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Note on the cortex of some exponential Lie groups

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Abstract. In this paper, we built a family of $4d$-dimensional two-step nilpotent Lie algebras $(\mathfrak{g}_d)_{d\geq 2}$ so that the cortex of the dual of each $\mathfrak{g}_d$ is a projective algebraic set. We also give a complete description of the cortex of the exponential connected and simply connected Lie group $G = \mathbb{R}^n \rtimes \mathbb{R}$.

Contents

1. Introduction

The cortex of general locally compact group G was defined in [9] as

$$\mathrm{cor}(G) = \{\pi \in \widehat{G}, \pi \text{ is not Hausdorff-separated from the identity representation } \mathbf{1}_G\},$$

where $\widehat{G}$ is the dual of G (set of equivalence classes of unitary irreducible representations of G). Note that $\widehat{G}$ is equipped with the topology of Fell which can be described in terms of weak containment (see [6]) and, in general, is not separated. However, if G is abelian, then $\widehat{G}$ is separated and hence $\mathrm{cor}(G) = \{\mathbf{1}_G\}$.

When G is a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, the Kirillov theory says that $\mathfrak{g}^*/\mathrm{Ad}^*(G)$ and $\widehat{G}$ are homeomorphic,

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where $\mathrm{Ad}^*(G)$ denotes the coadjoint representation of G on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. Hence, for this class of Lie groups, $\mathrm{cor}(G)$ can be identified with a certain $\mathrm{Ad}^*(G)$-invariant subset of $\mathfrak{g}^*$. From [2], one introduces the cortex of $\mathfrak{g}^*$ as

$$\mathrm{Cor}(\mathfrak{g}^*) = \{\ell = \lim_{m\to\infty} \mathrm{Ad}^*_{s_m}(\ell_m), \text{ where } \{s_m\} \subset G \text{ and } \{\ell_m\} \subset \mathfrak{g}^* \text{such that } \lim_{m\to\infty} \ell_m = 0\}$$

and we have $\pi_\ell \in \mathrm{cor}(G)$ if and only if $\ell \in \mathrm{Cor}(\mathfrak{g}^*)$. Note that in the case of general Lie groups, the two definitions are not so easily related. Motivated by this situation, the authors in [3] define the cortex $C_V(G)$ of a representation of a locally compact group G on a finite-dimensional vector space V as the set of all $v \in V$ for which $G.v$ and $\{0\}$ cannot be Hausdorff-separated in the orbit-space V/G. They give a precise description of $C_V(G)$ in the case $G = \mathbb{R}$. Moreover, they consider the subset $IC_V(G)$ of V consisting of the common zeroes of all G-invariant polynomials P on V with $P(0) = 0$. Note that when G is a nilpotent Lie group, one has $IC_V(G) \subset C_V(G)$ and they show that $IC_V(G) = C_V(G)$ when G is a nilpotent Lie group of the form $G = \mathbb{R}^n \rtimes \mathbb{R}$ and $V = \mathfrak{g}^*$ the dual of the Lie algebra $\mathfrak{g}$. This fails for a general nilpotent Lie group, even in the case of two-step nilpotent Lie group (see [2]). In [7], the authors show that the cortex of a connected and simply connected nilpotent Lie group is a semi-algebraic set. In [5] one gives an explicit description of the cortex of certain class of exponential Lie algebras (using the results of parametrization in [1]).

Fixing the class of two-step nilpotent Lie algebras, we see that each coadjoint orbit is a flat (affine) symplectic manifold, however the cortex of that class of Lie algebras may not be flat and in this paper, we give a generalization of the example given in [2] p. 210. Our example consists of a family of $4d$-dimensional two-step nilpotent Lie algebras $(\mathfrak{g}_d)_{d\geq 2}$ such that the cortex of each $\mathfrak{g}_d^*$ is the zero set of a homogeneous polynomial of degree d in the complement $\mathfrak{z}_d^\perp$ of the center $\mathfrak{z}_d$ of $\mathfrak{g}_d$. Finally we give some remarks on the cortex of $\mathbb{R}^n \rtimes \mathbb{R}$.

The paper is organized as follows: The next section is a review of the mathematics and basic tools used throughout the rest of the text. In the third section, we focus on the class of two-step nilpotent Lie algebras $\mathfrak{g}$, and we give a refinement of Theorem 4.5 ([1] p. 548) by which we give a description of the algebra of G-invariant polynomials on $\mathfrak{g}^*$ (G is the corresponding Lie group of $\mathfrak{g}$). Next we give an interesting example of a family of two-step nilpotent Lie algebras $(\mathfrak{g}_d)_{d\geq 2}$ for which the cortex of the dual $\mathfrak{g}_d^*$ of each $\mathfrak{g}_d$ is the zero set of homogeneous polynomials of degree d. In the final section, we consider the exponential nonnilpotent Lie group $G = \mathbb{R}^n \rtimes \mathbb{R}$ and we give a complete and explicit description of the cortex of the dual of its Lie algebra.

2. Background material and notations

If G is a locally compact group, Vershik and Karpushev [9] introduce the notion of cortex of G as the set of all unitary irreducible representations of

G that cannot be Hausdorff separated from the trivial representation. If G a Lie group with Lie algebra $\mathfrak{g}$, it's known that G acts on $\mathfrak{g}$ by the adjoint action denoted by Ad and on $\mathfrak{g}^*$ by the coadjoint action denoted by Ad^*. Following [3], we recall the following:

Definition 2.1. Let π be a continuous representation of a locally compact Lie group G on a finite-dimensional (real) space V we define

$$C_V(\pi) = \{v = \lim_{m\to\infty} \pi(s_m)v_m, \lim_{m\to\infty} v_m = 0, \{s_m\}_m \subset G\},$$

and the cortex of invariants of π as

$$IC_V(\pi) = \{v \in V : p(v) = p(0) \text{ for all } G\text{-invariant polynomials on } V\}.$$

In particular when G is a locally compact Lie group and π is the contragredient representation of G on the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G, one has:

