\nu\in B_j(\Lambda_\rho),\zeta,\xi\in B_k(\Lambda_\eta)} s_\mu t_\zeta \pi(x_{\mu,\zeta,\xi,\nu}) t_\xi^* s_\nu^* \right) s_{\nu'} t_{\xi'} = 0.\end{aligned}$$

As $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ is injective, one sees

$$\eta_{\zeta'}(\rho_{\mu'}(1))x_{\mu',\zeta',\xi',\nu'}\eta_{\xi'}(\rho_{\nu'}(1)) = 0$$

so that

$$S_{\mu'} T_{\zeta'} x_{\mu',\zeta',\xi',\nu'} T_{\xi'}^* S_{\nu'}^* = 0.$$

Hence we have

$$\sum_{\mu,\nu\in B_j(\Lambda_\rho),\zeta,\xi\in B_k(\Lambda_\eta)} S_\mu T_\zeta x_{\mu,\zeta,\xi,\nu} T_\xi^* S_\nu^* = 0.$$

Therefore $\tilde\pi$ is injective on $\mathcal{F}_{j,k}$. $\square$

We henceforth assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I) defined above. Take a unital increasing sequence $\{\mathcal{A}_l\}_{l \in \mathbb{Z}_+}$ of C^*-subalgebras of $\mathcal{A}$ as in the definition of condition (I). Recall that the algebra $\mathcal{F}^l_{j,k}$ for $j, k \le l$ is defined by

$$\mathcal{F}^l_{j,k} = C^*(S_\mu T_\zeta x T^*_\xi S^*_\nu : \mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta), x \in \mathcal{A}_l).$$

There exists an inclusion relation $\mathcal{F}^l_{j,k} \subset \mathcal{F}^{l'}_{j',k'}$ for $j \le j', k \le k'$ and $l \le l'$ through the identities (5.1), (5.2). Let $\mathcal{P}_{\pi,s,t}$ be the $*$-subalgebra of $\mathcal{O}_{\pi,s,t}$ algebraically generated by $\pi(x), s_\alpha, t_a$ for $x \in \mathcal{A}_l, l \in \mathbb{Z}_+,\ \alpha \in \Sigma^\rho, a \in \Sigma^\eta$.

Lemma 5.6. *Any element $x \in \mathcal{P}_{\pi,s,t}$ can be expressed in a unique way as*

$$\begin{aligned} x = & \sum_{|\nu|,|\xi| \ge 1} x_{-\xi,-\nu} t^*_\xi s^*_\nu + \sum_{|\zeta|,|\nu| \ge 1} t_\zeta x_{\zeta,-\nu} s^*_\nu + \sum_{|\mu|,|\xi| \ge 1} s_\mu x_{\mu,-\xi} t^*_\xi \\ & + \sum_{|\mu|,|\zeta| \ge 1} s_\mu t_\zeta x_{\mu,\zeta} + \sum_{|\xi| \ge 1} x_{-\xi} t^*_\xi + \sum_{|\nu| \ge 1} x_{-\nu} s^*_\nu \\ & + \sum_{|\mu| \ge 1} s_\mu x_\mu + \sum_{|\zeta| \ge 1} t_\zeta x_\zeta + x_0 \end{aligned}$$

where the above summations Σ are all finite sums and the elements

$$x_{-\xi,-\nu},\ x_{\zeta,-\nu},\ x_{\mu,-\xi},\ x_{\mu,\zeta},\ x_{-\xi},\ x_{-\nu},\ x_\mu,\ x_\zeta,\ x_0$$

for $\mu, \nu \in B_*(\Lambda_\rho), \zeta, \xi \in B_*(\Lambda_\eta)$ all belong to the dense subalgebra

$$\mathcal{P}_{\pi,s,t} \cap \mathcal{F}_{\pi,s,t}$$

which satisfy

$$\begin{aligned} x_{-\xi,-\nu} &= x_{-\xi,-\nu} \eta_\xi(\rho_\nu(1)), & x_{\zeta,-\nu} &= \eta_\zeta(1) x_{\zeta,-\nu} \rho_\nu(1), \\ x_{\mu,-\xi} &= \rho_\mu(1) x_{\mu,-\xi} \eta_\xi(1), & x_{\mu,\zeta} &= \eta_\zeta(\rho_\mu(1)) x_{\mu,\zeta}, \\ x_{-\xi} &= x_{-\xi} \eta_\xi(1), & x_{-\nu} &= x_{-\nu} \rho_\nu(1), \\ x_\mu &= \rho_\mu(1) x_\mu, & x_\zeta &= \eta_\zeta(1) x_\zeta. \end{aligned}$$

Proof. Put

$$\begin{aligned} x_{-\xi,-\nu} &= \mathcal{E}_{\rho,\eta}(x s_\nu t_\xi), & x_{\zeta,-\nu} &= \mathcal{E}_{\rho,\eta}(t^*_\zeta x s_\nu), \\ x_{\mu,-\xi} &= \mathcal{E}_{\rho,\eta}(s^*_\mu x t_\xi), & x_{\mu,\zeta} &= \mathcal{E}_{\rho,\eta}(t^*_\zeta s^*_\mu x), \\ x_{-\xi} &= \mathcal{E}_{\rho,\eta}(x t_\xi), & x_{-\nu} &= \mathcal{E}_{\rho,\eta}(x s_\nu), \\ x_\mu &= \mathcal{E}_{\rho,\eta}(s^*_\mu x), & x_\zeta &= \mathcal{E}_{\rho,\eta}(t^*_\zeta x), \\ x_0 &= \mathcal{E}_{\rho,\eta}(x). \end{aligned}$$

Then we have the desired expression of x. The elements

$$x_{-\xi,-\nu},\ x_{\zeta,-\nu},\ x_{\mu,-\xi},\ x_{\mu,\zeta},\ x_{-\xi},\ x_{-\nu},\ x_\mu,\ x_\zeta,\ x_0$$

for $\mu, \nu \in B_*(\Lambda_\rho), \zeta, \xi \in B_*(\Lambda_\eta)$ are automatically determined by the above formulae so that the expression is unique. □

Lemma 5.7. *For $h \in \mathcal{D}_{\rho,\eta} \cap \mathcal{A}_l'$ and $j, k \in \mathbb{Z}$ with $j + k \le l$, put*

$$h^{j,k} = \phi_\rho^j \circ \phi_\eta^k(h).$$

Then we have

(i) $h^{j,k} s_\mu = s_\mu h^{j-|\mu|,k}$ *for $\mu \in B_*(\Lambda_\rho)$ with $|\mu| \le j$.*
(ii) $h^{j,k} t_\zeta = t_\zeta h^{j,k-|\zeta|}$ *for $\zeta \in B_*(\Lambda_\eta)$ with $|\zeta| \le k$.*
(iii) $h^{j,k}$ *commutes with any element of $\mathcal{F}_{j,k}^l$.*

Proof. (i) It follows that for $\mu \in B_*(\Lambda_\rho)$ with $|\mu| \le j$

$$h^{j,k} s_\mu = \sum_{|\mu'|=|\mu|} s_{\mu'} \phi_\rho^{j-|\mu|}(\phi_\eta^k(h)) s_{\mu'}^* s_\mu = s_\mu \phi_\rho^{j-|\mu|}(\phi_\eta^k(h)) s_\mu^* s_\mu.$$

Since $h \in \mathcal{A}_l'$ and $\mathcal{A}_{j+k} \subset \mathcal{A}_l$, one has

$$\begin{aligned}
\phi_\rho^{j-|\mu|}(\phi_\eta^k(h)) s_\mu^* s_\mu &= \sum_{\nu \in B_{j-|\mu|}(\Lambda_\rho)} \sum_{\xi \in B_k(\Lambda_\eta)} s_\nu t_\xi h t_\xi^* s_\nu^* s_\mu^* s_\mu \\
&= \sum_{\nu \in B_{j-|\mu|}(\Lambda_\rho)} \sum_{\xi \in B_k(\Lambda_\eta)} s_\nu t_\xi h t_\xi^* s_\nu^* s_\mu^* s_\mu s_\nu t_\xi t_\xi^* s_\nu^* \\
&= \sum_{\nu \in B_{j-|\mu|}(\Lambda_\rho)} \sum_{\xi \in B_k(\Lambda_\eta)} s_\nu t_\xi \eta_\xi(\rho_{\mu\nu}(1)) h t_\xi^* s_\nu^* \\
&= \sum_{\nu \in B_{j-|\mu|}(\Lambda_\rho)} \sum_{\xi \in B_k(\Lambda_\eta)} s_\nu \rho_{\mu\nu}(1) t_\xi h t_\xi^* s_\nu^* \\
&= s_\mu^* s_\mu \phi_\rho^{j-|\mu|}(\phi_\eta^k(h)) = s_\mu^* s_\mu h^{j-|\mu|,k}
\end{aligned}$$

so that $h^{j,k} s_\mu = s_\mu h^{j-|\mu|,k}$.

(ii) Similarly we have $h^{j,k} t_\zeta = t_\zeta h^{j,k-|\zeta|}$ for $\zeta \in B_*(\Lambda_\eta)$ with $|\zeta| \le k$.

(iii) For $x \in \mathcal{A}_l, \mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta)$, we have

$$h^{j,k} s_\mu t_\zeta = s_\mu h^{0,k} t_\zeta = s_\mu t_\zeta h^{0,0} = s_\mu t_\zeta h.$$

It follows that

$$h^{j,k} s_\mu t_\zeta x t_\xi^* s_\nu^* = s_\mu t_\zeta h x t_\xi^* s_\nu^* = s_\mu t_\zeta x h t_\xi^* s_\nu^* = s_\mu t_\zeta x t_\xi^* s_\nu^* h^{j,k}.$$

Hence $h^{j,k}$ commutes with any element of $\mathcal{F}_{j,k}^l$. □

Lemma 5.8. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I). For $x \in \mathcal{P}_{\pi,s,t}$, let $x_0 = \mathcal{E}_{\rho,\eta}(x)$ as in Lemma 5.6. Then we have*

$$\|x_0\| \le \|x\|.$$

Proof. We may assume that the elements for $x \in \mathcal{P}_{\pi,s,t}$

$$x_{-\xi,-\nu},\ x_{\zeta,-\nu},\ x_{\mu,-\xi},\ x_{\mu,\zeta},\ x_{-\xi},\ x_{-\nu},\ x_\mu,\ x_\zeta,\ x_0$$

in Lemma 5.6 belong to $\tilde{\pi}(\mathcal{F}_{j_1,k_1}^{l_1})$ for some j_1, k_1, l_1 and $\mu, \nu \in \cup_{n=0}^{j_0} B_n(\Lambda_\rho)$, $\zeta, \xi \in \cup_{n=0}^{k_0} B_n(\Lambda_\eta)$ for some j_0, k_0. Take $j, k, l \in \mathbb{Z}_+$ such as

$$j \ge j_0 + j_1, \qquad k \ge k_0 + k_1, \qquad l \ge \max\{j + k, l_1\}.$$

By Lemma 5.5, $(\widetilde{\mathcal{A}}, \tilde{\rho}, \tilde{\eta}, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I). For any $\epsilon > 0$, the numbers j, k, l, and the element $x_0 \in \tilde{\pi}(\mathcal{F}^{l_1}_{j_1,k_1})$, one may find

$$g \in \tilde{\pi}(\mathcal{D}_{\rho,\eta}) \cap \pi(\mathcal{A}_l)'$$

with $0 \le g \le 1$ such that:

(i) $\|x_0 \phi^j_\rho \circ \phi^k_\eta(g)\| \ge \|x_0\| - \epsilon$.
(ii) $\phi^n_\rho(g)\phi^m_\eta(g) = \phi^n_\rho(\phi^m_\eta(g))g = \phi^n_\rho(g)g = \phi^m_\eta(g)g = 0$ for all $n = 1, 2, \dots, j$, $m = 1, 2, \dots, k$.

Put $h = g^{\frac{1}{2}}$ and $h^{j,k} = \phi^j_\rho \circ \phi^k_\eta(h)$. It follows that $\|x\| \ge \|h^{j,k} x h^{j,k}\|$ and

$$\|h^{j,k} x h^{j,k}\| = \|(1) + (2) + (3) + (4) + (5) + (6)\|$$

where the summands are given by

$$(1) \qquad \sum_{|\nu|,|\xi| \ge 1} h^{j,k} x_{-\xi,-\nu} t^*_\xi s^*_\nu h^{j,k}$$

$$(2) \qquad \sum_{|\zeta|,|\nu| \ge 1} h^{j,k} t_\zeta x_{\zeta,-\nu} s^*_\nu h^{j,k}$$

$$(3) \qquad \sum_{|\mu|,|\xi| \ge 1} h^{j,k} s_\mu x_{\mu,-\xi} t^*_\xi h^{j,k}$$

$$(4) \qquad \sum_{|\mu|,\zeta| \ge 1} h^{j,k} s_\mu t_\zeta x_{\mu,\zeta} h^{j,k}$$

$$(5) \qquad \sum_{|\xi| \ge 1} h^{j,k} x_{-\xi} t^*_\xi h^{j,k} + \sum_{|\nu| \ge 1} h^{j,k} x_{-\nu} s^*_\nu h^{j,k} + \sum_{|\mu| \ge 1} h^{j,k} s_\mu x_\mu h^{j,k} + \sum_{|\zeta| \ge 1} h^{j,k} t_\zeta x_\zeta h^{j,k}$$

$$(6) \qquad h^{j,k} x_0 h^{j,k}.$$

For (1), as $x_{-\xi,-\nu} \in \tilde{\pi}(\mathcal{F}^{l_1}_{j_1,k_1}) \subset \tilde{\pi}(\mathcal{F}^{l}_{j,k})$, one sees that $x_{-\xi,-\nu}$ commutes with $h^{j,k}$. Hence we have

$$h^{j,k} x_{-\xi,-\nu} t^*_\xi s^*_\nu h^{j,k} = x_{-\xi,-\nu} h^{j,k} t^*_\xi s^*_\nu h^{j,k} = x_{-\xi,-\nu} h^{j,k} h^{j-|\nu|,k-|\xi|} t^*_\xi s^*_\nu$$

and

$$\begin{aligned} h^{j,k} h^{j-|\nu|,k-|\xi|} (h^{j,k} h^{j-|\nu|,k-|\xi|})^* &= \phi^j_\rho(\phi^k_\eta(g)) \cdot \phi^{j-|\nu|}_\rho(\phi^{k-|\xi|}_\eta(g)) \\ &= \phi^{j-|\nu|}_\rho \circ \phi^{k-|\xi|}_\eta(\phi^{|\xi|}_\eta(\phi^{|\nu|}_\rho(g)g)) = 0 \end{aligned}$$

so that

$$h^{j,k} x_{-\xi,-\nu} t^*_\xi s^*_\nu h^{j,k} = 0.$$

For (2), as $x_{\xi,-\nu} \in \tilde{\pi}(\mathcal{F}^{l_1}_{j_1,k_1}) \subset \tilde{\pi}(\mathcal{F}^{l}_{j,k-|\xi|})$, one sees that $x_{\xi,-\nu}$ commutes with $h^{j,k-|\xi|}$. Hence we have

$$h^{j,k} t_\xi x_{\xi,-\nu} s^*_\nu h^{j,k} = t_\xi h^{j,k-|\xi|} x_{\xi,-\nu} h^{j-|\nu|,k} s^*_\nu = t_\xi x_{\xi,-\nu} h^{j,k-|\xi|} h^{j-|\nu|,k} s^*_\nu$$

and

$$\begin{aligned} h^{j,k-|\xi|}h^{j-|\nu|,k}(h^{j,k-|\xi|}h^{j-|\nu|,k})^* =&\phi_\rho^j(\phi_\eta^{k-|\zeta|}(g))\cdot\phi_\rho^{j-|\nu|}(\phi_\eta^k(g))\\ =&\phi_\rho^{j-|\nu|}\circ\phi_\eta^{k-|\zeta|}(\phi_\rho^{|\nu|}(g)\phi_\eta^{|\zeta|}(g))=0\end{aligned}$$

so that

$$h^{j,k}t_\xi x_{\xi,-\nu}s_\nu^* h^{j,k}=0.$$

For (3), as $x_{\mu,-\xi}\in\tilde{\pi}(\mathcal{F}_{j_1,k_1}^{l_1})\subset\tilde{\pi}(\mathcal{F}_{j-|\mu|,k}^{l})$, one sees that $x_{\mu,-\xi}$ commutes with $h^{j-|\mu|,k}$. Hence we have

$$h^{j,k}s_\mu x_{\mu,-\xi}t_\xi^* h^{j,k}=s_\mu h^{j-|\mu|,k}x_{\mu,-\xi}h^{j,k-|\xi|}t_\xi^*=s_\mu x_{\mu,-\xi}h^{j-|\mu|,k}h^{j,k-|\xi|}t_\xi^*$$

and

$$\begin{aligned} h^{j-|\mu|,k}h^{j,k-|\xi|}(h^{j-|\mu|,k}h^{j,k-|\xi|})^* =&\phi_\rho^{j-|\mu|}(\phi_\eta^k(g))\cdot\phi_\rho^j(\phi_\eta^{k-|\xi|}(g))\\ =&\phi_\rho^{j-|\mu|}\circ\phi_\eta^{k-|\xi|}(\phi_\eta^{|\xi|}(g)\phi_\rho^{|\mu|}(g))=0\end{aligned}$$

so that

$$h^{j,k}s_\mu x_{\mu,-\xi}t_\xi^* h^{j,k}=0.$$

For (4), as $x_{\mu,\zeta}\in\tilde{\pi}(\mathcal{F}_{j_1,k_1}^{l_1})\subset\tilde{\pi}(\mathcal{F}_{j-|\mu|,k-|\zeta|}^{l})$, one sees that $x_{\mu,\zeta}$ commutes with $h^{j-|\mu|,k-|\zeta|}$. Hence we have

$$h^{j,k}s_\mu t_\zeta x_{\mu,\zeta}h^{j,k}=s_\mu t_\zeta h^{j-|\mu|,k-|\zeta|}x_{\mu,\zeta}h^{j,k}=s_\mu t_\zeta x_{\mu,\zeta}h^{j-|\mu|,k-|\zeta|}h^{j,k}$$

and

$$\begin{aligned} h^{j-|\mu|,k-|\zeta|}h^{j,k}(h^{j-|\mu|,k-|\zeta|}h^{j,k})^* =&\phi_\rho^{j-|\mu|}(\phi_\eta^{k-|\zeta|}(g))\cdot\phi_\rho^j(\phi_\eta^k(g))\\ =&\phi_\rho^{j-|\mu|}\circ\phi_\eta^{k-|\zeta|}(g\phi_\rho^{|\mu|}(\phi_\eta^{|\zeta|}(g)))=0\end{aligned}$$

so that

$$h^{j,k}s_\mu t_\zeta x_{\mu,\zeta}h^{j,k}=0.$$

For (5), as $x_{-\xi}$ commutes with $h^{j,k}$, we have

$$h^{j,k}x_{-\xi}t_\xi^* h^{j,k}=x_{-\xi}h^{j,k}h^{j,k-|\xi|}t_\xi^*$$

and

$$\begin{aligned} h^{j,k}h^{j,k-|\xi|}(h^{j,k}h^{j,k-|\xi|})^* =&\phi_\rho^j(\phi_\eta^k(g))\cdot\phi_\rho^j(\phi_\eta^{k-|\xi|}(g))\\ =&\phi_\rho^j\circ\phi_\eta^{k-|\xi|}(\phi_\eta^{|\xi|}(g)g)=0\end{aligned}$$

so that

$$h^{j,k}x_{-\xi}t_\xi^* h^{j,k}=0.$$

We similarly see that

$$h^{j,k}x_{-\nu}s_\nu^* h^{j,k}=h^{j,k}s_\mu x_\mu h^{j,k}=h^{j,k}t_\zeta x_\zeta h^{j,k}=0.$$

Therefore we have

$$\|x\|\geq\|h^{j,k}x_0h^{j,k}\|=\|x_0(h^{j,k})^2\|=\|x_0\phi_\rho^j\circ\phi_\eta^k(g)\|\geq\|x_0\|-\epsilon.$$ □

By a similar argument to [8, 2.8 Proposition], one sees:

Corollary 5.9. *Assume $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I). There exists a conditional expectation $\mathcal{E}_{\pi,s,t} : \mathcal{O}_{\pi,s,t} \longrightarrow \mathcal{F}_{\pi,s,t}$ such that*

$$\mathcal{E}_{\pi,s,t} \circ \tilde{\pi} = \tilde{\pi} \circ \mathcal{E}_{\rho,\eta}.$$

Therefore we have

Proposition 5.10. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I). The $*$-homomorphism $\tilde{\pi} : \mathcal{O}^\kappa_{\rho,\eta} \longrightarrow \mathcal{O}_{\pi,s,t}$ defined by*

$$\tilde{\pi}(x) = \pi(x), \quad x \in \mathcal{A}, \qquad \tilde{\pi}(S_\alpha) = s_\alpha, \quad \alpha \in \Sigma^\rho, \qquad \tilde{\pi}(T_a) = t_a, \quad a \in \Sigma^\eta$$

becomes a surjective $*$-isomorphism, and hence the C^*-algebras $\mathcal{O}^\kappa_{\rho,\eta}$ and $\mathcal{O}_{\pi,s,t}$ are canonically $*$-isomorphic through $\tilde{\pi}$.

Proof. The map $\tilde{\pi} : \mathcal{F}_{\rho,\eta} \to \mathcal{F}_{\pi,s,t}$ is $*$-isomorphic and satisfies $\mathcal{E}_{\pi,s,t} \circ \tilde{\pi} = \tilde{\pi} \circ \mathcal{E}_{\rho,\eta}$. Since $\mathcal{E}_{\rho,\eta} : \mathcal{O}^\kappa_{\rho,\eta} \longrightarrow \mathcal{F}_{\rho,\eta}$ is faithful, a routine argument shows that the $*$-homomorphism $\tilde{\pi} : \mathcal{O}^\kappa_{\rho,\eta} \longrightarrow \mathcal{O}_{\pi,s,t}$ is actually a $*$-isomorphism. □

Hence the following uniqueness of the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ holds.

