Three-element bands in $\beta\mathbb{N}$

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ABSTRACT. Let $\mathbb{N}$ be the discrete additive semigroup of natural numbers and let $\beta\mathbb{N}$ be the Stone–Čech compactification of $\mathbb{N}$. The addition on $\mathbb{N}$ extends to an operation $+$ on $\beta\mathbb{N}$ making it a right topological semigroup, and to an operation $*$ making it a left topological semigroup. The semigroup $(\beta\mathbb{N}, *)$ is the opposite of the semigroup $(\beta\mathbb{N}, +)$:

$$p * q = q + p.$$

We list all 3-element idempotent semigroups that have algebraic copies in $(\beta\mathbb{N}, +)$. As a consequence we obtain that $(\beta\mathbb{N}, +)$ and $(\beta\mathbb{N}, *)$ are not algebraically isomorphic.

The addition of the discrete semigroup $\mathbb{N}$ of natural numbers extends to the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ so that for each $a \in \mathbb{N}$, the left translation

$$\beta\mathbb{N} \ni x \mapsto a + x \in \beta\mathbb{N}$$

is continuous, and for each $q \in \beta\mathbb{N}$, the right translation

$$\beta\mathbb{N} \ni x \mapsto x + q \in \beta\mathbb{N}$$

is continuous.

We take the points of $\beta\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$, identifying the principal ultrafilters with the points of $\mathbb{N}$. The topology of $\beta\mathbb{N}$ is generated by taking as a base the subsets of the form

$$\overline{A} = \{p \in \beta\mathbb{N} : A \in p\},$$

where $A \subseteq \mathbb{N}$. For $p, q \in \beta\mathbb{N}$, the ultrafilter $p + q$ has a base consisting of subsets of the form

$$\bigcup_{x \in A} (x + B_x),$$

where $A \in p$ and $B_x \in q$.

Being a compact Hausdorff right topological semigroup, $\beta\mathbb{N}$ has a smallest two sided ideal $K(\beta\mathbb{N})$ which is a disjoint union of minimal right ideals and a disjoint union of minimal left ideals. Every right (left) ideal of $\beta\mathbb{N}$ contains a minimal right (left) ideal. The intersection of a minimal right ideal and a minimal left ideal is a group, and all these groups are isomorphic. For every

minimal right ideal R, the set $E(R)$ of idempotents of R is a right zero semigroup ($xy = y$ for all x, y), and for every minimal left ideal L, $E(L)$ is a left zero semigroup ($xy = x$ for all x, y).

The semigroup $\beta\mathbb{N}$ is interesting both for its own sake and for its applications to Ramsey theory and to topological dynamics. An elementary introduction to $\beta\mathbb{N}$ can be found in [2].

The semigroup $\beta\mathbb{N}$ contains no nontrivial finite groups [2, Theorem 7.17]. However, it does contain bands (idempotent semigroups). For example, left zero semigroups, right zero semigroups, chains of idempotents (with respect to the order $x \le y$ if and only if $xy = yx = x$), and rectangular semigroups (direct products of a left zero semigroup and a right zero semigroup) [3]. Whether $\beta\mathbb{N}$ contains a finite semigroup distinct from bands is an old difficult question. It is equivalent to asking whether $\beta\mathbb{N}$ contains a 2-element semigroup with zero multiplication, and also to whether there is a nontrivial continuous homomorphism from $\beta\mathbb{N}$ to $\beta\mathbb{N} \setminus \mathbb{N}$ [5, 1].

A 2-element band is either left zero semigroup or right zero semigroup or chain of idempotents, and $\beta\mathbb{N}$ contains algebraic copies of each of them.

The list of 3-element bands $\{x, y, z\}$ is longer [4, Appendix A]:

(1) chain;
(2) semilattice ($xy = yx = z$);
(3) left zero semigroup;
(4) right zero semigroup;
(5) $\{x, y\}$ is left zero semigroup and z is identity;
(6) $\{x, y\}$ is right zero semigroup and z is identity;
(7) $\{x, y\}$ is left zero semigroup and z is zero;
(8) $\{x, y\}$ is right zero semigroup and z is zero;
(9) $\{x, y\}$ is left zero semigroup, z is right identity, and $zx = zy = x$;
(10) $\{x, y\}$ is right zero semigroup, z is left identity, and $xz = yz = x$.

