Circuits and Hurwitz action in finite root systems

Joel Brewster Lewis and Victor Reiner

ABSTRACT. In a finite real reflection group, two factorizations of a Coxeter element into an arbitrary number of reflections are shown to lie in the same orbit under the Hurwitz action if and only if they use the same multiset of conjugacy classes. The proof makes use of a surprising lemma, derived from a classification of the minimal linear dependences (matroid circuits) in finite root systems: any set of roots forming a minimal linear dependence with positive coefficients has a disconnected graph of pairwise acuteness.

CONTENTS

1. Introduction

Given a group W and set T of generators for W, consider factorizations $(t_1, t_2, \ldots, t_m)$ of a given element $g = t_1 \cdots t_m$ in W. When T is closed under conjugation, these factorizations carry a natural action of the Artin braid group on m strands called the *Hurwitz action*. Here the braid group generator σ_i acts on ordered factorizations by a *Hurwitz move*, interchanging two factors t_i, t_{i+1} while conjugating one by the other:

$$(1.1) \qquad \begin{array}{c} (t_1, \ldots, t_{i-1}, \quad t_i, \quad t_{i+1}, \quad t_{i+2}, \ldots, t_m) \quad \overset{\sigma_i}{\longmapsto} \\ (t_1, \ldots, t_{i-1}, \quad t_{i+1}, \quad t_i^{t_{i+1}}, \quad t_{i+2}, \ldots, t_m). \end{array}$$

(We use the notation $a^b := b^{-1}ab$ for conjugation in a group.) When W is a finite real reflection group of rank n and T is the set of all of its reflections, D. Bessis used a simple inductive argument to prove the following result about shortest factorizations of *Coxeter elements* (see Section 5 for the definition), which he called the *dual Matsumoto property*.

Bessis's Theorem ([Bes03, Prop. 1.6.1])**.** *Let W be a finite real reflection group of rank n and let c be a Coxeter element of W. The set of all shortest ordered factorizations $(t_1, \ldots, t_n)$ of $c = t_1 t_2 \cdots t_n$ as a product of reflections forms a single transitive orbit under the Hurwitz action.*

The original context for this result is the *dual Coxeter theory* developed by Bessis [Bes03] and Brady and Watt [Bra01, BW02]. It has since been extended to several other contexts:

- shortest reflection factorizations in *well-generated complex reflection groups* [Bes15, Prop. 7.6],
- shortest *primitive* factorizations in well-generated complex reflection groups [Rip10, Thm. 0.4], where primitivity means having at most one nonreflection factor,
- shortest reflection factorizations in not-necessarily-finite Coxeter groups [IS10, BaDSW14], and
- the classification in finite real reflection groups of the elements whose shortest reflection factorizations have a single Hurwitz orbit [BaGRW15].

However, the question of how Bessis's Theorem extends to *longer* reflection factorizations seems not to have been addressed. One obstruction to transitivity has been noted frequently [LaZ04, LeRS14, Rip10]: the Hurwitz action preserves the (unordered) m-element multiset of conjugacy classes of the factors. This multiset is called the *unordered passport* in type A by

Lando and Zvonkin [LaZ04, §5.4.2.2]. In considering reflection factorizations of a Coxeter element c whose length is strictly greater than the minimum (the rank n of W), it is possible for the factorizations to use different multisets of reflection conjugacy classes. When W is a finite real reflection group, we show that this is the only obstruction.

Theorem 1.1. *In a finite real reflection group, two reflection factorizations of a Coxeter element lie in the same Hurwitz orbit if and only if they share the same multiset of conjugacy classes.*

In particular, in the irreducible "oddly-laced types" (A_n, D_n, E_6, E_7, E_8, H_3, H_4, and $I_2(m)$ with m odd), there is only one conjugacy class of reflections, and hence the Hurwitz action is transitive.

We sketch here the proof of Theorem 1.1, which has three main steps. The first is a lemma, proven in Section 3, that one might paraphrase as asserting that "root circuits are acutely disconnected". Call a subset

$$C = \{\alpha_1, \ldots, \alpha_m\}$$

of a Euclidean space $(V, (\cdot, \cdot))$ a *minimal dependence* (or *circuit*) if there exist nonzero coefficients c_i such that $\sum_{i=1}^{m} c_i \alpha_i = 0$, and C is inclusion-minimal with respect to this property. Define its *acuteness graph* Γ_C to have vertices $\{1, 2, \ldots, m\}$ and an edge $\{i, j\}$ whenever $(c_i \alpha_i, c_j \alpha_j) > 0$.

Lemma 1.2. *In a finite not-necessarily-crystallographic root system, every circuit C has Γ_C disconnected.*

The second step (Section 4) uses Lemma 1.2 to prove a lemma on the *absolute (reflection) length* function

$$\ell_T(w) := \min\{\ell : w = t_1 t_2 \cdots t_\ell \text{ for some } t_i \in T\}.$$

Lemma 1.3. *For any reflection factorization $\mathbf{t} = (t_1, \ldots, t_m)$ of*

$$w = t_1 \cdots t_m$$

with $\ell_T(w) < m$, either $m = 2$, or there exists $\mathbf{t}' = (t_1', \ldots, t_m')$ in the Hurwitz orbit of $\mathbf{t}$ with $\ell_T(t_1' \cdots t_k') < k$ for some $k \leq m - 1$.

The third step, also in Section 4, iterates Lemma 1.3 to put reflection factorizations into a standard form.

Corollary 1.4. *If $\ell_T(w) = \ell$, then every factorization of w into m reflections lies in the Hurwitz orbit of some $\mathbf{t} = (t_1, \ldots, t_m)$ such that*

$$t_1 = t_2,$$
$$t_3 = t_4,$$
$$\vdots$$
$$t_{m-\ell-1} = t_{m-\ell},$$

and $(t_{m-\ell+1}, \ldots, t_m)$ is a shortest reflection factorization of w.

Section 5 then finishes off the proof of Theorem 1.1, using the case of Corollary 1.4 where w is a Coxeter element c, along with Bessis's Theorem above, and Bessis's observation that any reflection t can occur first in a shortest factorization of c. Section 6 collects a few remarks and questions suggested by this work.

We note that the proof of the crucial Lemma 1.2 is case-based and relies on large computer calculations. The remaining steps of the argument are case-independent (at least in the crystallographic case), so that one might hope to make the argument fully human-comprehensible by giving a case-free proof of Lemma 1.2.

Acknowledgements. The authors thank Guillaume Chapuy, Theodosios Douvropoulos, Vivien Ripoll, and Christian Stump for helpful conversations. They also thank Patrick Wegener both for helpful comments, and for the content of Section 6.1. Finally, the authors thank an anonymous referee for thorough and thoughtful comments.

2. Background and terminology

In this section, we review some standard definitions and facts about finite real reflection groups and root systems. Good references for this material are [BjB05, Chs. 1, 4], [Hum90], and [Arm09, §§2.1–2.2].

Definition 2.1. Let $(V, (\cdot, \cdot))$ be a finite-dimensional Euclidean space, that is, a real vector space $V \cong \mathbb{R}^n$ with a positive definite symmetric bilinear from $(\cdot, \cdot)$, whose associated norm $|v|$ is given by $|v|^2 = (v, v)$. For a nonzero vector α in V, the *reflection s_α* through the hyperplane $H = \alpha^\perp$ is the linear map given by the formula

$$(2.1) \qquad s_\alpha(v) = v - \frac{2(v, \alpha)}{|\alpha|^2}\alpha.$$

A *finite reflection group* is a finite subgroup W of $GL_n(\mathbb{R})$ generated by its subset $T \subset W$ of reflections.

Since reflections lie within the orthogonal group $O_n(\mathbb{R})$, so does W. That is, W preserves $(\cdot, \cdot)$.

Definition 2.2. A *(finite, reduced, not-necessarily-crystallographic) root system* associated to a finite reflection group W is any W-stable subset $\Phi \subset V$ consisting of a choice of two opposite normal vectors $\pm\alpha$ for each reflecting hyperplane H of a reflection t in T. We will assume W has no fixed vector in V, that is, the w-fixed spaces defined by $V^w := \{v \in V : w(v) = v\}$ satisfy $\bigcap_{w \in W} V^w = \{0\}$.

It is not hard to see that root systems Φ for W are parametrized by picking a representative t of each conjugacy class of reflection and choosing a scaling for the normal vectors $\pm\alpha$ to the reflecting hyperplane of t. On the other hand, one can axiomatize such root systems as follows: they are the

collections of finitely many nonzero vectors Φ spanning V with the property that $s_\alpha(\beta) \in \Phi$ for all $\alpha, \beta \in V$, and $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all α in Φ. In this case, one recovers W as the group generated by the reflections $\{s_\alpha : \alpha \in \Phi\}$.

Definition 2.3. An *open Weyl chamber* F for a finite reflection group W is a connected component of the complement within V of the union of the reflecting hyperplanes for all reflections t in T.

It turns out that W acts simply transitively on the set of Weyl chambers. Also, the closure $\overline{F}$ of any Weyl chamber F is a *fundamental domain* for the action of W on V: every W-orbit Wv on V has $|(Wv) \cap \overline{F}| = 1$.

Definition 2.4. The set Φ^+ of *positive roots* corresponding to a choice of an open Weyl chamber F is

$$\Phi^+ := \{\alpha \in \Phi : (\alpha, v) > 0 \text{ for all } v \text{ in } F\}.$$

The associated set of *simple roots* $\Pi \subset \Phi^+$ is the set of inward-pointing normal vectors to the walls of $\overline{F}$.

It is easily seen that $\Phi = \Phi^+ \sqcup (-\Phi^+)$. Less obvious are the following properties of the simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$:

- They are pairwise non-acute.
- They form an $\mathbb{R}$-basis for V.
- They contain W-orbit representatives for all of the roots.
- Every $\alpha \in \Phi^+$ has its unique expression $\alpha = \sum_{i=1}^n c_i \alpha_i$ with $c_i \geq 0$ for all i.

Definition 2.5. A finite reflection group W is called *reducible* if there exists a nontrivial orthogonal direct sum decomposition

$$V = V_1 \oplus V_2$$

respected by W.

Reducibility of the group W is equivalent to the existence of a nontrivial decomposition $\Phi = \Phi_1 \sqcup \Phi_2$ with $(\alpha_1, \alpha_2) = 0$ when $\alpha_i \in \Phi_i$ for $i = 1, 2$, in any (or every) root system Φ for W. It is also equivalent to the existence of a nontrivial decomposition $\Pi = \Pi_1 \sqcup \Pi_2$ with $(\alpha_1, \alpha_2) = 0$ when $\alpha_i \in \Pi_i$ for $i = 1, 2$, in any (or every) choice of simple roots Π for Φ. In this situation, $W = W_1 \times W_2$ where W_i is the subgroup generated by $\{s_\alpha : \alpha \in \Phi_i\}$, or by $\{s_\alpha : \alpha \in \Pi_i\}$.

There is a *classification* of finite irreducible reflection groups W. It contains four infinite families and six exceptional groups:

- type A_{n-1} for $n \geq 2$, where W is isomorphic to the symmetric group on n letters,
- type $\mathrm{B}_n/\mathrm{C}_n$ for $n \geq 2$, where W is the hyperoctahedral group of $n \times n$ signed permutation matrices,
- type D_n for $n \geq 4$, where W is an index-two subgroup of the hyperoctahedral group,

- type $\mathrm{I}_2(m)$ for $m \geq 5$, where W is the dihedral group of symmetries of a regular m-gon, and
- exceptional types $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8, \mathrm{F}_4, \mathrm{H}_3, \mathrm{H}_4$.

We will later need to consider the field extension K of $\mathbb{Q}$ that adjoins to $\mathbb{Q}$ the elements $\frac{(\beta,\alpha)}{|\alpha|^2}$ for all α, β in Φ. If we normalize all of the roots to the same length, then $\frac{(\beta,\alpha)}{|\alpha|^2} = \cos\left(\frac{2\pi}{m}\right)$ if the rotation $s_\alpha s_\beta$ has order m. This number is always algebraic, so we may assume that K is a number field, that is, a finite extension of $\mathbb{Q}$. We can sometimes do better, as in the following definition.