Theorem 5.11. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I). The C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ is the unique C^*-algebra subject to the relation $(\rho, \eta; \kappa)$. This means that if there exist a unital C^*-algebra $\mathcal{B}$, an injective $*$-homomorphism $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ and two families of partial isometries $s_\alpha, \alpha \in \Sigma^\rho$, $t_a, a \in \Sigma^\eta$ satisfying the following relations :*

$$\sum_{\beta \in \Sigma^\rho} s_\beta s_\beta^* = 1, \qquad \pi(x) s_\alpha s_\alpha^* = s_\alpha s_\alpha^* \pi(x), \qquad s_\alpha^* \pi(x) s_\alpha = \pi(\rho_\alpha(x)),$$

$$\sum_{b \in \Sigma^\eta} t_b t_b^* = 1, \qquad \pi(x) t_a t_a^* = t_a t_a^* \pi(x), \qquad t_a^* \pi(x) t_a = \pi(\eta_a(x))$$

$$s_\alpha t_b = t_a s_\beta \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta)$$

for $(\alpha, b) \in \Sigma^{\rho\eta}, (a, \beta) \in \Sigma^{\eta\rho}$ and $x \in \mathcal{A}$, $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, then the correspondence

$$x \in \mathcal{A} \longrightarrow \pi(x) \in \mathcal{B}, \quad S_\alpha \longrightarrow s_\alpha \in \mathcal{B}, \qquad T_a \longrightarrow t_a \in \mathcal{B}$$

extends to a $*$-isomorphism $\tilde{\pi}$ from $\mathcal{O}^\kappa_{\rho,\eta}$ onto the C^*-subalgebra $\mathcal{O}_{\pi,s,t}$ of $\mathcal{B}$ generated by $\pi(x), x \in \mathcal{A}$ and $s_\alpha, \alpha \in \Sigma$, $t_a, a \in \Sigma^\eta$.

For a C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$, let $\lambda_{\rho,\eta} : \mathcal{A} \to \mathcal{A}$ be the positive map on $\mathcal{A}$ defined by

$$\lambda_{\rho,\eta}(x) = \sum_{\alpha \in \Sigma^\rho, a \in \Sigma^\eta} \eta_a \circ \rho_\alpha(x), \qquad x \in \mathcal{A}.$$

Then $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to be *irreducible* if there exists no nontrivial ideal of $\mathcal{A}$ invariant under $\lambda_{\rho,\eta}$.

Corollary 5.12. *If $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I) and is irreducible, the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ is simple.*

Proof. Assume that there exists a nontrivial ideal $\mathcal{I}$ of $\mathcal{O}^{\kappa}_{\rho,\eta}$. Now suppose that $\mathcal{I} \cap \mathcal{A} = \{0\}$. As $S_\alpha^* S_\alpha = \rho_\alpha(1), T_a^* T_a = \eta_a(1) \in \mathcal{A}$, one knows that $S_\alpha, T_a \notin \mathcal{I}$ for all $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$. By the above theorem, the quotient map $q : \mathcal{O}^{\kappa}_{\rho,\eta} \longrightarrow \mathcal{O}^{\kappa}_{\rho,\eta}/\mathcal{I}$ must be injective so that $\mathcal{I}$ is trivial. Hence one sees that $\mathcal{I} \cap \mathcal{A} \neq \{0\}$ and it is invariant under $\lambda_{\rho,\eta}$. □

6. Concrete realization

In this section we will realize the C^*-algebra $\mathcal{O}^{\kappa}_{\rho,\eta}$ for $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ in a concrete way as a C^*-algebra constructed from a Hilbert C^*-bimodule. For $\gamma_i \in \Sigma^\rho \cup \Sigma^\eta$, put

$$\xi_{\gamma_i} = \begin{cases} \rho_{\gamma_i} & \text{if } \gamma_i \in \Sigma^\rho, \\ \eta_{\gamma_i} & \text{if } \gamma_i \in \Sigma^\eta. \end{cases}$$

A finite sequence of labels $(\gamma_1, \gamma_2, \dots, \gamma_k) \in (\Sigma^\rho \cup \Sigma^\eta)^k$ is said to be *concatenated labeled path* if $\xi_{\gamma_k} \circ \cdots \circ \xi_{\gamma_2} \circ \xi_{\gamma_1}(1) \neq 0$. For $m, n \in \mathbb{Z}_+$, let $L_{(n,m)}$ be the set of concatenated labeled paths $(\gamma_1, \gamma_2, \dots, \gamma_{m+n})$ such that symbols in Σ^ρ appear in $(\gamma_1, \gamma_2, \dots, \gamma_{m+n})$ n-times and symbols in Σ^η appear in $(\gamma_1, \gamma_2, \dots, \gamma_{m+n})$ m-times. We define a relation in $L_{(n,m)}$ for $i = 1, 2, \dots, n+m-1$. We write

$$(\gamma_1, \dots, \gamma_{i-1}, \gamma_i, \gamma_{i+1}, \gamma_{i+2}, \dots, \gamma_{m+n}) \underset{i}{\approx} (\gamma_1, \dots, \gamma_{i-1}, \gamma_i', \gamma_{i+1}', \gamma_{i+2}, \dots, \gamma_{m+n})$$

if one of the following two conditions holds:

(1) $(\gamma_i, \gamma_{i+1}) \in \Sigma^{\rho\eta}, (\gamma_i', \gamma_{i+1}') \in \Sigma^{\eta\rho}$ and $\kappa(\gamma_i, \gamma_{i+1}) = (\gamma_i', \gamma_{i+1}')$,

(2) $(\gamma_i, \gamma_{i+1}) \in \Sigma^{\eta\rho}, (\gamma_i', \gamma_{i+1}') \in \Sigma^{\rho\eta}$ and $\kappa(\gamma_i', \gamma_{i+1}') = (\gamma_i, \gamma_{i+1})$.

Denote by $\approx$ the equivalence relation in $L_{(n,m)}$ generated by the relations $\underset{i}{\approx}, i = 1, 2, \dots, n+m-1$. Let $\mathfrak{T}_{(n,m)} = L_{(n,m)}/\approx$ be the set of equivalence classes of $L_{(n,m)}$ under $\approx$. Denote by $[\gamma] \in \mathfrak{T}_{(n,m)}$ the equivalence class of $\gamma \in L_{(n,m)}$. Put the vectors $e = (1,0), f = (0,-1)$ in $\mathbb{R}^2$. Consider the set of all paths consisting of sequences of vectors e, f starting at the point $(-n, m) \in \mathbb{R}^2$ for $n, m \in \mathbb{Z}_+$ and ending at the origin. Such a path consists of n e-vectors and m f-vectors. Let $\mathfrak{P}_{(n,m)}$ be the set of all such paths from $(-n, m)$ to the origin. We consider the correspondence

$$\rho_\alpha \longrightarrow e \quad (\alpha \in \Sigma^\rho), \qquad \eta_a \longrightarrow f \quad (a \in \Sigma^\eta),$$

denoted by π. It extends a surjective map from $L_{(n,m)}$ to $\mathfrak{P}_{(n,m)}$ in a natural way. For a concatenated labeled path $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_{n+m}) \in L_{(n,m)}$, put the projection in $\mathcal{A}$

$$P_\gamma = (\xi_{\gamma_{n+m}} \circ \cdots \circ \xi_{\gamma_2} \circ \xi_{\gamma_1})(1).$$

We note that $P_\gamma \neq 0$ for all $\gamma \in L_{(n,m)}$.

Lemma 6.1. *For $\gamma, \gamma' \in L_{(n,m)}$, if $\gamma \approx \gamma'$, we have $P_\gamma = P_{\gamma'}$. Hence the projection $P_{[\gamma]}$ for $[\gamma] \in \mathfrak{T}_{(n,m)}$ is well-defined.*

Proof. If $\kappa(\alpha, b) = (a, \beta)$, one has $\eta_b \circ \rho_\alpha(1) = \rho_\beta \circ \eta_a(1) \neq 0$. Hence the assertion is obvious. □

Denote by $|\mathfrak{T}_{(n,m)}|$ the cardinal number of the finite set $\mathfrak{T}_{(n,m)}$. Let $e_t, t \in \mathfrak{T}_{(n,m)}$ be the standard complete orthonormal basis of $\mathbb{C}^{|\mathfrak{T}_{(n,m)}|}$. Define

$$H_{(n,m)} = \sum_{t \in \mathfrak{T}_{(n,m)}} {}^{\oplus} \mathbb{C} e_t \otimes P_t \mathcal{A}$$
$$\left(= \sum_{t \in \mathfrak{T}_{(n,m)}} {}^{\oplus} \mathrm{Span}\{ c e_t \otimes P_t x \mid c \in \mathbb{C}, x \in \mathcal{A} \} \right)$$

the direct sum of $\mathbb{C} e_t \otimes P_t \mathcal{A}$ over $t \in \mathfrak{T}_{(n,m)}$. $H_{(n,m)}$ has a structure of C^*-bimodule over $\mathcal{A}$ by setting

$$(e_t \otimes P_t x) y := e_t \otimes P_t x y,$$
$$\phi(y)(e_t \otimes P_t x) := e_t \otimes \xi_\gamma(y) x (= e_t \otimes P_t \xi_\gamma(y) x) \quad \text{for } x, y \in \mathcal{A}$$

where $t = [\gamma]$ for $\gamma = (\gamma_1, \dots, \gamma_{n+m})$ and $\xi_\gamma(y) = (\xi_{\gamma_{n+m}} \circ \cdots \circ \xi_{\gamma_2} \circ \xi_{\gamma_1})(y)$. Define an $\mathcal{A}$-valued inner product on $H_{(n,m)}$ by setting

$$\langle e_t \otimes P_t x \mid e_s \otimes P_s y \rangle := \begin{cases} x^* P_t y & \text{if } t = s, \\ 0 & \text{otherwise} \end{cases}$$

for $t, s \in \mathfrak{T}_{(n,m)}$ and $x, y \in \mathcal{A}$. Then $H_{(n,m)}$ becomes a Hilbert C^*-bimodule over $\mathcal{A}$. Put $H_{(0,0)} = \mathcal{A}$. Denote by F_κ the Hilbert C^*-bimodule over $\mathcal{A}$ defined by the direct sum:

$$F_\kappa = \sum_{(n,m) \in \mathbb{Z}_+^2} {}^{\oplus} H_{(n,m)}.$$

For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, the creation operators s_α, t_a on F_κ :

$$s_\alpha : H_{(n,m)} \longrightarrow H_{(n+1,m)}, \qquad t_a : H_{(n,m)} \longrightarrow H_{(n,m+1)}$$

are defined by

$$s_\alpha x = e_{[\alpha]} \otimes P_{[\alpha]} x, \qquad \text{for } x \in H_{(0,0)} (= \mathcal{A}),$$
$$s_\alpha (e_{[\gamma]} \otimes P_{[\gamma]} x) = \begin{cases} e_{[\alpha\gamma]} \otimes P_{[\alpha\gamma]} x & \text{if } \alpha\gamma \in L_{(n+1,m)}, \\ 0 & \text{otherwise,} \end{cases}$$
$$t_a x = e_{[a]} \otimes P_{[a]} x, \qquad \text{for } x \in H_{(0,0)} (= \mathcal{A}),$$
$$t_a (e_{[\gamma]} \otimes P_{[\gamma]} x) = \begin{cases} e_{[a\gamma]} \otimes P_{[a\gamma]} x & \text{if } a\gamma \in L_{(n,m+1)}, \\ 0 & \text{otherwise.} \end{cases}$$

For $y \in \mathcal{A}$ an operator $i_{F_\kappa}(y)$ on F_κ:

$$i_{F_\kappa}(y) : H_{(n,m)} \longrightarrow H_{(n,m)}$$

is defined by

$$i_{F_\kappa}(y)x = yx \qquad \text{for } x \in H_{(0,0)}(=\mathcal{A}),$$
$$i_{F_\kappa}(y)(e_{[\gamma]} \otimes P_{[\gamma]}x) = \phi(y)(e_{[\gamma]} \otimes P_{[\gamma]}x)(= e_{[\gamma]} \otimes \xi_\gamma(y)x).$$

Define the Cuntz–Toeplitz C^*-algebra for $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ by

$$\mathcal{T}^\kappa_{\rho,\eta} = C^*(s_\alpha, t_a, i_{F_\kappa}(y) \mid \alpha \in \Sigma^\rho, a \in \Sigma^\eta, y \in \mathcal{A})$$

as the C^*-algebra on F_κ generated by $s_\alpha, t_a, i_{F_\kappa}(y)$ for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta, y \in \mathcal{A}$.

Lemma 6.2. *For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, we have*

(i) $s^*_\alpha(e_{[\gamma]} \otimes P_{[\gamma]}x) = \begin{cases} \phi(\rho_\alpha(1))(e_{[\gamma']} \otimes P_{[\gamma']}x) & \text{if } \gamma \approx \alpha\gamma', \\ 0 & \text{otherwise.} \end{cases}$

(ii) $t^*_a(e_{[\gamma]} \otimes P_{[\gamma]}x) = \begin{cases} \phi(\eta_a(1))(e_{[\gamma']} \otimes P_{[\gamma']}x) & \text{if } \gamma \approx a\gamma', \\ 0 & \text{otherwise.} \end{cases}$

Proof. (i) For $\gamma \in L_{(n,m)}, \gamma' \in L_{(n-1,m)}$ and $\alpha \in \Sigma^\rho$, we have

$$\begin{aligned}\langle s^*_\alpha(e_{[\gamma]} \otimes P_{[\gamma]}x) \mid e_{[\gamma']} \otimes P_{[\gamma']}x'\rangle &= \langle e_{[\gamma]} \otimes P_{[\gamma]}x \mid e_{[\alpha\gamma']} \otimes P_{[\alpha\gamma']}x'\rangle \\ &= \begin{cases} x^*P_{[\alpha\gamma']}x & \text{if } \gamma \approx \alpha\gamma', \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

On the other hand,

$$\phi(\rho_\alpha(1))(e_{[\gamma']} \otimes P_{[\gamma']}x) = e_{[\gamma']} \otimes P_{[\alpha\gamma']}P_{\gamma'}x = e_{[\gamma']} \otimes P_{[\alpha\gamma']}x$$

so that

$$\langle \phi(\rho_\alpha(1))(e_{[\gamma']} \otimes P_{[\gamma']}x) \mid e_{[\gamma']} \otimes P_{[\gamma']}x'\rangle = x^*P_{[\alpha\gamma']}x'.$$

Hence we obtain the desired equality. Similarly we see (ii). □

The following lemma is straightforward.

Lemma 6.3. *For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and $\gamma \in L_{(n,m)}, x \in \mathcal{A}$, we have:*

(i)

$$s_\alpha s^*_\alpha(e_{[\gamma]} \otimes P_{[\gamma]}x) = \begin{cases} e_{[\gamma]} \otimes P_{[\gamma]}x) & \text{if } \gamma \approx \alpha\gamma' \text{ for some } \gamma' \in L_{(n-1,m)}, \\ 0 & \text{otherwise.} \end{cases}$$

(ii)

$$t_a t^*_a(e_{[\gamma]} \otimes P_{[\gamma]}x) = \begin{cases} e_{[\gamma]} \otimes P_{[\gamma]}x) & \text{if } \gamma \approx a\gamma' \text{ for some } \gamma' \in L_{(n,m-1)}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence we see:

Lemma 6.4.

(i) $1 - \sum_{\alpha \in \Sigma^\rho} s_\alpha s^*_\alpha =$ *the projection onto the subspace spanned by the vectors* $e_{[\gamma]} \otimes P_{[\gamma]}x$ *for* $\gamma \in \cup_{m=0}^\infty L_{(0,m)}, x \in \mathcal{A}$.

(ii) $1 - \sum_{a\in\Sigma^\eta} t_a t_a^*$ = *the projection onto the subspace spanned by the vectors* $e_{[\gamma]} \otimes P_{[\gamma]}x$ *for* $\gamma \in \cup_{n=0}^\infty L_{(n,0)}, x \in \mathcal{A}$.

Lemma 6.5. *For* $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ *and* $x \in \mathcal{A}$, *we have:*

(i) $s_\alpha^* x s_\alpha = \phi(\rho_\alpha(x))$ *and in particular* $s_\alpha^* s_\alpha = \phi(\rho_\alpha(1))$.

(ii) $t_a^* x t_a = \phi(\eta_a(x))$ *and in particular* $t_a^* t_a = \phi(\eta_a(1))$.

Proof. (i) It follows that for $\gamma \in L(n,m)$ with $\alpha\gamma \in L(n+1,m)$ and $y \in \mathcal{A}$,

$$\begin{aligned} s_\alpha^* x s_\alpha(e_{[\gamma]} \otimes P_{[\gamma]}y) &= s_\alpha^*(e_{[\alpha\gamma]} \otimes P_{[\alpha\gamma]}y\xi_{\alpha\gamma}(x)) \\ &= e_{[\gamma]} \otimes P_{[\gamma]}y\xi_\gamma(\rho_\alpha(x)) \\ &= \phi(\rho_\alpha(x))(e_{[\gamma]} \otimes P_{[\gamma]}y). \end{aligned}$$

If $\alpha\gamma \notin L(n+1,m)$, we have

$$s_\alpha(e_{[\gamma]} \otimes P_{[\gamma]}y) = 0, \qquad \phi(\rho_\alpha(x))(e_{[\gamma]} \otimes P_{[\gamma]}y) = 0.$$

Hence we see that $s_\alpha^* x s_\alpha = \phi(\rho_\alpha(x))$. Similarly we see (ii). □

Lemma 6.6. *For* $\alpha, \beta \in \Sigma^\rho$, $a, b \in \Sigma^\eta$ *we have:*

$$s_\alpha t_b = t_a s_\beta \qquad \textit{if } \kappa(\alpha, b) = (a, \beta). \tag{6.1}$$

Proof. For $\gamma \in L_{(n,m)}$ with $\alpha b\gamma, a\beta\gamma \in L_{(n+1,m+1)}$ and $x \in \mathcal{A}$, we have

$$\begin{aligned} s_\alpha t_b(e_{[\gamma]} \otimes P_{[\gamma]}x) &= e_{[\alpha b\gamma]} \otimes P_{[\alpha b\gamma]}y), \\ t_a s_\beta(e_{[\gamma]} \otimes P_{[\gamma]}x) &= (e_{[a\beta\gamma]} \otimes P_{[a\beta\gamma]}x). \end{aligned}$$

Since $\kappa(\alpha, b) = (a, \beta)$, the condition $\alpha b\gamma \in L_{(n+1,m+1)}$ is equivalent to the condition $a\beta\gamma \in L_{(n+1,m+1)}$. We then have $[\alpha b\gamma] = [a\beta\gamma]$ and $P_{[\alpha b\gamma]} = P_{[a\beta\gamma]}$. □

Let $\mathcal{I}_{\rho,\eta}^\kappa$ be the ideal of $\mathcal{T}_{\rho,\eta}^\kappa$ generated by the two projections:

$$1 - \sum_{\alpha\in\Sigma^\rho} s_\alpha s_\alpha^* \quad \text{and} \quad 1 - \sum_{a\in\Sigma^\eta} t_a t_a^*.$$

Let $\widehat{\mathcal{O}}_{\rho,\eta}^\kappa$ be the quotient C^*-algebra

$$\widehat{\mathcal{O}}_{\rho,\eta}^\kappa = \mathcal{T}_{\rho,\eta}^\kappa / \mathcal{I}_{\rho,\eta}^\kappa.$$

Let $\pi_{\rho,\eta} : \mathcal{T}_{\rho,\eta}^\kappa \longrightarrow \widehat{\mathcal{O}}_{\rho,\eta}^\kappa$ be the quotient map. Put

$$\widehat{S}_\alpha = \pi_{\rho,\eta}(s_\alpha), \quad \widehat{T}_a = \pi_{\rho,\eta}(t_a), \quad \hat{i}(x) = \pi_{\rho,\eta}(i_{(F_\kappa)})(x)$$

for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and $x \in \mathcal{A}$. By the above discussions, the following relations hold:

$$\sum_{\beta\in\Sigma^\rho} \widehat{S}_\beta \widehat{S}_\beta^* = 1, \qquad \hat{i}(x)\widehat{S}_\alpha\widehat{S}_\alpha^* = \widehat{S}_\alpha\widehat{S}_\alpha^*\hat{i}(x), \qquad \widehat{S}_\alpha^*\hat{i}(x)\widehat{S}_\alpha = \hat{i}(\rho_\alpha(x)),$$

$$\sum_{b\in\Sigma^\eta} \widehat{T}_b \widehat{T}_b^* = 1, \qquad \hat{i}(x)\widehat{T}_a\widehat{T}_a^* = \widehat{T}_a\widehat{T}_a^*\hat{i}(x), \qquad \widehat{T}_a^*\hat{i}(x)\widehat{T}_a = \hat{i}(\eta_a(x)),$$

$$\widehat{S}_\alpha \widehat{T}_b = \widehat{T}_a \widehat{S}_\beta \quad \text{if } \kappa(\alpha, b) = (a, \beta)$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$.

For $(z,w) \in \mathbb{T}^2$, the correspondence

$$e_{[\gamma]} \otimes P_{[\gamma]}x \in H_{(n,m)} \longrightarrow z^n w^m e_{[\gamma]} \otimes P_{[\gamma]}x \in H_{(n,m)}$$

yields a unitary representation of $\mathbb{T}^2$ on $H_{(n,m)}$, which extends to F_κ, denoted by $u_{(z,w)}$. Since

$$u_{(z,w)}\mathcal{T}^\kappa_{\rho,\eta}u^*_{(z,w)} = \mathcal{T}^\kappa_{\rho,\eta}, \qquad u_{(z,w)}\mathcal{I}^\kappa_{\rho,\eta}u^*_{(z,w)} = \mathcal{I}^\kappa_{\rho,\eta},$$

The map

$$X \in \mathcal{T}^\kappa_{\rho,\eta} \longrightarrow u_{(z,w)}Xu^*_{(z,w)} \in \mathcal{T}^\kappa_{\rho,\eta}$$

yields an action of $\mathbb{T}^2$ on the C^*-algebra $\widehat{\mathcal{O}}^\kappa_{\rho,\eta}$, which we denote by $\widehat{\theta}$. Similarly to the action θ on $\mathcal{O}^\kappa_{\rho,\eta}$, we may define the conditional expectation $\widehat{\mathcal{E}}_{\rho,\eta}$ from $\widehat{\mathcal{O}}^\kappa_{\rho,\eta}$ to the fixed point algebra $(\widehat{\mathcal{O}}^\kappa_{\rho,\eta})^{\widehat{\theta}}$ by taking the integration of the function $\widehat{\theta}_{(z,w)}(X)$ over $(z,w) \in \mathbb{T}^2$ for $X \in \widehat{\mathcal{O}}^\kappa_{\rho,\eta}$. Then as in the proof of Proposition 5.10, one may prove the following theorem.

Theorem 6.7. *The algebra $\widehat{\mathcal{O}}^\kappa_{\rho,\eta}$ is canonically $*$-isomorphic to the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ through the correspondences:*

$$S_\alpha \longrightarrow \widehat{S}_\alpha, \qquad T_a \longrightarrow \widehat{T}_a, \qquad x \longrightarrow \hat{i}(x)$$

for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and $x \in \mathcal{A}$.