The aim of this note is to show that

Theorem 1. *The semigroups* (1), (3), (4), (5), (6), (7), (9), *and* (10) *have algebraic copies in* $\beta\mathbb{N}$, *and* (2) *and* (8) *do not.*

A finite semigroup S is an *absolute coretract* if for every continuous homomorphism $f : T \to S$ of a compact Hausdorff right topological semigroup T onto S there is a homomorphism $g : S \to T$ such that $f \circ g = \mathrm{id}_S$. Since $\beta\mathbb{N}$ has closed subsemigroups that admit a continuous homomorphism onto any finite semigroup [2, Corollary 6.5], it follows that $\beta\mathbb{N}$ contains copies of any finite absolute coretract. The finite absolute coretracts are completely described, they are certain chains of rectangular semigroups [6, Section 10.4]. Using that description one can show that the semigroups (1), (3), (4), (5), (6), (9), and (10) are such. But this fact may also be established directly, not involving the whole description. Indeed, for (1), (3), and (4) this is [6, Lemma 10.2], and for (5), (6), (9), and (10) the following lemma.

Lemma 2. *The semigroups* (5), (6), (9), *and* (10) *are absolute coretracts.*

Proof. We restrict ourselves to proving the lemma for (5) and (9). Let $f : T \to S$ be a continuous homomorphism of a compact Hausdorff right topological semigroup T onto $S = \{x, y, z\}$. Pick an idempotent $r = g(z) \in f^{-1}(z)$, and then a minimal left ideal L of T contained in Tr.

Suppose that S is (5). Pick minimal right ideals R_x and R_y of T contained in $rf^{-1}(x)$ and $rf^{-1}(y)$, respectively. Let $p = g(x)$ and $q = g(y)$ be the identities of the groups $R_x \cap L$ and $R_y \cap L$.

Since $p \in R_x \subseteq rf^{-1}(x)$ and $q \in R_y \subseteq rf^{-1}(y)$, one has $f(p) = zx = x$ and $f(q) = zy = y$. Consequently, $f \circ g = \mathrm{id}_S$.

Clearly, $\{p, q\}$ is a left zero semigroup. Since $p, q \in L \subseteq Tr$, one has $pr = p$ and $qr = q$, and since $p \in R_x \subseteq rf^{-1}(x)$ and $q \in R_y \subseteq rf^{-1}(y)$, one has $rp = p$ and $rq = q$. It follows that g is a homomorphism.

Now suppose that S is (9). Pick a minimal right ideal R_y of T contained in $f^{-1}(y)$ and put $R_x = rR_y$. Let $p = g(x)$ and $q = g(y)$ be the identities of the groups $R_x \cap L$ and $R_y \cap L$.

Clearly, $f \circ g = \mathrm{id}_S$. To see that g is a homomorphism, it suffices to check that $rq = p$.

Since $rR_y = R_x$, one has $rq \in R_x$, so $rq \in R_x \cap L$. And since $rqrq = rqq = rq$, rq is an idempotent, so $rq = p$. $\square$

It remains to consider (2), (7), and (8).

Lemma 3. *The semigroups* (2) *and* (8) *have no copies in* $\beta\mathbb{N}$.

Proof. Assume on the contrary that (2) has a copy $\{p, q, r\}$ in $\beta\mathbb{N}$. Then we have that $p + q = q + p$. But then by [2, Corollary 6.21], either $q \in \beta\mathbb{N} + p$ or $p \in \beta\mathbb{N} + q$. The first possibility implies that $q + p = q$, a contradiction, and the second possibility implies that $p + q = p$, also a contradiction.