Definition 2.6. Say a root system Φ is *crystallographic* if

$$(2.2) \qquad\qquad \frac{2(\beta,\alpha)}{|\alpha|^2} \in \mathbb{Z} \text{ for all } \alpha, \beta \in \Phi.$$

Of course, if Φ is crystallographic, then $K = \mathbb{Q}$. Since rescaling roots within a W-orbit does not affect W itself (or any of the circuit properties to be discussed later), we always choose without further mention a crystallographic root system Φ for W when one is available. This means that Φ will be chosen crystallographic in all types except $\mathrm{H}_3, \mathrm{H}_4$ (where one can take $K = \mathbb{Q}[\sqrt{5}]$), and $\mathrm{I}_2(m)$ for $m \notin \{3, 4, 6\}$.

3. Circuit classification and proof of Lemma 1.2

The goal of this section is to prove Lemma 1.2 from the Introduction, which we recall here. Fix a finite (not-necessarily-crystallographic) root system Φ in a Euclidean space $(V, (\cdot, \cdot))$.

Definition 3.1. A finite subset $C = \{\alpha_1, \ldots, \alpha_m\} \subseteq V$ is a *circuit* if it has a nontrivial dependence $c_1\alpha_1 + \cdots + c_m\alpha_m = 0$, but no proper subset of C is dependent. Given a circuit C, the dependence coefficients $(c_1, \ldots, c_m) \in \mathbb{R}^m$ are uniquely determined up to simultaneous $\mathbb{R}$-scaling. Thus, one may define the *acuteness graph* Γ_C to have vertex set $\{1, 2, \ldots, m\}$ and an edge $\{i, j\}$ whenever $(c_i\alpha_i, c_j\alpha_j) > 0$.

Lemma 1.2. *In a finite root system, every circuit C has disconnected acuteness graph Γ_C.*

We will often abuse terminology by considering two circuits C, C' to be the *same* when they span the same set of lines $\{\mathbb{R}\alpha\}_{\alpha \in C} = \{\mathbb{R}\alpha'\}_{\alpha' \in C'}$, or have the same set of normal hyperplanes $\{\alpha^\perp\}_{\alpha \in C} = \{(\alpha')^\perp\}_{\alpha' \in C'}$. Note that in this case, $\Gamma_C = \Gamma_{C'}$. In fact, our figures below will depict slightly more graphical information about the circuits C, namely an *acuteness-obtuseness graph* that shows the $c_i\alpha_i$ labeling vertices, and these solid (acute) and

dotted (obtuse) edges:

$$
\begin{cases}
c_i\alpha_i \text{——} c_j\alpha_j & \text{when } (c_i\alpha_i, c_j\alpha_j) > 0, \\
c_i\alpha_i \text{……} c_j\alpha_j & \text{when } (c_i\alpha_i, c_j\alpha_j) < 0, \\
c_i\alpha_i \qquad c_j\alpha_j & \text{when } (c_i\alpha_i, c_j\alpha_j) = 0.
\end{cases}
$$

The acuteness graph Γ_C comes from erasing the dotted (obtuse) edges in the acuteness-obtuseness graph.

Our proof of Lemma 1.2 relies on a classification of circuits in finite root systems, which may be of independent interest. Such a classfication is essentially already provided in the classical types $A_{n-1}, B_n/C_n, D_n$ by Zaslavsky's theory of *signed graphs* [Zas82], and we rely on a computer calculation for the exceptional types.

Remark 3.2. A different sort of circuit classification in finite root systems was undertaken by Stembridge [Ste07], who defined the notion of an *irreducible circuit*. Say that a circuit $C = \{\alpha\} \cup I \subset \Phi$ is irreducible if α is in the positive linear span of I, and no proper subset of I has any elements of $\Phi \smallsetminus I$ in its positive linear span. Stembridge gave a classification, up to isometry, of the irreducible circuits in all finite root systems. Unfortunately, we did not see how to check Lemma 1.2 directly from the classification of irreducible circuits. See also Example 3.10 below.

Given a finite reflection group W and a choice of a root system Φ_W in $V \cong \mathbb{R}^n$ for W, one might attempt to classify all of the circuits $C \subset \Phi_W$ up to the action of W, that is, regarding $w(C)$ and C equivalent for all w in W. We will do slightly less, taking advantage of the following reduction.

Definition 3.3. Call a circuit $C \subset \Phi_W$ a *full circuit* if $\{s_\alpha : \alpha \in C\}$ generates the group W.

Remark 3.4. Every non-full circuit $C \subset \Phi_W$ lies in a root system $\Phi_{W'}$ corresponding to some proper subgroup $W' := \langle s_\alpha : \alpha \in C \rangle \subsetneq W$; then C will be a full circuit within $\Phi_{W'}$.

Furthermore, only *irreducible* root systems Φ_W contain full circuits C: if one has $V = V_1 \oplus V_2$ with $\Phi = \Phi_1 \sqcup \Phi_2$ and $W = W_1 \times W_2$ then the circuit $C \subset \Phi$ being inclusion-minimal forces $C \subset \Phi_i$ for either $i = 1$ or 2, and hence $\{s_\alpha : \alpha \in C\} \subset W_i$ for either $i = 1$ or 2.

In light of Remark 3.4, in order to prove Lemma 1.2, we will carry out the classification up to W-action only[1] for the full circuits in Φ_W. In particular, we only need to consider the irreducible finite root systems.

---

[1]In principle, one could fill in the rest of the classification data using, e.g., the work of Douglass–Pfeiffer–Röhrle [DPR13], which classifies the reflection subgroups of finite real reflection groups up to conjugacy.

3.1. Rank 1. Here $C = \Phi = \{\pm\alpha\}$, whose acuteness-obtuseness graph has two vertices and a dotted edge:

$$+\alpha \longleftarrow \quad \longrightarrow -\alpha \qquad\qquad +\alpha \cdots\cdots\cdots -\alpha$$

3.2. Rank 2: the dihedral types $\mathbf{I_2(m)}$. A full circuit $C = \{\alpha_1, \alpha_2, \alpha_3\}$ satisfies $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0$ for some scalars c_i. Taking the inner product of both sides of this equation with $c_i\alpha_i$ and noting that $(c_i\alpha_i, c_i\alpha_i) > 0$, one concludes that at most one of the other two inner products

$$(c_i\alpha_i, c_j\alpha_j), \quad (c_i\alpha_i, c_k\alpha_k)$$

where $\{i, j, k\} = \{1, 2, 3\}$ can be positive. Hence each vertex $i = 1, 2, 3$ is incident to at most one edge in the acuteness graph Γ_C on vertex set $\{1, 2, 3\}$, forcing Γ_C to be disconnected—see the typical pictures below.
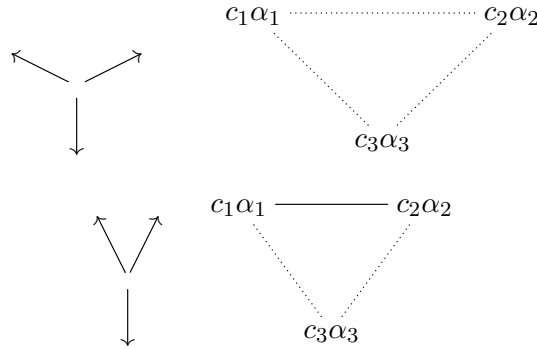
Although the rank 2 setting required no classification of the W-orbits of full circuits $C \subset \Phi_W$, such a classification is not hard. Consider the unordered triple $\{A_{12}, A_{13}, A_{23}\}$, where $\frac{\pi}{m}A_{ij}$ is the angular measure of the sector $\mathbb{R}_{\geq 0}c_i\alpha_i + \mathbb{R}_{\geq 0}c_j\alpha_j$, so that $A_{ij} \in \{1, 2, \cdots, m-1\}$ and

$$A_{12} + A_{13} + A_{23} = 2m.$$

One checks that C is a full circuit in Φ_W if and only if

$$g := \gcd\{A_{12}, A_{13}, A_{23}\} = 1;$$

otherwise C is full inside a sub-root system of type $I_2(m')$ with $m' := \frac{m}{g}$. Furthermore, if m is odd, the unordered triple $\{A_{12}, A_{13}, A_{23}\}$ completely determines the W-orbit of C, while for even m, there are exactly two W-orbits corresponding to each such triple, represented by circuits that differ from each other by a $\frac{\pi}{m}$ rotation.

Remark 3.5. The rank 2 case raises a reasonable question: does the conclusion of Lemma 1.2 have anything at all to do with root systems? In other words, is it possible that *any* minimal linearly dependent set of vectors $C = \{\alpha_1, \ldots, \alpha_m\}$ in a Euclidean space V has disconnected acuteness graph Γ_C? Unfortunately, this is *not* true for $\dim(V) \geq 3$. A result of Fiedler [Fie05, Thm. 2.5], stated in terms of of the *Gram matrix*

$((\alpha_i, \alpha_j))_{i,j=1,\ldots,m}$, asserts that C will have its *obtuseness graph* connected, and that one can in fact, find a circuit C with any prescribed set of obtuse pairs, orthogonal pairs, and acute pairs, as long as the obtuse pairs form a connected graph. When $\dim(V) \geq 3$, this means one can have both the obtuseness and acuteness graphs being connected. For example, one has a circuit $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$ with the following four vectors $(\alpha_i)_{i=1}^4$ in $\mathbb{R}^3$, having acuteness-obtuseness graph as shown:

$$C = \left(\alpha_1 = \begin{bmatrix} -6 \\ -3 \\ 0 \end{bmatrix}, \alpha_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \alpha_4 = \begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix} \right)$$

3.3. Type A_{n-1} for $n \geq 3$.

Consider $\mathbb{R}^n$ with its usual inner product $(\cdot, \cdot)$ making the basis vectors $e_1, \ldots, e_n$ orthonormal. Inside the codimension-one subspace $V = (e_1 + \cdots + e_n)^{\perp} \subset \mathbb{R}^n$, considered as a Euclidean space via the restriction of $(\cdot, \cdot)$, one has the *type A_{n-1} root system*

$$\Phi_{\mathrm{A}_{n-1}} = \{\pm(e_i - e_j) : 1 \leq i < j \leq n\}.$$

The Weyl group W is the symmetric group $\mathfrak{S}_n$, permuting the coordinates in $\mathbb{R}^n$ and preserving the subspace V. It is well-known (see, e.g., Oxley [Oxl92, Prop. 1.1.7]) and easily checked that circuits in $\Phi_{\mathrm{A}_{n-1}}$ are subsets of the form $\{e_{i_1} - e_{i_2}, e_{i_2} - e_{i_3}, \ldots, e_{i_{k-1}} - e_{i_k}, e_{i_k} - e_{i_1}\}$ for distinct elements $i_1, i_2, \ldots, i_k$ of $\{1, 2, \ldots, n\}$. Therefore full circuits in $\Phi_{\mathrm{A}_{n-1}}$ all lie in the W-orbit of

$$C = \{\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \ldots, \quad \alpha_{n-1} = e_{n-1} - e_n, \quad \alpha_n = e_n - e_1\},$$

whose minimal dependence is $\alpha_1 + \cdots + \alpha_n = 0$. Since $(\alpha_i, \alpha_j) \in \{-1, 0\}$ for each $i \neq j$, its acuteness graph Γ_C contains n vertices and no edges, and so is disconnected.