7. K-Theory machinery

Let us denote by $\mathcal{K}$ the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space. For a C^*-algebra $\mathcal{B}$, we denote by $M(\mathcal{B})$ its multiplier algebra. In this section, we will study K-theory groups $K_*(\mathcal{O}^\kappa_{\rho,\eta})$ for the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$. We fix a C^*-textile dynamical system $(\mathcal{A},\rho,\eta,\Sigma^\rho,\Sigma^\eta,\kappa)$. We define two actions

$$\hat{\rho} : \mathbb{T} \longrightarrow \mathrm{Aut}(\mathcal{O}^\kappa_{\rho,\eta}), \quad \hat{\eta} : \mathbb{T} \longrightarrow \mathrm{Aut}(\mathcal{O}^\kappa_{\rho,\eta})$$

of the circle group $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ to $\mathcal{O}^\kappa_{\rho,\eta}$ by setting

$$\hat{\rho}_z = \theta_{(z,1)}, \qquad \hat{\eta}_w = \theta_{(1,w)}, \qquad z,w \in \mathbb{T}.$$

They satisfy

$$\hat{\rho}_z \circ \hat{\eta}_w = \hat{\eta}_w \circ \hat{\rho}_z = \theta_{(z,w)}, \qquad z,w \in \mathbb{T}.$$

Set the fixed point algebras

$$(\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}} = \{x \in \mathcal{O}^\kappa_{\rho,\eta} \mid \hat{\rho}_z(x) = x \text{ for all } z \in \mathbb{T}\},$$
$$(\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\eta}} = \{x \in \mathcal{O}^\kappa_{\rho,\eta} \mid \hat{\eta}_w(x) = x \text{ for all } w \in \mathbb{T}\}.$$

For $x \in (\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}$, define the $\mathcal{O}^\kappa_{\rho,\eta}$-valued constant function

$$\widehat{x} \in L^1(\mathbb{T},\mathcal{O}^\kappa_{\rho,\eta}) \subset \mathcal{O}^\kappa_{\rho,\eta} \times_{\hat{\rho}} \mathbb{T}$$

from $\mathbb{T}$ by setting $\widehat{x}(z) = x, z \in \mathbb{T}$. Put $p_0 = \widehat{1}$. By [45], the algebra $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}$ is canonically isomorphic to $p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0$ through the map

$$j_\rho : x \in (\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \longrightarrow \widehat{x} \in p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0$$

which induces an isomorphism

$$j_{\rho*} : K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \longrightarrow K_i(p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0), \qquad i = 0, 1 \tag{7.1}$$

on their K-groups. By a similar manner to the proofs given in [23, Section 4], one may prove the following lemma.

Lemma 7.1.

(i) *There exists an isometry*

$$v \in M((\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \otimes \mathcal{K})$$

such that $vv^* = p_0 \otimes 1, v^*v = 1$.

(ii) $\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}$ *is stably isomorphic to* $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}$, *and similarly* $\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\eta} \mathbb{T}$ *is stably isomorphic to* $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat\eta}$.

(iii) *The inclusion* $\iota_{\hat\rho} : p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0 \hookrightarrow \mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}$ *induces an isomorphism*

$$\iota_{\hat\rho*} : K_i(p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0) \cong K_i(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}), \qquad i = 0, 1$$

on their K-groups.

Thanks to the lemma above, the isomorphism

$$\mathrm{Ad}(v^*) : x \in p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0 \otimes \mathcal{K} \longrightarrow v^*xv \in (\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \otimes \mathcal{K}$$

induces isomorphisms

$$\mathrm{Ad}(v^*)_* : K_i(p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0) \longrightarrow K_i(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}), \qquad i = 0, 1. \tag{7.2}$$

Let $\hat{\hat\rho}$ be the automorphism on $\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}$ for the positive generator of $\mathbb{Z}$ for the dual action of $\hat\rho$. By (7.1) and (7.2), we may define an isomorphism

$$\beta_{\rho,i} = j_{\rho*}^{-1} \circ \mathrm{Ad}(v^*)_*^{-1} \circ \hat{\hat\rho}_* \circ \mathrm{Ad}(v^*)_* \circ j_{\rho*} : K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \longrightarrow K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})$$

for $i = 0, 1$, so that the diagram is commutative:

$$\begin{array}{ccc}
K_i(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) & \xrightarrow{\hat{\hat\rho}_*} & K_i(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \\
\uparrow{\scriptstyle \mathrm{Ad}(v^*)_*} & & \uparrow{\scriptstyle \mathrm{Ad}(v^*)_*} \\
K_i(p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0) & & K_i(p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0) \\
\uparrow{\scriptstyle j_{\rho*}} & & \uparrow{\scriptstyle j_{\rho*}} \\
K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) & \xrightarrow{\beta_{\rho,i}} & K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}).
\end{array}$$

By [39] (cf. [15]), one has the six term exact sequence of K-theory:

$$\begin{array}{ccccc}
K_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) & \xrightarrow{\mathrm{id}-\hat{\hat\rho}_*} & K_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) & \xrightarrow{\iota_*} & K_0((\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \times_{\hat{\hat\rho}} \mathbb{Z}) \\
\delta \uparrow & & & & \downarrow \exp \\
K_1((\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \times_{\hat{\hat\rho}} \mathbb{Z}) & \xleftarrow[\iota_*]{} & K_1(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) & \xleftarrow[\mathrm{id}-\hat{\hat\rho}_*]{} & K_1(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}).
\end{array}$$

Since $(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \times_{\hat{\hat\rho}} \mathbb{Z} \cong \mathcal{O}^\kappa_{\rho,\eta} \otimes \mathcal{K}$ and $K_i(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \cong K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})$, one has:

Lemma 7.2. *The following six term exact sequence of K-theory holds:*

$$\begin{array}{ccccc}
K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) & \xrightarrow{\mathrm{id}-\beta_{\rho,0}} & K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) & \xrightarrow{\iota_*} & K_0(\mathcal{O}^\kappa_{\rho,\eta}) \\
\delta \uparrow & & & & \downarrow \exp \\
K_1(\mathcal{O}^\kappa_{\rho,\eta}) & \xleftarrow[\iota_*]{} & K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) & \xleftarrow[\mathrm{id}-\beta_{\rho,1}]{} & K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}).
\end{array}$$

Hence there exist short exact sequences for $i = 0, 1$:

$$\begin{aligned}
0 &\longrightarrow \mathrm{Coker}(\mathrm{id} - \beta_{\rho,i}) \textit{ in } K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\
&\longrightarrow K_i(\mathcal{O}^\kappa_{\rho,\eta}) \\
&\longrightarrow \mathrm{Ker}(\mathrm{id} - \beta_{\rho,i+1}) \textit{ in } K_{i+1}((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\
&\longrightarrow 0.
\end{aligned}$$

In the rest of this section, we will study the groups

$$\mathrm{Coker}(\mathrm{id} - \beta_{\rho,i}) \text{ in } K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}), \qquad \mathrm{Ker}(\mathrm{id} - \beta_{\rho,i+1}) \text{ in } K_{i+1}((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}).$$

The action $\hat\eta$ acts on the subalgebra $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}$, which we still denote by $\hat\eta$. Then the fixed point algebra $((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^{\hat\eta}$ of $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}$ under $\hat\eta$ coincides with $\mathcal{F}_{\rho,\eta}$. The above discussions for the action $\hat\rho : \mathbb{T} \longrightarrow \mathcal{O}^\kappa_{\rho,\eta}$ works for the action $\hat\eta : \mathbb{T} \longrightarrow (\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}$ as in the following way. For $y \in ((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^{\hat\eta}$, define the constant function $\widehat{y} \in L^1(\mathbb{T}, (\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \subset (\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}$ by setting $\widehat{y}(w) = y, w \in \mathbb{T}$. Putting $q_0 = \widehat{1}$, the algebra $((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^{\hat\eta}$ is canonically isomorphic to $q_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0$ through the map

$$j^\rho_\eta : y \in ((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^{\hat\eta} \longrightarrow \hat{y} \in q_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0$$

which induces an isomorphism

$$j^\rho_{\eta *} : K_i(((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^{\hat\eta}) \longrightarrow K_i(q_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0), \qquad i = 0, 1$$

on their K-groups. Similarly to Lemma 7.1, we have:

Lemma 7.3.

(i) *There exists an isometry*

$$u \in M(((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}) \otimes \mathcal{K})$$

such that $uu^* = q_0 \otimes 1, u^*u = 1$.

(ii) $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}$ *is stably isomorphic to* $((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^{\hat\eta}$.

(iii) *The inclusion*

$$\iota^{\hat\rho}_{\hat\eta} : q_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0 (= ((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^{\hat\eta} = \mathcal{F}_{\rho,\eta}) \hookrightarrow (\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}$$

induces an isomorphism

$$\iota^{\hat\rho}_{\hat\eta *} : K_i(q_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0) \cong K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}), \qquad i = 0,1$$

on their K-groups.

The isomorphism

$$\mathrm{Ad}(u^*) : y \in q_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0 \longrightarrow u^* y u \in (\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}$$

induces isomorphisms

$$\mathrm{Ad}(u^*)_* : K_i(q_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0) \cong K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}), \qquad i = 0,1.$$

Let $\hat{\hat\eta}_\rho$ be the automorphism on $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}$ for the positive generator of $\mathbb{Z}$ for the dual action of $\hat\eta$. Define an isomorphism

$$\gamma_{\eta,i} = j^{\rho-1}_{\eta *} \circ \mathrm{Ad}(u^*)^{-1}_* \circ \hat{\hat\eta}_{\rho *} \circ \mathrm{Ad}(u^*)_* \circ j^\rho_{\eta *} : K_i(\mathcal{F}_{\rho,\eta}) \longrightarrow K_i(\mathcal{F}_{\rho,\eta}), \qquad i = 0,1$$

such that the diagram is commutative for $i = 0,1$:

$$\begin{array}{ccc}
K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}) & \xrightarrow{\hat{\hat\eta}_{\rho *}} & K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}) \\
\uparrow \mathrm{Ad}(u^*)_* & & \uparrow \mathrm{Ad}(u^*)_* \\
K_i(q_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0) & & K_i(q_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0) \\
\uparrow j^\rho_{\eta *} & & \uparrow j^\rho_{\eta *} \\
K_i(((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^{\hat\eta}) & & K_i(((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^{\hat\eta}) \\
\| & & \| \\
K_i(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\gamma_{\eta,i}} & K_i(\mathcal{F}_{\rho,\eta}).
\end{array}$$

We similarly define an endomorphism $\gamma_{\rho,i} : K_i(\mathcal{F}_{\rho,\eta}) \longrightarrow K_i(\mathcal{F}_{\rho,\eta})$ by exchanging the rôles of ρ and η.

Under the equality $((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^{\hat\eta} = \mathcal{F}_{\rho,\eta}$, we have the following lemma which is similar to Lemma 7.2

Lemma 7.4. *The following six term exact sequence of K-theory holds:*

$$\begin{array}{ccccc}
K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\mathrm{id}-\gamma_{\eta,0}} & K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\iota_*} & K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\
\delta \uparrow & & & & \downarrow \exp \\
K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) & \xleftarrow[\iota_*]{} & K_1(\mathcal{F}_{\rho,\eta}) & \xleftarrow[\mathrm{id}-\gamma_{\eta,1}]{} & K_1(\mathcal{F}_{\rho,\eta}).
\end{array}$$

In particular, if $K_1(\mathcal{F}_{\rho,\eta}) = 0$*, we have*

$$K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}) = \mathrm{Coker}(\mathrm{id} - \gamma_{\eta,0}) \quad \textit{in } K_0(\mathcal{F}_{\rho,\eta}),$$
$$K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}) = \mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \quad \textit{in } K_0(\mathcal{F}_{\rho,\eta}).$$

Denote by $M_n(\mathcal{B})$ the $n \times n$ matrix algebra over a C^*-algebra $\mathcal{B}$, which is identified with the tensor product $\mathcal{B} \otimes M_n(\mathbb{C})$. The following lemmas hold.

Lemma 7.5. *For a projection* $q \in M_n((\mathcal{O}^\kappa_{\rho,\eta})^\rho)$ *and a partial isometry* $S \in \mathcal{O}^\kappa_{\rho,\eta}$ *such that*

$$\hat{\rho}_z(S) = zS \quad \textit{for } z \in \mathbb{T}, \qquad q(SS^* \otimes 1_n) = (SS^* \otimes 1_n)q,$$

we have

$$\beta^{-1}_{\rho,0}([(SS^* \otimes 1_n)q]) = [(S^* \otimes 1_n)q(S \otimes 1_n)] \quad \textit{in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}).$$

Proof. As q commutes with $SS^* \otimes 1_n$, $p = (S^* \otimes 1_n)q(S \otimes 1_n)$ is a projection in $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}$. Since $p \le S^*S \otimes 1_n$, By a similar argument to the proof of [23, Lemma 4.5], one sees that $\beta_{\rho,0}([p]) = [(S \otimes 1_n)p(S^* \otimes 1_n)]$ in $K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}})$. □

Lemma 7.6.

(i) *For a projection* $q \in M_n(\mathcal{F}_{\rho,\eta})$ *and a partial isometry* $T \in (\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}$ *such that*

$$\hat{\eta}_w(T) = wT \quad \textit{for } w \in \mathbb{T}, \qquad q(TT^* \otimes 1_n) = (TT^* \otimes 1_n)q,$$

we have

$$\gamma^{-1}_{\eta,0}([(TT^* \otimes 1_n)q]) = [(T^* \otimes 1_n)q(T \otimes 1_n)] \quad \textit{in } K_0(\mathcal{F}_{\rho,\eta}).$$

(ii) *For a projection* $q \in M_n(\mathcal{F}_{\rho,\eta})$ *and a partial isometry* $S \in (\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\eta}}$ *such that*

$$\hat{\rho}_z(S) = zS \quad \textit{for } z \in \mathbb{T}, \qquad q(SS^* \otimes 1_n) = (SS^* \otimes 1_n)q,$$

we have

$$\gamma^{-1}_{\rho,0}([(SS^* \otimes 1_n)q]) = [(S^* \otimes 1_n)q(S \otimes 1_n)] \quad \textit{in } K_0(\mathcal{F}_{\rho,\eta}).$$

Hence we have

Lemma 7.7. *The diagram*

$$\begin{array}{ccc} K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\mathrm{id}-\gamma_{\rho,0}} & K_0(\mathcal{F}_{\rho,\eta}) \\ \big\downarrow \iota_* & & \big\downarrow \iota_* \\ K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}) & \xrightarrow{\mathrm{id}-\beta_{\rho,0}} & K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}) \end{array}$$

is commutative.

Proof. By [35, Proposition 3.3], the map $\iota_* : K_0(\mathcal{F}_{\rho,\eta}) \longrightarrow K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}})$ is induced by the natural inclusion $\mathcal{F}_{\rho,\eta}(= ((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}})^\eta) \hookrightarrow (\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}$. For an element $[q] \in K_0(\mathcal{F}_{\rho,\eta})$ one may assume that $q \in M_n(\mathcal{F}_{\rho,\eta})$ for some $n \in \mathbb{N}$ so that one has

$$\begin{aligned}\gamma_{\rho,0}^{-1}([q]) &= \sum_{\alpha\in\Sigma^\rho} [(S_\alpha S_\alpha^* \otimes 1_n)q] \\ &= \sum_{\alpha\in\Sigma^\rho} [(S_\alpha^* \otimes 1_n)q(S_\alpha \otimes 1_n)] \\ &= \sum_{\alpha\in\Sigma^\rho} \beta_{\rho,0}^{-1}([q(S_\alpha S_\alpha^* \otimes 1_n)]) = \beta_{\rho,0}^{-1}([q])\end{aligned}$$

so that $\beta_{\rho,0}|_{K_0(\mathcal{F}_{\rho,\eta})} = \gamma_{\rho,0}$. $\square$

In the rest of this section, we assume that $K_1(\mathcal{F}_{\rho,\eta}) = 0$. The following lemma is crucial in our further discussions.

Lemma 7.8. *In the six term exact sequence in Lemma 7.4 with $K_1(\mathcal{F}_{\rho,\eta}) = 0$, we have the following commutative diagrams:*

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}) & \xrightarrow{\mathrm{id}-\beta_{\rho,1}} & K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}) \\ \delta\downarrow & & \delta\downarrow \\ K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\mathrm{id}-\gamma_{\rho,0}} & K_0(\mathcal{F}_{\rho,\eta}) \\ \mathrm{id}-\gamma_{\eta,0}\downarrow & & \mathrm{id}-\gamma_{\eta,0}\downarrow \\ K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\mathrm{id}-\gamma_{\rho,0}} & K_0(\mathcal{F}_{\rho,\eta}) \\ \iota_*\downarrow & & \iota_*\downarrow \\ K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}) & \xrightarrow{\mathrm{id}-\beta_{\rho,0}} & K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array} \tag{7.3}$$

Proof. It is well-known that δ-map is functorial (see [48, Theorem 7.2.5], [4, p.266 (LX)]). Hence the diagram of the upper square

$$\begin{array}{ccc} K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}) & \xrightarrow{\mathrm{id}-\beta_{\rho,1}} & K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}) \\ \delta\downarrow & & \delta\downarrow \\ K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\mathrm{id}-\gamma_{\rho,0}} & K_0(\mathcal{F}_{\rho,\eta}) \end{array}$$

is commutative. Since $\gamma_{\rho,0} \circ \gamma_{\eta,0} = \gamma_{\eta,0} \circ \gamma_{\rho,0}$, the diagram of the middle square

$$
\begin{array}{ccc}
K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\mathrm{id}-\gamma_{\rho,0}} & K_0(\mathcal{F}_{\rho,\eta}) \\
\big\downarrow {\scriptstyle \mathrm{id}-\gamma_{\eta,0}} & & \big\downarrow {\scriptstyle \mathrm{id}-\gamma_{\eta,0}} \\
K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\mathrm{id}-\gamma_{\rho,0}} & K_0(\mathcal{F}_{\rho,\eta})
\end{array}
\tag{7.4}
$$

is commutative. The commutativity of the lower square comes from the preceding lemma. □

We will describe the K-groups $K_*(\mathcal{O}^{\kappa}_{\rho,\eta})$ in terms of the kernels and cokernels of the homomorphisms $\mathrm{id}-\gamma_{\rho,0}$ and $\mathrm{id}-\gamma_{\eta,0}$ on $K_0(\mathcal{F}_{\rho,\eta})$. Recall that there exist two short exact sequences by Lemma 7.2:

$$
\begin{aligned}
0 &\longrightarrow \mathrm{Coker}(\mathrm{id}-\beta_{\rho,0}) \text{ in } K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \\
&\longrightarrow K_0(\mathcal{O}^{\kappa}_{\rho,\eta}) \\
&\longrightarrow \mathrm{Ker}(\mathrm{id}-\beta_{\rho,1}) \text{ in } K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \\
&\longrightarrow 0
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\longrightarrow \mathrm{Coker}(\mathrm{id}-\beta_{\rho,1}) \text{ in } K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \\
&\longrightarrow K_1(\mathcal{O}^{\kappa}_{\rho,\eta}) \\
&\longrightarrow \mathrm{Ker}(\mathrm{id}-\beta_{\rho,0}) \text{ in } K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \\
&\longrightarrow 0.
\end{aligned}
$$

As $\gamma_{\eta,0} \circ \gamma_{\rho,0} = \gamma_{\rho,0} \circ \gamma_{\eta,0}$ on $K_0(\mathcal{F}_{\rho,\eta})$, the homomorphisms $\gamma_{\rho,0}$ and $\gamma_{\eta,0}$ naturally act on $\mathrm{Coker}(\mathrm{id}-\gamma_{\eta,0}) = K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})$ and $\mathrm{Coker}(\mathrm{id}-\gamma_{\rho,0}) = K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})$ as endomorphisms respectively, which we denote by $\bar{\gamma}_{\rho,0}$ and $\bar{\gamma}_{\eta,0}$ respectively.

Lemma 7.9.

(i) *For $K_0(\mathcal{O}^{\kappa}_{\rho,\eta})$, we have*

$$
\begin{aligned}
&\mathrm{Coker}(\mathrm{id}-\beta_{\rho,0}) \textit{ in } K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \\
&\cong \mathrm{Coker}(\mathrm{id}-\bar{\gamma}_{\rho,0}) \textit{ in } K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}) \\
&\cong K_0(\mathcal{F}_{\rho,\eta})/((\mathrm{id}-\gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}))
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{Ker}(\mathrm{id}-\beta_{\rho,1}) \textit{ in } K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \\
&\cong \mathrm{Ker}(\mathrm{id}-\gamma_{\rho,0}) \textit{ in } (\mathrm{Ker}(\mathrm{id}-\gamma_{\eta,0}) \textit{ in } K_0(\mathcal{F}_{\rho,\eta})) \\
&\cong \mathrm{Ker}(\mathrm{id}-\gamma_{\rho,0}) \cap \mathrm{Ker}(\mathrm{id}-\gamma_{\eta,0}) \textit{ in } K_0(\mathcal{F}_{\rho,\eta}).
\end{aligned}
$$

(ii) *For $K_1(\mathcal{O}^\kappa_{\rho,\eta})$, we have*

$$\begin{aligned}&\operatorname{Coker}(\mathrm{id}-\beta_{\rho,1}) \text{ in } K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})\\ &\cong (\operatorname{Ker}(\mathrm{id}-\gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}))/(\mathrm{id}-\gamma_{\rho,0})(\operatorname{Ker}(\mathrm{id}-\gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}))\end{aligned}$$

and

$$\begin{aligned}&\operatorname{Ker}(\mathrm{id}-\beta_{\rho,0}) \text{ in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})\\ &\cong \operatorname{Ker}(\mathrm{id}-\bar\gamma_{\rho,0}) \text{ in } (K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})).\end{aligned}$$

Proof. (i) We will first prove the assertions for the group

$$\operatorname{Coker}(\mathrm{id}-\beta_{\rho,0}) \text{ in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}).$$

In the diagram (7.3), the exactness of the vertical arrows implies that ι_* is surjective so that

$$K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \cong \iota_*(K_0(\mathcal{F}_{\rho,\eta})) \cong K_0(\mathcal{F}_{\rho,\eta})/\operatorname{Ker}(\mathrm{id}-\gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}).$$

By the commutativity in the lower square in the diagram (7.3), one has

$$\begin{aligned}&\operatorname{Coker}(\mathrm{id}-\beta_{\rho,0}) \text{ in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})\\ &\cong \operatorname{Coker}(\mathrm{id}-\bar\gamma_{\rho,0}) \text{ in } (\operatorname{Coker}(\mathrm{id}-\gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}).)\end{aligned}$$

The latter group will be proved to be isomorphic to the group

$$K_0(\mathcal{F}_{\rho,\eta})/((\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})) + (\mathrm{id}-\gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})).$$

Put $H_{\rho,\eta} = (\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id}-\gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})$ the subgroup of $K_0(\mathcal{F}_{\rho,\eta})$ generated by $(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})$ and $(\mathrm{id}-\gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})$. Set the quotient maps

$$\begin{aligned}K_0(\mathcal{F}_{\rho,\eta}) &\xrightarrow{q_\eta} \quad K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})\\ &\xrightarrow{q_{(\mathrm{id}-\gamma_{\rho,0})}} \operatorname{Coker}(\mathrm{id}-\bar\gamma_{\rho,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})\end{aligned}$$

and

$$\begin{aligned}\Phi = q_{(\mathrm{id}-\gamma_{\rho,0})} \circ q_\eta : &K_0(\mathcal{F}_{\rho,\eta})\\ &\longrightarrow \operatorname{Coker}(\mathrm{id}-\bar\gamma_{\rho,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}).\end{aligned}$$

It suffices to show the equality $\operatorname{Ker}(\Phi) = H_{\rho,\eta}$. As $(\mathrm{id}-\gamma_{\rho,0})$ commutes with $(\mathrm{id}-\gamma_{\eta,0})$, one has

$$(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}) \subset \operatorname{Ker}(\Phi), \qquad (\mathrm{id}-\gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta}) \subset \operatorname{Ker}(\Phi).$$

Hence we have $H_{\rho,\eta} \subset \operatorname{Ker}(\Phi)$. On the other hand, for $g \in \operatorname{Ker}(\Phi)$, we have $g \in (\mathrm{id}-\bar\gamma_{\rho,0})(K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}))$ so that $g = (\mathrm{id}-\gamma_{\rho,0})[h]$ for some $[h] \in K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})$. Hence

$$g = (\mathrm{id}-\gamma_{\rho,0})h + (\mathrm{id}-\gamma_{\rho,0})(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})$$

so that $g \in H_{\rho,\eta}$. Hence we have $\operatorname{Ker}(\Phi) \subset H_{\rho,\eta}$ and $\operatorname{Ker}(\Phi) = H_{\rho,\eta}$.