Definition 2.2. We define the cortex of $\mathfrak{g}^*$ as

$$\mathrm{Cor}(\mathfrak{g}^*) = \big\{\lim_{m\to\infty} \mathrm{Ad}^*_{s_m}(\ell_m) \mid (s_m)_m \subset G,\ (\ell_m)_m \subset \mathfrak{g}^* \text{ with } \lim_{m\to\infty} \ell_m = 0\big\},$$

and the cortex of invariants

$$\mathrm{ICor}(\mathfrak{g}^*) = \{\ell \in \mathfrak{g}^* : p(\ell) = p(0),\ \text{for all } G\text{-invariant polynomial } p \text{ on } \mathfrak{g}^*\}.$$

When G is a nilpotent connected and simply connected Lie group, Kirillov's theory establishes a bijection between $\mathfrak{g}^*/\mathrm{Ad}^*(G)$ (the orbit space of the coadjoint representation of G on $\mathfrak{g}^*$) and $\widehat{G}$ (the unitary dual of G). More precisely, associated to $\ell \in \mathfrak{g}^*$ is an irreducible representation π_ℓ of G, and π_f and π_ℓ ($f \in \mathfrak{g}^*$) are equivalent if and only if $f \in \mathrm{Ad}^*(G)\ell$. The Kirillov correspondence is a homeomorphism provided that $\mathfrak{g}^*/\mathrm{Ad}^*(G)$ is endowed with the quotient topology [4]. In that case, the unitary dual $\widehat{G}$ of G can be parameterized via the orbit-method. More precisely, let $\ell \in \mathfrak{g}^*$ and $\mathfrak{p}_\ell$ be a Pukanszky polarization at ℓ, we define the representation $\pi_{\ell,\mathfrak{p}_\ell}$ by

$$\pi_{\ell,\mathfrak{p}_\ell} := \mathrm{ind}_{\mathcal{P}_\ell}^G \chi_\ell,$$

where $\mathcal{P}_\ell = \exp \mathfrak{p}_\ell$ and χ_ℓ is the unitary character associated with $\mathcal{P}_\ell$ given by

$$\chi_\ell(\exp X) = e^{-i\langle \ell, X\rangle}, \quad X \in \mathfrak{p}_\ell.$$

Then:

Theorem 2.1 (A. A. Kirillov). *Let G be a simply connected nilpotent real Lie group with Lie algebra $\mathfrak{g}$. If $\ell \in \mathfrak{g}^*$, there exists a polarization $\mathfrak{p}(\ell)$ of $\mathfrak{g}$ for ℓ such that the monomial representation $\pi_{\ell,\mathfrak{p}(\ell)} := \mathrm{ind}_{\exp \mathfrak{p}_\ell}^G \chi_\ell$ is irreducible and of trace class. If ℓ' is an element of $\mathfrak{g}^*$ which belongs to the coadjoint orbit of ℓ and $\mathfrak{p}_{\ell'}$ is a polarization of $\mathfrak{g}$ for ℓ', then the monomial representations $\pi_{\ell,\mathfrak{p}_\ell}$ and $\pi_{\ell',\mathfrak{p}_{\ell'}}$ are unitarily equivalent. Conversely, if $\mathfrak{h}$ and $\mathfrak{h}'$ are polarizations of $\mathfrak{g}$ for $\ell \in \mathfrak{g}^*$ and $\ell' \in \mathfrak{g}^*$ respectively such that the monomial representations $\pi_{\ell,\mathfrak{h}}$ and $\pi_{\ell',\mathfrak{h}'}$ of G are unitarily equivalent, then*

ℓ and ℓ' belong to the same coadjoint orbit of G in $\mathfrak{g}^*$. Finally, for each irreducible unitary representation π of G, there exists a unique coadjoint orbit $\mathcal{O}$ of G in $\mathfrak{g}^*$ such that for any linear from ℓ and each polarization $\mathfrak{h}$ of $\mathfrak{g}$ for ℓ, the representations π and $\operatorname{ind}_{\exp\mathfrak{h}}^{G}\chi_\ell$ are unitarily equivalent. Any irreducible unitary representation of G is strongly trace class. Moreover the mapping

$$\begin{array}{cccc} K: \mathfrak{g}^*/\operatorname{Ad}^*(G) & \longrightarrow & \widehat{G} \\ \mathcal{O}_\ell & \mapsto & [\pi_{\ell,\mathfrak{p}(\ell)}] \end{array}$$

is a homeomorphism (the Kirillov correspondence).

The above Kirillov's result was generalized immediately to the class known as exponential solvable Lie groups, the Kirillov correspondence is still a bijection. For more details, see [8]. With this in mind, we see that if G is an exponential Lie group, then $\pi := \pi_{\ell,\mathfrak{p}_\ell} \in \operatorname{cor}(G)$ (cortex of G) if and only if $\ell \in \operatorname{Cor}(\mathfrak{g}^*)$. However if G is exponential nonnilpotent, $\operatorname{ICor}(\mathfrak{g}^*)$ may not be defined.

Throughout, G will always denote a connected and simply connected Lie group with (real) Lie algebra $\mathfrak{g}$. We denote by $\mathfrak{z}$ the center of $\mathfrak{g}$ (if it exists) and $\mathfrak{g}^*$ denotes the dual of $\mathfrak{g}$. If $\ell \in \mathfrak{g}^*$, $\mathcal{O}_\ell$ denotes the coadjoint orbit of ℓ.

3. The two-step nilpotent Lie algebras

Let G be a connected and simply connected two-step nilpotent Lie group with Lie algebra $\mathfrak{g}$, then if $\mathcal{O}_\ell = \operatorname{Ad}^*(G)\ell$, one has

$$\mathcal{O}_\ell = \{\ell\} + T_\ell\mathcal{O}_\ell,$$

and

$$T_\ell\mathcal{O}_\ell = \mathfrak{g}(\ell)^\perp,$$

where $T_\ell\mathcal{O}_\ell$ is the tangent space of $\mathcal{O}_\ell$ at ℓ, by which we see that the coadjoint orbits in two-step nilpotent Lie algebras are flat (and symplectic) manifolds. In [2], the authors show the following:

Proposition 3.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of class 2 (i.e, $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]] = 0$), and let $G = \exp\mathfrak{g}$ be the associated Lie group. Denote by ad^* the coadjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^*$. Let $f \in \mathfrak{g}^*$. Then the corresponding representation π_f of G belongs to $\operatorname{cor}(G)$ if and only if f belongs to the closure of the subset $\{ad^*_X(\ell), X \in \mathfrak{g}, \ell \in \mathfrak{g}^*\}$ of $\mathfrak{g}^*$.*