Pictorially, one may associate to a subset of $\Phi_{\mathrm{A}_{n-1}}$ a graph on vertex set $\{1, 2, \ldots, n\}$ in which the roots $\pm(e_i - e_j)$ perpendicular to the hyperplane $x_i = x_j$ are associated with the edge $i \text{---} j$. Circuits then correspond to graphs that are cycles, and the circuit C above for $n = 4$ would be depicted as the graph on the left, with its acuteness-obtuseness graph shown to its right:
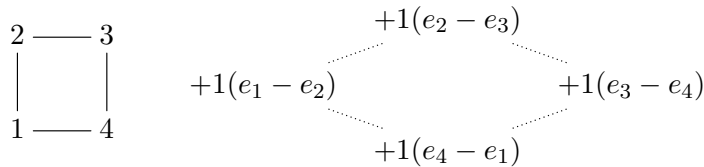
Remark 3.6. This W-orbit of full circuits C in type A where Γ_C has no edges at all generalizes to an interesting and well-known family of full circuits for each irreducible crystallographic root system Φ, which we describe here. Choose an open fundamental chamber F for W, with corresponding choice of positive roots Φ^+ and simple roots Π. Then there will always be either one or two roots in $\overline{F} \cap \Phi$, namely:

- the *highest root* α_0, whose unique expression $\alpha_0 = \sum_{i=1}^{n} h_i \alpha_i$ as a positive root simultaneously maximizes all the coefficients h_i (in particular $h_i > 0$ for each $i = 1, \ldots, n$);
- the *highest short root* $\alpha^* := (\alpha_0(\Phi^\vee))^\vee$, where $\alpha^\vee := \frac{2\alpha}{|\alpha|^2}$ and $\alpha_0(\Phi^\vee)$ is the highest root for the *dual* crystallographic root system $\Phi^\vee := \{\alpha^\vee : \alpha \in \Phi\}$. (When $\Phi^\vee = \Phi$, one has $\alpha_0 = \alpha^*$.)

Either of the roots $\beta = \alpha_0$ or $\beta = \alpha^*$ gives rise to a full circuit

$$C = \{-\beta\} \sqcup \Pi \subset \Phi$$

whose minimal dependence has the form $-\beta + c_1\alpha_1 + \cdots c_n\alpha_n = 0$. The acuteness graph Γ_C has no edges, since the simple roots are pairwise non-acute and since β in $\overline{F} \cap \Phi$ means that $(\beta, \alpha_i) \geq 0$ for all α_i in Π.

3.4. Type D_n for $n \geq 3$. The *type D_n root system*

$$\Phi_{D_n} = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\}$$

has Weyl group W which is an index-two reflection subgroup of the *hyperoctahedral group* $\mathfrak{S}_n^\pm$ of all *signed permutations* $e_i \mapsto \pm e_{w(i)}$; here w is a permutation of $\{1, 2, \ldots, n\}$. Specifically, $W = W(D_n)$ consists of those signed permutations in which there are *evenly many* indices i for which $e_i \mapsto -e_{w(i)}$.

Just as one can associate graphs whose edges correspond to pairs $\pm\alpha$ of roots in type A, Zaslavsky's theory of *signed graphs* [Zas82] associates to each root pair $\pm\alpha$ in Φ_{D_n} (or reflecting hyperplane $x_i = \pm x_j$) an edge $\{i, j\}$ on vertex set $\{1, 2, \ldots, n\}$ with a $\pm$ label:

- The roots $\alpha = \pm(e_i - e_j)$ with $\alpha^\perp$ defined by $x_i = +x_j$ give rise to *plus edges* $i \stackrel{+}{\rule{2em}{0.4pt}} j$.
- The roots $\alpha = \pm(e_i + e_j)$ with $\alpha^\perp$ defined by $x_i = -x_j$ give rise to *minus edges* $i \stackrel{-}{\rule{2em}{0.4pt}} j$.

Call a cycle in a signed graph *balanced* if it has an even number of minus edges, and *unbalanced* otherwise.

Proposition 3.7 (Zaslavsky [Zas82, Thm. 5.1(e)]). *A set of roots in Φ_{D_n} is a circuit if and only if its associated signed graph is one of the following types:*

    (i) *a balanced cycle;*

(ii) *two edge-disjoint unbalanced cycles, having either a path joining a vertex of one cycle to a vertex of the other, or else sharing exactly one vertex.*

The circuits of type (ii) in Proposition 3.7 are exemplified by the following full circuits. Given $i, j \geq 2$ such that $i + j \leq n + 1$, let $C(n; i, j)$ consist of two particular unbalanced cycles of sizes i, j, connected by a path having $n + 1 - (i + j)$ edges:

$$C(n; i, j)$$
$$:= \{ e_1 - e_2, \ e_2 - e_3, \ \ldots, \ e_{i-1} - e_i, \ -e_1 - e_i \}$$
$$\cup \{ e_i - e_{i+1}, \ e_{i+1} - e_{i+2}, \ \ldots, \ e_{n-j} - e_{n-j+1} \}$$
$$\cup \{ e_{n-j+1} - e_{n-j+2}, \ e_{n-j+2} - e_{n-j+3}, \ \ldots, \ e_{n-1} - e_n, \ e_{n-j+1} + e_n \}.$$
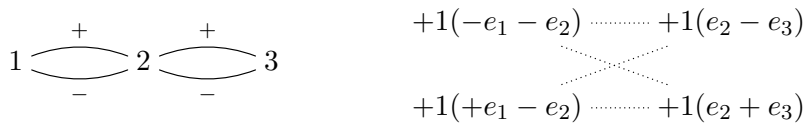
For example, the circuit $C(12; 4, 6) \subset \Phi_{D_{12}}$ corresponds to this signed graph:
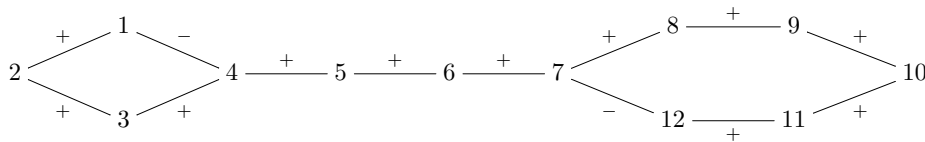
and this acuteness-obtuseness graph:
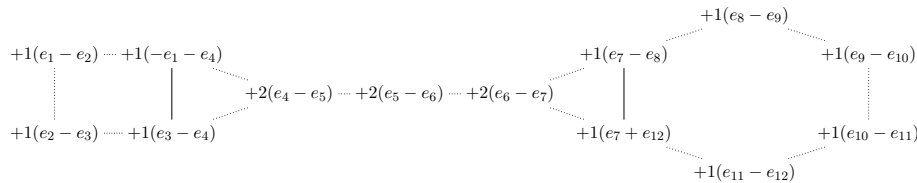
Note that the conditions $i, j \geq 2$ and $i + j \leq n + 1$ on $C(n; i, j)$ allow for various degenerate instances, including the most degenerate case $C(3; 2, 2)$ with the following signed graph and acuteness-obtuseness graph:

The action of the hyperoctahedral group $\mathfrak{S}_n^\pm$ on subsets of Φ_{D_n} induces an action on their signed graphs that Zaslavsky calls *switching*: the permutations $\mathfrak{S}_n \subset \mathfrak{S}_n^\pm$ simply permute the vertex labels on the signed graphs, while the sign change $e_i \mapsto -e_i$ swaps the two kinds of edges incident to vertex i, that is, it swaps $i \overset{+}{\rule{1.5em}{0.4pt}} j$ and $i \overset{-}{\rule{1.5em}{0.4pt}} j$ for any j. Note that this allows one to perform these changes of edge labels in signed graphs via the

switching $e_i \mapsto -e_i$:

(3.1)
$$k \;\underline{\;\;-\;\;}\; i \;\underline{\;\;-\;\;}\; j \quad\rightsquigarrow\quad k \;\underline{\;\;+\;\;}\; i \;\underline{\;\;+\;\;}\; j$$
$$k \;\underline{\;\;+\;\;}\; i \;\underline{\;\;-\;\;}\; j \quad\rightsquigarrow\quad k \;\underline{\;\;-\;\;}\; i \;\underline{\;\;+\;\;}\; j$$

Proposition 3.8. *Consider the set of full circuits in Φ_{D_n} under the action of $\mathfrak{S}_n^\pm$, and under the action of its subgroup $W(D_n)$. A system of orbit representatives for the $\mathfrak{S}_n^\pm$-action is*

$$\{C(n; i, j) : 2 \le i \le j \text{ and } i + j \le n + 1\}.$$

Upon restriction to the $W(D_n)$-action, the $\mathfrak{S}_n^\pm$-orbit of $C(n; i, j)$
- *is a single $W(D_n)$-orbit if n is odd or if either of i, j is even;*
- *breaks into two $W(D_n)$-orbits if n is even and both i, j are odd.*

Proof. Among the circuits described in Proposition 3.7, the balanced cycles (type (i)) are never full circuits in Φ_{D_n}: one can use the switching action to make them have all plus edges $i \;\underline{\;\;+\;\;}\; j$, and so the group generated by the associated reflections is conjugate to a subgroup of $\mathfrak{S}_n \subsetneq W(D_n)$.

It is easily seen that a circuit of type (ii) in Proposition 3.7, having two unbalanced cycles connected by a path, is full in Φ_{D_n} if and only if its set of vertices covers $\{1, 2, \ldots, n\}$. In this case, if its two disjoint cycles have sizes i, j with $i \le j$, then we claim it is in the $\mathfrak{S}_n^\pm$-orbit of $C(n; i, j)$. To see this, perform the following sequence of switchings:

- First, apply switches as in (3.1) to push all of the minus edges off of the path in the middle, and into the unbalanced cycles at either end.
- Then, in each cycle, similarly apply switches to push all of the minus edges into one consecutive string, touching the unique vertex in the cycle of degree three or more.
- Then, in each cycle, apply switches to change pairs of consecutive minus edges to plus, so that there is only one minus edge left and it touches the vertex of degree three or more.
- Finally, apply a permutation in $\mathfrak{S}_n$ to make the vertex labels match those of $C(n; i, j)$.

We next analyze the W-orbit structure where $W := W(D_n)$. Since

$$[\mathfrak{S}_n^\pm : W] = 2,$$

any $\mathfrak{S}_n^\pm$-orbit is either a single W-orbit, or splits as a union of two W-orbits. One way to show that a $\mathfrak{S}_n^\pm$-orbit remains a single W-orbit is to exhibit an element C of the orbit and some w in $\mathfrak{S}_n^\pm \setminus W$ with $w(C) = C$. Note that any circuit C is fixed by the element w_0 in $\mathfrak{S}_n^\pm$ that sends $e_i \mapsto -e_i$ for all $i = 1, 2, \ldots, n$, and when n is odd, w lies in $\mathfrak{S}_n^\pm \setminus W$. Thus no $\mathfrak{S}_n^\pm$-orbits split when n is odd. Also, if i is even, then the circuit $C(n; i, j)$ is fixed by the element w in $\mathfrak{S}_n^\pm \setminus W$ that sends $e_k \leftrightarrow -e_{i-k}$ for $1 \le k \le i - 1$ (in

particular $e_{\frac{i}{2}} \mapsto -e_{\frac{i}{2}}$). Thus the $\mathfrak{S}_n^{\pm}$-orbit of $C(n;i,j)$ does not split when i is even. A similar argument shows that it does not split when j is even.

It only remains to show that the $\mathfrak{S}_n^{\pm}$-orbit of $C(n;i,j)$ *does* split into two W-orbits when n is even but i,j are both odd. To do this, we describe a $\mathbb{Z}/2\mathbb{Z}$-valued W-invariant $\pi(C)$ of these circuits C. Consider the unique perfect matching M of the undirected graph for C. For example, M is shown here as the doubled edges for $(n,i,j) = (16,5,7)$:
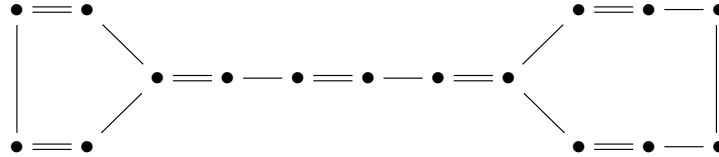
Define $\pi(C)$ to be the parity of the number of minus edges in the signed graph for C that lie in M. Applying elements of $\mathfrak{S}_n$ to C does not affect $\pi(C)$, but switches of the form $e_k \mapsto -e_k$ reverse $\pi(C)$. Thus both values $\pi(C)$ in $\mathbb{Z}/2\mathbb{Z}$ occur within the $\mathfrak{S}_n^{\pm}$-orbit of $C(n;i,j)$, while only one value occurs in each W-orbit.   □

Note that Proposition 3.8 immediately implies that full circuits $C \subset \Phi_{D_n}$ have disconnected acuteness graph Γ_C, since $\Gamma_{C(n;i,j)}$ has at least four vertices but at most two edges.