Now assume on the contrary that (8) has a copy $\{p, q, r\}$ in $\beta\mathbb{N}$. Then we have that $r + p = r + q$. It follows that either $p \in \beta\mathbb{N} + q$ or $q \in \beta\mathbb{N} + p$. The first possibility implies that $p + q = p$, and the second $q + p = q$, a contradiction. $\square$

Since a commutative band contains a copy of (2) if and only if it is not a chain, we obtain:

Corollary 4. *A finite commutative band has a copy in* $\beta\mathbb{N}$ *if and only if it is a chain.*

The next lemma finishes the proof of Theorem 1.

Lemma 5. *The semigroup* (7) *has a copy in* $\beta\mathbb{N}$.

To prove Lemma 5, we need some additional facts.

Lemma 6. *Let R be a minimal right ideal of* $\beta\mathbb{N}$. *Then there is $p \in \overline{E(R)}$ such that $p \notin \mathbb{Z}^* + \mathbb{Z}^*$.*

The proof of Lemma 6 is practically the same as that of [2, Theorem 8.22]. An ultrafilter $p \in \mathbb{Z}^*$ is *right cancelable* if $p \notin \mathbb{Z}^* + p$.

Lemma 7. *Let $p \in \mathbb{Z}^*$ be a right cancelable ultrafilter and let C_p denote the smallest closed subsemigroup of $\mathbb{Z}^*$ containing p. Then:*

(i) *$C_p \cap K(\beta\mathbb{Z}) = \emptyset$.*

(ii) *There is a continuous homomorphism of C_p onto $\beta\mathbb{N}$.*

The first statement of Lemma 7 is [2, Theorems 8.57], and the second [2, Theorem 8.51].

Notice that $K(\beta\mathbb{N}) = K(\beta\mathbb{Z}) \cap \beta\mathbb{N}$ [2, Theorem 1.65].

Proof of Lemma 5. Pick a minimal right ideal R of $\beta\mathbb{N}$ and $e \in E(R)$. Let $T = \{x \in \beta\mathbb{N} : x + e = e\}$. Notice that T is a closed subsemigroup of $\beta\mathbb{N}$ and $K(T) = E(R)$. By Lemma 6, there is $t \in \overline{K(T)}$ which is right cancelable in $\mathbb{Z}^*$. By Lemma 7, $C_t \cap K(T) = \emptyset$ and we can pick a left zero semigroup $\{p, q\}$ in C_t. Define $r \in K(T)$ by $r = e + p$. Then

$$r + p = e + p + p = e + p = r,$$
$$p + r = p + e + p = e + p = r,$$
$$r + q = e + p + q = e + p = r,$$
$$q + r = q + e + p = e + p = r.$$

Hence, the semigroup $\{p, q, r\}$ is as required.   □

Since (7) is not a subsemigroup of a finite absolute coretract [6, Proposition 11.12], we obtain:

Corollary 8. *There are finite bands in $\beta\mathbb{N}$ distinct from subsemigroups of finite absolute coretracts.*

We have extended the addition of natural numbers to an operation $+$ on $\beta\mathbb{N}$ so as to obtain a right topological semigroup. But one can equally well extend the addition to an operation $*$ on $\beta\mathbb{N}$ so as to obtain a left topological semigroup. The semigroup $(\beta\mathbb{N}, *)$ is the opposite of the semigroup $(\beta\mathbb{N}, +)$:

$$p * q = q + p.$$

Since $(\beta\mathbb{N}, +)$ does not contain (8), it follows that $(\beta\mathbb{N}, *)$ does not contain (7), which is the opposite of (8). But $(\beta\mathbb{N}, +)$ does contain (7). Thus, we obtain:

Corollary 9. *The semigroups $(\beta\mathbb{N}, +)$ and $(\beta\mathbb{N}, *)$ are not algebraically isomorphic.*

We conclude this note with the following question.

Question. Characterize all finite bands that have algebraic copies in $\beta\mathbb{N}$.

It would be interesting to answer this question even for 4-element bands (there are 46 of them).

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