From this we can conclude the following:

Corollary 3.2. *Let $\mathfrak{g}$ is a two-step nilpotent Lie algebra. If $T_\ell\mathcal{O}_\ell$ denotes the tangent space to the coadjoint orbit $\mathcal{O}_\ell$ at ℓ, then the $\operatorname{Cor}(\mathfrak{g}^*)$ is the closure in $\mathfrak{g}^*$ of the set*

$$\bigcup_{\ell\in\mathfrak{g}^*} T_\ell\mathcal{O}_\ell = \bigcup_{\mathcal{O}_\ell\in\mathfrak{g}^*/\operatorname{Ad}^*(G)} T\mathcal{O}_\ell,$$

where $T\mathcal{O}_\ell$ is the fiber tangent of $\mathcal{O}_\ell$ and $\mathfrak{g}^*/\operatorname{Ad}^*(G)$ is the space of coadjoint orbits in $\mathfrak{g}^*$.

Proof. Indeed, for any $\ell \in \mathfrak{g}^*$, one has

$$\{ad_X^*(\ell); X \in \mathfrak{g}\} = T_\ell \mathcal{O}_\ell,$$

and hence with Proposition (3.1), the conclusion yields. □

Here we give a refinement of Theorem 4.5 ([1] p. 548).

Proposition 3.3. *Let G be a two-step nilpotent Lie group with Lie algebra $\mathfrak{g}$, choose a real Jordan–Hölder basis $\{X_j\}$. Let $\mathcal{P}$ be the corresponding fine stratification of $\mathfrak{g}^*$, and let Ω be a layer belonging to $\mathcal{P}$. Then there is an explicit construction of an open set U in $\mathfrak{g}^*$ and real-valued functions $p_1, p_2, \dots, p_d, q_1, q_2, \dots, q_d$ on U , such that U contains Ω, and such that for each coadjoint orbit $\mathcal{O}_\ell$ in Ω, $p_{1|\mathcal{O}_\ell}, p_{2|\mathcal{O}_\ell}, \dots, p_{d|\mathcal{O}_\ell}, q_{1|\mathcal{O}_\ell}, q_{2|\mathcal{O}_\ell}, \dots, q_{d|\mathcal{O}_\ell}$ are real-valued, global canonical coordinates for $\mathcal{O}_\ell$. Moreover, for each $1 \le j \le n$, $0 \le u \le d$, there are rational functions $\alpha_{j,u}$ and $\beta_{j,u}$ such that for each $1 \le j \le n$ and $\ell \in \Omega$ one has*

$$\ell_j := \ell(X_j) = \sum_{u:\ j_u \le j} \alpha_{j,u}(\ell) p_u + \sum_{r=1}^{d} \beta_{j,u}(\ell) q_u.$$

Proof. Recall that the construction of p_r, q_r depends on the flag

$$(\mathfrak{g}_j = \mathrm{span}\{X_1, \dots, X_j\})_{1 \le j \le n}.$$

More precisely if $j_t = \min\{j_r,\ 1 \le r \le d\}$, then:

$$p_1^{(1)} = \ell_{i_t}, \quad q_1^{(1)} = \frac{\ell_{j_t}}{\ell[X_{j_t}, X_{i_t}]}.$$

Now suppose we have built $p_1^{(m)}, \dots, p_k^{(m)}, \dots, q_1^{(m)}, \dots, q_k^{(m)}$, then for $\mathfrak{g}_{m+1}$ one has either $m+1 \notin \mathbf{e}$ and in this case $p_r^{(m+1)} = p_r^{(m)}, q_r^{(m+1)} = q_r^{(m)}$ or $m+1 = j_{k+1} \in \mathbf{e}$ and in this case

$$q_r^{(m+1)}(\ell) = q_r^{(m)}(\exp -qX_{m+1}\ell) = q_r^{(m)}(\ell) - q\{x_{m+1}, q_r^{(m)}\},$$

and

$$p_r^{(m+1)}(\ell) = p_r^{(m)}(\exp -qX_{m+1}\ell) = p_r^{(m)}(\ell) - q\{x_{m+1}, p_r^{(m)}\},$$

with $q = \frac{y}{\ell[X_{m+1}, y]}$, where y is a G_m-invariant and non-G_{m+1}-invariant polynomial function such that $\{x_{m+1}, y\}$ is nonvanishing on Ω (here $G_j = \exp \mathfrak{g}_j$). □

Corollary 3.4. *Let $\mathbf{e} = \{e_1 < \dots < e_{2d}\}$ be the set of jump indices corresponding to the minimal layer in $\mathfrak{g}^*$. Let F be the cross-section mapping associated with the minimal layer Ω then $F(\ell) = (F_1(\ell), \dots, F_n(\ell))$ and let $\mathbf{e} = \{e_1 < \dots < e_{2d}\}$ be the corresponding jump indices then*

$$F_k(\ell) = \begin{cases} \ell_k, & \text{if } k = 1, \dots e_1 - 1; \\ 0, & \text{if } k \in \mathbf{e}; \\ \ell_k + \sum_{j: e_j \le k-1} a_j(\ell) \ell_{e_j}, & \text{if } k \notin \mathbf{e}, k \ge e_1, \end{cases}$$

where each of $a_1(\ell), \dots, a_{k-1}(\ell)$ is (nontrivial) a rational regular function on the minimal layer depending only upon $\ell^0 = \ell_{|\mathfrak{z}}$ ($\mathfrak{z}$ is the center of $\mathfrak{g}$).