3.5. Type $\mathbf{B_n/C_n}$ for $\boldsymbol{n \geq 3}$. Since we are only concerned with the hyperplanes and reflections associated to the roots, we are free to choose the crystallographic root system of type C_n:

$$\Phi_{C_n} := \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \sqcup \{\pm 2e_i : 1 \leq i \leq n\}.$$

Here $W = W(C_n) = \mathfrak{S}_n^{\pm}$ is the full hyperoctahedral group of $n \times n$ signed permutations $e_i \mapsto \pm e_{w(i)}$.

As in type D, Zaslavsky [Zas82] associates a signed graph to each subset of roots. Roots in Φ_{D_n} correspond to (signed) edges as before, and the pair $\pm 2e_i$ is depicted as a *self-loop* on vertex i, with a minus sign. Such a self-loop is considered an unbalanced cycle (with one edge). Then Proposition 3.7 remains correct as a characterization of circuits $C \subset \Phi_{C_n}$, that is, they are either of type (i) or (ii) mentioned there, allowing for self-loops as unbalanced cycles.

We thus extend the definition of the circuits $C(n;i,j)$ to allow $C(n;1,j)$ where $1 \leq j \leq n$:

$$C(n;1,j)$$
$$:= \{-2e_1\} \cup \{e_1 - e_2,\ e_2 - e_3,\ \ldots,\ e_{n-j} - e_{n-j+1}\}$$
$$\cup \{e_{n-j+1} - e_{n-j+2},\ e_{n-j+2} - e_{n-j+3},\ \ldots,\ e_{n-1} - e_n\ e_{n-j+1} + e_n\}.$$
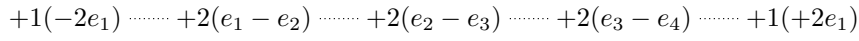
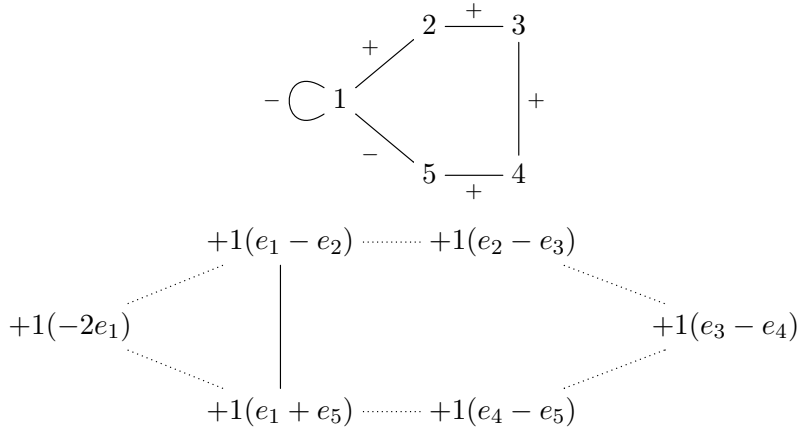The following example depicts $C(9; 1, 6)$ as a signed graph, as well as its acuteness-obtuseness graph:
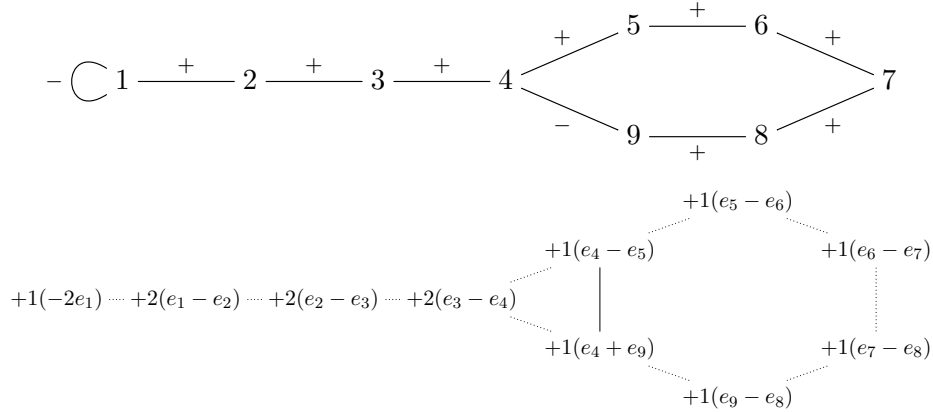
Note that the condition $1 \leq j \leq n$ on $C(n; 1, j)$ allows for various degenerate instances. As examples, in the case $j = 1$ we have the circuit $C(4; 1, 1)$

with acuteness-obtuseness graph

$$+1(-2e_1) \cdots +2(e_1 - e_2) \cdots +2(e_2 - e_3) \cdots +2(e_3 - e_4) \cdots +1(+2e_1)$$
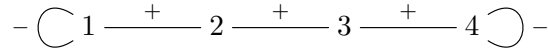
and in the case $j = n$ we have the circuit $C(5; 1, 5)$:

Proposition 3.9. *The set* $\{C(n; 1, j) : 1 \leq j \leq n\}$ *is a system of representatives for the* $\mathfrak{S}_n^{\pm}$*-orbits of full circuits in* Φ_{C_n}.

Proof. As before, among the circuits described in Proposition 3.7, those of type (i) (balanced cycles) are never full circuits. But now a circuit of type (ii), having two unbalanced cycles connected by a path, is full if and only if its set of vertices covers $\{1, 2, \ldots, n\}$ *and also* one of its balanced cycles has size one, i.e., is a self-loop. In this case, if its two disjoint cycles have sizes $1, j$, then we claim it is in the $\mathfrak{S}_n^{\pm}$-orbit of $C(n; 1, j)$. To see this, perform

switchings as in type D that push all of the minus edges off of the path in the middle and into the unbalanced cycle of size j, then toward one end of this cycle, and cancel them in pairs until only one is left; the result can then be relabeled by an element of $\mathfrak{S}_n$ to give $C(n; 1, j)$.   □

Note that Proposition 3.9 immediately implies that full circuits $C \subset \Phi_{C_n}$ have disconnected acuteness graph Γ_C, since $\Gamma_{C(n;1,j)}$ has at least three vertices but at most one edge.

3.6. Exceptional types. We outline our `Mathematica` computations verifying Lemma 1.2 in the exceptional types H_3, H_4, F_4, E_6, E_7, and E_8. This data is available at `http://nyjm.albany.edu/j/2016/CircuitData.zip` as auxilliary data files named in a logical way; e.g., data for type E_8 is in the file `E8.txt`. We first generated a set of W-orbit representatives for all bases of positive roots in each root system Φ_W. Given the list of W-orbit representatives for bases $B \subset \Phi_W^+$, we produced the W-orbit representatives for all circuits C by adding to each B a positive root $\alpha \in \Phi_W^+ \smallsetminus B$ in all possible ways, finding the unique circuit $C \subset B \cup \{\alpha\}$, and classifying all such C up to W-action. Non-full circuits were discarded. Finally, for each of these full circuits C, we computed the acuteness graph Γ_C and verified that it was disconnected.

The table below shows the number of orbits of bases and of full circuits in each of the exceptional types.

Type	# orbits of bases	# orbits of full circuits
H_3	11	15
H_4	96	416
F_4	35	22
E_6	39	17
E_7	311	142
E_8	1943	1717

In the case of E_8, this computation required several days to produce the 1943 W-orbits of bases in $\Phi_{E_8}^+$. To corroborate this data, we also produced the sizes of the stabilizers of each W-orbit representative B; these allowed us to compare with the calculations of De Concini–Procesi [DCP08], who found (e.g.) that there are 348607121625 total bases in $\Phi_{E_8}^+$.

Example 3.10. Some of the full circuits that we encountered are the irreducible circuits $C = \{\alpha\} \cup I$, discussed in Remark 3.2 above. Stembridge shows that what he calls the *apex* vector α has $(\alpha, \beta) > 0$ for each β in I, if the irreducible circuit C comes from a dependence of the form $(-1)\alpha + \sum_{\beta \in I} c_\beta \beta = 0$ with $c_\beta > 0$. Therefore α always gives rise to a vertex of Γ_C having an obtuse edge to every other vertex in the acuteness-obtuseness graph, and so becomes an isolated vertex of the acuteness graph Γ_C. We depict here the acuteness-obtuseness graphs for the irreducible circuits in type E_6, E_7, E_8, adapted from his figure [Ste07, Fig. 1], where the

apex α always appears in the center:
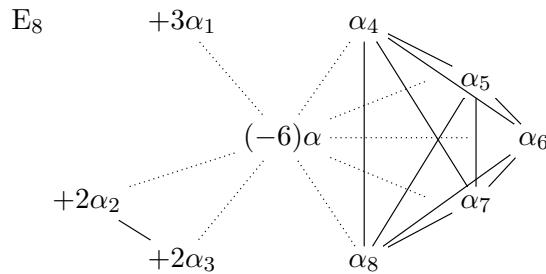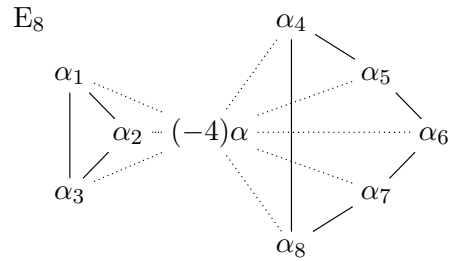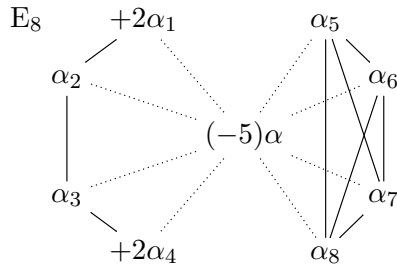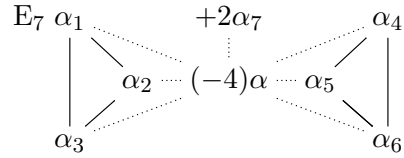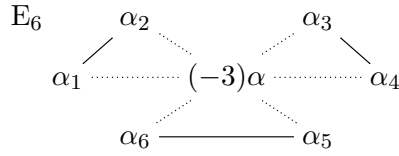
Example 3.11. In most cases it is extremely easy to recognize the disconnectedness of Γ_C: either it has an isolated vertex, or it has v vertices and fewer than $v - 1$ edges, or both. For example, in Stembridge's irreducible circuits $C = \{\alpha\} \cup I$, the apex vector α necessarily gives rise to an isolated vertex of Γ_C.

Meanwhile in type H_4, of the 419 full-rank circuit orbit representatives, there are only 25 with at least four edges (two each with five or six edges, 21 with four edges); of these, ten have an isolated vertex (including all of those with more than four edges) and the other fifteen consist of a disjoint triangle and edge.

Example 3.12. Here is another interesting example of an acuteness-obtuseness graph of a full circuit in E_8:
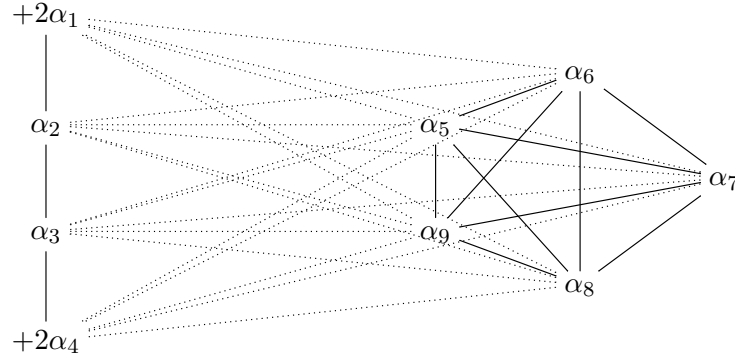
4. Non-minimal factorizations, and proofs of Lemma 1.3 and Corollary 1.4

Most of this section is devoted to the proof of the following lemma from the Introduction, recalled here, which we then use to prove Corollary 1.4.