We will second prove the assertions for the group

$$\mathrm{Ker}(\mathrm{id} - \beta_{\rho,1}) \text{ in } K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}).$$

In the diagram (7.3), the exactness of the vertical arrows implies that δ is injective and $\mathrm{Im}(\delta) = \mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0})$ so that we have

$$K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \cong \mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}). \tag{7.5}$$

By the commutativity in the upper square in the diagram (7.3), one has

$$\mathrm{Ker}(\mathrm{id}-\beta_{\rho,1}) \text{ in } K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \cong \mathrm{Ker}(\mathrm{id}-\gamma_{\rho,0}) \text{ in } (\mathrm{Ker}(\mathrm{id}-\gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})).$$

Since $\gamma_{\eta,0}$ commutes with $\gamma_{\rho,0}$ in $K_0(\mathcal{F}_{\rho,\eta})$, we have

$$\begin{aligned} &\mathrm{Ker}(\mathrm{id} - \gamma_{\rho,0}) \text{ in } (\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})) \\ &\cong \mathrm{Ker}(\mathrm{id} - \gamma_{\rho,0}) \cap \mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}). \end{aligned}$$

(ii) The assertions are similarly shown as in (i). □

Therefore we have:

Theorem 7.10. *Assume that $K_1(\mathcal{F}_{\rho,\eta}) = 0$. There exist short exact sequences:*

$$\begin{aligned} 0 &\longrightarrow K_0(\mathcal{F}_{\rho,\eta})/((\mathrm{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id} - \gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})) \\ &\longrightarrow K_0(\mathcal{O}^{\kappa}_{\rho,\eta}) \\ &\longrightarrow \mathrm{Ker}(\mathrm{id} - \gamma_{\rho,0}) \cap \mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \textit{ in } K_0(\mathcal{F}_{\rho,\eta}) \\ &\longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 &\longrightarrow (\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \textit{ in } K_0(\mathcal{F}_{\rho,\eta}))/(\mathrm{id} - \gamma_{\rho,0})(\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \textit{ in } K_0(\mathcal{F}_{\rho,\eta})) \\ &\longrightarrow K_1(\mathcal{O}^{\kappa}_{\rho,\eta}) \\ &\longrightarrow \mathrm{Ker}(\mathrm{id} - \bar{\gamma}_{\rho,0}) \textit{ in } (K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id} - \gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})) \\ &\longrightarrow 0. \end{aligned}$$

We may describe the above formulae as follows.

Corollary 7.11. *Suppose $K_1(\mathcal{F}_{\rho,\eta}) = 0$. There exist short exact sequences:*

$$\begin{aligned} 0 &\longrightarrow \mathrm{Coker}(\mathrm{id} - \bar{\gamma}_{\rho,0}) \textit{ in } (\mathrm{Coker}(\mathrm{id} - \gamma_{\eta,0}) \textit{ in } K_0(\mathcal{F}_{\rho,\eta})) \\ &\longrightarrow K_0(\mathcal{O}^{\kappa}_{\rho,\eta}) \\ &\longrightarrow \mathrm{Ker}(\mathrm{id} - \gamma_{\rho,0}) \textit{ in } (\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \textit{ in } K_0(\mathcal{F}_{\rho,\eta})) \\ &\longrightarrow 0 \end{aligned}$$

and

$$
\begin{aligned}
0 &\longrightarrow \mathrm{Coker}(\mathrm{id} - \gamma_{\rho,0}) \text{ in } ((\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})) \\
&\longrightarrow K_1(\mathcal{O}^{\kappa}_{\rho,\eta}) \\
&\longrightarrow \mathrm{Ker}(\mathrm{id} - \bar{\gamma}_{\rho,0}) \text{ in } (\mathrm{Coker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})) \\
&\longrightarrow 0.
\end{aligned}
$$

8. K-Theory formulae

In this section, we will present more useful formulae to compute the K-groups $K_i(\mathcal{O}^{\kappa}_{\rho,\eta})$ under a certain additional assumption on $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$. The additional condition on $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is the following:

Definition 8.1. A C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to *form square* if the C^*-subalgebra $C^*(\rho_\alpha(1) : \alpha \in \Sigma^\rho)$ of $\mathcal{A}$ generated by the projections $\rho_\alpha(1), \alpha \in \Sigma^\rho$ coincides with the C^*-subalgebra $C^*(\eta_a(1) : a \in \Sigma^\eta)$ of $\mathcal{A}$ generated by the projections $\eta_a(1), a \in \Sigma^\eta$.

Lemma 8.2. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ forms square. Put for $l \in \mathbb{Z}_+$*

$$
\mathcal{A}^\rho_l = C^*(\rho_\mu(1) : \mu \in B_l(\Lambda_\rho)), \qquad \mathcal{A}^\eta_l = C^*(\eta_\xi(1) : \xi \in B_l(\Lambda_\eta)).
$$

Then $\mathcal{A}^\rho_l = \mathcal{A}^\eta_l$.

Proof. By the assumption, we have $\mathcal{A}^\rho_1 = \mathcal{A}^\eta_1$. Hence the desired equality for $l = 1$ holds. Suppose that the equalities hold for all $l \le k$ for some $k \in \mathbb{N}$. For $\mu = (\mu_1, \mu_2, \dots, \mu_k, \mu_{k+1}) \in B_{k+1}(\Lambda_\rho)$ we have $\rho_\mu(1) = \rho_{\mu_{k+1}}(\rho_{\mu_1\mu_2\cdots\mu_k}(1))$ so that $\rho_\mu(1) \in \rho_{\mu_{k+1}}(\mathcal{A}^\rho_k)$. By the commutation relation (3.1), one sees that

$$
\rho_{\mu_{k+1}}(\mathcal{A}^\rho_k) \subset C^*(\eta_\xi(\rho_\alpha(1)) : \xi \in B_k(\Lambda_\eta), \alpha \in \Sigma^\rho).
$$

Since $C^*(\rho_\alpha(1) : \alpha \in \Sigma^\rho) = C^*(\eta_a(1) : a \in \Sigma^\eta)$, the algebra $C^*(\eta_\xi(\rho_\alpha(1)) : \xi \in B_k(\Lambda_\eta), \alpha \in \Sigma^\rho)$ is contained in $\mathcal{A}^\eta_{k+1}$ so that $\rho_{\mu_{k+1}}(\mathcal{A}^\eta_k) \subset \mathcal{A}^\eta_{k+1}$. This implies $\rho_\mu(1) \in \mathcal{A}^\eta_{k+1}$ so that $\mathcal{A}^\rho_{k+1} \subset \mathcal{A}^\eta_{k+1}$ and hence $\mathcal{A}^\rho_{k+1} = \mathcal{A}^\eta_{k+1}$. □

Therefore we have

Lemma 8.3. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ forms square. Put for $j, k \in \mathbb{Z}_+$*

$$
\begin{aligned}
\mathcal{A}_{j,k} &= C^*(\rho_\mu(\eta_\zeta(1)) : \mu \in B_j(\Lambda_\rho), \zeta \in B_k(\Lambda_\eta)) \\
&\quad (= C^*(\eta_\xi(\rho_\nu(1)) : \xi \in B_k(\Lambda_\eta), \nu \in B_j(\Lambda_\rho))).
\end{aligned}
$$

Then $\mathcal{A}_{j,k}$ is commutative and of finite dimensional such that

$$
\mathcal{A}_{j,k} = \mathcal{A}^\rho_{j+k} (= \mathcal{A}^\eta_{j+k}).
$$

Hence $\mathcal{A}_{j,k} = \mathcal{A}_{j',k'}$ if $j + k = j' + k'$.

Proof. Since $\eta_\zeta(1) \in Z_{\mathcal{A}}$ and $\rho_\mu(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}$, the algebra $\mathcal{A}_{j,k}$ belongs to the center $Z_{\mathcal{A}}$ of $\mathcal{A}$. By the preceding lemma, we have

$$
\mathcal{A}_{j,k} = C^*(\rho_\mu(\rho_\nu(1)) : \mu \in B_j(\Lambda_\rho), \nu \in B_k(\Lambda_\rho)) = \mathcal{A}^\rho_{j+k}. \qquad \square
$$

For $j, k \in \mathbb{Z}_+$, put $l = j + k$. We denote by $\mathcal{A}_l$ the commutative finite dimensional algebra $\mathcal{A}_{j,k}$. Put $m(l) = \dim \mathcal{A}_l$. Take the finite sequence of minimal projections $E_i^l, i = 1, 2, \dots, m(l)$ in $\mathcal{A}_l$ such that $\sum_{i=1}^{m(l)} E_i^l = 1$ and hence $\mathcal{A}_l = \oplus_{i=1}^{m(l)} \mathbb{C} E_i^l$. Since $\rho_\alpha(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$, there exists $A_{l,l+1}^\rho(i, \alpha, n)$, which takes 0 or 1, such that

$$\rho_\alpha(E_i^l) = \sum_{n=1}^{m(l+1)} A_{l,l+1}^\rho(i, \alpha, n) E_n^{l+1}, \qquad \alpha \in \Sigma^\rho,\ i = 1, \dots, m(l).$$

Similarly, there exists $A_{l,l+1}^\eta(i, a, n)$, which takes 0 or 1, such that

$$\eta_a(E_i^l) = \sum_{n=1}^{m(l+1)} A_{l,l+1}^\eta(i, a, n) E_n^{l+1}, \qquad a \in \Sigma^\eta,\ i = 1, \dots, m(l).$$

Set for $i = 1, \dots, m(l)$

$$\begin{aligned}\mathcal{F}_{j,k}(i) &= C^*(S_\mu T_\zeta E_i^l x E_i^l T_\xi^* S_\nu^* \mid \mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta), x \in \mathcal{A}), \\ &= C^*(T_\zeta S_\mu E_i^l x E_i^l S_\nu^* T_\xi^* \mid \mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta), x \in \mathcal{A}).\end{aligned}$$

Let $N_{j,k}(i)$ be the cardinal number of the finite set

$$\{(\mu, \zeta) \in B_j(\Lambda_\rho) \times B_k(\Lambda_\eta) \mid \rho_\mu(\eta_\zeta(1)) \ge E_i^l\}.$$

Since E_i^l is a central projection in $\mathcal{A}$, we have

Lemma 8.4. *For $j, k \in \mathbb{Z}_+$, put $l = j + k$. Then we have:*

(i) *$\mathcal{F}_{j,k}(i)$ is isomorphic to the matrix algebra*

$$M_{N_{j,k}(i)}(E_i^l \mathcal{A} E_i^l)(= M_{N_{j,k}(i)}(\mathbb{C}) \otimes E_i^l \mathcal{A} E_i^l)$$

over $E_i^l \mathcal{A} E_i^l$ for $i = 1, \dots, m(l)$.

(ii) *$\mathcal{F}_{j,k} = \mathcal{F}_{j,k}(1) \oplus \cdots \oplus \mathcal{F}_{j,k}(m(l))$.*

Proof. (i) For $(\mu, \zeta) \in B_j(\Lambda_\rho) \times B_k(\Lambda_\eta)$ with $S_\mu T_\zeta E_i^l \ne 0$, one has

$$\eta_\zeta(\rho_\mu(1)) E_i^l \ne 0$$

so that $\eta_\zeta(\rho_\mu(1)) \ge E_i^l$. Hence $(S_\mu T_\zeta E_i^l)^* S_\mu T_\zeta E_i^l = E_i^l$. One sees that the set

$$\{S_\mu T_\zeta E_i^l \mid (\mu, \zeta) \in B_j(\Lambda_\rho) \times B_k(\Lambda_\eta); S_\mu T_\zeta E_i^l \ne 0\}$$

consist of partial isometries which give rise to matrix units of $\mathcal{F}_{j,k}(i)$ such that $\mathcal{F}_{j,k}(i)$ is isomorphic to $M_{N_{j,k}(i)}(E_i^l \mathcal{A} E_i^l)$.

(ii) Since $\mathcal{A} = E_1^l \mathcal{A} E_1^l \oplus \cdots \oplus E_{m(l)}^l \mathcal{A} E_{m(l)}^l$, the assertion is easy. □

Define homomorphisms $\lambda_\rho, \lambda_\eta : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})$ by setting

$$\lambda_\rho([p]) = \sum_{\alpha \in \Sigma^\rho} [(\rho_\alpha \otimes 1_n)(p)], \qquad \lambda_\eta([p]) = \sum_{a \in \Sigma^\eta} [(\eta_a \otimes 1_n)(p)]$$

for a projection $p \in M_n(\mathcal{A})$ for some $n \in \mathbb{N}$. Recall that the identities (5.1), (5.2) give rise to the embeddings (5.3), which induce homomorphisms

$$K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j,k+1}), \qquad K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j+1,k}).$$

We still denote them by $\iota_{*,+1}, \iota_{+1,*}$ respectively.

Lemma 8.5. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ forms square. There exists an isomorphism*

$$\Phi_{j,k} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{A})$$

such that the following diagrams are commutative:

(i)

$$\begin{array}{ccc} K_0(\mathcal{F}_{j,k}) & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{j+1,k}) \\ \Phi_{j,k}\downarrow & & \Phi_{j+1,k}\downarrow \\ K_0(\mathcal{A}) & \xrightarrow{\lambda_\rho} & K_0(\mathcal{A}) \end{array}$$

(ii)

$$\begin{array}{ccc} K_0(\mathcal{F}_{j,k}) & \xrightarrow{\iota_{*,+1}} & K_0(\mathcal{F}_{j,k+1}) \\ \Phi_{j,k}\downarrow & & \Phi_{j,k+1}\downarrow \\ K_0(\mathcal{A}) & \xrightarrow{\lambda_\eta} & K_0(\mathcal{A}). \end{array}$$

Proof. Put for $i = 1, 2, \dots, m(l)$

$$P_i = \sum_{\mu \in B_j(\Lambda_\rho), \zeta \in B_k(\Lambda_\eta)} S_\mu T_\zeta E_i^l T_\zeta^* S_\mu^*.$$

Then P_i is a central projection in $\mathcal{F}_{j,k}$ such that $\sum_{i=1}^{m(l)} P_i = 1$. For $X \in \mathcal{F}_{j,k}$, one has $P_i X P_i \in \mathcal{F}_{j,k}(i)$ such that

$$X = \sum_{i=1}^{m(l)} P_i X P_i \in \bigoplus_{i=1}^{m(l)} \mathcal{F}_{j,k}(i).$$

Define an isomorphism

$$\varphi_{j,k} : X \in \mathcal{F}_{j,k} \longrightarrow \sum_{i=1}^{m(l)} P_i X P_i \in \bigoplus_{i=1}^{m(l)} \mathcal{F}_{j,k}(i)$$

which induces an isomorphism on their K-groups

$$\varphi_{j,k*} : K_0(\mathcal{F}_{j,k}) \longrightarrow \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i)).$$

Take and fix $\nu(i), \mu(i) \in B_j(\Lambda_\rho)$ and $\zeta(i), \xi(i) \in B_k(\Lambda_\eta)$ such that

$$T_{\xi(i)} S_{\nu(i)} = S_{\mu(i)} T_{\zeta(i)} \quad \text{and} \quad T_{\xi(i)} S_{\nu(i)} E_i^l \neq 0. \tag{8.1}$$

Hence $S^*_{\nu(i)}T^*_{\xi(i)}T_{\xi(i)}S_{\nu(i)} \ge E^l_i$. Since $\mathcal{F}_{j,k}(i)$ is isomorphic to

$$M_{N_{j,k(i)}}(\mathbb{C}) \otimes E^l_i \mathcal{A} E^l_i,$$

the embedding

$$\iota_{j,k}(i) : x \in E^l_i \mathcal{A} E^l_i \longrightarrow T_{\xi(i)}S_{\nu(i)} x S^*_{\nu(i)}T^*_{\xi(i)} \in \mathcal{F}_{j,k}(i)$$

induces an isomorphism on their K-groups

$$\iota_{j,k}(i)_* : K_0(E^l_i \mathcal{A} E^l_i) \longrightarrow K_0(\mathcal{F}_{j,k}(i)).$$

Put

$$\psi_{j,k} = \bigoplus_{i=1}^{m(l)} \iota_{j,k}(i) : \bigoplus_{i=1}^{m(l)} E^l_i \mathcal{A} E^l_i \longrightarrow \bigoplus_{i=1}^{m(l)} \mathcal{F}_{j,k}(i)$$

and hence we have an isomorphism

$$\psi_{j,k*} = \bigoplus_{i=1}^{m(l)} {\iota_{j,k}(i)}_* : \bigoplus_{i=1}^{m(l)} K_0(E^l_i \mathcal{A} E^l_i) \longrightarrow \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i)).$$

Since $K_0(\mathcal{A}) = \bigoplus_{i=1}^{m(l)} K_0(E^l_i \mathcal{A} E^l_i)$, we have an isomorphism

$$\Phi_{j,k} = {\psi_{j,k*}}^{-1} \circ \varphi_{j,k*} : K_0(\mathcal{F}_{j,k}) \overset{\varphi_{j,k*}}{\longrightarrow} \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i)) \overset{{\psi_{j,k*}}^{-1}}{\longrightarrow} K_0(\mathcal{A}).$$

(i) It suffices to show the following diagram

$$\begin{array}{ccc}
K_0(\mathcal{F}_{j,k}) & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{j+1,k}) \\
{\scriptstyle \varphi_{j,k*}}\downarrow & & {\scriptstyle \varphi_{j+1,k*}}\downarrow \\
\bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i)) & & \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j+1,k}(i)) \\
{\scriptstyle \psi_{j,k*}}\uparrow & & {\scriptstyle \psi_{j+1,k*}}\uparrow \\
K_0(\mathcal{A}) & \xrightarrow{\lambda_\rho} & K_0(\mathcal{A})
\end{array}$$

is commutative. For $x = \sum_{i=1}^{m(l)} E^l_i x E^l_i \in \mathcal{A}$, we have

$$\psi_{j,k}(x) = \sum_{i=1}^{m(l)} T_{\xi(i)}S_{\nu(i)} E^l_i x E^l_i S^*_{\nu(i)} T^*_{\xi(i)} = \sum_{i=1}^{m(l)} S_{\mu(i)}T_{\zeta(i)} E^l_i x E^l_i T^*_{\zeta(i)} S^*_{\mu(i)}.$$

Since $P_i T_{\xi(i)}S_{\nu(i)} E^l_i x E^l_i S^*_{\nu(i)} T^*_{\xi(i)} P_i = T_{\xi(i)}S_{\nu(i)} E^l_i x E^l_i S^*_{\nu(i)} T^*_{\xi(i)}$, we have

$$\varphi_{j,k}^{-1} \circ \psi_{j,k}(x) = \sum_{i=1}^{m(l)} T_{\xi(i)}S_{\nu(i)} E^l_i x E^l_i S^*_{\nu(i)} T^*_{\xi(i)}$$

so that

$$\iota_{+1,*} \circ \varphi_{j,k}^{-1} \circ \psi_{j,k}(x) = \sum_{\alpha \in \Sigma^\rho} \sum_{i=1}^{m(l)} T_{\xi(i)}S_{\nu(i)\alpha} \rho_\alpha(E^l_i x E^l_i) S^*_{\nu(i)\alpha} T^*_{\xi(i)}.$$

Since

$$S_{\nu(i)\alpha}\rho_\alpha(E_i^l x E_i^l)S^*_{\nu(i)\alpha} = \sum_{n=1}^{m(l+1)} A^\rho_{l,l+1}(i,\alpha,n)S_{\nu(i)\alpha}E_n^{l+1}\rho_\alpha(x)E_n^{l+1}S^*_{\nu(i)\alpha}$$

and $A^\rho_{l,l+1}(i,\alpha,n)S_{\nu(i)\alpha}E_n^{l+1} = S_{\nu(i)\alpha}E_n^{l+1}$, we have

$$\sum_{\alpha\in\Sigma^\rho} S_{\nu(i)\alpha}\rho_\alpha(E_i^l x E_i^l)S^*_{\nu(i)\alpha} = \sum_{n=1}^{m(l+1)}\sum_{\alpha\in\Sigma^\rho} S_{\nu(i)\alpha}E_n^{l+1}\rho_\alpha(x)E_n^{l+1}S^*_{\nu(i)\alpha}$$

so that

$$\iota_{+1,*}\circ\varphi_{j,k}^{-1}\circ\psi_{j,k}(x) = \sum_{\alpha\in\Sigma^\rho}\sum_{i=1}^{m(l)}\sum_{n=1}^{m(l+1)} T_{\xi(i)}S_{\nu(i)\alpha}E_n^{l+1}\rho_\alpha(x)E_n^{l+1}S^*_{\nu(i)\alpha}T^*_{\xi(i)}.$$