Proof. For each layer in $\mathfrak{g}^*$, the mapping $(\ell_{e_1}, \dots, \ell_{e_{2d}}) \mapsto (p_i(\ell), q_i(\ell))_{1\le i\le d}$ is a rational diffeomorphism whose inverse is also rational on any layer, then we consider the minimal layer and by Proposition 3.3, we can write

$$p_i(\ell) = \sum_j u_j(\ell_1, \dots, \ell_p)\ell_{e_j}, \quad q_i(\ell) = \sum_j v_j(\ell_1, \dots, \ell_p)\ell_{e_j}, \quad i = 1, \dots, d,$$

where u_j and v_j are rational regular functions on the minimal layer. Then after substituting each of the functions $(p_i, q_i)_i$ by the above expressions in the coordinate functions $(\ell_k)_{k\notin \mathbf{e}}$ we obtain the invariant functions of $\mathfrak{g}^*$ and this ends the proof. □

Corollary 3.5. *If $\mathfrak{g}$ is a two-step nilpotent Lie algebra and $\mathfrak{g}^*$ denotes its dual, then*

$$\mathrm{ICor}(\mathfrak{g}^*) = \{\ell \in \mathfrak{g}^* : \ell(Z) = 0 \quad \forall Z \in \mathfrak{z}\}.$$

Proof. The nontrivial coordinates of the cross-section mapping

$$(F_k(\ell))_{k\ge p, k\notin \mathbf{e}}$$

associated with the minimal layer can be written as

$$F_k(\ell) = \frac{B(\ell^0)\ell_k + A_k(\ell^0)}{B(\ell^0)}, \quad k \notin \mathbf{e}, k > p := e_1,$$

where each of $B(\ell^0)$ and $B(\ell^0)\ell_k + A_k(\ell^0), (k \ge p, k \notin \mathbf{e})$ is a nontrivial G-invariant polynomial on $\mathfrak{g}^*$, with $\ell^0 = \ell_{|\mathfrak{z}}$. Note that these polynomials are homogeneous and for each $k \ge p, k \notin \mathbf{e}$, one has

$$\deg(B(\ell^0)\ell_k + A_k(\ell^0)) = \deg(B(\ell^0)) + 1.$$

Finally the ring $\mathrm{Pol}(\mathfrak{g}^*)^G$ of G-invariant polynomials is spanned by the polynomials

$$\ell_1, \dots, \ell_p, B(\ell^0), \left(B(\ell^0)\ell_k + A_k(\ell^0)\right)_{k\ge p, k\notin \mathbf{e}},$$

and this ends the proof. □

3.1. Main example. In [2], one introduces an interesting example of 8-dimensional two-step nilpotent Lie algebra $\mathfrak{g}$ so that the corresponding cortex in $\mathfrak{g}^*$ is a projective algebraic set given by a quadric and such that $\mathrm{Cor}(\mathfrak{g}^*) \subsetneq \mathrm{ICor}(\mathfrak{g}^*)$. Here we give a generalization of that example. Let $d \in \mathbb{N}$ with $d \ge 2$ and let $\mathfrak{g}_d$ be the Lie algebra with basis

$$(Z_1, \dots, Z_d, Y_1, Y_2, \dots, Y_{2d-1}, Y_{2d}, X_1, \dots, X_d),$$

and nontrivial brackets

$$\begin{aligned} [X_i, Y_{2i-1}] &= Z_1, & i &= 1, \dots, d, \\ [X_k, Y_{2k}] &= Z_{k+1}, & k &= 1, \dots, d-1, \\ [X_d, Y_{2d}] &= Z_2 + \dots + Z_d. \end{aligned}$$

Let's denote the center of $\mathfrak{g}_d$ by $\mathfrak{z}_d = \mathrm{span}\{Z_1, \dots, Z_d\}$ and G_d the corresponding connected and simply connected Lie group.

Proposition 3.6. *For each Lie algebra $\mathfrak{g}_d$ $(d \geq 2)$, one has:*

(i) *The minimal layer in $\mathfrak{g}_d^*$ is given by*

$$\Omega_d = \{\ell \in \mathfrak{g}_d^* : \ell(Z_1) \neq 0\}.$$

(ii) *The coadjoint orbits in Ω_d are $2d$-dimensional and if*

$$\ell = \sum_{k=1}^{d} (\lambda_k Z_k^* + \beta_k X_k^*) + \sum_{k=1}^{2d} \gamma_k Y_k^* \in \Omega_d,$$
$$\xi = ((z_i)_{1 \leq i \leq d}, (y_j)_{1 \leq j \leq 2d}, (x_k)_{1 \leq k \leq d}) \in G\ell,$$

then

$$\xi = \begin{cases} z_k = \lambda_k, & \text{if } k = 1, \dots, d; \\ y_{2k-1} = \gamma_{2k-1} + s_j \lambda_1, & \text{if } k = 1, \dots, d-1; \\ y_{2k} = \gamma_{2k} + s_j \lambda_{k+1}, & \text{if } k = 1, \dots, d-1; \\ y_{2d-1} = \gamma_{2d-1} + s_d \lambda_1; & \\ y_{2d} = \gamma_{2d} + s_d(\lambda_2 + \cdots + \lambda_d); & \\ x_k = \beta_k + t_k & \text{if } k = 1, \dots, d. \end{cases}$$

(iii) *The algebra of G-invariant polynomials is*

$$\mathrm{Pol}(\mathfrak{g}_d^*)^{G_d} = \mathbb{R}[z_1, \dots, z_d, z_1 y_2 - z_2 y_1, \dots, z_1 y_{2d-2} - z_{d-1} y_{2d-3}, \\ z_1 y_{2d} - (z_2 + \cdots + z_d) y_{2d-1}].$$

Proof. Let $\mathcal{B}_d = (U_1, \dots, U_{4d})$ be the Jordan–Hölder basis defined by

$$U_i = \begin{cases} Z_i, & \text{if } 1 \leq i \leq d, \\ Y_{i-d}, & \text{if } d+1 \leq i \leq 3d; \\ X_{i-3d}, & \text{if } 3d+1 \leq i \leq 4d. \end{cases}$$

Using the methods of [1], we can see that the minimal layer in $\mathfrak{g}_d^*$ is

$$\Omega_d = \{\ell \in \mathfrak{g}^* : \ell(U_1) = \ell(Z_1) \neq 0\},$$

which corresponds to the set of jump indices $\mathbf{e_d} = \mathbf{i_d} \cup \mathbf{j_d}$ with

$$\mathbf{i_d} = \{d+1 < d+3 < \cdots < 3d-1\},$$
$$\mathbf{j_d} = \{3d+1, 3d+2, \dots, 4d\}.$$

Then by using the methods of [1] (the parametrization of coadjoint orbits) we can deduce the results of (ii) and (iii). □

Remark 3.1.