Lemma 1.3. *For any reflection factorization* $\mathbf{t} = (t_1, \ldots, t_m)$ *of*

$$w = t_1 \cdots t_m$$

with $\ell_T(w) < m$, *either* $m = 2$ *or there exists* $\mathbf{t}' = (t'_1, \ldots, t'_m)$ *in the Hurwitz orbit of* $\mathbf{t}$ *with* $\ell_T(t'_1 \cdots t'_k) < k$ *for some* $k \leq m - 1$.

An important tool will be Carter's characterization of minimal reflection factorizations.

Proposition 4.1 (Carter [Car72, Lem. 3]). *In a finite real reflection group* W, *one has* $\ell_T(s_{\alpha_1} \cdots s_{\alpha_k}) = k$ *if and only the roots* $\alpha_1, \ldots, \alpha_k$ *are linearly independent.*

In particular, this implies that the reflection length function $\ell_T : W \to \mathbb{N}$ only takes values in $\{0, 1, \ldots, \dim(V)\}$. A second important observation is the following.

Proposition 4.2. *A subsequence* $(t_{i_1}, \ldots, t_{i_k})$ *with* $1 \leq i_1 < \ldots < i_k \leq m$ *of a factorization* $\mathbf{t} = (t_1, \ldots, t_m)$ *of* $w = t_1 t_2 \cdots t_m$ *is always a prefix for some* $\mathbf{t}' = (t_{i_1}, \ldots, t_{i_k}, t'_{k+1}, t'_{k+2}, \ldots, t'_m)$ *in the Hurwitz orbit of* $\mathbf{t}$.

Proof. Starting with $\mathbf{t}$, apply $\sigma_{i_1-1}, \sigma_{i_1-2}, \ldots, \sigma_2, \sigma_1$ to move the t_{i_1} to the first position; then similarly apply $\sigma_{i_2-1}, \sigma_{i_2-2}, \ldots, \sigma_3, \sigma_2$ to move t_{i_2} to the second position, and so on. $\square$

Using Propositions 4.1 and 4.2 in the context of Lemma 1.3, one can assume without loss of generality that the reflection factorization

$$\mathbf{t} = (t_1, \ldots, t_m)$$

of $w = t_1 \cdots t_m$ with $\ell_T(w) < m$ corresponds via $t_i = s_{\alpha_i}$ to roots

$$\alpha_1, \alpha_2, \ldots, \alpha_m$$

that form a circuit $C = \{\alpha_1, \ldots, \alpha_m\} \subset \Phi$. Furthermore, in light of Remark 3.4, one can assume that C is a full circuit in Φ_W, and hence that Φ_W is irreducible.

Note that Lemma 1.3 can be checked trivially in rank 1, since

$$W = \langle s : s^2 = e \rangle.$$

The next subsection deals with rank 2, and the one following deals with ranks 3 and higher, relying ultimately on Lemma 1.2.

4.1. Rank 2. Lemma 1.3 is already interesting in rank 2, so that W is the dihedral group

$$W = W_m := \langle s, t : s^2 = t^2 = e, (st)^m = e \rangle$$

of type $I_2(m)$. Since full circuits C have size 3, the reductions above show that, to prove Lemma 1.3, it only remains to check the following assertion: any reflection factorization $\mathbf{t} = (t_1, t_2, t_3)$ of $w = t_1 t_2 t_3$ in W_m has a factorization of the form $\mathbf{t}' = (t', t', t'')$ in its Hurwitz orbit. In fact, one need only prove this same assertion for the *infinite dihedral group*

$$W_\infty = \langle s, t \mid s^2 = t^2 = e \rangle$$

of type $I_2(\infty)$. The reflections in W_∞ are the elements of odd length in the generating set s, t; we denote them by $T = \{t(n) := (st)^n s, n \in \mathbb{Z}\}$. (In particular, this means $s = t(0)$ and $t = t(-1)$.) The obvious surjection $W_\infty \twoheadrightarrow W_m$

- carries reflections in W_∞ to reflections in W_m, and
- sends Hurwitz moves σ_i on factorizations in W_∞ as in (1.1) to the same Hurwitz move in W_m.

There is a standard geometric model for W_∞ as generated by *affine reflections* of the real line $\mathbb{R}$: the reflection $t(n)$ reflects $\mathbb{R}$ across the point $x = n$ in $\mathbb{R}$. Thus the conjugation action

$$t(b)^{t(a)} = t(a) \cdot t(b) \cdot t(a) = t(2a - b)$$

corresponds to reflecting b across a on the line $\mathbb{R}$. Therefore, bearing in mind Proposition 4.2, it will suffice to show that, given an ordered triple of integers (a, b, c), one can eventually reach a triple having two of the integers equal via moves that reflect one of a, b across the other and swapping their positions within the triple, or doing the same with b, c. We give an algorithm that does this by reflecting one of the three values a, b, c across the median value, proceeding by induction on the (positive integer) length

$$M(a, b, c) := \max\{a, b, c\} - \min\{a, b, c\}$$

of the interval that they span, and eventually making two of them coincide.

Up to the irrelevant symmetries

$$(a, b, c) \mapsto (c, b, a) \quad \text{and} \quad (a, b, c) \mapsto (-a, -b, -c)$$

(the latter being achieved by reflection across 0), we may suppose that either $a \leq b \leq c$ or $a \leq c < b$. In the latter case, reflecting b across a produces $(2a - b, a, c)$ with $2a - b < a \leq c$, so we reduce to the former case. Since $a \leq b \leq c$, one has $M = c - a$. Let $m := \min\{b - a, c - b\}$. Without loss of generality, $M(a, b, c) > m > 0$, else we are done. If $m = b - a$, reflect a across b giving $(a', b', c') = (b, 2a - b, c)$ with
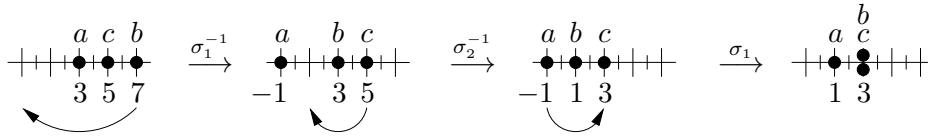
$$M(a', b', c') = c - b \leq m < M(a, b, c).$$

If $m = c - b$, reflect c across b giving $(a', b', c') = (a, 2b - c, b)$ with

$$M(a', b', c') = b - a \leq m < M(a, b, c).$$

In either case, we are done by induction.

Here is an illustration in the case $(a, b, c) = (3, 7, 5)$ (so that initially $a < c < b$):

4.2. Higher ranks. When W has rank at least three, we require a somewhat more subtle argument to prove Lemma 1.3. Given a factorization $w = t_1 t_2 \cdots t_m$ in which $\ell_T(w) < m$, there exists an m-tuple $(\alpha_1, \ldots, \alpha_m)$ of roots for which $t_i = s_{\alpha_i}$, and by Proposition 4.1 this m-tuple is linearly dependent.

Definition 4.3. A pair $(\mathbf{C}, \mathbf{c})$ where $\mathbf{C} = (\alpha_1, \ldots, \alpha_m)$ in Φ^m and

$$\mathbf{c} = (c_1, \ldots, c_m)$$

in $\mathbb{R}^m$ with $\sum_{i=1}^m c_i \alpha_i = 0$ will be called an m-*dependence* in Φ. Its *weight* is defined as

$$\mathrm{wt}(\mathbf{C}, \mathbf{c}) := \mathrm{wt}(\mathbf{c}) := \sum_{i=1}^m |c_i|.$$

Our proof strategy for Lemma 1.3 is to start with any nontrivial m-dependence $(\mathbf{C}, \mathbf{c})$ that accompanies a non-minimal factorization

$$w = t_1 t_2 \cdots t_m,$$

and try to apply Hurwitz moves that make $\mathrm{wt}(\mathbf{C}, \mathbf{c})$ strictly smaller. Then we work by induction to show that for $m > 2$, every m-dependence has in its Hurwitz orbit an m-dependence $(\mathbf{C}', \mathbf{c}')$ where one of the coefficients c_i' *vanishes*, so that a proper subset of the vectors in $\mathbf{C}$ is dependent. Bearing in mind Proposition 4.2, this would prove Lemma 1.3. There are at least three separate issues here:

(i) We need a well-defined Hurwitz action on the set of m-dependences (easy—see Proposition 4.4).

(ii) We need to know that some Hurwitz move applies that *lowers* $\mathrm{wt}(\mathbf{C}, \mathbf{c})$. Here we use Lemma 1.2.

(iii) We need to know that one cannot lower $\mathrm{wt}(\mathbf{C}, \mathbf{c})$ infinitely often. This is a fairly easy argument in the crystallographic case, but requires one further computation in types $\mathrm{H}_3, \mathrm{H}_4$.

We deal with issues (i), (ii), (iii) in the next three subsections.

4.2.1. Dealing with issue (i)**.** We lift Hurwitz moves on reflection factorizations to moves on m-dependences.

Proposition 4.4. *The Hurwitz move* $\mathbf{t} \xmapsto{\sigma_i} \mathbf{t}'$ *of* (1.1) *lifts to the following (invertible) Hurwitz move* σ_i *on the set of m-dependences in* Φ*: given*

$$(\mathbf{C} = (\alpha_i)_{i=1}^m, \mathbf{c})$$

corresponding to $\mathbf{t}$*, send it to* $(\mathbf{C}' = (\alpha_i')_{i=1}^m, \mathbf{c}')$ *having* $\alpha_j' = \alpha_j$ *and* $c_j' = c_j$ *for all* $j \neq i, i+1$*, and*

$$(4.1) \qquad \begin{pmatrix} \alpha_i & \alpha_{i+1} \\ \\ c_i & c_{i+1} \end{pmatrix} \xmapsto{\sigma_i} \begin{pmatrix} \alpha_{i+1} & s_{\alpha_{i+1}}(\alpha_i) \\ \\ c_{i+1} + \dfrac{2(\alpha_i, \alpha_{i+1})}{|\alpha_{i+1}|^2} c_i & c_i \end{pmatrix}.$$

Furthermore, the ith sign change involution ϵ_i *on* $(\mathbf{C}, \mathbf{c})$ *that replaces* $\alpha_i \mapsto -\alpha_i$ *and* $c_i \mapsto -c_i$ *satisfies*

$$(4.2) \qquad \begin{aligned} \sigma_i \epsilon_j &= \epsilon_j \sigma_i && \text{for } j \neq i, i+1, \\ \sigma_i \epsilon_i &= \epsilon_{i+1} \sigma_i, && \text{and} \\ \sigma_i \epsilon_{i+1} &= \epsilon_i \sigma_i. \end{aligned}$$

Proof. For any root α and any w in W one has $s_\alpha^w = w^{-1} s_\alpha w = s_{w^{-1}(\alpha)}$. Applying this with $\alpha = \alpha_i$ and $w = s_{\alpha_{i+1}} = w^{-1}$ shows that the pair $(\mathbf{C}', \mathbf{c}')$ defined in the statement corresponds to $\mathbf{t}' = \sigma_i(\mathbf{t})$. The fact that this pair is another m-dependence comes from $\sum_{i=1}^m c_i \alpha_i = 0$ and a calculation with the formula (2.1). The inverse $\sigma_i^{-1} : (\mathbf{C}, \mathbf{c}) \longmapsto (\mathbf{C}', \mathbf{c}')$ has the following formula: $\alpha_j' = \alpha_j$ and $c_j' = c_j$ for all $j \neq i, i+1$, and

$$(4.3) \qquad \begin{pmatrix} \alpha_i & \alpha_{i+1} \\ \\ c_i & c_{i+1} \end{pmatrix} \xmapsto{\sigma_i^{-1}} \begin{pmatrix} s_{\alpha_i}(\alpha_{i+1}) & \alpha_i \\ \\ c_{i+1} & c_i + \dfrac{2(\alpha_{i+1}, \alpha_i)}{|\alpha_i|^2} c_{i+1} \end{pmatrix}.$$

The relations in (4.2) are all straightforward to check. $\qquad\square$

Remark 4.5. We will not need it here, but a slightly laborious calculation shows that the permutation action of the operators σ_i on the set of m-dependences in Φ satisfies the usual braid relations, giving an action of the m-strand braid group on the set of m-dependences.