On the other hand,

$$\begin{aligned}\psi_{j,k}(\lambda_\rho(x)) &= \psi_{j,k}\left(\sum_{n=1}^{m(l+1)}\sum_{\alpha\in\Sigma^\rho} E_n^{l+1}\rho_\alpha(x)E_n^{l+1}\right)\\ &= \sum_{\alpha\in\Sigma^\rho}\sum_{i=1}^{m(l)}\sum_{n=1}^{m(l+1)} T_{\xi(i)}S_{\nu(i)\alpha}E_n^{l+1}\rho_\alpha(x)E_n^{l+1}S^*_{\nu(i)\alpha}T^*_{\xi(i)}.\end{aligned}$$

Therefore we have

$$\iota_{+1,*}\circ\varphi_{j,k}^{-1}\circ\psi_{j,k}(x) = \psi_{j,k}(\lambda_\rho(x)).$$

(ii) is symmetric to (i). □

Define the abelian groups of the inductive limits:

$$G_\rho = \lim\{\lambda_\rho : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})\}, \qquad G_\eta = \lim\{\lambda_\eta : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})\}.$$

Put the subalgebras of $\mathcal{F}_{\rho,\eta}$ for $j,k\in\mathbb{Z}_+$

$$\begin{aligned}\mathcal{F}_{\rho,k} &= C^*(T_\zeta S_\mu x S_\nu^* T_\xi^* \mid \mu,\nu\in B_*(\Lambda_\rho), |\mu|=|\nu|, \zeta,\xi\in B_k(\Lambda_\eta), x\in\mathcal{A})\\ &= C^*(T_\zeta y T_\xi^* \mid \zeta,\xi\in B_k(\Lambda_\eta), y\in\mathcal{F}_\rho),\\ \mathcal{F}_{j,\eta} &= C^*(S_\mu T_\zeta x T_\xi^* S_\nu^* \mid \mu,\nu\in B_j(\Lambda_\rho), \zeta,\xi\in B_*(\Lambda_\eta), |\zeta|=|\xi|, x\in\mathcal{A})\\ &= C^*(S_\mu y S_\nu^* \mid \mu,\nu\in B_j(\Lambda_\rho), y\in\mathcal{F}_\eta).\end{aligned}$$

By the preceding lemma, we have:

Lemma 8.6. *For $j,k\in\mathbb{Z}_+$, there exist isomorphisms*

$$\Phi_{\rho,k} : K_0(\mathcal{F}_{\rho,k}) \longrightarrow G_\rho, \qquad \Phi_{j,\eta} : K_0(\mathcal{F}_{j,\eta}) \longrightarrow G_\eta$$

such that the following diagrams are commutative:

(i)

$$\begin{array}{ccccccc} K_0(\mathcal{F}_{j,k}) & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{j+1,k}) & \xrightarrow{\iota_{+1,*}} & \cdots & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{\rho,k}) \\ {\scriptstyle\Phi_{j,k}}\big\downarrow & & {\scriptstyle\Phi_{j+1,k}}\big\downarrow & & & & {\scriptstyle\Phi_{\rho,k}}\big\downarrow \\ K_0(\mathcal{A}) & \xrightarrow{\lambda_\rho} & K_0(\mathcal{A}) & \xrightarrow{\lambda_\rho} & \cdots & \xrightarrow{\lambda_\rho} & G_\rho \end{array}$$

(ii)

$$\begin{array}{ccccccc} K_0(\mathcal{F}_{j,k}) & \xrightarrow{\iota_{*,+1}} & K_0(\mathcal{F}_{j,k+1}) & \xrightarrow{\iota_{*,+1}} & \cdots & \xrightarrow{\iota_{*,+1}} & K_0(\mathcal{F}_{j,\eta}) \\ {\scriptstyle\Phi_{j,k}}\big\downarrow & & {\scriptstyle\Phi_{j,k+1}}\big\downarrow & & & & {\scriptstyle\Phi_{j,\eta}}\big\downarrow \\ K_0(\mathcal{A}) & \xrightarrow{\lambda_\eta} & K_0(\mathcal{A}) & \xrightarrow{\lambda_\eta} & \cdots & \xrightarrow{\lambda_\eta} & G_\eta. \end{array}$$

Lemma 8.7. *If $\xi = (\xi_1, \dots, \xi_k) \in B_k(\Lambda_\eta), \nu = (\nu_1, \dots, \nu_j) \in B_j(\Lambda_\rho)$ satisfy the condition $\rho_\nu(\eta_\xi(1)) \geq E_i^l$ for some $i = 1, \dots, m(l)$ with $l = j + k$, then $T^*_{\xi_1}T_\xi S_\nu E_i^l = T_{\bar\xi} S_\nu E_i^l$ where $\bar\xi = (\xi_2, \dots, \xi_k)$.*

Proof. Since $T^*_{\xi_1}T_\xi = T^*_{\xi_1}T_{\xi_1}T_{\bar\xi}T^*_{\bar\xi}T_{\bar\xi} = T_{\bar\xi}T^*_{\bar\xi}T^*_{\xi_1}T_{\xi_1}T_{\bar\xi} = T_{\bar\xi}T^*_\xi T_\xi$, we have

$$T^*_{\xi_1}T_\xi S_\nu E_i^l = T_{\bar\xi}S_\nu S_\nu^* T_\xi^* T_\xi S_\nu E_i^l = T_{\bar\xi}S_\nu \rho_\nu(\eta_\xi(1))E_i^l = T_{\bar\xi}S_\nu E_i^l. \qquad \square$$

Let us denote by γ_ρ, γ_η the endomorphisms $\gamma_{\rho,0}, \gamma_{\eta,0}$ on $K_0(\mathcal{F}_{\rho,\eta})$ appeared in Lemma 7.6, respectively.

Lemma 8.8. *For $k, j \in \mathbb{Z}_+$, we have:*

(i) *The restriction of γ_η^{-1} to $K_0(\mathcal{F}_{j,k})$ makes the following diagram commutative:*

$$\begin{array}{ccccc} K_0(\mathcal{F}_{j,k}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{j,k-1}) & \xrightarrow{\iota_{*,+1}} & K_0(\mathcal{F}_{j,k}) \\ {\scriptstyle\Phi_{j,k}}\big\downarrow & & & & {\scriptstyle\Phi_{j,k}}\big\downarrow \\ K_0(\mathcal{A}) & & \xrightarrow{\lambda_\eta} & & K_0(\mathcal{A}). \end{array}$$

(ii) *The restriction of γ_ρ^{-1} to $K_0(\mathcal{F}_{j,k})$ makes the following diagram commutative:*

$$\begin{array}{ccccc} K_0(\mathcal{F}_{j,k}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{j-1,k}) & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{j,k}) \\ {\scriptstyle\Phi_{j,k}}\big\downarrow & & & & {\scriptstyle\Phi_{j,k}}\big\downarrow \\ K_0(\mathcal{A}) & & \xrightarrow{\lambda_\rho} & & K_0(\mathcal{A}). \end{array}$$

Proof. (i) Put $l = j + k$. Take a projection $p \in M_n(\mathcal{A})$ for some $n \in \mathbb{N}$. Since $\mathcal{A} \otimes M_n(\mathbb{C}) = \sum_{i=1}^{m(l)} {}^{\oplus}(E_i^l \otimes 1)(\mathcal{A} \otimes M_n)(E_i^l \otimes 1)$, by putting

$$p_i^l = (E_i^l \otimes 1)p(E_i^l \otimes 1) \in M_n(E_i^l \mathcal{A} E_i^l),$$

we have $p = \sum_{i=1}^{m(l)} p_i^l$. Take

$$\xi(i) = (\xi_1(i), \dots, \xi_k(i)) \in B_k(\Lambda_\eta), \quad \nu(i) = (\nu_1(i), \dots, \nu_j(i)) \in B_j(\Lambda_\rho)$$

as in (8.1) so that $\rho_{\nu(i)}(\eta_{\xi(i)}(1)) \geq E_i^l$ and put $\bar{\xi}(i) = (\xi_2(i), \dots, \xi_k(i))$ so that $\xi(i) = \xi_1(i)\bar{\xi}(i)$. We have

$$\psi_{j,k*}([p]) = \sum_{i=1}^{m(l)} \oplus [(T_{\xi(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)] \in \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i)).$$

As

$$(T_{\xi(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n) \leq T_{\xi_1(i)} T_{\xi_1(i)}^* \otimes 1_n,$$

by the preceding lemma we have

$$T_{\xi_1(i)}^* T_{\xi(i)} S_{\nu(i)} E_i^l = T_{\bar{\xi}(i)} S_{\nu(i)} E_i^l$$

so that by Lemma 7.6

$$\gamma_\eta^{-1}([(T_{\xi(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)] = [(T_{\bar{\xi}(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S_{\nu(i)}^* T_{\bar{\xi}(i)}^* \otimes 1_n)].$$

Hence $K_0(\mathcal{F}_{j,k})$ goes to $K_0(\mathcal{F}_{j,k-1})$ by the homomorphism γ_η^{-1}. Take $\mu(i) \in B_j(\Lambda_\rho), \bar{\zeta}(i) \in B_{k-1}(\Lambda_\eta)$ such that $T_{\bar{\xi}(i)} S_{\nu(i)} = S_{\mu(i)} T_{\bar{\zeta}(i)}$ for $i = 1, \dots, m(l)$. The element

$$\begin{aligned} &\sum_{i=1}^{m(l)} [(T_{\bar{\xi}(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S_{\nu(i)}^* T_{\bar{\xi}(i)}^* \otimes 1_n)] \\ &= \sum_{i=1}^{m(l)} [(S_{\mu(i)} T_{\bar{\zeta}(i)} \otimes 1_n) p_i^l (T_{\bar{\zeta}(i)}^* S_{\mu(i)}^* \otimes 1_n)] \in K_0(\mathcal{F}_{j,k-1}) \end{aligned}$$

goes to

$$\sum_{i=1}^{m(l)} \sum_{a \in \Sigma^\eta} [(S_{\mu(i)} T_{\bar{\zeta}(i)a} \otimes 1_n)(T_a^* \otimes 1_n) p_i^l (T_a \otimes 1_n)(T_{\bar{\zeta}(i)a}^* S_{\mu(i)}^* \otimes 1_n)] \in K_0(\mathcal{F}_{j,k})$$

by $\iota_{*,+1}$. The latter one is expressed as

(8.2)

$$\sum_{h=1}^{m(l)} \oplus \sum_{i=1}^{m(l)} \sum_{a \in \Sigma^\eta} [(S_{\mu(i)} T_{\bar{\zeta}(i)a} \otimes 1_n) E_h^l (T_a^* \otimes 1_n) p_i^l (T_a \otimes 1_n) E_h^l (T_{\bar{\zeta}(i)a}^* S_{\mu(i)}^* \otimes 1_n)]$$

in $\bigoplus_{h=1}^{m(l)} K_0(\mathcal{F}_{j,k}(h))$. On the other hand, we have

$$\begin{aligned} \lambda_\eta([p]) &= \sum_{a \in \Sigma^\eta} [(T_a^* \otimes 1_n) p (T_a \otimes 1_n)] \\ &= \sum_{h=1}^{m(l)} \oplus \sum_{a \in \Sigma^\eta} [E_h^l (T_a^* \otimes 1_n) p (T_a \otimes 1_n) E_h^l] \in \bigoplus_{h=1}^{m(l)} K_0(E_h^l \mathcal{A} E_h^l), \end{aligned}$$

which is expressed as

$$\begin{aligned}
&\sum_{h=1}^{m(l)}{}^{\oplus} \sum_{a\in\Sigma^\eta} [(T_{\xi(h)}S_{\nu(h)}E_h^l \otimes 1_n)(T_a^* \otimes 1_n)p(T_a \otimes 1_n)(E_h^l S_{\nu(h)}^* T_{\xi(h)}^* \otimes 1_n)] \\
&= \sum_{h=1}^{m(l)}{}^{\oplus} \sum_{a\in\Sigma^\eta} \sum_{i=1}^{m(l)} \Big[(T_{\xi(h)}S_{\nu(h)}E_h^l \otimes 1_n)(T_a^* \otimes 1_n) \\
&\quad \cdot p_i^l(T_a \otimes 1_n)(E_h^l S_{\nu(h)}^* T_{\xi(h)}^* \otimes 1_n)\Big]
\end{aligned}$$

in $\bigoplus_{h=1}^{m(l)} K_0(\mathcal{F}_{j,k}(h))$. Take $\mu'(h) \in B_j(\Lambda_\rho), \zeta'(h) \in B_k(\Lambda_\eta)$ such that $T_{\xi(h)}S_{\nu(h)} = S_{\mu'(h)}T_{\zeta'(h)}$ so that the above element is

(8.3)

$$\sum_{h=1}^{m(l)}{}^{\oplus} \sum_{i=1}^{m(l)} \sum_{a\in\Sigma^\eta} [(S_{\mu'(h)}T_{\zeta'(h)}E_h^l \otimes 1_n)(T_a^* \otimes 1_n)p_i^l(T_a \otimes 1_n)(E_h^l T_{\zeta'(h)}^* S_{\nu'(h)}^* \otimes 1_n)]$$

in $\bigoplus_{h=1}^{m(l)} K_0(\mathcal{F}_{j,k}(h))$. Since for $h, i = 1, \dots, m(l), a \in \Sigma^\eta$ their classes of the K-groups coincide such as

$$\begin{aligned}
&[(S_{\mu(i)}T_{\bar\zeta(i)a} \otimes 1_n)E_h^l(T_a^* \otimes 1_n)p_i^l(T_a \otimes 1_n)E_h^l(T_{\bar\zeta(i)a}^* S_{\mu(i)}^* \otimes 1_n)] \\
&= [(S_{\mu'(h)}T_{\zeta'(h)}E_h^l \otimes 1_n)(T_a^* \otimes 1_n)p_i^l(T_a \otimes 1_n)(E_h^l T_{\zeta'(h)}^* S_{\nu'(h)}^* \otimes 1_n)] \\
&\in K_0(\mathcal{F}_{j,k}(h)),
\end{aligned}$$

the element of (8.2) is equal to the element of (8.3) in $K_0(\mathcal{F}_{j,k})$. Thus (i) holds.

(ii) is similar to (i). □

We note that for $j, k \in \mathbb{Z}_+$,

$$\begin{aligned}
K_0(\mathcal{F}_{\rho,k}) &= \lim_j \{\iota_{+1,*} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j+1,k})\}, \\
K_0(\mathcal{F}_{j,\eta}) &= \lim_k \{\iota_{*,+1} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j,k+1})\}.
\end{aligned}$$

The following lemma is direct.

Lemma 8.9. *For $k, j \in \mathbb{Z}_+$, the following diagrams are commutative:*

(i)

$$\begin{array}{ccc}
K_0(\mathcal{F}_{j,k}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{j,k-1}) \\
{\scriptstyle\iota_{+1,*}}\big\downarrow & & {\scriptstyle\iota_{+1,*}}\big\downarrow \\
K_0(\mathcal{F}_{j+1,k}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{j+1,k-1}).
\end{array}$$

Hence γ_η^{-1} yields a homomorphism from $K_0(\mathcal{F}_{\rho,k})$ to $K_0(\mathcal{F}_{\rho,k-1})$.

(ii)

$$\begin{array}{ccc} K_0(\mathcal{F}_{j,k}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{j-1,k}) \\ {\scriptstyle \iota_{*,+1}}\downarrow & & {\scriptstyle \iota_{*,+1}}\downarrow \\ K_0(\mathcal{F}_{j,k+1}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{j-1,k+1}). \end{array}$$

Hence γ_ρ^{-1} yields a homomorphism from $K_0(\mathcal{F}_{j,\eta})$ to $K_0(\mathcal{F}_{j-1,\eta})$.

The homomorphisms

$$\iota_{+1,*} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j+1,k}), \qquad \iota_{*,+1} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j,k+1})$$

are naturally induce homomorphisms

$$K_0(\mathcal{F}_{j,\eta}) \longrightarrow K_0(\mathcal{F}_{j+1,\eta}), \qquad \iota_{*,+1} : K_0(\mathcal{F}_{\rho,k}) \longrightarrow K_0(\mathcal{F}_{\rho,k+1})$$

which we denote by $\iota_{+1,\eta},\ \iota_{\rho,+1}$ respectively. They are also induced by the identities (5.1), (5.2) respectively.

Lemma 8.10. *For $k, j \in \mathbb{Z}_+$, the following diagrams are commutative:*

(i)

$$\begin{array}{ccc} K_0(\mathcal{F}_{\rho,k}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{\rho,k-1}) \\ {\scriptstyle \iota_{\rho,+1}}\downarrow & & {\scriptstyle \iota_{\rho,+1}}\downarrow \\ K_0(\mathcal{F}_{\rho,k+1}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{\rho,k}). \end{array}$$

(ii)

$$\begin{array}{ccc} K_0(\mathcal{F}_{j,\eta}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{j-1,\eta}) \\ {\scriptstyle \iota_{+1,\eta}}\downarrow & & {\scriptstyle \iota_{+1,\eta}}\downarrow \\ K_0(\mathcal{F}_{j+1,\eta}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{j,\eta}). \end{array}$$

Proof. (i) As in the proof of Lemma 8.9, one may take an element of $K_0(\mathcal{F}_{\rho,k})$ as in the following form:

$$\sum_{i=1}^{m(l)}{}^{\oplus}[(T_{\xi(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)] \in \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i))$$

for some projection $p \in M_n(\mathcal{A})$ and j, l with $l = j + k$, where

$$p_i^l = (E_i^l \otimes 1)p(E_i^l \otimes 1) \in M_n(E_i^l \mathcal{A} E_i^l).$$

Let $\xi(i) = \xi_1(i)\bar{\xi}(i)$ with $\xi_1(i) \in \Sigma^\eta$, $\bar{\xi}(i) \in B_{k-1}(\Lambda_\eta)$. One may assume that $T_{\xi(i)}S_{\nu(i)} \neq 0$ so that $T_{\bar{\xi}(i)}S_{\nu(i)} = S_{\nu(i)'}T_{\bar{\xi}(i)'}$ for some $\nu(i)' \in B_j(\Lambda_\rho), \bar{\xi}(i)' \in$

$B_{k-1}(\Lambda_\eta)$. As in the proof of Lemma 8.9, one has

$$\begin{aligned}&\gamma_\eta^{-1}([(T_{\xi(i)}S_{\nu(i)}\otimes 1_n)p_i^l(S^*_{\nu(i)}T^*_{\xi(i)}\otimes 1_n)]\\ &=[(T_{\bar\xi(i)}S_{\nu(i)}\otimes 1_n)p_i^l(S^*_{\nu(i)}T^*_{\bar\xi(i)}\otimes 1_n)]\\ &=[(S_{\nu(i)'}T_{\bar\xi(i)'}\otimes 1_n)p_i^l(S^*_{\nu(i)'}T^*_{\bar\xi(i)'}\otimes 1_n)].\end{aligned}$$

Hence we have

$$\begin{aligned}&\iota_{*,+1}\circ\gamma_\eta^{-1}([(T_{\xi(i)}S_{\nu(i)}\otimes 1_n)p_i^l(S^*_{\nu(i)}T^*_{\xi(i)}\otimes 1_n)]\\ &=\iota_{*,+1}([S_{\nu(i)'}T_{\bar\xi(i)'}\otimes 1_n)p_i^l(T^*_{\bar\xi(i)'}S^*_{\nu(i)'}\otimes 1_n])\\ &=\sum_{b\in\Sigma^\eta}[(S_{\nu(i)'}T_{\bar\xi(i)'b}\otimes 1_n)(T_b^*\otimes 1_n)p_i^l(T_b\otimes 1_n)(T^*_{\bar\xi(i)'b}S^*_{\nu(i)'}\otimes 1_n)].\end{aligned}$$

On the other hand, the equality $T_{\xi(i)}S_{\nu(i)}=T_{\xi(i)_1}S_{\nu(i)'}T_{\bar\xi(i)'}$ implies

$$\begin{aligned}&\iota_{*,+1}([(T_{\xi(i)}S_{\nu(i)}\otimes 1_n)p_i^l(S^*_{\nu(i)}T^*_{\xi(i)}\otimes 1_n)]\\ &=\sum_{b\in\Sigma^\eta}[(T_{\xi(i)_1}S_{\nu(i)'}T_{\bar\xi(i)'b}\otimes 1_n)(T_b^*\otimes 1_n)p_i^l(T_b\otimes 1_n)(T^*_{\bar\xi(i)'b}S^*_{\nu(i)'}T^*_{\xi(i)_1}\otimes 1_n)]\end{aligned}$$

and hence

$$\begin{aligned}&\gamma_\eta^{-1}\circ\iota_{*,+1}([(T_{\xi(i)}S_{\nu(i)}\otimes 1_n)p_i^l(S^*_{\nu(i)}T^*_{\xi(i)}\otimes 1_n)]\\ &=\sum_{b\in\Sigma^\eta}\gamma_\eta^{-1}\Big([(T_{\xi(i)_1}S_{\nu(i)'}T_{\bar\xi(i)'b}\otimes 1_n)(T_b^*\otimes 1_n)\\ &\qquad\cdot p_i^l(T_b\otimes 1_n)(T^*_{\bar\xi(i)'b}S^*_{\nu(i)'}T^*_{\xi(i)_1}\otimes 1_n)]\Big)\\ &=\sum_{b\in\Sigma^\eta}[(S_{\nu(i)'}T_{\bar\xi(i)'b}\otimes 1_n)(T_b^*\otimes 1_n)p_i^l(T_b\otimes 1_n)(T^*_{\bar\xi(i)'b}S^*_{\nu(i)'}\otimes 1_n)].\end{aligned}$$

(ii) The proof is completely symmetric to the above proof. ☐

Since the homomorphisms $\lambda_\rho,\lambda_\eta:K_0(\mathcal{A})\longrightarrow K_0(\mathcal{A})$ are mutually commutative, the map λ_η induces a homomorphism on the inductive limit $G_\rho=\lim\{\lambda_\rho:K_0(\mathcal{A})\longrightarrow K_0(\mathcal{A})\}$ and similarly λ_ρ does on the inductive limit G_η. They are still denoted by $\lambda_\rho,\lambda_\eta$ respectively.