(i) If Ω_d is the minimal layer given as above, then the canonical coordinates on Ω_d (see [1]) are given by

$$p_i(\ell) = x_i, q_i(\ell) = \frac{y_{2i-1}}{z_1}, \quad i = 1, \dots, d.$$

(ii) The cross-section Σ_d is given by

$$\Sigma_d = \left(\sum_{k \notin \mathbf{e}} \mathbb{R}U_k^*\right) \cap \Omega_d = \left(\sum_{k=1}^{d} \mathbb{R}Z_k^* + \mathbb{R}Y_{2k}^*\right) \cap \Omega_d.$$

(iii) The cross-section mapping $F_d : \Omega_d \to \Sigma_d$ is as follows

$$\begin{aligned} F_d(z_i, y_j, x_k) = &\sum_{i=1}^{d} z_i Z_i^* \\ &+ \sum_{i=1}^{d-1} \left(y_{2i} - \frac{z_{i+1}}{z_i} y_{2i-1}\right) Y_{2i}^* + \left(y_{2d} - \frac{z_2 + \cdots + z_d}{z_1}\right) Y_{2d}^*, \end{aligned}$$

where $(Z_1^*, \dots, Z_d^*, Y_1^*, \dots, Y_{2d}^*, X_1^*, \dots, X_d^*)$ is the dual basis of $\mathcal{B}$.

Proposition 3.7. *Let's denote $\ell = \sum_{i=1}^{d}(z_i Z_i^* + x_i X_i^*) + \sum_{j=1}^{2d} y_j Y_j^* \in \mathfrak{g}^*$ by $\ell = (z_i, y_j, x_k)$, where $(Z_1^*, \dots, Z_d^*, Y_1^*, \dots, Y_{2d}^*, X_1^*, \dots, X_d^*)$ is the dual basis in $\mathfrak{g}_d^*$. Then the cortex of $\mathfrak{g}_d^*$ is the projective algebraic set given by*

$$\begin{aligned} \mathrm{Cor}(\mathfrak{g}_d^*) = \Bigg\{ &\ell = (z_i, y_j, x_k) : \ z_1 = \cdots = z_d = \\ &= y_{2d-1}\left(\sum_{i=1}^{d-1} y_{2i} \prod_{j=1, j\neq i}^{d-1} y_{2j-1}\right) - y_{2d} \prod_{j=1}^{d-1} y_{2j-1} = 0 \Bigg\}. \end{aligned}$$

Proof. Note that since Ω_d is dense in $\mathfrak{g}_d^*$ (Zariski open subset in $\mathfrak{g}_d^*$) then

$$\mathrm{Cor}(\mathfrak{g}_d^*) = \left\{ \lim_m \mathrm{Ad}^*_{\exp X_m} \ell_m, \ (\ell_m) \in \mathfrak{g}_d^*, \ (\ell_m)_m \in \Omega_d, \text{ and } \lim_m \ell_m = 0 \right\}.$$

On the other hand if $\mathcal{O}_\ell = G\ell$, then the tangent space $T_\ell \mathcal{O}_\ell$ at ℓ is

$$T_\ell \mathcal{O}_\ell = \{ad_X^*(\ell), \ \ell \in \Omega_d, X \in \mathrm{Vect}\{Y_{2k-1}, X_k, \ 1 \leq k \leq d\}\}.$$

Now if $\ell = (\lambda_i, \gamma_j, \beta_k) \in \Omega_d$ and $\xi \in T_\ell \mathcal{O}_\ell$, with

$$\xi = \sum_{i=1}^{d}(z_i Z_i^* + x_i X_i^*) + \sum_{j=1}^{2d} y_i Y_i^*,$$

then

$$\begin{cases} z_i = 0, & \text{if } i = 1, \dots, d; \\ y_{2j-1} = s_j \lambda_1, & \text{if } j = 1, \dots, d-1; \\ y_{2j} = s_j \lambda_{j+1}, & \text{if } j = 1, \dots, d-1; \\ y_{2d-1} = s_d \lambda_1; \\ y_{2d} = s_d(\lambda_2 + \cdots + \lambda_d); \\ x_k = t_k, & \text{if } k = 1, \dots, d. \end{cases}$$

From which we can see that $\xi = (z_i, y_j, x_k) \in T_\ell \mathcal{O}$ if and only if

$$\begin{cases} z_i = 0, & \text{if } i = 1, \dots, d; \\ y_{2j} = y_{2j-1} \frac{\lambda_{j+1}}{\lambda_1}, & \text{if } j = 1, \dots, d-1; \\ y_{2d} = y_{2d-1} \frac{\lambda_2 + \cdots + \lambda_d}{z_1}. \end{cases}$$

with y_{2j-1}, x_j are free variables in $\mathbb{R}$ $(j = 1, \dots, d)$. Then we see that a.e. $\xi \in T_\ell \mathcal{O}$ satisfies

$$\frac{y_{2d}}{y_{2d-1}} = \sum_{j=1}^{d-1} \frac{y_{2j}}{y_{2j-1}},$$

and hence

$$\mathrm{Cor}(\mathfrak{g}_d^*) = \Bigg\{ \ell = (z_i, y_j, x_k) \in \mathfrak{g}_d^* : z_i = 0,$$
$$y_{2d-1} \left(\sum_{i=1}^{d-1} y_{2i} \prod_{j=1, j \neq i}^{d-1} y_{2j-1} \right) - y_{2d} \prod_{j=1}^{d-1} y_{2j-1} = 0 \Bigg\}. \quad \square$$

Corollary 3.8. *For each integer $d \geq 2$ if $\mathfrak{z}_d$ denotes the center of the Lie algebra $\mathfrak{g}_d$, then*

$$\mathrm{Cor}(\mathfrak{g}_d^*) \subsetneqq \mathrm{ICor}(\mathfrak{g}_d^*) = \mathfrak{z}_d^\perp.$$