4.2.2. Dealing with issue (ii)**.** We begin by studying how the two Hurwitz moves σ_i, σ_i^{-1} affect the weight of a dependence. Note that the sign change involution ϵ_i has no effect on the weight of a dependence.

Proposition 4.6. *Consider a nontrivial m-dependence* $(\mathbf{C}, \mathbf{c})$ *for* $m \geq 3$, *with* $\mathbf{C} = (\alpha_1, \ldots, \alpha_m)$ *supported on a circuit* $C = \{\alpha_1, \ldots, \alpha_m\}$.

(a) *If* $(c_i \alpha_i, c_{i+1} \alpha_{i+1}) = 0$, *then*

$$\mathrm{wt}(\sigma_i(\mathbf{C}, \mathbf{c})) = \mathrm{wt}(\sigma_i^{-1}(\mathbf{C}, \mathbf{c})) = \mathrm{wt}(\mathbf{C}, \mathbf{c}).$$

(b) *If* $(c_i \alpha_i, c_{i+1} \alpha_{i+1}) > 0$, *then both*

$$\mathrm{wt}(\sigma_i(\mathbf{C}, \mathbf{c})) > \mathrm{wt}(\mathbf{C}, \mathbf{c}) \quad and \quad \mathrm{wt}(\sigma_i^{-1}(\mathbf{C}, \mathbf{c})) > \mathrm{wt}(\mathbf{C}, \mathbf{c}).$$

(c) *If* $(c_i \alpha_i, c_{i+1} \alpha_{i+1}) < 0$, *then either*

$$\mathrm{wt}(\sigma_i(\mathbf{C}, \mathbf{c})) < \mathrm{wt}(\mathbf{C}, \mathbf{c}) \quad or \quad \mathrm{wt}(\sigma_i^{-1}(\mathbf{C}, \mathbf{c})) < \mathrm{wt}(\mathbf{C}, \mathbf{c}).$$

Proof. Since C is a circuit, all entries of $\mathbf{c}$ are nonzero. Assertion (a) follows because $c_i, c_{i+1} \neq 0$ imply $(\alpha_i, \alpha_{i+1}) = 0$, so that $\sigma_i^{\pm 1}$ simply permute the coefficients.

In arguing assertions (b), (c), it is convenient to have all entries $c_j > 0$ in $\mathbf{c}$. One can reduce to this case by applying sign change operations ϵ_i that negate some of the α_j, using the relations (4.2).

Then from (4.1), one has

$$(4.4) \qquad \mathrm{wt}(\sigma_i(\mathbf{C}, \mathbf{c})) - \mathrm{wt}(\mathbf{C}, \mathbf{c}) = c_i' - c_{i+1}$$

where

$$c_i' := \left| c_{i+1} + \frac{2 (\alpha_i, \alpha_{i+1})}{|\alpha_{i+1}|^2} c_i \right|.$$

For assertion (b), note that $c_i, c_{i+1} > 0$ implies that $(\alpha_i, \alpha_{i+1}) > 0$, and hence

$$c_i' - c_{i+1} = \frac{2 (\alpha_i, \alpha_{i+1})}{|\alpha_{i+1}|^2} c_i > 0,$$

so that $\mathrm{wt}(\sigma_i(\mathbf{C}, \mathbf{c})) > \mathrm{wt}(\mathbf{C}, \mathbf{c})$. Moreover the same holds when σ_i is replaced by σ_i^{-1}, since this only has the effect of switching i and $i+1$ everywhere in (4.4).

For assertion (c), let us assume that $(c_i \alpha_i, c_{i+1} \alpha_{i+1}) < 0$ and that both

$$(4.5) \qquad \mathrm{wt}(\sigma_i(\mathbf{C}, \mathbf{c})) \geq \mathrm{wt}(\mathbf{C}, \mathbf{c}) \quad and \quad \mathrm{wt}(\sigma_i^{-1}(\mathbf{C}, \mathbf{c})) \geq \mathrm{wt}(\mathbf{C}, \mathbf{c}),$$

in order to reach a contradiction. Note that

$$\mathrm{wt}(\sigma_i(\mathbf{C}, \mathbf{c})) \geq \mathrm{wt}(\mathbf{C}, \mathbf{c})$$

$$\Longleftrightarrow \left| c_{i+1} + \frac{2\,(\alpha_i, \alpha_{i+1})}{|\alpha_{i+1}|^2} c_i \right| \geq c_{i+1}$$

$$\Longleftrightarrow \left| \frac{2\,(\alpha_i, \alpha_{i+1})}{|\alpha_{i+1}|^2} \right| c_i \geq 2c_{i+1} \qquad \text{(since } c_i, c_{i+1} > 0 \text{ and } (\alpha_i, \alpha_{i+1}) < 0),$$

$$\Longleftrightarrow \left| \frac{(\alpha_i, \alpha_{i+1})}{|\alpha_{i+1}|^2} \right| \geq \frac{c_{i+1}}{c_i}.$$

Similarly, swapping the indices $i, i+1$, one has

$$\mathrm{wt}(\sigma_i^{-1}(\mathbf{C}, \mathbf{c})) \geq \mathrm{wt}(\mathbf{C}, \mathbf{c}) \quad \Longleftrightarrow \quad \left| \frac{(\alpha_i, \alpha_{i+1})}{|\alpha_i|^2} \right| \geq \frac{c_i}{c_{i+1}}.$$

Therefore the assumption (4.5) implies that

$$\left(\frac{(\alpha_i, \alpha_{i+1})}{|\alpha_i| \cdot |\alpha_{i+1}|} \right)^2 = \left| \frac{(\alpha_i, \alpha_{i+1})}{|\alpha_{i+1}|^2} \right| \cdot \left| \frac{(\alpha_i, \alpha_{i+1})}{|\alpha_i|^2} \right| \geq \frac{c_{i+1}}{c_i} \cdot \frac{c_i}{c_{i+1}} = 1.$$

Cauchy–Schwarz then forces $\alpha_{i+1} = \pm\alpha_i$, contradicting $\mathbf{C} = (\alpha_1, \ldots, \alpha_m)$ being a circuit with $m \geq 3$. $\qquad\square$

With this in hand, issue (ii) is dealt with by the following result.

Proposition 4.7. *Given a nontrivial m-dependence $(\mathbf{C}, \mathbf{c})$ for $m \geq 3$, with $\mathbf{C} = (\alpha_1, \ldots, \alpha_m)$ supported on a full circuit $C = \{\alpha_1, \ldots, \alpha_m\} \subset \Phi$, there exists another m-dependence $(\mathbf{C}', \mathbf{c}')$ in its Hurwitz orbit such that*

$$\mathrm{wt}(\mathbf{c}') < \mathrm{wt}(\mathbf{c}).$$

Proof. As before, $c_i \neq 0$ for all i since C is a circuit. The acuteness graph Γ_C is disconnected by Lemma 1.2, so one has a nontrivial decomposition $\{1, 2, \ldots, m\} = I \sqcup J$ in which $c_i\alpha_i, c_j\alpha_j$ are nonacute for every $(i, j) \in I \times J$. They cannot always be *orthogonal*, else C would not be a circuit. Hence, there exists at least one pair $(i_*, j_*) \in I \times J$ for which $c_{i_*}\alpha_{i_*}, c_{j_*}\alpha_{j_*}$ are (strictly) obtuse. Assume $i_* < j_*$ without loss of generality.

Let $I = \{i_1 < i_2 < \cdots\}$ and $J = \{j_1 < j_2 < \cdots\}$, and imagine the process of sorting the sequence $(1, 2, 3, \ldots, m)$ into the linear order $(j_1, j_2, \ldots, i_1, i_2, \ldots)$ using adjacent transpositions s_k (i.e., s_k swaps the entries in *positions* $k, k+1$) so that at each step, the transposed values $\{i, j\}$ satisfy $(i, j) \in I \times J$. Since this process starts with i_* left of j_* and ends with i_* right of j_*, there must exist some *first* step in this process where one uses some s_{k_0} to swap a pair $(i, j) \in I \times J$ having $c_i\alpha_i, c_j\alpha_j$ obtuse. All of the previous steps swap pairs of orthogonal roots, and hence lift to a corresponding sequence of Hurwitz moves σ_i applied to $(\mathbf{C}, \mathbf{c})$ that only re-order the entries. The product of these moves is a (re-ordered) m-dependence $(\mathbf{C}'\mathbf{c}')$ having $\mathrm{wt}(\mathbf{c}') = \mathrm{wt}(\mathbf{c})$. However, at the next step, Lemma 4.6(c) shows that one of

the two lifts $\sigma_{k_0}^{\pm 1}$ of s_{k_0} will have $\mathrm{wt}(\sigma_{k_0}^{\pm 1}(\mathbf{C}', \mathbf{c}')) < \mathrm{wt}(\mathbf{C}', \mathbf{c}') = \mathrm{wt}(\mathbf{C}, \mathbf{c})$, as desired. $\qquad\square$

4.2.3. Dealing with issue (iii)**.** We need to know that, after starting with an an m-dependence and applying a sequence of Hurwitz moves that decrease its weight at each step, one cannot return to the same m-dependence.

Proposition 4.8. *Fix a circuit C in a finite root system Φ of rank at least 3, and two m-dependences $(\mathbf{C}, \mathbf{c}), (\mathbf{C}', \mathbf{c}')$ supported on C. If the two m-dependences are in the same Hurwitz orbit then $\mathrm{wt}(\mathbf{c}') = \mathrm{wt}(\mathbf{c})$.*

Proof. Note that the hypotheses and conclusion of the proposition are unaffected by rescaling $\mathbf{c}$.

Let K be the finite extension of $\mathbb{Q}$ generated by $\frac{2(\alpha, \beta)}{|\alpha|^2}$ for all roots α, β in Φ. Every root α is in the W-orbit of a root in Π, and hence by (2.1) in the K-subspace of V generated by Π. Thus, we may rescale $\mathbf{c}$ so that it lies in K^m. Clearing denominators, we can assume $\mathbf{c}$ lies in $\mathfrak{o}^m$, where $\mathfrak{o}$ is the ring of integers within K.

We further claim that one can assume both that $\mathfrak{o}$ is a *principal ideal domain*, and that it contains all of the algebraic numbers $\frac{2(\alpha, \beta)}{|\alpha|^2}$ for α, β in Φ. To see this claim, note that our finite root systems Φ of rank at least 3 have been chosen either to be

- crystallographic, so that $\mathfrak{o} = \mathbb{Z} \subset \mathbb{Q} = K$, with $\frac{2(\alpha, \beta)}{|\alpha|^2}$ in $\mathbb{Z}$, or
- type H_3, H_4, so that $\mathfrak{o} = \mathbb{Z}\left[(1 + \sqrt{5})/2\right] \subset \mathbb{Q}[\sqrt{5}] = K$, with $\frac{2(\alpha, \beta)}{|\alpha|^2} \in \left\{2\cos(\frac{2\pi}{5}), 2\cos(\frac{4\pi}{5})\right\} \cup \mathbb{Z} \subset \mathfrak{o}$.

Therefore the ideal $I = (\mathbf{c})$ of $\mathfrak{o}$ generated by the entries of $\mathbf{c}$ is a *principal ideal $I = (g)$* in $\mathfrak{o}$, where $g := \gcd(\mathbf{c})$ is uniquely defined up to scaling by units in $\mathfrak{o}^\times$. The formulas (4.1), (4.3) show that the Hurwitz moves $\sigma_i^\pm$ do not change I.