Lemma 8.11. *For $k,j\in\mathbb{Z}_+$, the following diagrams are commutative:*

(i)

$$\begin{array}{ccccc} K_0(\mathcal{F}_{\rho,k}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{\rho,k-1}) & \xrightarrow{\iota_{\rho,+1}} & K_0(\mathcal{F}_{\rho,k}) \\ {\scriptstyle\Phi_{\rho,k}}\downarrow & & & & {\scriptstyle\Phi_{\rho,k}}\downarrow \\ G_\rho & & \xrightarrow{\lambda_\eta} & & G_\rho. \end{array}$$

(ii)

$$\begin{array}{ccccc}
K_0(\mathcal{F}_{j,\eta}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{j-1,\eta}) & \xrightarrow{\iota_{+1,\eta}} & K_0(\mathcal{F}_{j,\eta}) \\
\Phi_{j,\eta}\downarrow & & & & \Phi_{j,\eta}\downarrow \\
G_\eta & & \xrightarrow{\lambda_\rho} & & G_\eta .
\end{array}$$

Proof. (i) As in the proof of Lemma 8.8 and Lemma 8.10 one may take an element of $K_0(\mathcal{F}_{\rho,k})$ as in the following form:

$$\sum_{i=1}^{m(l)}{}^{\oplus}[(T_{\xi(i)}S_{\nu(i)}\otimes 1_n)p_i^l(S_{\nu(i)}^*T_{\xi(i)}^*\otimes 1_n)] \in \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i))$$

for some projection $p \in M_n(\mathcal{A})$ and j,l with $l = j+k$, where

$$p_i^l = (E_i^l \otimes 1)p(E_i^l \otimes 1).$$

Keep the notations as in the proof of Lemma 8.8, we have

$$\begin{aligned}
&\iota_{*,+1}\circ\gamma_\eta^{-1}([(T_{\xi(i)}S_{\nu(i)}\otimes 1_n)p_i^l(S_{\nu(i)}^*T_{\xi(i)}^*\otimes 1_n)]) \\
&\qquad = \sum_{b\in\Sigma^\eta}[(S_{\nu(i)'}T_{\bar{\xi}(i)'b}\otimes 1_n)(T_b^*\otimes 1_n)p_i^l(T_b\otimes 1_n)(T_{\bar{\xi}(i)'b}^*S_{\nu(i)'}^*\otimes 1_n)]
\end{aligned}$$

so that

$$\begin{aligned}
&\Phi_{\rho,k}\circ\iota_{*,+1}\circ\gamma_\eta^{-1}([[(T_{\xi(i)}S_{\nu(i)}\otimes 1_n)p_i^l(S_{\nu(i)}^*T_{\xi(i)}^*\otimes 1_n)] \\
&= \sum_{b\in\Sigma^\eta}\Phi_{\rho,k}([S_{\nu(i)'}T_{\bar{\xi}(i)'b}\otimes 1_n)(T_b^*\otimes 1_n)p_i^l(T_b\otimes 1_n)(T_{\bar{\xi}(i)'b}^*S_{\nu(i)'}^*\otimes 1_n)]) \\
&= \sum_{b\in\Sigma^\eta}[(T_b^*\otimes 1_n)p_i^l(T_b\otimes 1_n)] \\
&= \lambda_\eta([p_i^l]) = (\lambda_\eta\circ\Phi_{\rho,k})([[(T_{\xi(i)}S_{\nu(i)}\otimes 1_n)p_i^l(S_{\nu(i)}^*T_{\xi(i)}^*\otimes 1_n)]).
\end{aligned}$$

Therefore we have $\Phi_{\rho,k}\circ\iota_{\rho,+1}\circ\gamma_\eta^{-1} = \lambda_\eta\circ\Phi_{\rho,k}$.

(ii) The proof is completely symmetric to the above proof. ☐

Put for $j,k\in\mathbb{Z}_+$

$$\begin{aligned}
G_{\rho,k} &= K_0(\mathcal{F}_{\rho,k})(\cong G_\rho = \lim\{\lambda_\rho : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})\}), \\
G_{j,\eta} &= K_0(\mathcal{F}_{j,\eta})(\cong G_\eta = \lim\{\lambda_\eta : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})\}).
\end{aligned}$$

The map $\lambda_\eta : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})$ naturally gives rise to a homomorphism from $G_{\rho,k}$ to $G_{\rho,k+1}$ which we will still denote by λ_η. Similarly we have a homomorphism λ_ρ from $G_{j,\eta}$ to $G_{j+1,\eta}$.

Lemma 8.12. *For $k,j\in\mathbb{Z}_+$, the following diagrams are commutative:*

(i)

$$\begin{array}{ccc} K_0(\mathcal{F}_{\rho,k}) & \xrightarrow{\iota_{\rho,+1}} & K_0(\mathcal{F}_{\rho,k+1}) \\ \| & & \| \\ G_{\rho,k} & \xrightarrow{\lambda_\eta} & G_{\rho,k+1}. \end{array}$$

(ii)

$$\begin{array}{ccc} K_0(\mathcal{F}_{j,\eta}) & \xrightarrow{\iota_{+1,\eta}} & K_0(\mathcal{F}_{j+1,\eta}) \\ \| & & \| \\ G_{j,\eta} & \xrightarrow{\lambda_\rho} & G_{j+1,\eta}. \end{array}$$

We denote the abelian group $K_0(\mathcal{F}_{\rho,\eta})$ by $G_{\rho,\eta}$. Since

$$\begin{aligned} K_0(\mathcal{F}_{\rho,\eta}) &= \lim_k \{\iota_{\rho,+1} : K_0(\mathcal{F}_{\rho,k}) \longrightarrow K_0(\mathcal{F}_{\rho,k+1})\} \\ &= \lim_j \{\iota_{+1,\eta} : K_0(\mathcal{F}_{j,\eta}) \longrightarrow K_0(\mathcal{F}_{j+1,\eta})\}, \end{aligned}$$

one has

$$G_{\rho,\eta} = \lim_k \{\lambda_\eta : G_{\rho,k} \longrightarrow G_{\rho,k+1}\} = \lim_j \{\lambda_\rho : G_{j,\eta} \longrightarrow G_{j+1,\eta}\}.$$

Define two endomorphisms

$$\begin{aligned} &\sigma_\eta \text{ on } G_{\rho,\eta} = \lim_k \{\lambda_\eta : G_{\rho,k} \longrightarrow G_{\rho,k+1}\} \quad \text{and} \\ &\sigma_\rho \text{ on } G_{\rho,\eta} = \lim_j \{\lambda_\rho : G_{j,\eta} \longrightarrow G_{j+1,\eta}\} \end{aligned}$$

by setting

$$\begin{aligned} &\sigma_\rho : [g,k] \in G_{\rho,k} \longrightarrow [g,k-1] \in G_{\rho,k-1} \text{ for } g \in G_\rho \text{ and} \\ &\sigma_\eta : [h,j] \in G_{j,\eta} \longrightarrow [h,j-1] \in G_{j-1,\eta} \text{ for } h \in G_\eta. \end{aligned}$$

Therefore we have:

Lemma 8.13.

(i) *There exists an isomorphism $\Phi_{\rho,\infty} : K_0(\mathcal{F}_{\rho,\eta}) \longrightarrow G_{\rho,\eta}$ such that the following diagrams are commutative:*

$$\begin{array}{ccc} K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{\rho,\eta}) \\ \Phi_{\rho,\infty}\big\downarrow & & \Phi_{\rho,\infty}\big\downarrow \\ G_{\rho,\eta} & \xrightarrow{\sigma_\eta} & G_{\rho,\eta} \end{array}$$

and hence

$$\begin{array}{ccc} K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{id-\gamma_\eta^{-1}} & K_0(\mathcal{F}_{\rho,\eta}) \\ \Phi_{\rho,\infty}\big\downarrow & & \Phi_{\rho,\infty}\big\downarrow \\ G_{\rho,\eta} & \xrightarrow{id-\sigma_\eta} & G_{\rho,\eta}. \end{array}$$

(ii) *There exists an isomorphism* $\Phi_{\infty,\eta} : K_0(\mathcal{F}_{\rho,\eta}) \longrightarrow G_{\rho,\eta}$ *such that the following diagrams are commutative:*

$$\begin{array}{ccc} K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{\rho,\eta}) \\ {\scriptstyle\Phi_{\infty,\eta}}\downarrow & & {\scriptstyle\Phi_{\infty,\eta}}\downarrow \\ G_{\rho,\eta} & \xrightarrow{\sigma_\rho} & G_{\rho,\eta} \end{array}$$

and hence

$$\begin{array}{ccc} K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{id-\gamma_\rho^{-1}} & K_0(\mathcal{F}_{\rho,\eta}) \\ {\scriptstyle\Phi_{\infty,\eta}}\downarrow & & {\scriptstyle\Phi_{\infty,\eta}}\downarrow \\ G_{\rho,\eta} & \xrightarrow{id-\sigma_\rho} & G_{\rho,\eta}. \end{array}$$

Let us denote by $J_{\mathcal{A}}$ the natural embedding $\mathcal{A} = \mathcal{F}_{0,0} \hookrightarrow \mathcal{F}_{\rho,\eta}$, which induces a homomorphism $J_{\mathcal{A}*} : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{F}_{\rho,\eta})$.

Lemma 8.14. *The homomorphism* $J_{\mathcal{A}*} : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{F}_{\rho,\eta})$ *is injective such that*

$$J_{\mathcal{A}*} \circ \lambda_\rho = \gamma_\rho^{-1} \circ J_{\mathcal{A}*} \quad \textit{and} \quad J_{\mathcal{A}*} \circ \lambda_\eta = \gamma_\eta^{-1} \circ J_{\mathcal{A}*}.$$

Proof. We will first show that the endomorphisms $\lambda_\rho, \lambda_\eta$ on $K_0(\mathcal{A})$ are both injective. Put a projection $Q_\alpha = S_\alpha S_\alpha^*$ and a subalgebra $\mathcal{A}_\alpha = \rho_\alpha(\mathcal{A})$ of $\mathcal{A}$ for $\alpha \in \Sigma^\rho$. Then the endomorphism ρ_α on $\mathcal{A}$ extends to an isomorphism from $\mathcal{A}Q_\alpha$ onto $\mathcal{A}_\alpha$ by setting $\rho_\alpha(x) = S_\alpha^* x S_\alpha, x \in \mathcal{A}Q_\alpha$ whose inverse is $\phi_\alpha : \mathcal{A}_\alpha \longrightarrow \mathcal{A}Q_\alpha$ defined by $\phi_\alpha(y) = S_\alpha y S_\alpha^*, y \in \mathcal{A}_\alpha$. Hence the induced homomorphism $\rho_{\alpha*} : K_0(\mathcal{A}Q_\alpha) \longrightarrow K_0(\mathcal{A}_\alpha)$ is an isomorphism. Since $\mathcal{A} = \bigoplus_{\alpha\in\Sigma^\rho} Q_\alpha\mathcal{A}$, the homomorphism

$$\sum_{\alpha\in\Sigma^\rho} \phi_{\alpha*} \circ \rho_{\alpha*} : K_0(\mathcal{A}) \longrightarrow \bigoplus_{\alpha\in\Sigma^\rho} K_0(Q_\alpha\mathcal{A})$$

is an isomorphism, one may identify $K_0(\mathcal{A}) = \bigoplus_{\alpha\in\Sigma^\rho} K_0(Q_\alpha\mathcal{A})$. Let $g \in K_0(\mathcal{A})$ satisfy $\lambda_\rho(g) = 0$. Put $g_\alpha = \phi_{\alpha*} \circ \rho_{\alpha*}(g) \in K_0(Q_\alpha\mathcal{A})$ for $\alpha \in \Sigma^\rho$ so that $g = \sum_{\alpha\in\Sigma^\rho} g_\alpha$. As $\rho_{\beta*} \circ \phi_{\alpha*} = 0$ for $\beta \neq \alpha$, one sees $\rho_{\beta*}(g_\alpha) = 0$ for $\beta \neq \alpha$. Hence

$$0 = \lambda_\rho(g) = \sum_{\beta\in\Sigma^\rho} \sum_{\alpha\in\Sigma^\rho} \rho_{\beta*}(g_\alpha) = \sum_{\alpha\in\Sigma^\rho} \rho_{\alpha*}(g_\alpha) \in \bigoplus_{\alpha\in\Sigma^\rho} K_0(\mathcal{A}_\alpha).$$

It follows that $\rho_{\alpha*}(g_\alpha) = 0$ in $K_0(\mathcal{A}_\alpha)$. Since $\rho_{\alpha*} : K_0(Q_\alpha\mathcal{A}) \longrightarrow K_0(\mathcal{A}_\alpha)$ is isomorphic, one sees that $g_\alpha = 0$ in $K_0(\mathcal{A}Q_\alpha)$ for all $\alpha \in \Sigma^\rho$. This implies that $g = \sum_{\alpha\in\Sigma^\rho} g_\alpha = 0$ in $K_0(\mathcal{A})$. Therefore the endomorphism λ_ρ on $K_0(\mathcal{A})$ is injective, and similarly so is λ_η.

By the previous lemma, there exists an isomorphism $\Phi_{j,k} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{A})$ such that the diagram

$$\begin{array}{ccc} K_0(\mathcal{F}_{j,k}) & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{j+1,k}) \\ \Phi_{j,k}\downarrow & & \Phi_{j+1,k}\downarrow \\ K_0(\mathcal{A}) & \xrightarrow{\lambda_\rho} & K_0(\mathcal{A}) \end{array}$$

is commutative so that the embedding $\iota_{+1,*} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j+1,k})$ is injective, and similarly $\iota_{*,+1} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j,k+1})$ is injective. Hence for $n, m \in \mathbb{N}$, the homomorphism

$$\iota_{n,m} : K_0(\mathcal{A}) = K_0(\mathcal{F}_{0,0}) \longrightarrow K_0(\mathcal{F}_{n,m})$$

defined by the compositions of $\iota_{+1,*}$ and $\iota_{*,+1}$ is injective. By [44, Theorem 6.3.2 (iii)], one knows $\mathrm{Ker}(J_{\mathcal{A}*}) = \cup_{n,m\in\mathbb{N}}\mathrm{Ker}(\iota_{n,m})$, so that $\mathrm{Ker}(J_{\mathcal{A}*}) = 0$. □

We henceforth identify the group $K_0(\mathcal{A})$ with its image $J_{\mathcal{A}*}(K_0(\mathcal{A}))$ in $K_0(\mathcal{F}_{\rho,\eta})$. As in the above proof, not only $K_0(\mathcal{A})(= K_0(\mathcal{F}_{0,0}))$ but also the groups $K_0(\mathcal{F}_{j,k})$ for j, k are identified with subgroups of $K_0(\mathcal{F}_{\rho,\eta})$ via injective homomorphisms from $K_0(\mathcal{F}_{j,k})$ to $K_0(\mathcal{F}_{\rho,\eta})$ induced by the embeddings of $\mathcal{F}_{j,k}$ into $\mathcal{F}_{\rho,\eta}$. We note that

$$(\mathrm{id} - \gamma_\eta)K_0(\mathcal{F}_{\rho,\eta}) = (\mathrm{id} - \gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta}),$$
$$(\mathrm{id} - \gamma_\rho)K_0(\mathcal{F}_{\rho,\eta}) = (\mathrm{id} - \gamma_\rho^{-1})K_0(\mathcal{F}_{\rho,\eta})$$

and

$$\begin{aligned} &\mathrm{Ker}(\mathrm{id} - \gamma_\rho) \cap \mathrm{Ker}(\mathrm{id} - \gamma_\eta) \text{ in } K_0(\mathcal{F}_{\rho,\eta}) \\ &= \mathrm{Ker}(\mathrm{id} - \gamma_\rho^{-1}) \cap \mathrm{Ker}(\mathrm{id} - \gamma_\eta^{-1}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}). \end{aligned}$$

Denote by $(\mathrm{id} - \gamma_\rho)K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id} - \gamma_\eta)K_0(\mathcal{F}_{\rho,\eta})$ the subgroup of $K_0(\mathcal{F}_{\rho,\eta})$ generated by $(\mathrm{id} - \gamma_\rho)K_0(\mathcal{F}_{\rho,\eta})$ and $(\mathrm{id} - \gamma_\eta)K_0(\mathcal{F}_{\rho,\eta})$.

Lemma 8.15. *Any element in $K_0(\mathcal{F}_{\rho,\eta})$ is equivalent to some element of $K_0(\mathcal{A})$ modulo the subgroup $(\mathrm{id} - \gamma_\rho)K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id} - \gamma_\eta)K_0(\mathcal{F}_{\rho,\eta})$.*

Proof. For $g \in K_0(\mathcal{F}_{\rho,\eta})$, we may assume that $g \in K_0(\mathcal{F}_{j,k})$ for some $j, k \in \mathbb{Z}_+$. As γ_ρ^{-1} commutes with γ_η^{-1}, one sees that $(\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(g) \in K_0(\mathcal{A})$. Put $g_1 = \gamma_\rho^{-1}(g)$ so that

$$g - (\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(g) = g - \gamma_\rho^{-1}(g) + g_1 - (\gamma_\rho^{-1})^{j-1} \circ (\gamma_\eta^{-1})^k(g_1).$$

We inductively see that $g - (\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(g)$ belongs to the subgroup

$$(\mathrm{id} - \gamma_\rho)K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id} - \gamma_\eta)K_0(\mathcal{F}_{\rho,\eta}).$$ □

Denote by $(\mathrm{id} - \lambda_\rho)K_0(\mathcal{A}) + (\mathrm{id} - \lambda_\eta)K_0(\mathcal{A})$ the subgroup of $K_0(\mathcal{A})$ generated by $(\mathrm{id} - \lambda_\rho)K_0(\mathcal{A})$ and $(\mathrm{id} - \lambda_\eta)K_0(\mathcal{A})$.

Lemma 8.16. *If* $g \in K_0(\mathcal{A})$ *belongs to*

$$(\mathrm{id} - \gamma_\rho^{-1})K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id} - \gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta}),$$

then g *belongs to* $(\mathrm{id} - \lambda_\rho)K_0(\mathcal{A}) + (\mathrm{id} - \lambda_\eta)K_0(\mathcal{A})$.

Proof. By the assumption that $g \in (\mathrm{id} - \gamma_\rho^{-1})K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id} - \gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta})$, there exist $h_1, h_2 \in K_0(\mathcal{F}_{\rho,\eta})$ such that $g = (\mathrm{id} - \gamma_\rho^{-1})(h_1) + (\mathrm{id} - \gamma_\eta^{-1})(h_2)$. We may assume that $h_1, h_2 \in K_0(\mathcal{F}_{j,k})$ for large enough $j, k \in \mathbb{Z}_+$. Put $e_i = (\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(h_i)$ which belongs to $K_0(\mathcal{F}_{0,0})(= K_0(\mathcal{A}))$ for $i = 0, 1$. It follows that

$$\lambda_\rho^j \circ \lambda_\eta^k(g) = (\mathrm{id} - \lambda_\eta)(e_1) + (\mathrm{id} - \lambda_\rho)(e_2).$$

Since $g \in K_0(\mathcal{A})$ and $\lambda_\rho^j \circ \lambda_\eta^k(g) \in (\mathrm{id} - \lambda_\eta)K_0(\mathcal{A}) + (\mathrm{id} - \lambda_\rho)K_0(\mathcal{A})$, as in the proof of Lemma 8.15, by putting $g^{(n)} = \lambda_\rho^n(g), g^{(n,m)} = \lambda_\eta^m(g^{(n)}) \in K_0(\mathcal{A})$ we have

$$\begin{aligned}
&g - \lambda_\rho^j \circ \lambda_\eta^k(g) \\
&= g - \lambda_\rho(g) + g^{(1)} - \lambda_\rho(g^{(1)}) + g^{(2)} - \lambda_\rho(g^{(2)}) + \cdots + g^{(j-1)} - \lambda_\rho(g^{(j-1)}) \\
&\quad + g^{(j)} - \lambda_\eta(g^{(j)}) + g^{(j,1)} - \lambda_\eta(g^{(j,1)}) + g^{(j,2)} - \lambda_\eta(g^{(j,2)}) + \cdots \\
&\quad + g^{(j,k-1)} - \lambda_\eta(g^{(j,k-1)}) \\
&= (\mathrm{id} - \lambda_\rho)(g + g^{(1)} + \cdots + g^{(j-1)}) + (\mathrm{id} - \lambda_\eta)(g^{(j)} + g^{(j,1)} + \cdots + g^{(j,k-1)})
\end{aligned}$$

so that g belongs to the subgroup $(\mathrm{id} - \lambda_\eta)K_0(\mathcal{A}) + (\mathrm{id} - \lambda_\rho)K_0(\mathcal{A})$. □

Hence we obtain the following lemma for the cokernel.

Lemma 8.17. *The quotient group*

$$K_0(\mathcal{F}_{\rho,\eta})/((\mathrm{id} - \gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id} - \gamma_\rho^{-1})K_0(\mathcal{F}_{\rho,\eta}))$$

is isomorphic to the quotient group

$$K_0(\mathcal{A})/((\mathrm{id} - \lambda_\eta)K_0(\mathcal{A}) + (\mathrm{id} - \lambda_\rho)K_0(\mathcal{A})).$$

Proof. Surjectivity of the quotient map

$$K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{F}_{\rho,\eta})/((\mathrm{id} - \gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id} - \gamma_\rho^{-1})K_0(\mathcal{F}_{\rho,\eta}))$$

comes from Lemma 8.15. Its kernel coincides with

$$(\mathrm{id} - \lambda_\eta)K_0(\mathcal{A}) + (\mathrm{id} - \lambda_\rho)K_0(\mathcal{A})$$

by the preceding lemma. □

For the kernel, we have:

Lemma 8.18. *The subgroup*

$$\mathrm{Ker}(\mathrm{id} - \gamma_\eta^{-1}) \cap \mathrm{Ker}(\mathrm{id} - \gamma_\rho^{-1}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})$$

is isomorphic to the subgroup

$$\mathrm{Ker}(\mathrm{id} - \lambda_\eta) \cap \mathrm{Ker}(\mathrm{id} - \lambda_\rho) \text{ in } K_0(\mathcal{A})$$

through $J_{\mathcal{A}*}$.