Proof. The ring of G-invariant polynomials on $\mathfrak{g}^*$ is given by

$$\mathrm{Pol}(\mathfrak{g}_d^*)^{G_d} =$$
$$\mathbb{R}\left[z_1, \dots, z_d, \left(z_1 y_{2i} - z_{i+1} y_{2i-1} \right)_{1 \leq i \leq d-1}, z_1 y_{2d} - (z_2 + \cdots + z_d) y_{2d-1} \right],$$

where G_d is the connected and simply connected (nilpotent) Lie group corresponding to $\mathfrak{g}_d$. Thus

$$\{\ell \in \mathfrak{g}_d^* : P(\ell) = P(0), \forall P \in \mathrm{Pol}(\mathfrak{g}_d^*)^{G_d}\} = \mathfrak{z}_d^\perp,$$

where

$$\mathfrak{z}_d^\perp = \{\ell \in \mathfrak{g}_d^* : \ell(Z) = 0, \forall Z \in \mathfrak{z}_d\}.$$

hence with Proposition 3.7, we conclude that

$$\mathrm{Cor}(\mathfrak{g}_d^*) \subsetneqq \mathrm{ICor}(\mathfrak{g}_d^*) = \mathfrak{z}_d^\perp. \quad \square$$

4. The Lie group $\mathbb{R}^n \rtimes \mathbb{R}$

In [3], in there is a study of the cortex of the nilpotent Lie group

$$G = \mathbb{R}^n \rtimes \mathbb{R},$$

the authors show that

$$\begin{aligned}\mathrm{Cor}(\mathfrak{g}^*) &= \mathrm{ICor}(\mathfrak{g}^*)\\ &= \{\ell \in \mathfrak{g}^* : P(\ell) = P(0),\ P \text{ is } G \text{ invariant polynomial on } \mathfrak{g}^*\}.\end{aligned}$$

The definition of $\mathrm{ICor}(\mathfrak{g}^*)$ may not exist if G is not nilpotent but we can define G-invariant (or semi-invariant) functions. Let's consider the following example:

Example 4.1. Let (X_1, X_2, A) be a basis in $\mathfrak{g}$ with

$$[A, X_1,] = X_1, [A, X_2] = -2X_2.$$

Let's identify $\mathfrak{g}^*$ with $\mathbb{R}^3$ under the dual basis (X_1^*, X_2^*, A^*), and denote $x = (x_1, x_2, a) \in \mathfrak{g}^*$, then the minimal layer is

$$\Omega = \{\ell = (\ell_1, \ell_2, a) \in \mathfrak{g}^* : \ell_1 \neq 0\}.$$

If $\ell = (\ell_1, \ell_2, a) \in \mathfrak{g}^*$, then the coadjoint orbit of ℓ is given by

$$\mathcal{O}_\ell = \{x \in \mathfrak{g}^* : x = (\ell_1 e^t, \ell_2 e^{-2t}, a + s), t, s \in \mathbb{R}\},$$

that is,

$$\mathcal{O}_\ell = \{x = (x_1, x_2, x_3) \in \mathfrak{g}^* : \mathrm{sign}(x_1) = \mathrm{sign}(\ell_1), x_1^2 x_2 = \ell_1^2 \ell_2, x_3 \in \mathbb{R}\}.$$

We can check that the cortex of $\mathfrak{g}^*$ is given by

$$\mathrm{Cor}(\mathfrak{g}^*) = \{\ell = (\ell_1, \ell_2, \ell_3) \in \mathfrak{g}^* : \ell_1 \ell_2 = 0\}.$$

On other hand, the cross-section mapping is as follows

$$F : \Omega \to \Omega, \quad \ell \mapsto \left(\mathrm{sign}(\ell_1) = \frac{\ell_1}{|\ell_1|}, \ell_1^2 \ell_2, 0\right),$$

from which we see the existence of G-invariant polynomial $p(x) = x_1^2 x_2$ and we see that

$$\mathrm{Cor}(\mathfrak{g}^*) = \{\ell \in \mathfrak{g}^* : p(\ell) = 0\}.$$

In this example if we let $[A, X_1] = X_1, [A, X_2] = -\sqrt{2} X_2$ then there are no G-invariant polynomials on $\mathfrak{g}^*$, however the function $x_1^{\sqrt{2}} x_2$ is G-invariant and the cortex is still the same. This example can be generalized. To this end, if $\mathfrak{g} = \mathbb{R}^n \oplus \mathbb{R}A$, we denote $sp(adA) = \{\lambda_1, \ldots, \lambda_n\}$ the set of eigenvalues of adA, and for $\lambda \in sp(adA)$, we set.

$$E_\lambda = \bigcup_{m \in \mathbb{N}} \ker(adA - \lambda)^m,$$

and

$$E^+ = \bigcup_{\lambda \in sp(adA), \Re(\lambda) > 0} E_\lambda,$$

$$E^- = \bigcup_{\lambda \in sp(adA), \Re(\lambda)<0} E_\lambda.$$

Then we have the following:

Proposition 4.2. *Let $G = \mathbb{R}^n \rtimes \mathbb{R}$ be the Lie group whose Lie algebra $\mathfrak{g} = \mathbb{R}^n \oplus \mathbb{R}A$. Suppose that adA is diagonalizable, and let $sp(adA) = \{\lambda_1, \dots, \lambda_n\}$ denote the set of eigenvalues of adA.*

(a) *If $\{\Re(\lambda_j)\}_{1\le j\le n} \subset (0,\infty)$ or $\{\Re(\lambda_j)\}_{1\le j\le n} \subset (-\infty,0)$ then*

$$\mathrm{Cor}(\mathfrak{g}^*) = \mathfrak{g}^*.$$

(b) *If $\prod_{j=1}^n \Re(\lambda_j) < 0$. Then the cortex of $\mathfrak{g}^*$ is the union of two vector spaces. More precisely*

$$\mathrm{Cor}(\mathfrak{g}^*) = (V^+ + \mathbb{R}A^*) \cup (V^- + \mathbb{R}A^*),$$

where

$$V^+ = (E^+)^*, \quad V^- = (E^-)^*.$$

Proof. If $sp(adA) = \{\lambda_1, \dots, \lambda_n\}$ denotes the set of eigenvalues of adA (restricted to $\mathbb{R}^n$). Then identifying $\mathfrak{g}$ with $\mathbb{R}^{n+1}$ (respectively $\mathbb{C}^{n+1}$ if some of the eigenvalues of adA are nonreal), the coadjoint orbit of any $\ell \in \mathfrak{g}^*$ is parameterized as follows

$$\mathcal{O}_\ell = \{(\ell_1 e^{\lambda_1 t}, \dots, \ell_n e^{\lambda_n t}, \ell_{n+1} + s), \quad t, s \in \mathbb{R}\}.$$