Now assume we are given $(\mathbf{C}, \mathbf{c}), (\mathbf{C}', \mathbf{c}')$ as in the hypothesis of the proposition, and permute indices of $\mathbf{C}$ so that $\mathbf{C}' = \mathbf{C}$ as (ordered) circuits. The uniqueness of the dependence up to scaling forces $\mathbf{c}' = k\mathbf{c}$ for some $k \in K^\times$, and hence

$$\mathrm{wt}(\mathbf{C}', \mathbf{c}') = |k| \cdot \mathrm{wt}(\mathbf{C}, \mathbf{c}).$$

The above discusion shows that, additionally, $\gcd(\mathbf{c}) = \gcd(\mathbf{c}')$ in $\mathfrak{o}$, so that $\mathbf{c}' = k\mathbf{c}$ forces k to lie in $\mathfrak{o}^\times$. Thus, in the crystallographic case, we are done since $\mathfrak{o}^\times = \mathbb{Z}^\times = \{\pm 1\}$, so $|k| = 1$.

In the noncrystallographic H_3, H_4 cases, we still need to rule out the possibility that the unit k in $\mathfrak{o}^\times$ has $|k| \neq 1$. This would mean that the Hurwitz orbit of the m-dependence $(\mathbf{C}, \mathbf{c})$ contains *infinitely* many other elements, namely those whose weights are scaled by $1, |k|, |k|^2, \ldots$. However, we used a computer to check that this does not happen: for every W-orbit of full circuits C in Φ_{H_3}, Φ_{H_4}, as classified in Section 3.6, we linearly ordered C in all ways to form $\mathbf{C}$, picked coefficients $\mathbf{c}$ (uniquely up to scaling) to

create an m-dependence $(\mathbf{C}, \mathbf{c})$, applied all Hurwitz moves $\sigma_i^{\pm}$ to generate new dependences, then repeated with the new dependences. *A priori* this could have run indefinitely, but in fact it always terminated with a finite list, proving the claim. $\qquad\square$

4.2.4. Proof of Lemma 1.3 in ranks at least 3. Given a reflection factorization $\mathbf{t} = (t_1, \ldots, t_m)$ of $w = t_1 \cdots t_m$ with $\ell_T(w) < m$ and $m \geq 3$, we want to show there exists $\mathbf{t}' = (t'_1, \ldots, t'_m)$ in the Hurwitz orbit of $\mathbf{t}$ with $\ell_T(t'_1 \cdots t'_k) < k$ for some $k \leq m - 1$.

As mentioned earlier, using Propositions 4.1 and 4.2, we may assume without loss of generality that the tuple $\mathbf{C} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ of roots corresponding to $(t_1, \ldots, t_m)$ via $t_i = s_{\alpha_i}$ is supported on a circuit $C = \{\alpha_1, \ldots, \alpha_m\}$ in Φ. Furthermore, as in Section 3, one can also assume that Φ_W is irreducible, and that C is a full circuit in Φ_W.

Pick coefficients $\mathbf{c}$ that make

$$(\mathbf{C}, \mathbf{c})(=: (\mathbf{C}^{(0)}, \mathbf{c}^{(0)}))$$

an m-dependence. Then Proposition 4.7 shows that that there exists an m-dependence $(\mathbf{C}^{(1)}, \mathbf{c}^{(1)})$ in its Hurwitz orbit having

$$\mathrm{wt}(\mathbf{C}^{(1)}, \mathbf{c}^{(1)}) < \mathrm{wt}(\mathbf{C}^{(0)}, \mathbf{c}^{(0)}).$$

Repeat this process, producing a sequence of m-dependences $(\mathbf{C}^{(i)}, \mathbf{c}^{(i)})$ in the Hurwitz orbit of $(\mathbf{C}, \mathbf{c})$, with *strictly decreasing* sequence of weights. If it ever happens that some coefficient $c_j^{(i)} = 0$, so that some proper subsequence of $\mathbf{C}^{(i)}$ is dependent, then we are done by Propositions 4.1 and 4.2. However, this *must* happen: otherwise each $\mathbf{C}^{(i)}$ is supported on a circuit $C^{(i)} \subset \Phi$, of which there are only finitely many, so $C^{(i)} = C^{(j)}$ for some $i < j$, contradicting Proposition 4.8.

This completes the proof of Lemma 1.3 in rank at least three, and hence in all ranks.

4.3. Proof of Corollary 1.4. Recall the statement of Corollary 1.4 from the Introduction.

Corollary 1.4. *If $\ell_T(w) = \ell$, then every factorization of w into m reflections lies in the Hurwitz orbit of some $\mathbf{t} = (t_1, \ldots, t_m)$ such that*

$$t_1 = t_2,$$
$$t_3 = t_4,$$
$$\vdots$$
$$t_{m-\ell-1} = t_{m-\ell},$$

and $(t_{m-\ell+1}, \ldots, t_m)$ is a shortest reflection factorization of w.

Proof. Induct on m, with trivial base case $m = 0$. In the inductive step for $m > 0$, given a reflection factorization $w = t_1 t_2 \cdots t_m$, either

$$\ell := \ell_T(w) = m,$$

in which case we are done, or there exists some smallest index i for which $\ell_T(t_1 t_2 \cdots t_i) < i$. By applying Lemma 1.3 repeatedly, we may assume that $i = 2$. This means that $t_1 = t_2$, and we are done by applying the induction to the factorization $w = t_3 t_4 \cdots t_m$. $\square$

5. Coxeter elements and the proof of Theorem 1.1

Theorem 1.1 is a statement about factorizations of Coxeter elements. We recall their definition and a few properties here, before proving the theorem.

Definition 5.1. Given a finite real reflection group W with root system Φ, one defines a *Coxeter element* to be any element of W of the form

$$c = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_n},$$

where $(\alpha_1, \ldots, \alpha_n)$ is any ordering of any choice of simple roots

$$\Pi = \{\alpha_1, \ldots, \alpha_n\}$$

for Φ.

It turns out (see, e.g., [Hum90, §3.16]) that all Coxeter elements c lie within the same W-conjugacy class. We mention here a few other important properties of Coxeter elements that we will use. One is Bessis's Theorem [Bes03, Prop. 1.6.1] from the Introduction, asserting that any two shortest reflection factorizations $c = t_1 t_2 \ldots t_n$ lie in the same Hurwitz orbit. It has the following non-obvious corollary.

Corollary 5.2. *In a finite real reflection group, any two shortest reflection factorizations of a Coxeter element use the same multiset of reflection conjugacy classes.*

(Specifically, it is the multiset of conjugacy classes of the simple root reflections $(s_\alpha)_{\alpha \in \Pi}$, two of which lie in the same W-conjugacy class if and only if they have a path of odd-labeled edges between them in the *Coxeter diagram* for W; see [BjB05, Exer. 1.16].)

We will also need the following lemma used by Bessis in the proof of his theorem.

Lemma 5.3 (Bessis [Bes03, Lem. 1.4.2]). *For every Coxeter element c and reflection t in W there exists a shortest reflection factorization $c = t_1 t_2 \cdots t_n$ starting with $t_1 = t$.*

Remark 5.4. In fact, Bessis's result [Bes03, Lem. 1.4.2] asserts something much stronger. The weaker consequence that we need above easily generalizes to the following assertion: given an element w in a finite real reflection group W of rank n having $\ell_T(w) = n$, every reflection t in W occurs as the

first element $t = t_1$ in at least one shortest factorization $w = t_1 t_2 \cdots t_n$. This is true since $\ell_T(tw) = n - 1$ (because $|\ell_T(v) - \ell_T(tv)| = 1$ and $\ell_T(v) \leq n$ for all v in W), so any shortest factorization $tw = t_2 t_3 \cdots t_n$ gives such a shortest factorization $w = t \cdot t_2 \cdots t_n$.

Combining Bessis's Theorem from the Introduction with Lemma 5.3 gives the following.

Corollary 5.5. *Fix a reflection t and a Coxeter element c. Then every shortest reflection factorization $c = t_1 t_2 \cdots t_n$ lies in the Hurwitz orbit of such a factorization that starts with t.*

We can now prove Theorem 1.1 from the Introduction, whose statement we recall here.

Theorem 1.1. *In a finite real reflection group, two reflection factorizations of a Coxeter element lie in the same Hurwitz orbit if and only if they share the same multiset of conjugacy classes.*

Proof. The "only if" statement is clear, as Hurwitz moves do not affect the multiset of conjugacy classes.

For the "if" statement, given two reflection factorizations $\mathbf{t} = (t_1, \ldots, t_m)$ and $\mathbf{t}' = (t_1', \ldots, t_m')$ of c having the same multiset of reflection conjugacy classes, we show that they lie in the same Hurwitz orbit via induction on m. By Corollary 1.4, we may assume that both $\mathbf{t}, \mathbf{t}'$ consist of a sequence of $(m - n)/2$ pairs of equal reflections, followed by shortest factorizations $\hat{\mathbf{t}}, \hat{\mathbf{t}}'$ of c:

$$\mathbf{t} = (t_1, t_1, \ t_3, t_3, \ \ldots, \ t_{m-n-1}, t_{m-n-1}, \ \hat{\mathbf{t}}),$$
$$\mathbf{t}' = (t_1', t_1', \ t_3', t_3', \ \ldots, \ t_{m-n-1}', t_{m-n-1}', \ \hat{\mathbf{t}}').$$

It would suffice to show that the Hurwitz orbit of $\mathbf{t}$ contains a factorization that starts with (t_1', t_1'), since one could then apply induction after restricting $\mathbf{t}, \mathbf{t}'$ to their last $m - 2$ positions $\{3, 4, \ldots, m\}$.

To this end, we first claim that one of the pairs (t_i, t_i) (in positions $i, i+1$) of adjacent equal reflections in $\mathbf{t}$ has t_i in the same conjugacy class as t_1'; this is so because Corollary 5.2 implies $\hat{\mathbf{t}}, \hat{\mathbf{t}}'$ share the same multiset of conjugacy classes, and it is a hypothesis of the theorem that $\mathbf{t}, \mathbf{t}'$ share the same multiset of conjugacy classes.

Via a sequence of Hurwitz moves of the form σ_k^{-1}, one can move the two copies (t_i, t_i) in $\mathbf{t}$ to the right, stopping just before $\hat{\mathbf{t}}$, giving an element in the Hurwitz orbit of $\mathbf{t}$ whose last $n + 2$ positions are

$$(t_i, t_i, \hat{\mathbf{t}}).$$

Since t_i is W-conjugate to t_1', one can choose w in W and a reflection factorization $w = r_1 \cdots r_k$ such that

$$t_1' \quad = \quad w^{-1} t_i w \quad = \quad t_i^w \quad = \quad t_i^{r_1 r_2 \cdots r_k}.$$

By Corollary 5.5, one can apply Hurwitz moves to $\hat{\mathbf{t}}$ and make it start with
the reflection r_1. Thus the last $n+2$ positions in the factorization now look
like

$$(t_i, t_i, r_1, \hat{\hat{\mathbf{t}}}).$$

Apply the Hurwitz moves of the form σ_k that move r_1 two steps left, chang-
ing the factorization to

$$(r_1, t_i^{r_1}, t_i^{r_1}, \hat{\hat{\mathbf{t}}}).$$

Then apply Hurwitz moves of the form σ_k that move both of $(t_i^{r_1}, t_i^{r_1})$ one
step to the left, changing it to

$$(t_i^{r_1}, t_i^{r_1}, r_1^{t_i^{r_1} t_i^{r_1}}, \hat{\hat{\mathbf{t}}}) \quad = \quad (t_i^{r_1}, t_i^{r_1}, r_1, \hat{\hat{\mathbf{t}}}),$$

where the suffix $(r_1, \hat{\hat{\mathbf{t}}})$ is still a shortest factorization of c. Repeating this
process with $r_2, r_3, \ldots, r_k$ in place of r_1 gives a factorization whose last $n+2$
positions have the form

$$(t_1', t_1', \tilde{\mathbf{t}})$$

for some shortest factorization $\tilde{\mathbf{t}}$ of c. Then applying a sequence of moves of
the form σ_k gives a factorization in the Hurwitz orbit of $\mathbf{t}$ that moves (t_1', t_1')
to the first two positions, as desired. $\qquad\square$

6. Remarks

6.1. Quasi-Coxeter elements. Baumeister, Gobet, Roberts, and Wege-
ner [BaGRW15] define a *quasi-Coxeter element* c in a finite reflection group
W to be an element having a shortest reflection factorization $c = t_1 t_2 \cdots t_n$
for which $\{t_1, t_2, \ldots, t_n\}$ generates W. For example, Coxeter elements as in
Definition 5.1 have this property. P. Wegener has pointed out that our proof
of Theorem 1.1 generalizes to prove the following.