Proof. For $g \in \mathrm{Ker}(\mathrm{id} - \gamma_\eta^{-1}) \cap \mathrm{Ker}(\mathrm{id} - \gamma_\rho^{-1})$ in $K_0(\mathcal{F}_{\rho,\eta})$, one may assume that $g \in K_0(\mathcal{F}_{j,k})$ for some $j, k \in \mathbb{Z}_+$ so that $g = (\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(g) \in K_0(\mathcal{A})$. Since $\lambda_\eta = \gamma_\eta^{-1}$ and $\lambda_\rho = \gamma_\rho^{-1}$ on $K_0(\mathcal{A})$ under the identification between $J_{\mathcal{A}*}(K_0(\mathcal{A}))$ and $K_0(\mathcal{A})$ via $J_{\mathcal{A}*}$, one has that $g \in \mathrm{Ker}(\mathrm{id}-\lambda_\eta) \cap \mathrm{Ker}(\mathrm{id}-\lambda_\rho)$ in $K_0(\mathcal{A})$. The converse inclusion relation

$$\mathrm{Ker}(\mathrm{id} - \lambda_\eta) \cap \mathrm{Ker}(\mathrm{id} - \lambda_\rho) \subset \mathrm{Ker}(\mathrm{id} - \gamma_\eta^{-1}) \cap \mathrm{Ker}(\mathrm{id} - \gamma_\rho^{-1})$$

is clear through the above identification. □

Therefore the short exact sequence for $K_0(\mathcal{O}^\kappa_{\rho,\eta})$ in Theorem 7.10 is restated as the following proposition.

Proposition 8.19. *Assume that* $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ *forms square and*

$$K_1(\mathcal{F}_{\rho,\eta}) = \{0\}.$$

Then there exists a short exact sequence:

$$\begin{aligned} 0 &\longrightarrow K_0(\mathcal{A})/((\mathrm{id} - \lambda_\eta)K_0(\mathcal{A}) + (\mathrm{id} - \lambda_\rho)K_0(\mathcal{A})) \\ &\longrightarrow K_0(\mathcal{O}^\kappa_{\rho,\eta}) \\ &\longrightarrow \mathrm{Ker}(\mathrm{id} - \lambda_\eta) \cap \mathrm{Ker}(\mathrm{id} - \lambda_\rho) \ \textit{in}\ K_0(\mathcal{A}) \\ &\longrightarrow 0. \end{aligned}$$

Let $\mathcal{F}_\rho$ be the fixed point algebra $(\mathcal{O}_\rho)^{\hat{\rho}}$ of the C^*-algebra $\mathcal{O}_\rho$ by the gauge action $\hat{\rho}$ for the C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma^\rho)$. The algebra $\mathcal{F}_\rho$ is isomorphic to the subalgebra $\mathcal{F}_{\rho,0}$ of $\mathcal{F}_{\rho,\eta}$ in a natural way. As in the proof of Lemma 8.15, the group $K_0(\mathcal{F}_{\rho,0})$ is regarded as a subgroup of $K_0(\mathcal{F}_{\rho,\eta})$ and the restriction of γ_η^{-1} to $K_0(\mathcal{F}_{\rho,0})$ satisfies $\gamma_\eta^{-1}(K_0(\mathcal{F}_{\rho,0})) \subset K_0(\mathcal{F}_{\rho,0})$ so that γ_η^{-1} yields an endomorphism on $K_0(\mathcal{F}_\rho)$, which we still denote by γ_η^{-1}.

For the group $K_1(\mathcal{O}^\kappa_{\rho,\eta})$, we provide several lemmas.

Lemma 8.20.

(i) *Any element in* $K_0(\mathcal{F}_{\rho,\eta})$ *is equivalent to some element of* $K_0(\mathcal{F}_{\rho,0})(= K_0(\mathcal{F}_\rho))$ *modulo the subgroup* $(\mathrm{id} - \gamma_\eta)K_0(\mathcal{F}_{\rho,\eta})$.

(ii) *If* $g \in K_0(\mathcal{F}_{\rho,0})(= K_0(\mathcal{F}_\rho))$ *belongs to* $(\mathrm{id} - \gamma_\eta)K_0(\mathcal{F}_{\rho,\eta})$, *then* g *belongs to* $(\mathrm{id} - \gamma_\eta)K_0(\mathcal{F}_\rho)$.

As γ_ρ commutes with γ_η on $K_0(\mathcal{F}_{\rho,\eta})$, it naturally acts on the quotient group $K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id} - \gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta})$. We denote it by $\bar{\gamma}_\rho$. Similarly λ_ρ naturally induces an endomorphism on $K_0(\mathcal{A})/(\mathrm{id} - \lambda_\eta)K_0(\mathcal{A})$. We denote it by $\bar{\lambda}_\rho$.

Lemma 8.21.

(i) *The quotient group* $K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta})$ *is isomorphic to the quotient group* $K_0(\mathcal{F}_\rho)/(\mathrm{id} - \gamma_\eta^{-1})K_0(\mathcal{F}_\rho)$, *that is also isomorphic to the quotient group* $K_0(\mathcal{A})/(\mathrm{id} - \lambda_\eta)K_0(\mathcal{A})$.

(ii) *The kernel of* $\mathrm{id} - \bar{\gamma}_\rho$ *in* $K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id} - \gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta})$ *is isomorphic to the kernel of* $\mathrm{id} - \bar{\lambda}_\rho$ *in* $K_0(\mathcal{A})/(\mathrm{id} - \lambda_\eta)K_0(\mathcal{A})$.

Proof. (i) The fact that the three quotient groups

$$K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id} - \gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta}),$$
$$K_0(\mathcal{F}_{\rho})/(\mathrm{id} - \gamma_\eta^{-1})K_0(\mathcal{F}_{\rho}),$$
$$K_0(\mathcal{A})/(\mathrm{id} - \lambda_\eta)K_0(\mathcal{A}),$$

are naturally isomorphic is similarly proved to the previous discussions.

(ii) The kernel $\mathrm{Ker}(\mathrm{id} - \bar{\gamma}_\rho)$ in $K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id} - \gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta})$ is isomorphic to the kernel $\mathrm{Ker}(\mathrm{id} - \bar{\gamma}_\rho)$ in $K_0(\mathcal{F}_{\rho})/(\mathrm{id} - \gamma_\eta^{-1})K_0(\mathcal{F}_{\rho})$ which is isomorphic to the kernel $\mathrm{Ker}(\mathrm{id} - \bar{\lambda}_\rho)$ in $K_0(\mathcal{A})/(\mathrm{id} - \lambda_\eta)K_0(\mathcal{A})$. □

Lemma 8.22. *The kernel of* $\mathrm{id} - \gamma_\rho$ *in* $K_0(\mathcal{F}_{\rho,\eta})$ *is isomorphic to the kernel of* $\mathrm{id} - \gamma_\rho$ *in* $K_0(\mathcal{F}_\rho)$ *that is also isomorphic to the kernel of* $\mathrm{id} - \lambda_\eta$ *in* $K_0(\mathcal{A})$ *such that the quotient group*

$$(\mathrm{Ker}(\mathrm{id} - \gamma_\eta) \textit{ in } K_0(\mathcal{F}_{\rho,\eta}))/(\mathrm{id} - \gamma_\rho)(\mathrm{Ker}(\mathrm{id} - \gamma_\eta) \textit{ in } K_0(\mathcal{F}_{\rho,\eta}))$$

is isomorphic to the quotient group

$$(\mathrm{Ker}(\mathrm{id} - \lambda_\eta) \textit{ in } K_0(\mathcal{A}))/(\mathrm{id} - \lambda_\rho)(\mathrm{Ker}(\mathrm{id} - \lambda_\eta) \textit{ in } K_0(\mathcal{A})).$$

Proof. The proofs are similar to the previous discussions. □

Therefore the short exact sequence for $K_1(\mathcal{O}^\kappa_{\rho,\eta})$ in Theorem 7.10 is restated as the following proposition.

Proposition 8.23. *Assume that* $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ *forms square and*

$$K_1(\mathcal{F}_{\rho,\eta}) = \{0\}.$$

Then there exists a short exact sequence:

$$\begin{aligned} 0 &\longrightarrow (\mathrm{Ker}(\mathrm{id} - \lambda_\eta) \textit{ in } K_0(\mathcal{A}))/(\mathrm{id} - \lambda_\rho)(\mathrm{Ker}(\mathrm{id} - \lambda_\eta) \textit{ in } K_0(\mathcal{A})) \\ &\longrightarrow K_1(\mathcal{O}^\kappa_{\rho,\eta}) \\ &\longrightarrow \mathrm{Ker}(\mathrm{id} - \bar{\lambda}_\rho) \textit{ in } (K_0(\mathcal{A})/(\mathrm{id} - \lambda_\eta)K_0(\mathcal{A})) \\ &\longrightarrow 0. \end{aligned}$$

We give a condition on $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ which makes $K_1(\mathcal{F}_{\rho,\eta}) = \{0\}$.

Lemma 8.24. *Suppose that a* C^**-textile dynamical system*

$$(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$$

forms square and satisfies $K_1(\mathcal{A}) = \{0\}$. *Then* $K_1(\mathcal{F}_{\rho,\eta}) = \{0\}$.

Proof. The algebra $\mathcal{F}_{\rho,\eta}$ is an inductive limit C^*-algebra of subalgebras $\mathcal{F}_{j,k}$ with inclusion maps (5.3). Let $E_i^l, i = 1, \ldots, m(l)$ be the minimal projections in $\mathcal{A}_l$ as in Lemma 8.4, which are central in $\mathcal{A}$ such that $\sum_{i=1}^{m(l)} E_i^l = 1$. By Lemma 8.4, we have

$$K_1(\mathcal{F}_{j,k}) = \bigoplus_{i=1}^{m(l)} K_1(\mathcal{F}_{j,k}(i)) = \bigoplus_{i=1}^{m(l)} K_1(E_i^l \mathcal{A} E_i^l) = K_1(\mathcal{A})$$

so that the condition $K_1(\mathcal{A}) = \{0\}$ implies $K_1(\mathcal{F}_{\rho,\eta}) = \{0\}$. □

A a C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to *have trivial* K_1 if $K_1(\mathcal{A}) = \{0\}$.

Consequently we reach the following K-theory formulae for the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ by Proposition 8.19 and Proposition 8.23.

Theorem 8.25. *Suppose that a C^*-textile dynamical system*

$$(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$$

forms square having trivial K_1. Then there exist short exact sequences for their K-groups as in the following way:

$$\begin{aligned} 0 &\longrightarrow K_0(\mathcal{A})/((\mathrm{id} - \lambda_\eta)K_0(\mathcal{A}) + (\mathrm{id} - \lambda_\rho)K_0(\mathcal{A})) \\ &\longrightarrow K_0(\mathcal{O}^\kappa_{\rho,\eta}) \\ &\longrightarrow \mathrm{Ker}(\mathrm{id} - \lambda_\eta) \cap \mathrm{Ker}(\mathrm{id} - \lambda_\rho) \text{ in } K_0(\mathcal{A}) \\ &\longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 &\longrightarrow (\mathrm{Ker}(\mathrm{id} - \lambda_\eta) \text{ in } K_0(\mathcal{A}))/(\mathrm{id} - \lambda_\rho)(\mathrm{Ker}(\mathrm{id} - \lambda_\eta) \text{ in } K_0(\mathcal{A})) \\ &\longrightarrow K_1(\mathcal{O}^\kappa_{\rho,\eta}) \\ &\longrightarrow \mathrm{Ker}(\mathrm{id} - \bar{\lambda}_\rho) \text{ in } (K_0(\mathcal{A})/(\mathrm{id} - \lambda_\eta)K_0(\mathcal{A})) \\ &\longrightarrow 0 \end{aligned}$$

where the endomorphisms $\lambda_\rho, \lambda_\eta : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})$ are defined by

$$\lambda_\rho([p]) = \sum_{\alpha \in \Sigma^\rho} [\rho_\alpha(p)] \in K_0(\mathcal{A}) \text{ for } [p] \in K_0(\mathcal{A}),$$

$$\lambda_\eta([p]) = \sum_{a \in \Sigma^\eta} [\eta_a(p)] \in K_0(\mathcal{A}) \text{ for } [p] \in K_0(\mathcal{A}).$$

9. Examples

9.1. LR-textile $\boldsymbol{\lambda}$-graph systems. A symbolic matrix

$$\mathcal{M} = [\mathcal{M}(i,j)]_{i,j=1}^N$$

is a matrix whose components consist of formal sums of elements of an alphabet Σ, such as

$$\mathcal{M} = \begin{bmatrix} a & a+c \\ c & 0 \end{bmatrix} \qquad \text{where } \Sigma = \{a, b, c\}.$$

$\mathcal{M}$ is said to be essential if there is no zero column or zero row. $\mathcal{M}$ is said to be left-resolving if for each column a symbol does not appear in two different rows. For example, $\begin{bmatrix} a & a+b \\ c & 0 \end{bmatrix}$ is left-resolving, but $\begin{bmatrix} a & a+b \\ c & b \end{bmatrix}$ is not left-resolving because of b at the second column. We assume that symbolic matrices are always essential and left-resolving. We denote by $\Sigma^{\mathcal{M}}$ the alphabet Σ of the symbolic matrix $\mathcal{M}$.

Let $\mathcal{M} = [\mathcal{M}(i,j)]_{i,j=1}^N$ and $\mathcal{M}' = [\mathcal{M}'(i,j)]_{i,j=1}^N$ be $N \times N$ symbolic matrices over $\Sigma^{\mathcal{M}}$ and $\Sigma^{\mathcal{M}'}$ respectively. Suppose that there is a bijection $\kappa : \Sigma^{\mathcal{M}} \longrightarrow \Sigma^{\mathcal{M}'}$. Following Nasu's terminology [34] we say that $\mathcal{M}$ and $\mathcal{M}'$ are equivalent under specification κ, or simply, specified equivalent if $\mathcal{M}'$ can be obtained from $\mathcal{M}$ by replacing every symbol $\alpha \in \Sigma^{\mathcal{M}}$ by $\kappa(\alpha) \in \Sigma^{\mathcal{M}'}$. That is if $\mathcal{M}(i,j) = \alpha_1 + \cdots + \alpha_n$, then $\mathcal{M}'(i,j) = \kappa(\alpha_1) + \cdots + \kappa(\alpha_n)$. We write this situation as $\mathcal{M} \overset{\kappa}{\cong} \mathcal{M}'$ (see [34]).

For a symbolic matrix $\mathcal{M} = [\mathcal{M}(i,j)]_{i,j=1}^N$ over $\Sigma^{\mathcal{M}}$, we set for $\alpha \in \Sigma^{\mathcal{M}}, i,j = 1,\dots,N$

$$A^{\mathcal{M}}(i,\alpha,j) = \begin{cases} 1 & \text{if } \alpha \text{ appears in } \mathcal{M}(i,j), \\ 0 & \text{otherwise.} \end{cases}$$

Put an $N \times N$ nonnegative matrix $A^{\mathcal{M}} = [A^{\mathcal{M}}(i,j)]_{i,j=1}^N$ by setting

$$A^{\mathcal{M}}(i,j) = \sum_{\alpha \in \Sigma^{\mathcal{M}}} A^{\mathcal{M}}(i,\alpha,j).$$

Let $\mathcal{A}$ be an N-dimensional commutative C^*-algebra $\mathbb{C}^N$ with minimal projections $E_1,\dots,E_N$ such that

$$\mathcal{A} = \mathbb{C}E_1 \oplus \cdots \oplus \mathbb{C}E_N.$$

We set for $\alpha \in \Sigma^{\mathcal{M}}$:

$$\rho_\alpha^{\mathcal{M}}(E_i) = \sum_{j=1}^N A^{\mathcal{M}}(i,\alpha,j)E_j, \qquad i = 1,\dots,N.$$

Then we have a C^*-symbolic dynamical system $(\mathcal{A}, \rho^{\mathcal{M}}, \Sigma^{\mathcal{M}})$.

Let $\mathcal{M} = [\mathcal{M}(i,j)]_{i,j=1}^N$ and $\mathcal{N} = [\mathcal{N}(i,j)]_{i,j=1}^N$ be $N \times N$ symbolic matrices over $\Sigma^{\mathcal{M}}$ and $\Sigma^{\mathcal{N}}$ respectively. We have two C^*-symbolic dynamical systems $(\mathcal{A}, \rho^{\mathcal{M}}, \Sigma^{\mathcal{M}})$ and $(\mathcal{A}, \rho^{\mathcal{N}}, \Sigma^{\mathcal{N}})$. Put

$$\Sigma^{\mathcal{M}\mathcal{N}} = \{(\alpha, b) \in \Sigma^{\mathcal{M}} \times \Sigma^{\mathcal{N}} \mid \rho_b^{\mathcal{N}} \circ \rho_\alpha^{\mathcal{M}} \neq 0\},$$
$$\Sigma^{\mathcal{N}\mathcal{M}} = \{(a, \beta) \in \Sigma^{\mathcal{N}} \times \Sigma^{\mathcal{M}} \mid \rho_\beta^{\mathcal{M}} \circ \rho_a^{\mathcal{N}} \neq 0\}.$$

Suppose that there is a bijection κ from $\Sigma^{\mathcal{M}\mathcal{N}}$ to $\Sigma^{\mathcal{N}\mathcal{M}}$ such that κ yields a specified equivalence

$$\mathcal{M}\mathcal{N} \overset{\kappa}{\cong} \mathcal{N}\mathcal{M} \tag{9.1}$$

and fix it.

Proposition 9.1. *Keep the above situations. The specified equivalence* (9.1) *induces a specification* $\kappa : \Sigma^{\mathcal{M}\mathcal{N}} \longrightarrow \Sigma^{\mathcal{N}\mathcal{M}}$ *such that*

$$\rho^{\mathcal{N}}_b \circ \rho^{\mathcal{M}}_\alpha = \rho^{\mathcal{M}}_\beta \circ \rho^{\mathcal{N}}_a \quad \textit{if} \quad \kappa(\alpha, b) = (a, \beta). \tag{9.2}$$

Hence $(\mathcal{A}, \rho^{\mathcal{M}}, \rho^{\mathcal{N}}, \Sigma^{\mathcal{M}}, \Sigma^{\mathcal{N}}, \kappa)$ *gives rise to a* C^**-textile dynamical system which forms square having trivial* K_1.

Proof. Since $\mathcal{M}\mathcal{N} \overset{\kappa}{\cong} \mathcal{N}\mathcal{M}$, one sees that for $i, j = 1, 2, \dots, N$,

$$\kappa(\mathcal{M}\mathcal{N}(i, j)) = \mathcal{N}\mathcal{M}(i, j).$$

For $(\alpha, b) \in \Sigma^{\mathcal{M}\mathcal{N}}$, there exists $i, k = 1, 2, \dots, N$ such that

$$\rho^{\mathcal{N}}_b \circ \rho^{\mathcal{M}}_\alpha(E_i) \ge E_k.$$

As $\kappa(\alpha, b)$ appears in $\mathcal{N}\mathcal{M}(i, k)$, by putting $(a, \beta) = \kappa(\alpha, b)$, we have

$$\rho^{\mathcal{M}}_\beta \circ \rho^{\mathcal{N}}_a(E_i) \ge E_k.$$

Hence $\kappa(\alpha, b) \in \Sigma^{\mathcal{N}\mathcal{M}}$. One indeed sees that $\rho^{\mathcal{N}}_b \circ \rho^{\mathcal{M}}_\alpha = \rho^{\mathcal{M}}_\beta \circ \rho^{\mathcal{N}}_a$ by the relation $\mathcal{M}\mathcal{N} \overset{\kappa}{\cong} \mathcal{N}\mathcal{M}$. □

Two symbolic matrices satisfying (9.1) give rise to an LR textile system that has been introduced by Nasu (see [34]). Textile systems introduced by Nasu give a strong tool to analyze automorphisms and endomorphisms of topological Markov shifts. The author has generalized LR-textile systems to LR-textile λ-graph systems which consist of two pairs of sequences $(\mathcal{M}, I) = (\mathcal{M}_{l,l+1}, I_{l,l+1})_{l\in\mathbb{Z}_+}$ and $(\mathcal{N}, I) = (\mathcal{N}_{l,l+1}, I_{l,l+1})_{l\in\mathbb{Z}_+}$ such that

$$\mathcal{M}_{l,l+1}\mathcal{N}_{l+1,l+2} \overset{\kappa}{\cong} \mathcal{N}_{l,l+1}\mathcal{M}_{l+1,l+2}, \qquad l \in \mathbb{Z}_+ \tag{9.3}$$

through a specification κ ([28]). We denote the LR-textile λ-graph system by $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$. Denote by $\mathfrak{L}^{\mathcal{M}}$ and $\mathfrak{L}^{\mathcal{N}}$ the associated λ-graph systems respectively. Since $\mathfrak{L}^{\mathcal{M}}$ and $\mathfrak{L}^{\mathcal{N}}$ have common sequences $V^{\mathcal{M}}_l = V^{\mathcal{N}}_l, l \in \mathbb{Z}_+$ of vertices which denoted by $V_l, l \in \mathbb{Z}_+$, and its common inclusion matrices $I_{l,l+1}, l \in \mathbb{Z}_+$. Hence $\mathfrak{L}^{\mathcal{M}}$ and $\mathfrak{L}^{\mathcal{N}}$ form square in the sense of [28, p.170]. Let $(\mathcal{A}_{\mathcal{M}}, \rho^{\mathcal{M}}, \Sigma^{\mathcal{M}})$ and $(\mathcal{A}_{\mathcal{N}}, \rho^{\mathcal{N}}, \Sigma^{\mathcal{N}})$ be the associated C^*-symbolic dynamical systems with the λ-graph systems $\mathfrak{L}^{\mathcal{M}}$ and $\mathfrak{L}^{\mathcal{N}}$ respectively. Since both the algebras $\mathcal{A}_{\mathcal{M}}$ and $\mathcal{A}_{\mathcal{N}}$ are the C^*-algebras of inductive limit of the system $I^*_{l,l+1} : C(V_l) \to C(V_{l+1}), l \in \mathbb{Z}_+$, they are identical, which is denoted by $\mathcal{A}$. It is easy to see that the relation (9.3) implies

$$\rho^{\mathcal{M}}_\alpha \circ \rho^{\mathcal{N}}_b = \rho^{\mathcal{N}}_a \circ \rho^{\mathcal{M}}_\beta \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta). \tag{9.4}$$

Proposition 9.2. *An LR-textile* λ*-graph system* $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ *yields a* C^**-textile dynamical system* $(\mathcal{A}, \rho^{\mathcal{M}}, \rho^{\mathcal{N}}, \Sigma^{\mathcal{M}}, \Sigma^{\mathcal{N}}, \kappa)$ *which forms square. Conversely, a* C^**-textile dynamical system* $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ *which forms square yields* *an LR-textile λ-graph system $\mathcal{T}_{\mathcal{K}^{\mathcal{M}^\rho}_{\mathcal{M}^\eta}}$ such that the associated C^*-textile dynamical system written $(\mathcal{A}_{\rho,\eta}, \rho^{\mathcal{M}^\rho}, \rho^{\mathcal{M}^\eta}, \Sigma^{\mathcal{M}^\rho}, \Sigma^{\mathcal{M}^\rho}, \kappa)$ is a subsystem of $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ in the sense that the relations:*

$$\mathcal{A}_{\rho,\eta} \subset \mathcal{A}, \qquad \rho|_{\mathcal{A}_{\rho,\eta}} = \rho^{\mathcal{M}^\rho}, \qquad \eta|_{\mathcal{A}_{\rho,\eta}} = \rho^{\mathcal{M}^\eta}$$

hold.