Since $\Re(\lambda_j) \neq 0, j = 1, \dots, n$, then for any $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ the linear system

$$\begin{pmatrix} e^{\lambda_1 t} & 0 & \dots \\ 0 & \ddots & 0 \\ \dots & 0 & e^{\lambda_n t} \end{pmatrix} \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix},$$

has a unique solution $(\ell_1, \dots, \ell_n)^\top$ with

$$\|(\ell_1, \dots, \ell_n)^\top\| = \|(e^{-\lambda_1 t}\alpha_1, \dots, e^{-\lambda_n t}\alpha_n)^\top\|.$$

(a) If $\{\Re(\lambda_j)\}_{1\le j\le n} \subset (0,\infty)$ or $\{\Re(\lambda_j)\}_{1\le j\le n} \subset (-\infty,0)$ then for any $(\alpha_1, \dots, \alpha_n, \beta) \in \mathfrak{g}^*$ it exists $\{x^{(m)} = x(t_m, \ell^{(m)})\}_m \in \mathfrak{g}^*$ with $\{\ell^{(m)}\}_m \subset \Omega$ and $\lim_{m:\Re(\lambda_1)t_m\to\infty} \ell^{(m)} = 0$ such that

$$\lim_{m:\Re(\lambda_1)t_m\to\infty} x^{(m)} = (\alpha_1, \dots, \alpha_n, \beta),$$

and hence the cortex is all of $\mathfrak{g}^*$.

(b) In that case, let's rearrange the basis $(X_1, \dots, X_n)$ in $\mathbb{C}^n$ such that the matrix of adA in this basis is $\mathrm{diag}(\lambda_1, \dots, \lambda_{k_0}, \lambda_{k_0+1}, \dots, \lambda_n)$ with

$$\Re(\lambda_1) > 0, \dots, \Re(\lambda_{k_0}) > 0, \Re(\lambda_{k_0+1}) < 0, \dots, \Re(\lambda_n) < 0,$$

then for any $x = (x_1, \dots, x_n, x_{n+1}) \in \mathcal{O}_\ell$ one has

$$\begin{cases} |\ell_1|^{\lambda_k} |x_k|^{\lambda_1} = |\ell_k|^{\lambda_1} |x_1|^{\lambda_k}, & k \le k_0, \\ |x_k|^{\lambda_1} |x_1|^{-\lambda_k} = |\ell_k|^{\lambda_1} |\ell_1|^{-\lambda_k}, & k \ge k_0 + 1. \end{cases}$$

Hence one has

$$\begin{aligned}\operatorname{Cor}(\mathfrak{g}^*) = \{\ell \in \mathfrak{g}^* : \ell = (\ell_1, \dots, \ell_{k_0}, 0, \dots, 0, \ell_{n+1})\} \\ \cup \{\ell \in \mathfrak{g}^* : \ell = (0, \dots, 0, \ell_{k_0+1}, \dots, \ell_n, \ell_{n+1})\}.\end{aligned}$$ □

Remark 4.1. Let $\mathfrak{g} = \mathbb{R}^n \oplus \mathbb{R}A$ be a real Lie algebra. Suppose that there exists a basis $(X_1, \dots, X_n)$ in $\mathbb{R}^n$ such that

$$[A, X_j] = m_j X_j, \quad j = 1, \dots, n,$$

with $\{m_1, \dots, m_n\} \subset \mathbb{R}^\times$.

(a) If $\{\frac{m_k}{m_1}\}_{1 \le k \le n} \in \mathbb{N}$, then any generic coadjoint orbit of ℓ ($\ell_1 \neq 0$) is given by

$$\mathcal{O}_\ell = \left\{ x = (x_1, \dots, x_n, x_{n+1}) \in \mathfrak{g}^* : x_1 \ell_1 > 0, x_k = \frac{\ell_k}{\ell_1^{\frac{m_k}{m_1}}} x_1^{\frac{m_k}{m_1}}, \quad k = 2, \dots, n, x_{n+1} \in \mathbb{R} \right\},$$

and hence, it is an open semi-algebraic subset in $\mathfrak{g}^*$.

(b) Now suppose that $\{m_1, \dots, m_n\} \subset \mathbb{Z}^\times$ with $\prod_{j=1}^n m_j < 0$. We can assume the existence of a basis $(X_1, \dots, X_n)$ in $\mathbb{R}^n$ so that with respect to this basis the matrix of adA is

$$adA = \operatorname{diag}(m_1, \dots, m_{k_0}, m_{k_0+1}, \dots, m_n)$$

with $m_1 > 0, \dots, m_{k_0} > 0, m_{k_0+1} < 0, \dots, m_n < 0$. Then for any $x = (x_1, \dots, x_n, x_{n+1}) \in \mathcal{O}_\ell$ (with $\ell_1 \neq 0$) one has

$$\begin{cases} x_1 \ell_1 > 0, \\ \ell_1^{m_j} x_j^{m_1} = \ell_j^{m_1} x_1^{m_j}, & j = 2, \dots, k_0, \\ x_1^{-m_j} x_j^{m_1} = \ell_1^{-m_j} \ell_j^{m_1}, & j = k_0 + 1, \dots n. \end{cases}$$

On other hand, the polynomials

$$p_{i,j}(\ell) = \ell_i^{-m_j} \ell_j^{m_i}, \quad i = 1, \dots, k_0, \quad j = k_0 + 1, \dots, n$$

are G-invariant on $\mathfrak{g}^*$ and the cortex is the union of two vector spaces given by:

$$\operatorname{Cor}(\mathfrak{g}^*) = \{\ell \in \mathfrak{g}^* : p_{i,j}(\ell) = 0 \quad \forall 1 \le i \le k_0, k_0 + 1 \le j \le n\}.$$