Theorem 6.1. *In a finite real reflection group, two reflection factorizations
of a quasi-Coxeter element lie in the same Hurwitz orbit if and only if they
share the same multiset of conjugacy classes.*

Proof sketch. The crucial Corollary 1.4 applies to *any* element of W. Also,
Bessis's Theorem from the Introduction, asserting transitivity of the Hurwitz
action for shortest reflection factorization of Coxeter elements, was general-
ized to quasi-Coxeter elements as [BaGRW15, Thm. 1.1]. This implies the
analogue of Corollary 5.2, replacing the word "Coxeter element" by "quasi-
Coxeter element." Remark 5.4 shows that the conclusion of Lemma 5.3
applies also to every quasi-Coxeter element c, because the quasi-Coxeter
property is easily seen to imply that $\ell_T(c) = n$. The rest of the proof of
Theorem 1.1 uses only these properties. $\qquad\square$

In fact, the quasi-Coxeter property seems to go to the heart of Hurwitz
transitivity for factorizations of arbitrary length. For example, in a Cox-
eter group W having only one reflection conjugacy class, if one is given

a non-quasi-Coxeter element w, one can choose a reflection factorization $w = t_1 t_2 \cdots t_m$ such that $W' := \langle t_1, \ldots, t_m \rangle \subsetneq W$. Then for any reflection t in $W \setminus W'$, the two factorizations $w = t_1 \cdots t_m \cdot t \cdot t = t_1 \cdots t_m \cdot t_1 \cdot t_1$ of length $m + 2$ use the same multiset of reflection conjugacy classes, but necessarily lie in different Hurwitz orbits.

6.2. Affine Weyl groups. Note that the crucial Lemma 1.3 holds for the smallest case of an *affine Weyl group*, namely, the infinite dihedral group W_∞ of type $I_2(\infty)$ from Section 4.1. It is also not hard to check that Theorem 1.1 holds verbatim for this group W_∞, raising the following question.

Question 6.2. Does Theorem 1.1 hold verbatim for affine Weyl groups? Other non-finite Coxeter groups?

6.3. Complex reflection groups. As mentioned in the Introduction, Bessis extended his theorem on shortest factorizations from real reflection groups to *well-generated complex reflection groups*, where the notion of *Coxeter elements* still makes sense; see [Bes15]. In fact, all evidence points to the following verbatim generalization of Theorem 1.1.

Conjecture 6.3. *In a well-generated finite complex reflection group, two reflection factorizations of a Coxeter element lie in the same Hurwitz orbit if and only if they share the same multiset of conjugacy classes.*

We discuss some of the evidence for Conjecture 6.3 here. Just as with real reflection groups, there is a classification of all finite complex reflection groups acting irreducibly, due to Shephard and Todd. It contains one infinite family $G(de, e, n)$ for $n, d, e \geq 1$, and 34 exceptional groups. The group $G(de, e, n)$ consists of all $n \times n$ matrices which are *monomial* (that is, having exactly one nonzero entry in each row and column) and whose nonzero entries are deth roots of unity, with their product a dth root of unity.[2]

Although every real reflection group is *well-generated* in the sense of having a generating set consisting of n reflections, this is not true for all complex reflection groups W acting on $\mathbb{C}^n$. For example, within the infinite family $G(de, e, n)$, this fails when $d, e, n \geq 2$; only the subfamilies $G(d, 1, n), G(e, e, n)$ are well-generated.

The first author has verified Conjecture 6.3 via a direct argument in $G(d, 1, n)$. We have verified it via computer for the factorizations $c = t_1 t_2 \cdots t_m$ with $m \leq n + 3$ in the following well-generated groups acting irreducibly on $\mathbb{C}^n$: $G(e, e, n)$ with $(n, e) = (3, 3), (3, 4), (3, 5), (3, 6), (4, 3)$ and Shephard-Todd's exceptional types G_4, G_5, G_6, G_8.

Regarding proof techniques, one might hope that Lemma 1.3 generalizes to all well-generated groups. Unfortunately, this is not the case, even in the infinite family $G(d, 1, n)$. For example, consider $W := G(d, 1, 2)$ with

---

[2]This contains as special cases the real types A_{n-1} as $G(1, 1, n)$ (restricted to act on the hyperplane $(1, 1, \ldots, 1)^\perp$), types B_n/C_n as $G(2, 1, n)$, type D_n as $G(2, 2, n)$, and type $I_2(m)$ as $G(m, m, 2)$.

$d > 2$, and let ζ be a primitive dth root of unity. The following factorization $w = t_1 t_2 t_3$

$$
\begin{matrix}
w & = & t_1 & t_2 & t_3 \\
\begin{bmatrix} \zeta^2 & 0 \\ 0 & \zeta^{-1} \end{bmatrix} & = & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & \zeta^{-1} \\ \zeta & 0 \end{bmatrix} & \begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix}
\end{matrix}
$$

in W is not shortest, as $\ell_T(w) = 2$. However, one can check that for any (t_1', t_2', t_3') within the Hurwitz orbit of (t_1, t_2, t_3), the prefix (t_1', t_2') is a shortest factorization of $t_1' t_2'$. It is not clear what might replace Lemma 1.3 in a proof of Conjecture 6.3.

References

[Arm09]   ARMSTRONG, DREW. Generalized noncrossing partitions and combinatorics of Coxeter groups. *Mem. Amer. Math. Soc.* **202** (2009), no. 949, x+159 pp. ISBN: 978-0-8218-4490-8. MR2561274, Zbl 1191.05095, arXiv:math/0611106, doi: 10.1090/S0065-9266-09-00565-1.

[BaDSW14] BAUMEISTER, BARBARA; DYER, MATTHEW; STUMP, CHRISTIAN; WEGENER, PATRICK. A note on the transitive Hurwitz action on decompositions of parabolic Coxeter elements. *Proc. Amer. Math. Soc. Ser. B* **1** (2014), 149–154. MR3294251, Zbl 1343.20041, arXiv:1402.2500, doi: 10.1090/S2330-1511-2014-00017-1.

[BaGRW15] BAUMEISTER, BARBARA; GOBET, THOMAS; ROBERTS, KIERAN; WEGENER, PATRICK. On the Hurwitz action in finite Coxeter groups. Preprint, 2015. arXiv:1512.04764.

[Bes03]   BESSIS, DAVID. The dual braid monoid. *Ann. Sci. École Norm. Sup. (4)* **36** (2003), no. 5, 647–683. MR2032983, Zbl 1064.20039, arXiv:math/0101158, doi: 10.1016/j.ansens.2003.01.001.

[Bes15]   BESSIS, DAVID. Finite complex reflection arrangements are $K(\pi, 1)$. *Ann. of Math. (2)* **181** (2015), no. 3, 809–904. MR3296817, Zbl 06446405, arXiv:math/0610777, doi: 10.4007/annals.2015.181.3.1.

[BjB05]   BJÖRNER, ANDERS; BRENTI, FRANCESCO. Combinatorics of Coxeter groups. Graduate Texts in Mathematics, 231. *Springer, New York*, 2005. xiv+363 pp. ISBN: 978-3540-442387; 3-540-44238-3. MR2133266, Zbl 1110.05001, doi: 10.1007/3-540-27596-7.

[Bra01]   BRADY, THOMAS. A partial order on the symmetric group and new $K(\pi, 1)$'s for the braid groups. *Adv. Math.* **161** (2001), no. 1, 20–40. MR1857934, Zbl 1011.20040, doi: 10.1006/aima.2001.1986.

[BW02]   BRADY, THOMAS; WATT, COLUM. $K(\pi, 1)$'s for Artin groups of finite type. Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part I, (Haifa, 2000), *Geom. Dedicata* **94**, (2002), 225–250. MR1950880, Zbl 1053.20034, arXiv:math/0102078, doi: 10.1023/A:1020902610809.

[Car72]   CARTER, R. W. Conjugacy classes in the Weyl group. *Compositio Math.* **25** (1972), 1–59. MR0318337, Zbl 0254.17005.

[DCP08]   DE CONCINI, C.; PROCESI, C. The zonotope of a root system. *Transform. Groups* **13** (2008), no. 3–4, 507–526. MR2452603, Zbl 1170.52002.

[DPR13]   DOUGLASS, J. MATTHEW; PFEIFFER, GÖTZ; RÖHRLE, GERHARD. On reflection subgroups of finite Coxeter groups. *Comm. Algebra* **41** (2013), no. 7, 2574–2592. MR3169410, Zbl 1282.20040, arXiv:1101.5893, doi: 10.1080/00927872.2012.661005.

[Fie05]   FIEDLER, MIROSLAV. Matrices and graphs in Euclidean geometry. *Electron. J. Linear Algebra* **14** (2005), 51–58. MR2202433, Zbl 1092.05044, doi: 10.13001/1081-3810.1177.

[Hum90]   HUMPHREYS, JAMES E. Reflection groups and Coxeter groups. Cambridge Studies in Advanced Mathematics, 29. *Cambridge University Press, Cambridge*, 1990. xii+204 pp. ISBN: 0-521-37510-X. MR1066460, Zbl 0725.20028, doi: 10.1017/CBO9780511623646.

[IS10]   IGUSA, KIYOSHI; SCHIFFLER, RALF. Exceptional sequences and clusters. *J. Algebra* **323** (2010), no. 8, 2183–2202. MR2596373, Zbl 1239.16019, arXiv:0901.2590, doi: 10.1016/j.jalgebra.2010.02.003.

[LaZ04]   LANDO, SERGEI K.; ZVONKIN, ALEXANDER K. Graphs on surfaces and their applications. Encyclopaedia of Mathematical Sciences, 141. Low-Dimensional Topology, II. *Springer-Verlag, Berlin*, 2004. xvi+455 pp. ISBN: 3-540-00203-0. MR2036721, Zbl 1040.05001, doi: 10.1007/978-3-540-38361-1.

[LeRS14]   LEWIS, J. B.; REINER, V.; STANTON, D. Reflection factorizations of Singer cycles. *J. Algebraic Combin.* **40** (2014), no. 3, 663–691. MR3265228, Zbl 1308.05108, arXiv:1308.1468, doi: 10.1007/s10801-014-0502-0.

[Oxl92]   OXLEY, JAMES G. Matroid theory. Oxford Science Publications. *The Clarendon Press, Oxford University Press, New York*, 1992. xii+532 pp. ISBN: 0-19-853563-5. MR1207587, Zbl 0784.05002.

[Rip10]   RIPOLL, VIVIEN. Orbites d'Hurwitz des factorisations primitives d'un élément de Coxeter. *J. Algebra* **323** (2010), no. 5, 1432–1453. MR2584963, Zbl 1223.20031, arXiv:0903.3604, doi: 10.1016/j.jalgebra.2009.12.010.

[Ste07]   STEMBRIDGE, JOHN R. Irreducible circuits and Coxeter arrangements. *J. Combin. Theory Ser. A* **114** (2007), no. 7, 1220–1237. MR2353120, Zbl 1124.52013, doi: 10.1016/j.jcta.2007.01.005.

[Zas82]   ZASLAVSKY, THOMAS. Signed graphs. *Discrete Appl. Math.* **4** (1982), no. 1, 47–74. MR0676405, Zbl 0476.05080, doi: 10.1016/0166-218X(82)90033-6.

(Joel Brewster Lewis) SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455, USA
jblewis@umn.edu

(Victor Reiner) SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455, USA
reiner@math.umn.edu

This paper is available via `http://nyjm.albany.edu/j/2016/22-63.html`.