Proof. Let $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ be an LR-textile λ-graph system. As in the above discussions, we have a C^*-textile dynamical system $(\mathcal{A}, \rho^{\mathcal{M}}, \rho^{\mathcal{N}}, \Sigma^{\mathcal{M}}, \Sigma^{\mathcal{N}}, \kappa)$. Conversely, let $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ be a C^*-textile dynamical system which forms square. Put for $l \in \mathbb{N}$

$$\mathcal{A}_l^\rho = C^*(\rho_\mu(1) : \mu \in B_l(\Lambda_\rho)), \qquad \mathcal{A}_l^\eta = C^*(\eta_\xi(1) : \xi \in B_l(\Lambda_\eta)).$$

Since $\mathcal{A}_l^\rho = \mathcal{A}_l^\eta$ and they are commutative and of finite dimensional, the algebra

$$\mathcal{A}_{\rho,\eta} = \overline{\cup_{l\in\mathbb{Z}_+}\mathcal{A}_l^\rho} = \overline{\cup_{l\in\mathbb{Z}_+}\mathcal{A}_l^\eta}$$

is a commutative AF-subalgebra of $\mathcal{A}$. It is easy to see that both $(\mathcal{A}_{\rho,\eta}, \rho, \Sigma^\rho)$ and $(\mathcal{A}_{\rho,\eta}, \eta, \Sigma^\eta)$ are C^*-symbolic dynamical systems such that

$$\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta) \tag{9.5}$$

By [27], there exist λ-graph systems $\mathfrak{L}^\rho$ and $\mathfrak{L}^\eta$ whose C^*-symbolic dynamical systems are $(\mathcal{A}_{\rho,\eta}, \rho, \Sigma^\rho)$ and $(\mathcal{A}_{\rho,\eta}, \eta, \Sigma^\eta)$ respectively. Let $(\mathcal{M}^\rho, I^\rho)$ and $(\mathcal{M}^\eta, I^\eta)$ be the associated symbolic matrix systems. It is easy to see that the relation (9.5) implies

$$\mathcal{M}^\rho_{l,l+1}\mathcal{M}^\eta_{l+1,l+2} \overset{\kappa}{\cong} \mathcal{M}^\eta_{l,l+1}\mathcal{M}^\rho_{l+1,l+2}, \qquad l \in \mathbb{Z}_+.$$

Hence we have an LR-textile λ-graph system $\mathcal{T}_{\mathcal{K}^{\mathcal{M}^\rho}_{\mathcal{M}^\eta}}$. It is direct to see that the associated C^*-textile dynamical system is $(\mathcal{A}_{\rho,\eta}, \rho|_{\mathcal{A}_{\rho,\eta}}, \eta|_{\mathcal{A}_{\rho,\eta}}, \Sigma^\rho, \Sigma^\eta, \kappa)$. □

Let A be an $N \times N$ matrix with entries in nonnegative integers. We may consider a directed graph $G_A = (V_A, E_A)$ with vertex set V_A and edge set E_A. The vertex set V_A consists of N vertices which we denote by $\{v_1, \dots, v_N\}$. We equip $A(i,j)$ edges from the vertex v_i to the vertex v_j. Denote by E_A the set of the edges. Let $\Sigma^A = E_A$ and the labeling map $\lambda_A : E_A \longrightarrow \Sigma^A$ be defined as the identity map. Then we have a labeled directed graph denoted by G_A as well as a symbolic matrix $\mathcal{M}_A = [\mathcal{M}_A(i,j)]_{i,j=1}^N$ by setting

$$\mathcal{M}_A(i,j) = \begin{cases} e_1 + \cdots + e_n & \text{if } e_1, \dots, e_n \text{ are edges from } v_i \text{ to } v_j, \\ 0 & \text{if there is no edge from } v_i \text{ to } v_j. \end{cases}$$

Let B be an $N \times N$ matrix with entries in nonnegative integers such that

$$AB = BA. \tag{9.6}$$

The equality (9.6) implies that the cardinal numbers of the sets of the pairs of directed edges

$$\Sigma^{AB}(i,j) = \{(e,f) \in E_A \times E_B \mid s(e) = v_i, t(e) = s(f), t(f) = v_j\} \text{ and}$$
$$\Sigma^{BA}(i,j) = \{(f,e) \in E_B \times E_A \mid s(f) = v_i, t(f) = s(e), t(e) = v_j\}$$

coincide with each other for each v_i and v_j. We put $\Sigma^{AB} = \cup_{i,j=1}^N \Sigma^{AB}(i,j)$ and $\Sigma^{BA} = \cup_{i,j=1}^N \Sigma^{BA}(i,j)$ so that one may take a bijection $\kappa : \Sigma^{AB} \longrightarrow \Sigma^{BA}$ which gives rise to a specified equivalence $\mathcal{M}_A\mathcal{M}_B \overset{\kappa}{\cong} \mathcal{M}_B\mathcal{M}_A$. We then have a C^*-textile dynamical system

$$(\mathcal{A}, \rho^{\mathcal{M}_A}, \rho^{\mathcal{M}_B}, \Sigma^A, \Sigma^B, \kappa)$$

which we denote by

$$(\mathcal{A}, \rho^A, \rho^B, \Sigma^A, \Sigma^B, \kappa).$$

The associated C^*-algebra is denoted by $\mathcal{O}^\kappa_{A,B}$. The algebra $\mathcal{O}^\kappa_{A,B}$ depends on the choice of a specification $\kappa : \Sigma^{AB} \longrightarrow \Sigma^{BA}$. The algebras are 2-graph algebras of Kumjian and Pask [19]. They are also C^*-algebras associated to textile systems studied by V. Deaconu [9]. By Theorem 8.25, we have:

Proposition 9.3. *Keep the above situations. There exist short exact sequences:*

$$\begin{aligned} 0 &\longrightarrow \mathbb{Z}^N/((1-A)\mathbb{Z}^N + (1-B)\mathbb{Z}^N) \\ &\longrightarrow K_0(\mathcal{O}^\kappa_{A,B}) \\ &\longrightarrow \operatorname{Ker}(1-A) \cap \operatorname{Ker}(1-B) \ \textit{in } \mathbb{Z}^N \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 &\longrightarrow (\operatorname{Ker}(1-B) \ \textit{in } \mathbb{Z}^N)/(1-A)(\operatorname{Ker}(1-B) \ \textit{in } \mathbb{Z}^N) \\ &\longrightarrow K_1(\mathcal{O}^\kappa_{A,B}) \\ &\longrightarrow \operatorname{Ker}(1-A) \ \textit{in } \ \mathbb{Z}^N/(1-B)\mathbb{Z}^N \longrightarrow 0. \end{aligned}$$

We consider 1×1 matrices $[N]$ and $[M]$ with its entries N and M respectively for $1 < N, M \in \mathbb{N}$. Let G_N be a directed graph with one vertex and N directed self-loops. Similarly we consider a directed graph G_M with M directed self-loops at the vertex. The self-loops are denoted by $\Sigma^N = \{e_1, \dots, e_N\}$ and $\Sigma^M = \{f_1, \dots, f_M\}$ respectively. As a specification κ, we take the exchanging map $(e,f) \in \Sigma^N \times \Sigma^M \longrightarrow (f,e) \in \Sigma^M \times \Sigma^N$ which we will fix. Put

$$\rho^N_{e_i}(1) = 1, \qquad \rho^M_{f_j}(1) = 1 \qquad \text{for } i = 1, \dots, N, \ j = 1, \dots, M.$$

Then we have a C^*-textile dynamical system

$$(\mathbb{C}, \rho^N, \rho^M, \Sigma^N, \Sigma^M, \kappa).$$

The associated C^*-algebra is denoted by $\mathcal{O}^\kappa_{N,M}$.

Lemma 9.4. $\mathcal{O}^{\kappa}_{N,M} = \mathcal{O}_N \otimes \mathcal{O}_M$.

Proof. Let $s_i, i = 1, \dots, N$ and $t_j, i = 1, \dots, M$ be the generating isometries of the Cuntz algebra $\mathcal{O}_N$ and those of of $\mathcal{O}_M$ respectively which satisfy

$$\sum_{i=1}^{N} s_i s_i^* = 1, \qquad \sum_{j=1}^{M} t_j t_j^* = 1.$$

Let $S_i, i = 1, \dots, N$ and $T_j, i = 1, \dots, M$ be the generating isometries of $\mathcal{O}^{\kappa}_{N,M}$ satisfying

$$\sum_{i=1}^{N} S_i S_i^* = 1, \qquad \sum_{j=1}^{M} T_j T_j^* = 1$$

and

$$S_i T_j = T_j S_i, \qquad i = 1, \dots, N, \quad j = 1, \dots, M.$$

The universality of $\mathcal{O}^{\kappa}_{N,M}$ subject to the relations and that of the tensor product $\mathcal{O}_N \otimes \mathcal{O}_M$ ensure us that the correspondence $\Phi : \mathcal{O}_{N,M} \longrightarrow \mathcal{O}_N \otimes \mathcal{O}_M$ given by $\Phi(S_i) = s_i \otimes 1,\ \Phi(T_j) = 1 \otimes t_j$ yields an isomorphism. □

Although we may easily compute the K-groups $K_*(\mathcal{O}^{\kappa}_{M,N})$ by using the Künneth formula for $K_i(\mathcal{O}_N \otimes \mathcal{O}_M)$ ([46]), we will compute them by Proposition 9.3 as in the following way.

Proposition 9.5 (cf. [19])**.** *For $1 < N, M \in \mathbb{N}$, the C^*-algebra $\mathcal{O}^{\kappa}_{N,M}$ is simple, purely infinite, such that*

$$K_0(\mathcal{O}^{\kappa}_{N,M}) \cong K_1(\mathcal{O}^{\kappa}_{N,M}) \cong \mathbb{Z}/d\mathbb{Z}$$

where $d = gcd(N-1, M-1)$ the greatest common divisor of $N-1, M-1$.

Proof. It is easy to see that the group $\mathbb{Z}/((N-1)\mathbb{Z}+(N-1)\mathbb{Z})$ is isomorphic to $\mathbb{Z}/d\mathbb{Z}$. As $\mathrm{Ker}(N-1) = \mathrm{Ker}(M-1) = 0$ in $\mathbb{Z}$, we see that

$$K_0(\mathcal{O}^{\kappa}_{N,M}) \cong \mathbb{Z}/d\mathbb{Z}.$$

It is elementary to see that the subgroup

$$\{[k] \in \mathbb{Z}/(M-1)\mathbb{Z} \mid (N-1)k \in (M-1)\mathbb{Z}\}$$

of $\mathbb{Z}/(M-1)\mathbb{Z}$ is isomorphic to $\mathbb{Z}/d\mathbb{Z}$. Hence we have

$$K_1(\mathcal{O}^{\kappa}_{N,M}) \cong \mathbb{Z}/d\mathbb{Z}.$$

□

We will generalize the above examples from the view point of tensor products.

9.2. Tensor products. Let $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ be C^*-symbolic dynamical systems. We will construct a C^*-textile dynamical system by taking tensor product. Put

$$\bar{\mathcal{A}} = \mathcal{A}^\rho \otimes \mathcal{A}^\eta, \qquad \bar{\rho}_\alpha = \rho_\alpha \otimes \mathrm{id}, \qquad \bar{\eta}_a = \mathrm{id} \otimes \eta_a, \qquad \Sigma^{\bar{\rho}} = \Sigma^\rho, \qquad \Sigma^{\bar{\eta}} = \Sigma^\eta$$

for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, where $\otimes$ means the minimal C^*-tensor product $\otimes_{\min}$. For $(\alpha, a) \in \Sigma^\rho \times \Sigma^\eta$, we see $\eta_b \circ \rho_\alpha(1) \neq 0$ if and only if $\eta_b(1) \neq 0, \rho_\alpha(1) \neq 0$, so that

$$\Sigma^{\bar{\rho}\bar{\eta}} = \Sigma^\rho \times \Sigma^\eta \quad \text{and similarly} \quad \Sigma^{\bar{\eta}\bar{\rho}} = \Sigma^\eta \times \Sigma^\rho.$$

Define $\bar{\kappa} : \Sigma^{\bar{\rho}\bar{\eta}} \longrightarrow \Sigma^{\bar{\eta}\bar{\rho}}$ by setting $\bar{\kappa}(\alpha, b) = (b, \alpha)$.

Lemma 9.6. *$(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is a C^*-textile dynamical system.*

Proof. By [2], we have $Z_{\bar{\mathcal{A}}} = Z_{\mathcal{A}^\rho} \otimes Z_{\mathcal{A}^\eta}$ so that

$$\bar{\rho}_\alpha(Z_{\bar{\mathcal{A}}}) \subset Z_{\bar{\mathcal{A}}}, \quad \alpha \in \Sigma^{\bar{\rho}} \quad \text{and} \quad \bar{\rho}_a(Z_{\bar{\mathcal{A}}}) \subset Z_{\bar{\mathcal{A}}}, \quad a \in \Sigma^{\bar{\eta}}.$$

We also have $\sum_{\alpha \in \Sigma^{\bar{\rho}}} \bar{\rho}_\alpha(1) = \sum_{\alpha \in \Sigma^\rho} \rho_\alpha(1) \otimes 1 \geq 1$, and similarly

$$\sum_{a \in \Sigma^{\bar{\eta}}} \bar{\eta}_(1) \geq 1$$

so that both families $\{\bar{\rho}_\alpha\}_{\alpha \in \Sigma^{\bar{\rho}}}$ and $\{\bar{\eta}_a\}_{a \in \Sigma^{\bar{\eta}}}$ of endomorphisms are essential. Since $\{\rho_\alpha\}_{\alpha \in \Sigma^\rho}$ is faithful on $\mathcal{A}^\rho$, the homomorphism

$$x \in \mathcal{A}^\rho \longrightarrow \sum_{\alpha \in \Sigma^\rho}{}^{\oplus} \rho_\alpha(x) \in \sum_{\alpha \in \Sigma^\rho}{}^{\oplus} \mathcal{A}^\rho$$

is injective so that the homomorphism

$$x \otimes y \in \mathcal{A}^\rho \otimes \mathcal{A}^\eta \longrightarrow \sum_{\alpha \in \Sigma^\rho}{}^{\oplus} \rho_\alpha(x) \otimes y \in \sum_{\alpha \in \Sigma^\rho}{}^{\oplus} \mathcal{A}^\rho \otimes \mathcal{A}^\eta$$

is injective. This implies that $\{\bar{\rho}_\alpha\}_{\alpha \in \Sigma^{\bar{\rho}}}$ is faithful. Similarly, so is $\{\bar{\eta}_a\}_{a \in \Sigma^{\bar{\eta}}}$. Hence $(\bar{\mathcal{A}}, \bar{\rho}, \Sigma^{\bar{\rho}})$ and $(\bar{\mathcal{A}}, \bar{\eta}, \Sigma^{\bar{\eta}})$ are both C^*-symbolic dynamical systems. It is direct to see that $\bar{\eta}_b \circ \bar{\rho}_\alpha = \bar{\rho}_\alpha \circ \bar{\eta}_b$ for $(\alpha, b) \in \Sigma^{\bar{\rho}\bar{\eta}}$. Therefore $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is a C^*-textile dynamical system. ∎

We call $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ the tensor product between $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$. Denote by $S_\alpha, \alpha \in \Sigma^{\bar{\rho}}$, $T_a, a \in \Sigma^{\bar{\eta}}$ the generating partial isometries of the C^*-algebra $\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}}$ for the C^*-textile dynamical system

$$(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa}).$$

By the universality for the algebra $\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}}$ subject to the relations $(\bar{\rho}, \bar{\eta}; \bar{\kappa})$, the algebra $\mathcal{D}_{\bar{\rho},\bar{\eta}}$ is isomorphic to the tensor product $\mathcal{D}_\rho \otimes \mathcal{D}_\eta$ through the correspondence

$$S_\mu T_\xi (x \otimes y) T_\xi^* S_\mu^* \longleftrightarrow S_\mu x S_\mu^* \otimes T_\xi y T_\xi^*$$

for $\mu \in B_*(\Lambda_\rho), \xi \in B_*(\Lambda_\eta), \ x \in A^\rho, y \in \mathcal{A}^\eta$.

Lemma 9.7. *Suppose that $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are both free (resp. AF-free). Then the tensor product $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is free (resp. AF-free).*

Proof. Suppose that $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are both free. There exist increasing sequences $\mathcal{A}_l^\rho, l \in \mathbb{Z}_+$ and $\mathcal{A}_l^\eta, l \in \mathbb{Z}_+$ of C^*-subalgebras of $\mathcal{A}^\rho$ and $\mathcal{A}^\eta$ satisfying the conditions of their freeness respectively. Put

$$\bar{\mathcal{A}}_l = \mathcal{A}_l^\rho \otimes \mathcal{A}_l^\eta, \quad l \in \mathbb{Z}_+.$$

It is clear that:

(1) $\bar{\rho}_\alpha(\bar{\mathcal{A}}_l) \subset \bar{\mathcal{A}}_{l+1}, \alpha \in \Sigma^{\bar{\rho}}$ and $\bar{\eta}_a(\bar{\mathcal{A}}_l) \subset \bar{\mathcal{A}}_{l+1}, a \in \Sigma^{\bar{\eta}}$ for $l \in \mathbb{Z}_+$.

(2) $\cup_{l\in\mathbb{Z}_+}\bar{\mathcal{A}}_l$ is dense in $\bar{\mathcal{A}}$.

We will show that the condition (3) for $\bar{\mathcal{A}}$ in Definition 5.3 holds. Take and fix arbitrary $j, k, l \in \mathbb{N}$ with $j + k \le l$. For $j \le l$, one may take a projection $q_\rho \in \mathcal{D}_\rho \cap \mathcal{A}_l^{\rho\prime}$ satisfying the condition (3) of the freeness of $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$, and similarly for $k \le l$, one may take a projection $q_\eta \in \mathcal{D}_\eta \cap \mathcal{A}_l^{\eta\prime}$. Put $q = q_\rho \otimes q_\eta \in \mathcal{D}_\rho \otimes \mathcal{D}_\eta (= \mathcal{D}_{\bar{\rho},\bar{\eta}})$ so that $q \in \mathcal{D}_{\bar{\rho},\bar{\eta}} \cap \bar{\mathcal{A}}_l'$. As the maps $\Phi_l^\rho : x \in \mathcal{A}_l^\rho \longrightarrow q_\rho x \in q_\rho \mathcal{A}_l^\rho$ and $\Phi_l^\eta : y \in \mathcal{A}_l^\eta \longrightarrow q_\eta x \in q_\eta \mathcal{A}_l^\eta$ are both isomorphisms, the tensor product

$$\Phi_l^\rho \otimes \Phi_l^\eta : x \otimes y \in \mathcal{A}_l^\rho \otimes \mathcal{A}_l^\eta \longrightarrow (q_\rho \otimes q_\eta)(x \otimes y) \in (q_\rho \otimes q_\eta)(\mathcal{A}_l^\rho \otimes \mathcal{A}_l^\eta)$$

is isomorphic. Hence $qa \ne 0$ for $0 \ne a \in \bar{\mathcal{A}}_l$. It is straightforward to see that q satisfies the condition (3) (ii) of Definition 5.3. Therefore the tensor product $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is free. It is obvious to see that if both $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are AF-free, then $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is AF-free. □

Proposition 9.8. *Suppose that $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are both free. Then the C^*-algebra $\mathcal{O}_{\bar{\rho},\bar{\eta}}^{\bar{\kappa}}$ for the tensor product C^*-textile dynamical system $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is isomorphic to the minimal tensor product $\mathcal{O}_\rho \otimes \mathcal{O}_\eta$ of the C^*-algebras between $\mathcal{O}_\rho$ and $\mathcal{O}_\eta$. If in particular, $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are both irreducible, the C^*-algebra $\mathcal{O}_{\bar{\rho},\bar{\eta}}^{\bar{\kappa}}$ is simple.*

Proof. Suppose that $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are both free. By the preceding lemma, the tensor product $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is free and hence satisfies condition (I). Let $s_\alpha, \alpha \in \Sigma^\rho$ and $t_a, a \in \Sigma^\eta$ be the generating partial isometries of the C^*-algebras $\mathcal{O}_\rho$ and $\mathcal{O}_\eta$ respectively. Let $S_\alpha, \alpha \in \Sigma^{\bar{\rho}}$ and $T_a, a \in \Sigma^{\bar{\eta}}$ be the generating partial isometries of the C^*-algebra $\mathcal{O}_{\bar{\rho},\bar{\eta}}^{\bar{\kappa}}$. By the uniqueness of the algebra $\mathcal{O}_{\bar{\rho},\bar{\eta}}^{\bar{\kappa}}$ with respect to the relations $(\bar{\rho}, \bar{\eta}; \bar{\kappa})$, the correspondence

$$S_\alpha \longrightarrow s_\alpha \otimes 1 \in \mathcal{O}_\rho \otimes \mathcal{O}_\eta, \qquad T_a \longrightarrow 1 \otimes t_a \in \mathcal{O}_\rho \otimes \mathcal{O}_\eta$$

naturally gives rise to an isomorphism from $\mathcal{O}_{\bar{\rho},\bar{\eta}}^{\bar{\kappa}}$ onto the tensor product $\mathcal{O}_\rho \otimes \mathcal{O}_\eta$.

If in particular, $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are both irreducible, the C^*-algebras $\mathcal{O}_\rho$ and $\mathcal{O}_\eta$ are both simple so that $\mathcal{O}_{\bar{\rho},\bar{\eta}}^{\bar{\kappa}}$ is simple. □

We remark that the tensor product $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ does not necessarily form square. The K-theory groups $K_*(\mathcal{O}_{\bar{\rho},\bar{\eta}}^{\bar{\kappa}})$ are computed from the Künneth formulae for $K_*(\mathcal{O}_\rho \otimes \mathcal{O}_\eta)$ [46].

10. Concluding remark

In [31], a different construction of C^*-algebra written $\mathcal{O}_{\mathcal{H}_\kappa}$ from C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is studied by using a 2-dimensional analogue of Hilbert C^*-bimodule. The C^*-algebra $\mathcal{O}_{\mathcal{H}_\kappa}$ is different from the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ in the present paper (see also [33], [32]).

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