Corollary 4.3. *Let $\mathfrak{g} = \mathbb{R}^n \oplus \mathbb{R}A$ be a real Lie algebra. Let's denote $sp(adA) = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{C}$ the set of eigenvalues of adA. If*

$$\{\Re(\lambda_j)\}_{1 \le j \le n} \subset (0, \infty) \quad \text{or} \quad \{\Re(\lambda_j)\}_{1 \le j \le n} \subset (-\infty, 0),$$

then

$$\operatorname{Cor}(\mathfrak{g}^*) = \mathfrak{g}^*.$$

Proof. First let's suppose that the real endomorphism adA has a single eigenvalue $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. According to λ is real or complex, we can suppose the existence of a basis in $\mathbb{R}^n$ (resp. in $\mathbb{C}^n$) such that the matrix of adA is written in a Jordan block form:

$$adA = J_\lambda = \begin{pmatrix} \lambda & 1 & 0 & \dots \\ 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \lambda \end{pmatrix}$$

The coadjoint orbit of $\ell = (\ell_1, \dots, \ell_n, \ell_{n+1})$ is given by

$$\mathcal{O}_\ell = \left\{ \left(\left(x_k = e^{\lambda t} \sum_{j \geq 1, i+j=k} \frac{t^i}{i!} \ell_j \right)_{1 \leq k \leq n}, x_{n+1} = \ell_{n+1} + s \right), t, s \in \mathbb{R} \right\}.$$

Now let's remark that for any $\alpha \in \mathbb{R}^n$(resp. in $\mathbb{C}^n$), since $\Re(\lambda) \neq 0$, the linear system

$$e^{\lambda t} \begin{pmatrix} 1 & 0 & \dots & 0 \\ t & 1 & 0 & \ddots \\ \vdots & \ddots & \ddots & 0 \\ \frac{t^{n-1}}{(n-1)!} & \dots & t & 1 \end{pmatrix} \begin{pmatrix} \ell_1 \\ \ell_2 \\ \vdots \\ \ell_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

has a unique solution and if we let

$$M(t) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ t & 1 & 0 & \ddots \\ \vdots & \ddots & \ddots & 0 \\ \frac{t^{n-1}}{(n-1)!} & \dots & t & 1 \end{pmatrix},$$

then $M(t)$ is a unipotent matrix whose inverse $M^{-1}(t) = (p_{i,j}(t))_{1 \leq i,j \leq n}$ is also unipotent and all its entries $p_{i,j}(t)$ are polynomial functions in t, then

$$[\ell_1, \dots, \ell_n]^\perp = e^{-\lambda t} M^{-1}(t) [\alpha_1, \dots, \alpha_n]^\perp$$

and

$$\|[\ell_1, \dots, \ell_n]^\perp\| = e^{-\Re(\lambda)t} F(t),$$

where $F(t)^2$ is polynomial function in t and then

$$\lim_{\Re(\lambda)t \to \infty} e^{-\Re(\lambda)t} F(t) = 0,$$

For instance for any $\alpha = [\alpha_1, \dots, \alpha_n] \in \mathbb{R}^n$ if λ is real (resp. $\alpha \in \mathbb{C}^n$ if $\lambda \in \mathbb{C} \setminus i\mathbb{R}$) and $\{t_m\}_m \subset \mathbb{R}$ such that $\lim_{m\to\infty} t_m \Re(\lambda) = \infty$ it exists $\{\ell_1^{(m)}, \dots, \ell_n^{(m)}\}_m$ such that

$$\lim_{m \to \infty} e^{\lambda t_m} M(t_m) [\ell_1^{(m)}, \dots, \ell_n^{(m)}]^\perp = [\alpha_1, \dots, \alpha_n]^\perp$$

and

$$\lim_{m\to\infty} \ell_1^{(m)} = \cdots = \lim_{m\to\infty} \ell_n^{(m)} = 0.$$

This shows that the cortex of $\mathfrak{g}^*$ coincides with $\mathfrak{g}^*$. Finally if adA has more then one single eigenvalue, we can write $adA = \mathrm{diag}(J_{\lambda_1}, \ldots, J_{\lambda_k})$ where each J_λ is a Jordan block matrix. □

Remark 4.2. Let $\mathfrak{g} = \mathbb{R}X_1 \oplus \mathbb{R}X_2 \oplus \mathbb{R}A$ with

$$[A, X_1] = X_1, [A, X_2] = X_2.$$

In this example the cortex of $\mathfrak{g}^*$ is $\mathfrak{g}^*$. The cross-section mapping of the minimal layer is given by

$$F(\ell_1, \ell_2, \ell_3) = \left(\frac{\ell_1}{|\ell_1|}, \frac{\ell_2}{|\ell_1|}, 0\right), \quad \ell_1 \neq 0.$$

On other hand, we remark that the rational function $r(\ell_1, \ell_2, \ell_3) = \frac{\ell_2}{\ell_1}$ is G-invariant on $\Omega = \{\ell = (\ell_1, \ell_2, \ell_3) \in \mathfrak{g}^* : \ell_1 \neq 0\}$ and

$$\mathrm{Cor}(\mathfrak{g}^*) \supsetneq \overline{\{\ell \in \Omega : r(\ell) = 0\}}.$$

Corollary 4.4. *Let $\mathfrak{g} = \mathbb{R}^n \oplus \mathbb{R}A$, be a real Lie algebra. Let's denote $sp(adA) = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$ the set of eigenvalues of adA. If $\prod_{j=1}^n \Re(\lambda_j) < 0$, then the cortex of $\mathfrak{g}^*$ is the union of two vector spaces. More precisely, with the notations of Proposition* 4.2*, one has*

$$\mathrm{Cor}(\mathfrak{g}^*) = (V^+ + \mathbb{R}A^*) \cup (V^- + \mathbb{R}A^*).$$

Remark 4.3. Let $\mathfrak{g} = \mathbb{R}^n \oplus \mathbb{R}A$ be a real Lie algebra, and assume that all the eigenvalues of adA are purely imaginary. Let's denote $\mathfrak{h} = \mathbb{R}^n \oplus \mathbb{R}N$ where N is the nilpotent part in the Jordan decomposition of A, then by [3], one has

$$\mathrm{Cor}(\mathfrak{g}^*) = \mathrm{Cor}(\mathfrak{h}^*).$$

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